J. Math. Soc. Japan
Vol. 69, No. 1 (2017) pp. 1–24
doi: 10.2969/jmsj/06910001

Darboux curves on surfaces I

By Ronaldo Garcia, Rémi Langevin and Paweł Walczak

(Received Apr. 11, 2014)
(Revised Mar. 20, 2015)

Abstract. In 1872, G. Darboux defined a family of curves on surfaces of $\mathbb{R}^3$ which are preserved by the action of the Möbius group and share many properties with geodesics. Here, we characterize these curves under the view point of Lorentz geometry and prove that they are geodesics in a 3-dimensional sub-variety of a quadric Λ^4 contained in the 5-dimensional Lorentz space $\mathbb{R}^5_1$ naturally associated to the surface. We construct a new conformal object: the Darboux plane-field $\mathcal{D}$ and give a condition depending on the conformal principal curvatures of the surface which guarantees its integrability. We show that $\mathcal{D}$ is integrable when the surface is a special canal.

1. Introduction.

Here, we study a family of curves on a surface called *Darboux curves*[1]. They are characterized by a relation between the geometry of the curve and of the surface: the osculating sphere to a Darboux curve is always tangent to the surface which contains the curve. This definition means that Darboux curves are conformally defined. The study of these curves started with Darboux at the end of 19^{th} century, and was continued until now. See [**Da1**], [**En**], [**Ha**], [**Pe**], [**Ri**], [**Sa1**], [**Sa2**], [**Se**].

Our main tool is the space of oriented spheres. It is a quadric Λ^4 contained in the 5-dimensional Lorentz space $\mathbb{R}^5_1$. Given a surface $M \subset \mathbb{S}^3$ (or in $\mathbb{R}^3$), the spheres tangent to M form a 3-dimensional variety (a manifold with singularities) contained in Λ^4. These constructions have already been used by Lie, Darboux and Klein (see [**Li**], [**Da2**] and [**Kle**]). Here, we consider only spheres which have contact of saddle type with M, and denote the set of all such spheres by $V(M)$. It is, except above umbilical points, an interval bundle with light-like fibers. We show here that the osculating spheres along a Darboux curve form a geodesic in $V(M)$ endowed with the semi-Riemannian metric induced from the Lorentz quadratic form in $\mathbb{R}^5_1$. From the two geodesics passing through a point of $V(M)$, we construct an intriguing plane-field $\mathcal{D}$, integrable for a few classical families of surfaces.

2. Preliminaries.

2.1. The set of spheres in $\mathbb{S}^3$.

In the Lorentz space $\mathbb{R}^5_1$ endowed with the Lorentz quadratic form $\mathcal{L}$. We will use the terminology of relativity theory to qualify vectors of $\mathbb{R}^5_1$: v is space-like if $\mathcal{L}(v) > 0$,

---

2010 *Mathematics Subject Classification.* Primary 53A30; Secondary 53C12, 53C50, 57R30.
Key Words and Phrases. Darboux curves, conformal geometry, space of spheres.

[1]Steven Verpoort called the attention of the authors on the fact that the term “Darboux curve” has different meanings in geometry. Note that some authors call our “Darboux curves” D-curves.

time-like if $\mathcal{L}(v) < 0$ and light-like if $\mathcal{L}(v) = 0$. A subspace will be space-like if it contains only space-like non-zero vectors, time-like if it contains some time-like non-zero vectors, and light-like if it contains some light-like non-zero vectors but no time-like non-zero vectors.

The points of the de Sitter quadric $\Lambda^4 \subset \mathbb{R}^5_1$ given by $\mathcal{L} = 1$ represent oriented spheres of $\mathbb{S}^3$ (see [**H-J**] and [**La-Wa**]).

The points of $\mathbb{S}^3$ are represented by the intersection of the light cone $\mathcal{L}ight$ given by $\mathcal{L} = 0$ and the hyperplane $\{x_0 = 1\}$.

It is convenient to have a formula giving the point $\sigma \in \Lambda^4$ in terms of the Riemannian geometry of the corresponding oriented sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ and a point m on it. For that we need the unit vector n tangent to $\mathbb{S}^3 \subset \mathcal{L}ight$ and normal to Σ at m and the geodesic curvature of Σ, that is the geodesic curvature k_g of any geodesic circle on Σ.

Proposition 1. *The point $\sigma \in \Lambda^4$ corresponding to the sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ is given by*

$$\sigma = k_g m + n. \tag{1}$$

Remark. The same formula holds for spheres in the Euclidean space $\mathbb{E}^3$ seen as a section of the light cone by an affine hyperplane parallel to an hyperplane tangent to the light cone, k_g is now the curvature of a geodesic circle of Σ.

The proof of Proposition 1 can be found in [**H-J**] and [**La-Oh**].

2.2. Curves in Λ^4 and canal surfaces.

A differentiable curve $\gamma = \gamma(t)$ is called *space-like* if at each point its tangent vector $\dot{\gamma}(t)$ is space-like, that is $\mathcal{L}(\dot{\gamma}) > 0$; it is called *time-like* if at each point its tangent vector $\dot{\gamma}(t)$ is time-like, that is $\mathcal{L}(\dot{\gamma}) < 0$ (see Figure 1). When the curve is time-like, the corresponding spheres are nested. When the curve is space-like, the family of spheres Σ_t associated to the points $\gamma(t)$ defines an envelope which is a surface, union of circles called the *characteristic circles* of the surface; notice that the surface may have singular cuspidal edges. From now on we will suppose that the space-like curve γ is parameterized by arc-length, that is $|\mathcal{L}(\dot{\gamma})| = 1$. There is one characteristic circle Γ_{Car} on each sphere

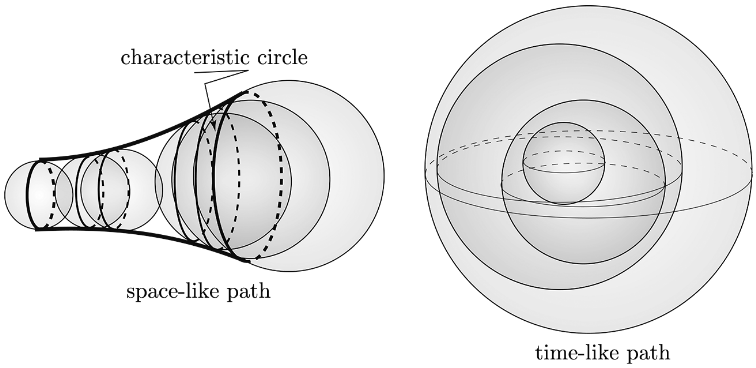

Figure 1. Spheres corresponding to a space-like and to a time-like paths.

Σ_t of the family and it is the intersection of Σ_t and the sphere $\dot{\Sigma}_t = [\mathrm{Span}(\dot{\gamma}(t))]^\perp \cap \mathbb{S}^3$.

We call such an envelope of a one-parameter family of spheres a *canal surface.* The sphere $\Sigma(t)$ is tangent to the canal surface along the characteristic circle except at maybe two points where the surface is singular. As any curve on a sphere is a line of principal curvature, all the characteristic circles are also lines of principal curvature on the canal surface.

We say that the curve $\gamma(t)$, the one-parameter family of spheres $\Sigma(t)$ and the canal surface, envelope of the family, *correspond.* Reciprocally, when a one-parameter family of spheres admits an envelope, the corresponding curve is space-like, as the existence of an envelope forces nearby spheres to intersect. One can refer, for example, to [**Ba-La-Wa**] for some proofs concerning canal surfaces.

An extra condition is necessary to guarantee that the envelope is immersed. The *geodesic curvature vector* $\vec{k_g} = \ddot{\gamma}(t) + \gamma(t)$ should be time-like. We call the envelope of the spheres Σ_t corresponding to the points of such a curve γ a *regular canal surface.*

LEMMA 2. *Let $\gamma = \{\gamma(t)\} \subset \Lambda^4$ be a space-like curve which has space-like geodesic curvature vector. Then the canal surface, envelope of the spheres $\Sigma(t) = (\gamma(t))^\perp \cap \mathbb{S}^3$, has two cuspidal edges. The two points of the cuspidal edges belonging to the characteristic circle $Car(t)$ of the canal are $\{m_1(t) \cup m_2(t)\} = [\mathbb{R}\gamma(t) \oplus \mathbb{R}\dot{\gamma}(t) \oplus \mathbb{R}\ddot{\gamma}(t)]^\perp \cap \mathbb{S}^3$; the characteristic circles are tangent to the two cuspidal edges.*

Here, let us just prove that the characteristic circles are tangent to the curves $m_i(t)$. We need to prove that $\dot{m}_i(t)$ is tangent to $Car(t)$, that is tangent to $\Sigma(t)$ and to $\dot{\Sigma}(t)$. This is the case if $\mathcal{L}(\dot{m}_i(t), \gamma(t)) = \mathcal{L}(\dot{m}_i(t), \dot{\gamma}(t)) = 0$. As the point $m_i(t)$ belong to the three spheres $\Sigma(t), \dot{\Sigma}(t)$ and $\ddot{\Sigma}(t)$, we know that $\mathcal{L}(m_i(t), \gamma(t)) = \mathcal{L}(m_i(t), \dot{\gamma}(t)) = \mathcal{L}(m_i(t), \ddot{\gamma}(t)) = 0$. Differentiating the first two Lorentz scalar products, and using the previous equalities, we get the desired relations $\mathcal{L}(\dot{m}_i(t), \gamma(t)) = \mathcal{L}(\dot{m}_i(t), \dot{\gamma}(t)) = 0$.

Notice that when a regular point μ of the envelope tends to a point $m_i(t)$ of the singular locus, the tangent plane at μ tends to the tangent plane at $m_i(t)$ to the sphere $\Sigma(t)$.

When the geodesic curvature vector $\vec{k_g}$ is light-like, we call the curve $\gamma \subset \Lambda^4$ a *drill.* Then generically it is the family of osculating spheres to a curve $C \subset \mathbb{S}^3$ (see [**Tho**] and [**La-So**]). In that case, the characteristic circles coincide with the osculating circles of the curve.

3. Spheres and surfaces.

3.1. Local conformal invariants of surfaces.

Assume that S is a surface which is *umbilic free*, that is, that the principal curvatures $k_1(x) \geq k_2(x)$ of S are different at any point x of S. We will keep the convention $k_1 \geq k_2$ throughout the paper and refer to the principal direction associated to k_1 as *first* principal direction. Let X_1 and X_2 be unit vector fields tangent to the curvature lines corresponding to, respectively, k_1 and k_2. Put $\mu = (k_1 - k_2)/2$. Since more than 100 years, it is known ([**Tr**], see also [**CSW**]) that the vector fields $\xi_i = X_i/\mu$ and the coefficients θ_i $(i = 1, 2)$ in

$$[\xi_1, \xi_2] = -\frac{1}{2}(\theta_2\xi_1 + \theta_1\xi_2) \tag{2}$$

are invariant under arbitrary (orientation preserving) conformal transformation of $\mathbb{R}^3$. (In fact, they are invariant under arbitrary conformal change of the Riemannian metric on the ambient space. This follows form the known (see [**La-Wa**, page 142], for instance) relation $\tilde{A} = e^{-\phi}(A - g(\nabla\phi, N) \times \mathrm{Id})$ between the Weingarten operators of a surface S with respect to conformally equivalent Riemannian metrics $\tilde{g} = e^{2\phi}g$ on the ambient space; here $\nabla\phi$ and N denote, respectively, the g-gradient of ϕ and the g-unit normal to S. Elementary calculation involving Codazzi equations shows that

$$\theta_1 = \frac{1}{\mu^2} \cdot X_1(k_1) \quad \text{and} \quad \theta_2 = \frac{1}{\mu^2} \cdot X_2(k_2). \tag{3}$$

The quantities θ_i $(i = 1, 2)$ are called *conformal principal curvatures* of S.

Let ω_1, ω_2 be the 1-forms dual to the vectors ξ_1, ξ_2.

Lemma 3. *The equalities*

$$d\omega_1 = \frac{1}{2}\theta_2\omega_1 \wedge \omega_2, \quad d\omega_2 = \frac{1}{2}\theta_1\omega_1 \wedge \omega_2 \tag{4}$$

hold.

Proof. Standard calculation. For example

$$\begin{aligned} d\omega_1(\xi_1, \xi_2) &= \xi_1(\omega_1(\xi_2)) - \xi_2(\omega_1(\xi_1)) - \omega_1([\xi_1, \xi_2]) \\ &= \frac{1}{2}\omega_1(\theta_2\xi_1 + \theta_1\xi_2) = \frac{1}{2}\theta_2. \end{aligned}$$

□

3.2. Spheres tangent to a surface.

From Proposition 1, we see that the points in Λ^4 corresponding to a pencil of spheres tangent to a surface M at a point m form two parallel light-rays (one for each choice of normal vector n). Let us now choose the normal vector n, and consider the spheres $\Sigma_{m,k}$, $k \in \mathbb{R}$, associated to the points $\sigma_{m,k} = km + n$ (see Formula 1). All the spheres $\Sigma_{m,k}$, but for at most two, have either a center contact or a saddle contact with M (see Figures 2 and 3).

When m is not an umbilic, the exceptional spheres have curvature k_g equal to one of the principal curvatures k_1 and k_2 of M at m; they are called osculating spheres.

We need to consider the germ at m of the intersection of a sphere tangent to M at m and M, that we will call *local intersection.*

When $k \notin [k_1, k_2]$, the local intersection of $\Sigma_{m,k}$ and M near the origin reduces to a point, the origin (*center contact*).

When $k \in]k_1, k_2[$, the local intersection of $\Sigma_{m,k}$ and M near the origin consists of the germs of two curves intersecting transversely at m (*saddle contact*).

When $k = k_1$ or $k = k_2$, the local intersection of $\Sigma_{m,k}$ and M near the origin is the germ of a curve singular at m; the singularity is in generically of cuspidal type.

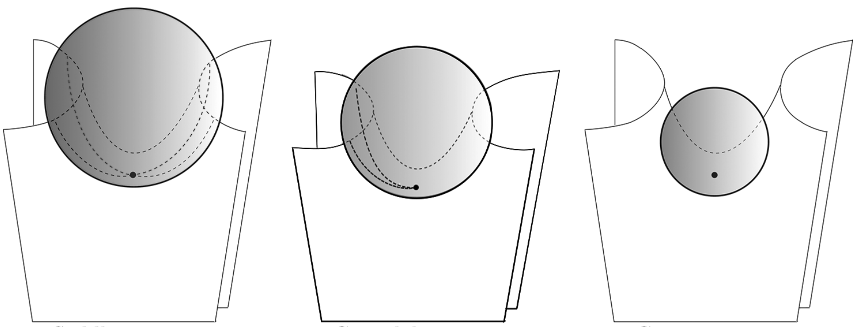

Figure 2. Possible contacts of a sphere and a surface.

The points of Λ^4 corresponding to osculating spheres associated to k_1 form a surface $\mathcal{O}_1$. We complete the surface with the osculating spheres at umbilics of M. In the same way we get a second surface $\mathcal{O}_2$ which intersects $\mathcal{O}_1$ only at the osculating spheres at umbilics.

When $k \in]k_1, k_2[$ tends to k_1, the two tangents at the common point to the intersection $\Sigma_k \cap M$ tend to the principal direction associated to k_1. This shows that the principal directions are conformally defined.

Let us denote by $\mathcal{F}_0$ the foliation by lines of curvature associated to k_1. It is also a conformal object.

We will use the direction tangent to the leaves of the foliation $\mathcal{F}_0$ as reference for angles in the set of lines in the planes tangent to M.

We can now define a one-parameter family of foliations $\mathcal{F}_\alpha$, $\alpha \in [-\pi/2, \pi/2]$ (or $\alpha \in \mathbb{P}_1$). $\mathcal{F}_\alpha$ is the foliation the leaves of which make a constant angle α with the leaves of $\mathcal{F}_0$. In particular, the foliation associated to k_2 is $\mathcal{F}_{\pi/2} = \mathcal{F}_{-\pi/2}$. We will study all these foliations in another article [**GLW2**].

Definition 4. The 3-manifold $V(M) \subset \Lambda^4$ is the set of spheres having a saddle contact with the surface $M \subset \mathbb{S}^3$.

It is a submanifold of Λ^4 with boundary which is the union of the two surfaces $\mathcal{O}_1$ and $\mathcal{O}_2$. Away from the umbilical points of M, it fibers over M with fibers being intervals of the light-rays of spheres tangent to M bounded by the two osculating spheres $o_1(m)$ and $o_2(m)$, $m \in M$. When M has umbilical points, the two folds $\mathcal{O}_1$ and $\mathcal{O}_2$ meet at osculating spheres at umbilical points of M.

The points σ_α of a fiber I_m can be expressed as linear combinations of the two osculating spheres $o_1(m)$ and $o_2(m)$: $\sigma_\alpha = (\cos^2\alpha)\, o_1(m) + (\sin^2\alpha)\, o_2(m)$.

Euler's computation of the normal curvatures $k_n = k_n(m, \ell)$ of sections of a surface by normal planes intersecting the tangent plane T_mM along a line ℓ making angle α with the first principal direction $\mathcal{F}_0$ implies

Proposition 5. *Let $k_n = k_1\cos^2\alpha + k_2\sin^2\alpha$. Then the angle of the two directions $\ell_{\pm\alpha}$ tangent at m to $\Sigma_{m,k} \cap M$ with the principal direction corresponding to k_1 is $\pm\alpha$.*

When the reference to the angle α is useful, we will also use the notation $k_n = k_n(m, \alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$ for the normal curvature and $\Sigma_{m,\alpha}$ instead of $\Sigma_{m,k}$ for the sphere of curvature $k_n(m, \alpha)$ tangent to M at the point m.

The interval bundle $V(M)$ is closely related with the projective tangent bundle $\mathbb{P}T_1(M)$. Indeed, let us chose as origin on a fiber of $\mathbb{P}T_1(M)$ the direction of the first principal direction. The "antipodal" direction of the first principal direction on the fiber P_m of $\mathbb{P}T_1(M)$ above m is the second principal direction. Sending the direction making an angle α with the first principal direction and the one making an angle $-\alpha$ to the point $\sigma_\alpha \in I_m$ "folds" the circle P_m on the interval I_m.

It is proved in [**La-Oh**] that the points of Λ^4 corresponding to the osculating spheres associated to k_1 along a line of curvature associated to k_1, that is along a leaf of $\mathcal{F}_0$, form a light-like curve. Therefore $\mathcal{F}_0$ lifts to a foliation $\tilde{\mathcal{F}}_0$ of the surface $\mathcal{O}_1$ by light-like curves.

Notice that a one-parameter family of spheres tangent to a curve can never be time-like, as a time-like curve gives locally nested spheres which cannot be tangent to a curve. Therefore a curve in $\mathcal{O}_1$ formed of osculating spheres to M along a curve C have a space-like tangent vector, except when the curve C is tangent to $\mathcal{F}_0$. This implies that, at a regular point of $\mathcal{O}_1$, the restriction to $T\mathcal{O}_1$ of the Lorentz metric is degenerated. The kernel at a point $\sigma \in \mathcal{O}_1$, a sphere tangent to M at m, is the light direction $Span(m)$.

DEFINITION 6. A point $m \in M$ is a *ridge point* for k_1 if m is a critical point for the restriction of k_1 to the line of curvature for k_1 through m.

A classical reference about ridge points is [**Po**].

The lift of a ridge point to $\mathcal{O}_1$ is in general a cusp of a leaf L of $\tilde{\mathcal{F}}_0$. To see that, let us parameterize a leaf L of $\mathcal{F}_0$ near a ridge point using a regular parameter on the corresponding line of curvature; as, at a ridge point, $k_1' = 0$, we get at the lift of the ridge point, $\sigma' = 0$.

In general, ridge points form curves in M that we call just *ridges*. The lift to $\mathcal{O}_1$ of a ridge for k_1 is in general a cuspidal edge of the surface $\mathcal{O}_1$.

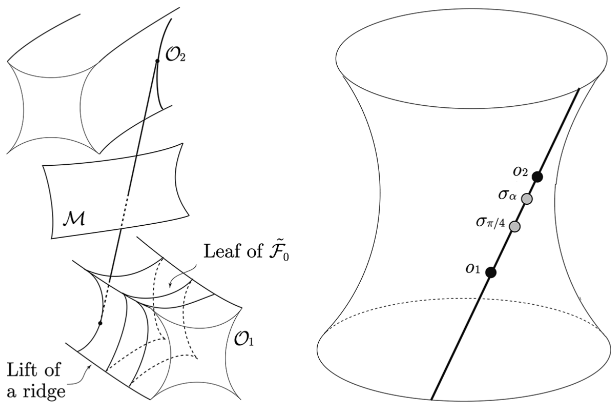

Figure 3. A leaf of the foliation $\tilde{\mathcal{F}}_0$ and a light segment of $V(M)$ between two osculating spheres.

The equations of the two families of ridges are $\theta_1 = 0$ and $\theta_2 = 0$; let us define $\zeta_1 = \xi_1(\theta_1)(p_0)$ and $\zeta_2 = \xi_2(\theta_2)(p_0)$.

THEOREM 7. *Let p_0 be a point of a ridge of M corresponding to the principal foliation $\mathcal{F}_0$ such that $\zeta_1(p_0) \neq 0$. Then the ridge R containing p_0 is locally a regular curve transverse to $\mathcal{F}_0$; the boundary component of $V(M)$ corresponding to $\alpha = 0$ has a cuspidal edge along $\pi^{-1}(R)$. The same is true for ridges associated to the principal foliation $\mathcal{F}_{\pi/2}$.*

REMARK. The generic singularities of $\partial V(M) \subset \Lambda^4$ need not a priori to be the same as the generic singularities of the focal set $Focal(M) \subset \mathbb{S}^3$ or $Focal(M) \subset \mathbb{R}^3$. The projection $\partial V(M) \to Focal(M)$ may a priori "unfold" some singularities.

PROOF. As $\theta_1(p_0) = 0$ and $\zeta_1(p_0) \neq 0$, it follows from the implicit function theorem that the ridge is transverse to the corresponding principal foliation. The ridge can be parameterized, in a local principal chart (u, v) of M, by $R_1(v) = (U(v), v)$ defined implicitly by $\theta_1(u, v) = 0$.

Let $\varphi(u, v, \alpha)$, $0 \leq \alpha \leq \pi/2 \in V(m)$ be the point of Λ^4 corresponding to the sphere to M at $m(u, v)$ and of curvature $k_n = k_1(u, v)\cos^2\alpha + k_2(u, v)\sin^2\alpha$. The boundary of $V(M)$ is parameterized by $\varphi_1(u, v) = \varphi(u, v, 0) = k_1(u, v)m(u, v) + \vec{n}(u, v)$ and $\varphi_2(u, v) = \varphi(u, v, \pi/2) = k_2(u, v)m(u, v) + \vec{n}(u, v)$. The maps φ_1 and φ_2 have rank 1 at the corresponding ridge points. Therefore we recognize cuspidal edges on the boundary of $V(M)$ since $(\varphi_1)_u(R_1(v)) = 0$, $(\varphi_1)_v(R_1(v)) = ((k_1 - k_2)m_v + (k_1)_v m)(R_1(v)) \neq 0$ and $(\varphi_1)_{uu}(R_1(v)) = (k_1)_{uu}(R_1(v))m(R_1(v)) \neq 0$. $\square$

REMARK. For an open and dense set of compact surfaces with the C^r-topology of Whitney, $r \geq 4$, the set of ridge points is the union of regular curves outside the umbilical points. See [**Br-Gi-Ta**].

From the proof above it follows that ridges are conformally defined. The correspondence between singular points of $\mathcal{O}_i$ and ridges can be seen directly observing that the osculating spheres along their line of principal curvature are stationary at a ridge point.

PROPOSITION 8. *At regular points, $V(M)$ inherits from Λ^4 a semi-Riemannian metric. In other words its tangent 3-space at any regular point is light-like.*

PROOF. Let us prove that, at each point $\sigma \in V(M)$, $T_\sigma V(M)$ is contained in $T_m\mathcal{L}ight = (\mathbb{R}m)^\perp$. For that, consider two curves in $V(M)$ of origin σ which project on two lines of curvature on M orthogonal at m. Suppose that their respective arc-lengths are u and v. Then, differentiating $\sigma = km + n$ respectively with respect to u and v, we get

$$\sigma_u = (km + n)_u = k_u m + kX_1 - k_1 X_1$$

$$\sigma_v = (km + n)_v = k_v m + kX_2 - k_2 X_2$$

The vectors X_1 and X_2 are tangent to $M \subset \mathbb{S}^3 \subset \mathcal{L}ight$, they are therefore contained in $T_m\mathcal{L}ight = (\mathbb{R}m)^\perp$. The restriction of $\mathcal{L}$ to $T_m\mathcal{L}ight$ is degenerated. The restriction of

$\mathcal{L}$ to any subspace of $T_m\mathcal{L}ight$ containing the light direction $\mathbb{R}m$, as $T_\sigma V(M)$, is therefore also degenerated.

Then the direction $\mathbb{R}m, m = \pi(\sigma)$, of the light ray through the point σ is the kernel direction of the restriction of the Lorentz metric to $T_\sigma V(M)$. The direction normal to the tangent space $T_\sigma V(M)$ is also $(\mathbb{R}m)$. □

3.3. Spheres tangent to a surface along a curve.

Let us consider now an arbitrary curve $C \subset M$.

The restriction of $V(M)$ to C forms a two-dimensional surface in Λ^4 which is a light-ray interval bundle $V(C)$ over C out of the umbilical points of M which may belong to C. From $V(M)$, we get on $V(C)$ the induced semi-Riemannian metric.

In this text we will use the symbol $'$ for derivatives with respect to parameterizations of curves contained in $\mathbb{R}^3$ or $\mathbb{S}^3$, (often the parameter is the arc length), and $^\bullet$ for derivatives with respect to a parameterization of a curve in Λ^4 (often the parameter is an arc length, but now for the metric induced by the Lorentz form).

DEFINITION 9. We denote by $\Sigma_{m,\ell}$ the sphere tangent to the surface M at the point m such than one branch of the intersection $\Sigma_{m,\ell} \cap M$ is tangent to the direction ℓ (*canonical* sphere associated to the direction ℓ). We will also use the notation $\Sigma_{m,v}$ when the direction ℓ is generated by a non-zero vector v.

REMARK. We can rephrase Proposition 5, saying that, if ℓ_α is the line of T_mM making at m the angle α with $\mathcal{F}_0$, then $\Sigma_{m,k} = \Sigma_{m,\ell_\alpha}$ when $k = k_1 \cos^2\alpha + k_2 \sin^2\alpha$.

Given a curve $C \subset M$ such that the tangent vector to C at $c(t)$ is contained in ℓ, $\Sigma_{c(t),c'(t)}$ is, among the spheres tangent to M at m, the one which have the best contact at m with the curve C.

At each point $m \in C$ such that the tangent to C is not a principal direction, there is a unique sphere $\Sigma_C(m)$ such that one branch of the local intersection $\Sigma_C(m) \cap M$ is tangent to C at m. If the tangent to C at m is a principal direction, we take $\Sigma_C(m) = o_i(m)$, the osculating sphere of the surface corresponding to the principal direction. We call the family of spheres $\Sigma_C(m)$ the *canonical family along* C, and denote it by $CanSec(C)$; the envelope $CanCan(C)$ of the spheres $\Sigma_C(m) \in CanSec(C)$ is called *the canonical canal* corresponding to $C \subset M$.

The point $CanSec(C)(m) \in V(C)$ of the *canonical section* of $V(C)$ corresponds to the sphere $\Sigma_C(m)$.

PROPOSITION 10. *The canonical section $CanSec(C)$ of $V(C)$ satisfies the following properties.*

i) *The geodesic curvature vector of the curve $CanSec(C) \subset V(M)$ satisfies $\overrightarrow{k_g} \in T_mV(M)$, and therefore is (space-or-light)-like.*
ii) *The section $CanSec(C)$ is a geodesic in $V(C)$.*
iii) *$CanSec(C)(m) = k_n \cdot m + n$, where k_n is the normal curvature of M at $m \in M$ in the direction of the vector $c'(m)$; n is as usual the unit vector normal at m to $M \subset \mathbb{S}^3$, and tangent to $\mathbb{S}^3 \subset \mathcal{L}ight \subset \mathbb{R}^5_1$.*

In the proof, we will need a lemma from [**La-So**].

LEMMA 11. *A curve $\Gamma = \{\gamma(t)\} \subset \mathbb{S}^3$ has contact of order $\geq k$ with a sphere Σ corresponding to σ if and only if*

$$\sigma \perp \operatorname{span}(\gamma(t), \gamma'(t), \dots, \gamma^{(k)}(t))$$

PROOF. The sphere Σ is the zero level of the restriction to $\mathbb{S}^3$ of the function $f(x) = \mathcal{L}(x, \sigma)$. Then the contact of $\Gamma(t)$ and Σ has the same order as the zero of $(f \circ \gamma)(t) = \mathcal{L}(\gamma(t), \sigma)$. □

PROOF OF PROPOSITION 10. The condition defining the sphere $\Sigma_C(m)$ implies that the order of contact of C and $\Sigma_C(m)$ is at least 2, one more that the order of contact of $\Sigma_C(m) \cap M$ and C, which is at least one. To verify this property, notice that, in terms of the arc-length of a branch of $\Sigma_C(m) \cap M$, or equivalently of the arc-length s on C, the angle of $\Sigma(m)$ with M along $\Sigma_C(m) \cap M$ is of the order of s, if not smaller. The distance to C is of the order of s^2, if not smaller. Therefore the order of the distance of a point of C to $\Sigma_C(m)$ is of order s^3 if not smaller. This means that C and the sphere $\Sigma_C(m)$ have contact of order at least 2 at m.

As the sphere Σ_C has contact order at least 2 with the curve C, $CanSec(C)(m)$ is orthogonal to m, $\dot{m}$ and $\ddot{m}$.

We will now denote by σ the section $CanSec(C)$ parameterized by arc-length in Λ^4.

Differentiating the relation $\mathcal{L}(\sigma(m), \dot{m}) = 0$ and using the relation $\mathcal{L}(\sigma(m), \ddot{m}) = 0$ we get $\mathcal{L}(\dot{\sigma}(m), \dot{m}) = 0$. Differentiating the relation $\mathcal{L}(\dot{\sigma}(m), m) = 0$, and using the relation $\mathcal{L}(\dot{\sigma}(m), \dot{m}) = 0$, we get $\mathcal{L}(\ddot{\sigma}(m), m) = 0$.

Recall that (see Subsection 2.2) the geodesic curvature vector $\vec{k_g}(m)$ is the orthogonal projection of $\ddot{\sigma}(m)$ on $T_{\sigma(m)}\Lambda^4$. One has $\vec{k_g}(m) = \ddot{\sigma}(m) + \sigma(m)$. It is therefore orthogonal to the line $\mathbb{R}m$, and therefore belongs to $(\mathbb{R}m)^{\perp} = T_{\sigma(m)}V(M)$, proving item i) of the proposition. As the tangent space to $V(M)$ is everywhere light-like, any direction different from the light-direction is space-like.

The geodesic curvature vector $\vec{k_g}$ is orthogonal to $\mathbb{R}m$ and to $\dot{\sigma}(m)$, it is therefore orthogonal to $T_{\sigma(m)}V(C) = \mathbb{R}m \oplus \mathbb{R}\,\dot{\sigma}(m)$, proving that $CanSec(C)$ is a geodesic in $V(C)$, proving item ii).

Item iii) is just Formula (1) of Proposition 1, as the sphere such that one branch of the local intersection with M at m is tangent to C at m has curvature k_n. □

In order to compute explicitly the vector $\vec{k_g}$ and to prove Proposition 12 below, we will use the Darboux frame T, N_1, n, m of the curve $C \subset M \subset \mathbb{S}^3 \subset \mathcal{L}ight$, where T is the unit tangent vector to C, N_1 is the unit vector tangent to M normal to C compatible with the orientation of M and n the unit vector normal to M and tangent to $\mathbb{S}^3$.

PROPOSITION 12. *The section $CanSec(C)$ is the shortest of the sections of $V(C)$.*

PROOF. Using again the formula $\sigma = k_n\, m + n$, where k_n is the normal curvature of M in the direction tangent to C at m, we see that, when σ is the section $CanSec(C)$

$$|\sigma'| = |k_n' \, m + k_n \, m' + n'| = |k_n' \, m - \tau_g N_1| = |\tau_g|, \tag{5}$$

where the *geodesic torsion* τ_g is defined by the following formula

$$\tau_g = -\langle \nabla_T \, n \mid N_1 \rangle. \tag{6}$$

Observe that our formula (5) gives an interpretation of the geodesic torsion of a curve $C \subset M$ as the rotation speed of the canonical family of spheres tangent to M along C.

In order to compute the geodesic curvature vector of the canonical section $CanSec(C)$, we need to use its parameterization by arc-length in Λ^4. Then $\overset{\bullet}{\sigma} = \sigma'(1/\tau_g) = (1/\tau_g)(k_n' \, m - \tau_g N_1)$.

Differentiating once more, we get

$$\overset{\bullet\bullet}{\sigma} = -\frac{\overset{\bullet}{\tau_g}}{\tau_g^3}(k_n' \, m - \tau_g N_1) + \frac{1}{\tau_g^2}[k_n'' m + k_n' T - \tau_g' N_1 - \tau_g(-k_g T + \tau_g n)].$$

Recall that k_g is the geodesic curvature of $C \subset M$, while $\vec{k_g}$ is the geodesic curvature vector of the curve $CanSec(C) \subset \Lambda^4$.

Simplifying, we get, as $\vec{k_g} = \overset{\bullet\bullet}{\sigma} + \sigma$,

$$\overset{\bullet\bullet}{\sigma} = \phi m + \frac{1}{\tau_g^2}[(k_n' + \tau_g k_g)T - \tau_g^2 n]$$

and

$$\vec{k_g} = \psi m + \frac{1}{\tau_g^2}(k_n' + \tau_g k_g)T \tag{7}$$

where ϕ and ψ are some real functions.

As the formula shows that the vector $\vec{k_g}$ is orthogonal to m and to $\overset{\bullet}{\sigma} = (1/\tau_g)(k_n' \, m - \tau_g N_1)$, we verify again that the curve $CanSec(C)$ is a geodesic on $V(C)$.

For another family of spheres tangent to M along C, in particular for another section of $V(C)$, where $\sigma = k \, m + n$, we see that spheres tangent to a surface along a curve form a space-like curve in Λ^4; explicitly we get

$$|\overset{\bullet}{\sigma}| = \left| \frac{1}{\tau_g}[(k - k_n)' m \, + \, (k - k_n)T \, + \, \tau_g N_1] \right|. \tag{8}$$

Since m, T and N_1 are mutually orthogonal in $\mathbb{R}^5_1$, this proves the fact that the section $CanSec(C)$ has minimal length among sections. Formula (8) shows also that no other section of $V(C)$ is of critical length. $\square$

Remark. $\tau_g ds$ is the differential of the rotation of the sphere $\Sigma_{c(t),\overset{\bullet}{c}(t)}$ (see Definition 9) along the curve C.

In fact, the sphere $\Sigma_{c(t),\overset{\bullet}{c}(t)}$ has a tangent movement which is a rotation of "axis"

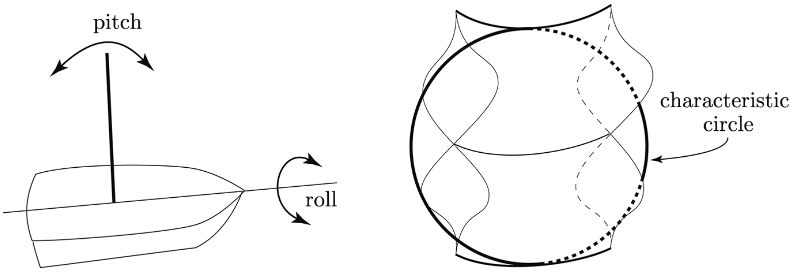

Figure 4. Pitch and roll (left). A canal with (locally) two cuspidal edges which is the envelope associated to a space-like curve with space-like geodesic curvature (right).

the characteristic circle of the family, which is tangent to C. We see that changing the sphere tangent to the surface along C changes the "pitch" term but not the "roll" term (see Figure 4). The characteristic circles of the canal corresponding to a non-canonical section are not in general tangent to C.

The characteristic circle of the envelope of the family $\Sigma_{c(t),c'(t)}$ is the intersection of $span(\sigma, \overset{\bullet}{\sigma})^{\perp}$ and the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$; as the vector T is orthogonal to m, n and N_1, and therefore to σ and σ' (and also $\overset{\bullet}{\sigma}$), the characteristic circle is tangent at m to C. Other families of spheres will have their characteristic circles transverse to C; in particular if the spheres are the planes tangent to M along C the characteristic lines of the family of planes are transverse to C when C is never tangent to an asymptotic direction. When C is tangent to a principal direction, the characteristic line of the family of tangent planes is orthogonal to C.

Proposition 13. *Let C be a curve contained in the surface $M \subset \mathbb{S}^3$. Suppose that it is nowhere tangent to a principal direction. Then the curve C is one fold of the singular locus of the canal surface $CanCan(C)$ defined by $CanSec(C)$.*

Proof. We already proved (Proposition 10) that the curve $CanSec(C)$ is space-like with a space-like geodesic acceleration at every point. The envelope $CanCan(C)$ is therefore a singular canal with (locally) two cuspidal edges (see Figure 4). The condition defining the canonical sphere $\Sigma_{c(t),c'(t)}$ and Remark 3.3 guarantees that the characteristic circle of the family $\Sigma_{c(t),c'(t)}$ is tangent to C, which is therefore a singular curve of the canal. □

4. Darboux curves: characterization in $V(M)$ and equations.

All the spheres containing the osculating circle to the curve C at $m \in C$ form a pencil. They all have contact order at least 2 (generically 2) with the curve. The osculating sphere of C belongs to this pencil and has contact of order at least 3 (generically 3) with the curve. Analytically, this means that if $C : J \ni t \mapsto c(t) \in \mathbb{R}^3$ is a parameterization of our curve, an $t_0 \in J$ and $a \in \mathbb{R}^3$, then the sphere of center a and radius $r = \|a - c(t_0)$ *osculates* C at $m = c(t_0)$ whenever the function $f_a : t \mapsto \|c(t) - a\|^2$ satisfies $f_a(t_0) = f'_a(t_0) = f''(t_0) = f'''(t_0) = 0$. The above shows that the osculating sphere is a conformally defined object and makes the following definition more natural.

DEFINITION 14. *Darboux curves* on a surface $M \subset \mathbb{R}^3$ or $M \subset \mathbb{S}^3$ are curves such that everywhere the osculating sphere is tangent to the surface. Darboux curves in $V(M)$ are families of osculating spheres to Darboux curves in M.

Observe that the definition of Darboux curves involves only spheres and contact order, so this notion belongs to conformal geometry.

We will now show that the osculating spheres along a Darboux curve form a geodesic in $V(M)$.

Recall that a *drill* (see Subsection 2.2) is a curve in the space of spheres the geodesic curvature of which is light-like at each point. Generically, points of a drill are osculating spheres to the curve $C \in \mathbb{S}^3$ defined by the geodesic acceleration vectors $\vec{k_g}$ of the drill.

We see that if we can find drills in $V(M)$ we find geodesics. In fact we find that way all geodesics which are not fibers of $V(M)$.

4.1. A geometric relation satisfied by Darboux curves.

THEOREM 15. *A curve $C \subset M$ is a Darboux curve if and only if the section $CanSec(C) \subset V(M)$ is a geodesic. This happens if and only if the curve $C \subset M$ satisfies the equation*

$$k_n' + \tau_g k_g = 0. \tag{9}$$

where we differentiate with respect to the arc-length of C.

We will prove the theorem after two remarks.

REMARK. The light rays of $V(M)$ are also geodesics. Segments of geodesics of $V(M)$ which are not tangent to light rays define an arc of curve $C \subset M$, and therefore, when $\vec{k_g}$ is light-like, C is a piece of Darboux curve on M.

REMARK. The Darboux curve in $V(M)$ associated to C is a particular case of $CanCan(C)$-curve when C is a Darboux curve of M; in that case, the only singular curve of the envelope of the spheres corresponding to $CanCan(C)$ is C (see [**Tho**], [**Ba-Wh**] and the proof below).

PROOF OF THEOREM 15. Let Σ be a sphere tangent to M at m (with saddle contact), and σ be the corresponding point of Λ^4. We have seen that the direction normal to $T_\sigma V(M)$ is $\mathbb{R}m$. Thomsen ([**Tho**], see also [**La-So**]) proved that the osculating spheres to a curve form a curve $\gamma \subset \Lambda^4$ with light-like geodesic acceleration vectors. Moreover, the geodesic acceleration at a point (a sphere osculating the curve at m) is on the light-ray $\mathbb{R}m$. Conversely, a curve γ in Λ^4 with geodesic acceleration light-like everywhere provides a curve $C \subset \mathbb{S}^3$ or $C \subset \mathbb{R}^3$ such that the osculating spheres correspond to the points of γ.

Therefore Darboux curves in $V(M)$ have light-like geodesic acceleration $\vec{k}_g$. At each $m \in C$, $\vec{k}_g$ is proportional to the light-like vector $m \in \mathbb{S}^3 \subset \mathcal{L}ight$. As the normal to $V(M)$ at a sphere σ tangent to M at m is the line $\sigma + \mathbb{R}m$, these curves are geodesics of $V(M)$. Reciprocally, a geodesic $\sigma(t)$ of $V(M)$ should have its geodesic acceleration (as curve of Λ^4) orthogonal to $T_\sigma V(M)$ everywhere.

This means that when σ is tangent to M at m, this geodesic acceleration is proportional to m.

As a Darboux curve in $V(M)$ is a particular case of the canonical section of spheres tangent to M along a curve, it satisfies the equation $\vec{k_g} = \psi m + (1/\tau_g^2)(k_n' + \tau_g k_g)T$ (compare with (7)). The T-component of $\vec{k_g}$ should be zero, therefore Darboux curves satisfy (9). □

A Darboux curve $D : t \mapsto d(t)$ have a better contact with the intersections $\Sigma_{d(t),d'(t)} \cap M$ of the spheres $\Sigma_{d(t),d'(t)}$ with M than "ordinary" curves $C : t \mapsto c(t)$ with the intersection $\Sigma_{c(t),c'(t)} \cap M$. In fact we have the following

Proposition 16. *A curve $C : t \mapsto c(t)$ is a Darboux curve if and only if one branch of the intersection $\Sigma_{c(t),\dot{c}(t)} \cap M$ has, at the point $c(t)$, the same geodesic curvature as the curve C.*

Proof. When the direction ℓ defined by $c'(t)$ is not a principal direction, there is a unique sphere tangent to M which has contact with C of best order, the others having contact of order one with C: it is the sphere $\Sigma_t = \Sigma_{c(t),c'(t)}$ (see Subsection 3.2, Proposition 5 and Definition 9).

When C is a Darboux curve, the sphere Σ_t osculates C and the contact order of Σ_t with C should be at least 3. This implies that C and one branch of $\Sigma_t \cap M$ should have the same geodesic curvature.

Otherwise, suppose that the curve is parameterized by arc-length. Let us show that the distance of the point $c(t+h) \in C$ and the sphere Σ_t, as a function of h defined in the neighborhood of 0, has a principal term equivalent $|h^3|$. Let $\bar{C} = \{\bar{c}(t+h)\}$ be the branch of $\Sigma_t \cap M$; the point $\bar{c}(t+h)$ is the (local) orthogonal projection of the point $c(t+h)$ on $\bar{C}$. As $\bar{C}$ has a contact of order 1 with C, the distance $d(c(t+h)), \bar{c}(t+h)$ has a principal term equivalent $|h^2|$.

As the tangent plane to M turns with speed τ_g along the intersection $\Sigma_t \cap M$, the vertical distance between $c(t) \in C$ and Σ_t, estimated starting from a point $\bar{c}(t)$ of $\Sigma_t \cap M$ close to $c(t)$, is of order $h \cdot (d(\bar{c}(t+h), c(t+h)))$, therefore has a principal term of the order of $|h^3|$. As the sphere Σ_t is osculating the curve C when it is a Darboux curve, this principal term should be of the order of $|h^4|$ (or negligible compared to $|h^4|$). □

4.2. Differential Equation of Darboux curves in a principal chart.

We have seen that, in $V(M)$, the Darboux curves are geodesics and form almost a flow: exactly two Darboux curves pass through every point of the interior of $V(M)$. To get a flow, we should "unfold" the intervals of light-ray fibering $V(M)$ into circles, obtaining a flow on $\mathbb{P}(TM)$. We keep, through the point $(m, \alpha), \alpha \in \mathbb{P}_1$, the inverse image of the two Darboux curves in $V(M)$ starting at the point σ_α which projects on the Darboux curve making the angle α (defined mod π) with the first principal direction of curvature, that is with $\mathcal{F}_0$ (see Subsection 3.3). In fact we will consider, in order to compute using an angle $\alpha \in \mathbb{S}^1$, the unit tangent bundle of M, T^1M, double cover of $\mathbb{P}(TM)$; it covers four times the regular points of $V(M)$.

Consider a local principal chart (u, v) in a surface $M \subset \mathbb{R}^3$, that is a chart obtained taking two lines of curvature intersecting at $m_0 \in M$, as $v = 0$ and $u = 0$ axes, and imposing that the leaves of $\mathcal{F}_0$ are given by $v = const.$ while those of $\mathcal{F}_{\pi/2}$ by $u = const.$.

In this chart, the first and second fundamental forms read, respectively, as $I = Edu^2 + Fdv^2$ and $II = edu^2 + gdv^2$.

With these notation, the principal curvatures are given by $k_1 = e/E$ and $k_2 = g/G$ (see [**St**]).

We denote by s the arc-length of a curve C and by s_c, *mean-sphere length*, the arc-length of the curve in $\mathcal{M}$ corresponding to C, that is the set of points of Λ^4 corresponding to mean spheres tangent to M along C (see Subsection 3.2, in particular Figure 3). Length elements ds_c and ds are related by $ds_c/ds = \mu = (k_1 - k_2)/2$.

Theorem 17. *Let (u, v) be a principal chart and $C : s \mapsto c(s)$ a Darboux curve parameterized by Euclidean arc length s or conformal arc length s_c. Let α be the angle between C and the principal direction $\partial/\partial u$. The angle α satisfies the following differential equation*

$$6 \sin\alpha \cos\alpha \, \frac{d\alpha}{ds_c} = \theta_1 \cos^3\alpha + \theta_2 \sin^3\alpha. \tag{10}$$

Proof. Recall that the functions θ_i are the conformal principal curvatures defined in Subsection 3.1. In our principal chart, the conformal principal curvatures and the conformal arc-length are given by

$$\begin{aligned} \theta_1 &= \frac{\xi_1(k_1)}{\mu} = \frac{X_1(k_1)}{\mu^2} = \frac{4}{\sqrt{E}} \frac{\partial k_1/\partial u}{(k_1 - k_2)^2}, \\ \theta_2 &= \frac{\xi_2(k_2)}{\mu} = \frac{X_1(k_1)}{\mu^2} = \frac{4}{\sqrt{G}} \frac{\partial k_2/\partial v}{(k_1 - k_2)^2}, \\ ds_c &= \mu ds. \end{aligned}$$

Let $C : s \mapsto c(s) = (u(s), v(s))$. Then

$$c'(s) = (u', v') = \left(\frac{\cos\alpha}{\sqrt{E}}, \frac{\sin\alpha}{\sqrt{G}} \right)$$

as the tangent to the curve C makes the angle α with the principal foliation $\mathcal{F}_0$.

Recall classical relations (see [**St**] and [**Sp**]), where $k_{g,1}$ and $k_{g,2}$ are the geodesic curvatures of the coordinate curves, which are lines of principal curvature

$$k_n(\alpha) = k_1 \cos^2\alpha + k_2 \sin^2\alpha, \text{ (Euler)} \tag{11}$$

$$\tau_g = (k_2 - k_1) \cos\alpha \sin\alpha, \tag{12}$$

and

$$k_g = \frac{d\alpha}{ds} + k_{g,1} \cos\alpha + k_{g,2} \sin\alpha. \text{ (Liouville)} \tag{13}$$

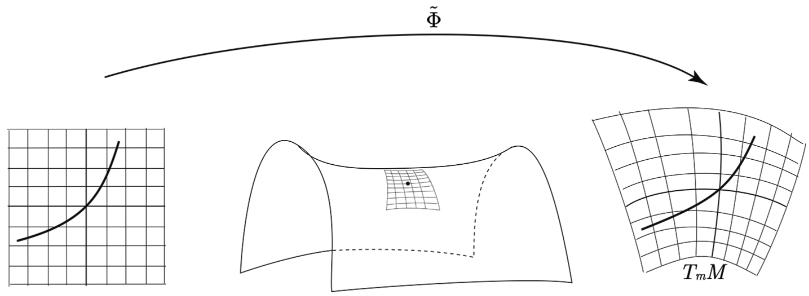

Figure 5. Geodesic curvature of the image of an arc by a principal chart projected on T_mM.

To obtain (12), recall that

$$\tau_g(m,w) = -\langle dN(m)w, \bar{w}\rangle = II(w,\bar{w})$$

where $w \in T_mM$ is a unit tangent vector and $w^{\perp} \in T_mM$ is unit and orthogonal to w and $\{w, w^{\perp}\}$ is a positive basis of T_mM. In order to obtain (13), one needs to compare the curvature $d\alpha/ds$ of the arc at the origin of the (u,v) principal chart and the curvature k_g of its image by the diffeomorphism Φ. It is enough to perform the computation for the diffeomorphism $\tilde{\Phi} = P_{T_mM} \circ \Phi$ obtained composing Φ with the orthogonal projection P_{T_mM} of M on T_mM. Notice that $\tilde{\Phi}$ is tangent to the identity at the origin.

Consider now the orthogonal, unit and positively oriented vector fields X_1 and X_2 tangent to the coordinate curves and denote the arc length of these curves by s_1 and s_2 respectively. Let $N = X_1 \wedge X_2$ be the unit normal. Then $t := c' = \cos\alpha X_1 + \sin\alpha X_2$ and $N \wedge t = -\sin\alpha X_1 + \cos\alpha X_2$.

It follows that

$$\begin{aligned} t' &= [-\sin\alpha X_1 + \cos\alpha X_2]\alpha' + \cos\alpha X_1' + \sin\alpha X_2' \\ &= \alpha' N \wedge t + \cos\alpha\left[\frac{dX_1}{ds_1}\cos\alpha + \frac{dX_1}{ds_2}\sin\alpha\right] + \sin\alpha\left[\frac{dX_2}{ds_1}\cos\alpha + \frac{dX_2}{ds_2}\sin\alpha\right] \end{aligned}$$

The geodesic curvatures of the coordinate curves are given by $k_{g,1} = \langle dX_1/ds_1, X_2\rangle$ and $k_{g,2} = \langle dX_2/ds_2, -X_1\rangle$.

Differentiating the equations $\langle X_1, X_2\rangle = 0$, $\langle X_1, X_1\rangle = 1$ and $\langle X_2, X_2\rangle = 1$ we obtain that

$$k_{g,1} = -\left\langle \frac{dX_2}{ds_1}, X_1\right\rangle, \quad k_{g,2} = \left\langle \frac{dX_1}{ds_2}, X_2\right\rangle, \quad \left\langle \frac{dX_1}{ds_2}, X_1\right\rangle = 0 \text{ and } \left\langle \frac{dX_2}{ds_1}, X_2\right\rangle = 0.$$

Therefore,

$$\begin{aligned} \left\langle \frac{dX_1}{ds_1}, N \wedge t\right\rangle &= k_{g,1}\cos\alpha, & \left\langle \frac{dX_1}{ds_2}, N \wedge t\right\rangle &= k_{g,2}\cos\alpha \\ \left\langle \frac{dX_2}{ds_1}, N \wedge t\right\rangle &= -k_{g,1}\sin\alpha, & \left\langle \frac{dX_2}{ds_2}, N \wedge t\right\rangle &= k_{g,2}\sin\alpha \end{aligned}$$

Then, after simplification, we obtain the equality

$$k_g = \langle t', N \wedge t \rangle = \alpha' + k_{g,1} \cos\alpha + k_{g,2} \sin\alpha.$$

Therefore,

$$\begin{aligned} \frac{dk_n}{ds} &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3\alpha + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} \cos^2\alpha \sin\alpha + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} \cos\alpha \sin^2\alpha \\ &+ \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3\alpha + 2(k_2 - k_1) \cos\alpha \sin\alpha \frac{d\alpha}{ds} \end{aligned}$$

We have seen in Subsection 4.1 that the differential equation of Darboux curves is given by $k_n' + k_g \tau_g = 0$, therefore we get

$$\begin{aligned} & k_n' + k_g \tau_g \\ &= \left[k_{g,1}(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} \right] \cos^2\alpha \sin\alpha + \left[k_{g,2}(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} \right] \cos\alpha \sin^2\alpha \\ &\quad + 3(k_2 - k_1) \cos\alpha \sin\alpha \frac{d\alpha}{ds} + \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3\alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3\alpha = 0 \end{aligned} \tag{14}$$

Given any orthogonal chart ($F = 0$), the infinitesimal rate of contraction of one foliation by coordinate lines is equal to the geodesic curvature of the leaves of the orthogonal foliation (see Figure 6 and [**Poin**]), that is

$$G_u = 2G\sqrt{E} k_{g,2} \quad \text{and} \quad E_v = -2E\sqrt{G} k_{g,1}, \tag{15}$$

see [**St**, p. 113]. When the leaves of the two orthogonal foliations are the lines of curvature, a first geometrical interpretation of these equations is that geodesic curvatures provide the infinitesimal holonomy of the principal foliations, more precisely the geodesic curvature $k_{g,2}$ is the infinitesimal holonomy of $\mathcal{F}_0$ and $k_{g,1}$ is the infinitesimal holonomy of $\mathcal{F}_{\pi/2}$, see [**So**]. Notice that the Codazzi equations in a principal chart are given by

$$\frac{\partial k_1}{\partial v} = \frac{E_v}{2E}(k_2 - k_1), \qquad \frac{\partial k_2}{\partial u} = \frac{G_u}{2G}(k_1 - k_2), \tag{16}$$

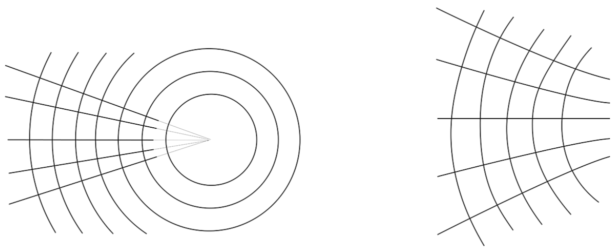

Figure 6. Infinitesimal rates of contraction and geodesic curvatures.

Equivalently,

$$\frac{\partial k_1/\partial v}{k_2-k_1} = \frac{E_v}{2E} = k_{g,1}\sqrt{G}, \qquad \frac{\partial k_2/\partial u}{k_1-k_2} = \frac{G_u}{2G} = k_{g,2}\sqrt{E}.$$

Therefore,

$$k_{g,1}(k_2-k_1) + \frac{1}{\sqrt{G}}\frac{\partial k_1}{\partial v} = -\frac{E_v}{2E\sqrt{G}}(k_2-k_1) + \frac{1}{\sqrt{G}}\frac{E_v}{2E}(k_2-k_1) = 0$$
$$k_{g,2}(k_2-k_1) + \frac{1}{\sqrt{E}}\frac{\partial k_2}{\partial u} = \frac{G_u}{2G\sqrt{E}}(k_2-k_1) + \frac{1}{\sqrt{E}}\frac{G_u}{2G}(k_1-k_2) = 0.$$

We can now see that, in (14), the coefficient at $\cos^2\alpha\sin\alpha$ and $\cos\alpha\sin^2\alpha$ are equal to 0; therefore

$$3(k_1-k_2)\sin\alpha\cos\alpha\,\frac{d\alpha}{ds} = \frac{1}{\sqrt{E}}\frac{\partial k_1}{\partial u}\cos^3\alpha + \frac{1}{\sqrt{G}}\frac{\partial k_2}{\partial v}\sin^3\alpha. \tag{17}$$

Recall that $(\partial k_1/\partial u)/\sqrt{E} = X_1(k_1)$ and $(\partial k_2/\partial v)/\sqrt{G} = X_2(k_2)$; then Equation (10) is obtained expressing the above in terms of the conformal curvatures θ_1 and θ_2 (see Subsection 3.1), where

$$\theta_1 = \frac{\xi_1(k_1)}{\mu} = \frac{X_1(k_1)}{\mu^2} = \frac{4}{\sqrt{E}}\frac{\partial k_1/\partial u}{(k_1-k_2)^2},$$
$$\theta_2 = \frac{\xi_2(k_2)}{\mu} = \frac{X_2(k_2)}{\mu^2} = \frac{4}{\sqrt{G}}\frac{\partial k_2/\partial v}{(k_1-k_2)^2},$$

and of the arc length s_c of the image of the curve in $\mathcal{M}$ which is related to the Euclidean length s by $ds_c = \mu ds$. □

Remark.

(i) In a principal chart the Codazzi equations can be written as

$$k_{g,1} = -\frac{\xi_2(k_1)}{2}, \qquad k_{g,2} = \frac{\xi_1(k_2)}{2}.$$

(ii) A curve $C = \{c(s)\}$ has contact of third order with the associated osculating sphere, tangent to the surface, when

$$\langle c', c'\rangle[2\langle N', c''\rangle + \langle N'', c'\rangle] - 3\langle c', N'\rangle\langle c', c''\rangle = 0. \tag{18}$$

This equation can be used to obtain the differential equation of Darboux curves in any chart (u,v), see [**Sa1**].

(iii) Let $\theta(s)$ be the angle between the unit normal N of the surface M and the principal normal n of a curve $C : s \mapsto c(s)$ parameterized by arc length s. Then C is a

Darboux curve on M if and only if $k' \cos\theta + k\tau \sin\theta = 0$, where k and τ are the curvature and torsion of C. Indeed, observe just that $k_n = k\cos\theta$, $k_g = k\sin\theta$ and $\tau_g = \tau + \theta'$. Direct substitution in (9) $k_n' + k_g\tau_g = 0$ leads to the result, see also [**Ve**].

5. A plane-field on $V(M)$.

In this section, we consider a natural plane-field associated to the Darboux curves and its integrability in terms of conformal invariants. Then we provide a relation of that with isothermicity of surfaces.

Two tangent vectors $\mathcal{D}_1$ and $\mathcal{D}_2$ to the two Darboux curves through the point $(m, \alpha) \in V(M)$ define a plane in $T_{m,\alpha}V(M)$. These planes define a plane-field $\mathcal{D}$. It will be called *Darboux plane-field.* As a branch of the intersection of a sphere $\sigma_{m,\alpha}$ and M, and the Darboux curve tangent at m to this branch have the same osculating circle, the vector, say $\mathcal{D}_1$, is also tangent to the pencils of spheres through this osculating circle. The plane $\mathcal{D}$ is therefore tangent at $(m, \alpha) \in V(M)$ to the two curves corresponding to pencils through the osculating circles to the two branches.

The next proposition gives an explicit parameterization of $V(M)$ that we will use to give an analytic expression for the Darboux plane-field $\mathcal{D}$.

PROPOSITION 18. *Consider a principal chart (u, v) and let $m(u, v)$ be the corresponding parameterization of $M \subset \mathbb{S}^3$; we denote by $n = n(u, v) \in T_m\mathbb{S}^3 \subset \mathbb{R}^5_1$ the normal vector at m to M. The set of spheres $V(M) \subset \Lambda^4$ is parameterized by*

$$\varphi(u, v, \alpha) = k_n(\alpha)m(u, v) + n(u, v) \tag{19}$$

with $k_n(\alpha) = k_1(u, v)\cos^2\alpha + k_2(u, v)\sin^2\alpha$.

REMARK. Notice that the sum $k_n(\alpha)m(u, v) + n(u, v)$ is performed in $\mathbb{R}^5_1$, even when we use the paraboloid model of the Euclidean space $\mathbb{R}^3$, section of the light cone $\mathcal{L}ight$ by a hyperplane parallel to a light direction instead of $\mathbb{S}^3 \subset \mathcal{L}ight \subset \mathbb{R}^5_1$. The vector $n(u, v)$ is therefore not contained in the paraboloid.

PROOF. Just, apply (1) of Proposition 1 and Proposition 5 to the spheres $\Sigma_{m,\alpha}$ and observe that the transformation φ sends the osculating sphere with radius $1/k_n(\alpha)$, tangent to M at the point $m(u, v)$ to a point in Λ^4, therefore $\mathcal{L}(m, m) = 0, \mathcal{L}(m, N) = 0$, and $n_u = -k_1 m_u$, $n_v = -k_2 m_v$. and

$$\begin{aligned}
\varphi_u &= \left[\frac{\partial k_1}{\partial u}\cos^2\alpha + \frac{\partial k_2}{\partial u}\sin^2\alpha\right]m + (k_n - k_1)m_u \\
\varphi_v &= \left[\frac{\partial k_1}{\partial v}\cos^2\alpha + \frac{\partial k_2}{\partial v}\sin^2\alpha\right]m + (k_n - k_2)m_v \qquad (20)\\
\varphi_\alpha &= [(k_2 - k_1)\cos\alpha\sin\alpha]m.
\end{aligned}$$

Consequently, $D\varphi$ has rank 3 at any $\alpha \in (0, \pi/2)$. $\square$

Proposition 19. *Consider again a principal chart (u, v). The Darboux plane-field $\mathcal{D}$ is defined locally by the vector fields*

$$\begin{aligned} \mathcal{D}_1^c &= \xi_1 + \frac{1}{6}\theta_1 \frac{\cos\alpha}{\sin\alpha}\frac{\partial}{\partial\alpha} \\ \mathcal{D}_2^c &= \xi_2 + \frac{1}{6}\theta_2 \frac{\sin\alpha}{\cos\alpha}\frac{\partial}{\partial\alpha}. \end{aligned} \tag{21}$$

Therefore $\mathcal{D}$ coincides with the kernel of the differential 1-form

$$\Omega = \cos^2\alpha\, \theta_1\omega_1 + \sin^2\alpha\, \theta_2\omega_2 - 6\sin\alpha\cos\alpha d\alpha, \tag{22}$$

where ω_1 and ω_2 are 1-forms dual to the conformal vector fields ξ_1 and ξ_2.

Proof. We shall derive (21) from the differential equation (17) of Darboux curves in a principal chart (u, v).

A unit vector $(u', v') \in T_pM$ making an angle α with the principal direction $\partial/\partial u$ is given by $(\cos\alpha/\sqrt{E})(\partial/\partial u) + (\sin\alpha/\sqrt{G})(\partial/\partial v)$.

Therefore the differential equation given by equation (17) is equivalent to

$$u' = \frac{du}{ds} = \frac{\cos\alpha}{\sqrt{E}}, \quad v' = \frac{dv}{ds} = \frac{\sin\alpha}{\sqrt{G}},$$

$$\alpha' = \frac{d\alpha}{ds} = -\frac{1}{3(k_1-k_2)\sin\alpha\cos\alpha}\left[\frac{1}{\sqrt{E}}\frac{\partial k_1}{\partial u}\cos^3\alpha + \frac{1}{\sqrt{G}}\frac{\partial k_2}{\partial v}\sin^3\alpha\right].$$

So, the lift of the Darboux curves to the unitary tangent bundle is given by the vector field

$$\mathcal{D}_1 = \frac{\cos\alpha}{\sqrt{E}}\frac{\partial}{\partial u} + \frac{\sin\alpha}{\sqrt{G}}\frac{\partial}{\partial v} + \left[\frac{\partial k_1/\partial u}{3\sqrt{E}(k_1-k_2)}\frac{\cos^2\alpha}{\sin\alpha} + \frac{\partial k_2/\partial v}{3\sqrt{G}(k_1-k_2)}\frac{\sin^2\alpha}{\cos\alpha}\right]\frac{\partial}{\partial\alpha}. \tag{23}$$

Consider the involution $\iota(u, v, \alpha) = (u, v, -\alpha)$ and the induced vector field $D_2 = \iota_*(D_1)$. So,

$$\mathcal{D}_2 = \frac{\cos\alpha}{\sqrt{E}}\frac{\partial}{\partial u} - \frac{\sin\alpha}{\sqrt{G}}\frac{\partial}{\partial v} - \left[-\frac{\partial k_1/\partial u}{3\sqrt{E}(k_1-k_2)}\frac{\cos^2\alpha}{\sin\alpha} + \frac{\partial k_2/\partial v}{3\sqrt{G}(k_1-k_2)}\frac{\sin^2\alpha}{\cos\alpha}\right]\frac{\partial}{\partial\alpha}.$$

Our Darboux plane-field $\mathcal{D}$ is spanned by $\{\mathcal{D}_1, \mathcal{D}_2\}$. and we may define a new pair of vector fields generating $\mathcal{D}$: $\bar{\mathcal{D}}_1 = (1/\cos\alpha)(\mathcal{D}_1 + \mathcal{D}_2)$ and $\bar{\mathcal{D}}_2 = (1/\sin\alpha)(\mathcal{D}_1 - \mathcal{D}_2)$. Then,

$$\bar{\mathcal{D}}_1 = \frac{2}{\sqrt{E}}\cdot\frac{\partial}{\partial u} + \frac{2\partial k_1/\partial u}{3\sqrt{E}(k_1-k_2)}\cdot\frac{\cos\alpha}{\sin\alpha}\cdot\frac{\partial}{\partial\alpha}$$

and

$$\bar{\mathcal{D}}_2 = \frac{2}{\sqrt{G}} \cdot \frac{\partial}{\partial v} + \frac{2\partial k_2/\partial v}{3\sqrt{G}(k_1 - k_2)} \cdot \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\partial}{\partial\alpha}.$$

Using the conformal vector fields $\xi_1 = 2((\partial/\partial u)/\sqrt{E}(k_1 - k_2))$, $\xi_2 = 2((\partial/\partial v)/\sqrt{G}(k_1 - k_2))$ and the conformal principal curvatures $\theta_i = (2/(k_1 - k_2))\xi_i(k_i)$, we obtain a new base given by

$$\mathcal{D}_1^c = \frac{\bar{\mathcal{D}}_1}{k_1 - k_2} = \xi_1 + \frac{1}{6}\theta_1 \frac{\cos\alpha}{\sin\alpha}\frac{\partial}{\partial\alpha}$$
$$\mathcal{D}_2^c = \frac{\bar{\mathcal{D}}_1}{k_1 - k_2} = \xi_2 + \frac{1}{6}\theta_2 \frac{\sin\alpha}{\cos\alpha}\frac{\partial}{\partial\alpha}.$$

This ends the proof of the first part.

Evaluating the form Ω on the two vectors $\mathcal{D}_1^c$ and $\mathcal{D}_2^c$ we obtain the second part of the statement. □

Next, let us observe that

$$\begin{aligned}\Omega \wedge d\Omega = \sin\alpha\cos\alpha[&-5\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2)\\ &+ 3\cos 2\alpha(\xi_2(\theta_1) + \xi_1(\theta_2))]\omega_1 \wedge \omega_2 \wedge d\alpha.\end{aligned} \tag{24}$$

Indeed, since $d\theta_1 = \xi_1(\theta_1)\omega_1 + \xi_2(\theta_1)\omega_2$ and $d\theta_2 = \xi_1(\theta_2)\omega_1 + \xi_2(\theta_2)\omega_2$, Equation (4) yields that

$$d\theta_1 \wedge \omega_1 + \theta_1 d\omega_1 = \left(\frac{\theta_1\theta_2}{2} - \xi_2(\theta_1)\right)\omega_1 \wedge \omega_2,$$
$$d\theta_2 \wedge \omega_2 + \theta_2 d\omega_2 = \left(\frac{\theta_1\theta_2}{2} + \xi_1(\theta_2)\right)\omega_1 \wedge \omega_2.$$

Therefore,

$$\begin{aligned}d\Omega = &\left(\frac{\theta_1\theta_2}{2} - \xi_2(\theta_1)\cos^2\alpha + \xi_1(\theta_2)\sin^2\alpha\right)\omega_1 \wedge \omega_2\\ &+ 2\sin\alpha\cos\alpha(\theta_2 d\alpha \wedge \omega_2 - \theta_1 d\alpha \wedge \omega_1)\end{aligned}$$

A straightforward calculation leads to Proposition 24.

Theorem 20. *The Darboux plane-field $\mathcal{D}$ is integrable if and only if*

$$\xi_1(\theta_2) = -\frac{5}{6}\theta_1\theta_2, \qquad \xi_2(\theta_1) = \frac{5}{6}\theta_1\theta_2. \tag{25}$$

Proof. The theorem is a direct consequence of (24). The Darboux plane-field $\mathcal{D}$ is integrable if and only if $\Omega \wedge d\Omega = 0$ (Frobenius theorem). The equality

$$[-5\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) + 3\cos 2\alpha(\xi_2(\theta_1) + \xi_1(\theta_2))] = 0,$$

holds for all α if and only if $-5\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) = 0$ and $\xi_2(\theta_1) + \xi_1(\theta_2) = 0$. Direct calculations lead to the condition stated in (25). □

COROLLARY 21. *Consider a principal chart (u, v) and principal curvatures $k_1 > k_2$. Then the criterium of integrability of the Darboux plane-field $\mathcal{D}$ is given by*:

$$\begin{aligned} \frac{\partial^2 k_1}{\partial u \partial v} &= \frac{1}{3} \cdot \frac{1}{k_1 - k_2} \frac{\partial k_1}{\partial u} \left(3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v} \right) \\ \frac{\partial^2 k_2}{\partial u \partial v} &= \frac{1}{3} \cdot \frac{1}{k_1 - k_2} \frac{\partial k_2}{\partial v} \left(\frac{\partial k_1}{\partial u} - 3\frac{\partial k_2}{\partial u} \right) \end{aligned} \tag{26}$$

PROOF. The result follows from (25), the formulae defining θ_i's and ξ_i's and the Codazzi equations (16). □

5.1. Integrability of Darboux plane-field on isothermic surfaces. Now, we shall establish a relation between the integrability of the Darboux plane-field with the property of being isothermic. In the case of canal surfaces we get a complete equivalence.

The class of isothermic surfaces was considered by Darboux [**Da3**] and Calapso [**Ca**] among the others. For more recent works see, for example, [**H-J**] and the references therein.

DEFINITION 22. A surface M is called isothermic if there is a locally conformal parameterization of the surface by curvature lines.

PROPOSITION 23. *Consider a surface M such that the Darboux plane field $\mathcal{D}$ is integrable. Then M is isothermic.*

PROOF. Let ξ_1 and ξ_2 be the conformal principal vector fields and the dual one-forms ω_1 and ω_2 defined by $\omega_i(\xi_j) = \delta_{ij}$. As the definition of isothermicity is local, we can suppose that all the forms are defined in a simply connected domain.

Since $|\xi_1| = |\xi_2| \neq 0$, M has a locally conformal parameterization by curvature lines if and only if there exists a function $h(u, v)$ such that $[h\xi_1, h\xi_2] = 0$.

Since $[\xi_1, \xi_2] = -(1/2)\theta_2\xi_1 - (1/2)\theta_1\xi_2$, direct calculation shows that

$$[h\xi_1, h\xi_2] = -h\left[\frac{1}{2}h\theta_2 + \xi_2(h)\right]\xi_1 + h\left[-\frac{1}{2}h\theta_1 + \xi_1(h)\right]\xi_2.$$

So the surface is isothermic when there exists a function $h > 0$ such that

$$\xi_1(h) = \frac{1}{2}h\theta_1 \text{ and } \xi_2(h) = -\frac{1}{2}h\theta_2. \tag{27}$$

Putting $h = e^H$ we reduce (27) to

$$\xi_1(H) = \frac{1}{2}\theta_1 \text{ and } \xi_2(H) = -\frac{1}{2}\theta_2.$$

Consider the one form $\omega = (\theta_1\omega_1 - \theta_2\omega_2)/2$ and write $d\theta_i = \xi_1(\theta_i)\omega_1 + \xi_2(\theta_i)\omega_2$. Then, $d\omega = -(1/2)[\xi_2(\theta_1) + \xi_1(\theta_2)]\omega_1 \wedge \omega_2$. So $\omega = dH$ if and only if ω is a closed form and this is guaranteed by the condition $\xi_1(\theta_2) + \xi_2(\theta_1) = 0$, direct consequence of the integrability condition of the plane-field $\mathcal{D}$. So, solutions $h = e^H$ of (27) exist and the surface is isothermic. □

Theorem 24. *Let M be a canal surface. Then M is isothermic if and only if the Darboux plane-field $\mathcal{D}$ is integrable.*

Proof. In a canal surface one of the conformal principal curvatures, say θ_2 vanishes identically. Therefore if M is isothermic, then $\xi_2(\theta_1) = 0$. The conditions of integrability of $\mathcal{D}$ are given by: $-5\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) = 0$ and $\xi_2(\theta_1) + \xi_1(\theta_2) = 0$. Therefore, when $\theta_2 = 0$ these two conditions are equivalent to $\xi_2(\theta_1) = 0$ and the result follows. The converse is given by Proposition 23. □

Remark. The authors of [**Ba-La-Wa**] call canals satisfying $\theta_2 = 0$ and $\xi_2(\theta_1) = 0$ *special*. Musso and Nicoldi [**M-N**] proved that any Willmore canal is isothermic, and therefore special in this sense.

In [**GLW1**] the authors will prove that the Darboux plane-field is integrable when M is a quadric.

The integrability of the Darboux plane field can be proved also on the helicoid. It would be interesting to characterize geometrically all the surfaces such that $\mathcal{D}$ is integrable (see Proposition 23).

Acknowledgements. The authors are grateful to the faculty and staff of Institut de Mathématiques de Bourgogne (UMR CNRS **5584**), Université de Bourgogne Franche Comté, Instituto de Matemática e Estatística, Universidade Federal de Goiás and Wydział Matematyki i Informatyki Uniwersytet Łódzki for the hospitality during their visits. The authors were supported by Pronex/FAPEG/CNPq Proc. 20121026 7000803, European Union (grant n^o ICA1-CT-2002-70017/8),
Polish National Science Center (grant n^o 6065/B/HO3/2011/40) and
France-Brésil agreements (n^o 70017).

References

[Ba-Wh] T. Banchoff and J. White, The behavior of the total twist and self-linking number of a closed space curve under inversions, Math. Scand., **36** (1975), 254–262.

[Ba-La-Wa] A. Bartoszek, R. Langevin and P. G. Walczak, Special canal surfaces of $\mathbb{S}^3$, Bull. Braz. Math. Soc. New Series, **42** (2011), 301–320.

[Br-Gi-Ta] J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: focal sets, ridges and umbilics, Math. Proc. Cambridge Philos. Soc., **125** (1999), 243–268.

[CSW] G. Cairns, R. Sharpe and L. Webb, Conformal invariants for curves and surfaces in three dimensional space forms, Rocky Mountain Jour. of Math., **24** (1994), 933–959.

[Ca] P. Calapso, Sulla superficie a linee di curvatura isoterme, Rend. Circ. Mat. Palermo, **17** (1903), 275–286.

[Da1] G. Darboux, Des courbes tracées sur une surface, dont la sphère osculatrice est tangente en chaque point à la surface, Note Comptes Rendus Acad. Scien. Paris, tome **LXXIII** (1872), 732–736.

[Da2] G. Darboux, Leçons sur la théorie générale des surfaces, Gauthier-Villars Paris, 1887.

[Da3] G. Darboux, Sur les surfaces isothermiques, Ann. Sci. École Norm. Sup., **16** (1899), 91–508.

[En] A. Enneper, Bemerkungen über die Differentialgleichung einer Art von Curven auf Flächen, Göttinger Nachrichten, 1891, 577–583.

[GLW1] R. Garcia, R. Langevin and P. Walczak, Darboux curves on surfaces II, the dynamical viewpoint, Bulletin of Brazilian Math. Soc., **47** (2016), 1119–1154.

[GLW2] R. Garcia, R. Langevin and P. Walczak, Foliations by curves making a constant angle with principal directions, Ann. Polon. Math., **113** (2015), 165–173.

[Haa] J. Haantjes, Conformal differential geometry, V, Special surfaces, (Dutch. German, English, French summaries) Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde, **52** (1943), 322–331.

[Ha] J. G. Hardy, Darboux lines on surfaces, Amer. Journal of Math., **20** (1898), 283–292.

[H-J] U. Hertrich–Jeromin, Introduction to Möbius Differential Geometry, London Math. Soc. Lecture Notes, **300**, Cambridge University Press (2003).

[Kle] F. Klein, Lectures on mathematics, Macmillan and company, (1894), reprint AMS Chelsea publishing Providence, Rhode Island, 2011 and 2000.

[La-Oh] R. Langevin and J. O'Hara, Conformal arc-length as 1/2 dimensional length of the set of osculating circles, Comment. Math. Helvetici, **85** (2010), 273–312.

[La-So] R. Langevin and G. Solanes, Conformal geometry of curves and the length of canals, Adv. Geom., **11**, N^o 4, (2001), 585–601.

[La-Wa] R. Langevin and P. G. Walczak, Conformal geometry of foliations, Geom. Dedicata, **132** (2008), 135–178.

[Li] S. Lie, Über Complexe, insbesondere Linien-und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen, Math. Annalen, **5** (1872) 145–256.

[M-N] E. Musso and L. Nicoldi, Willmore canal surfaces in Euclidean space, Rend. Istit. Mat. Univ. Trieste, **31** (1999), 177–202.

[Pe] A. Pell, D-lines on Quadrics, Trans. Amer. Math. Soc., **1** (1900), 315–322.

[Poin] H. Poincaré, Mémoires sur les courbes définies par une équation différentielle, J. Math. Pure et Appl. (Série 3), **7** (1881), 375–422.

[Po] I. R. Porteous, Geometric Differentiation, Cambridge Univ. Press, 2001.

[Ri] M. Ribaucour, Propriétés de courbes tracées sur les surfaces, Note Comptes Rendus Acad. Scien. Paris, Tome **LXXX** (1875), 642–645.

[Sa1] L. A. Santaló, Curvas extremales de la torsion total y curvas-D, Publ. Inst. Mat. Univ. Nac. Litoral., 1941, 131–156.

[Sa2] L. A. Santaló, Curvas D sobre conos, Select Works of L. A. Santaló, Springer Verlag, 2009, 317–325.

[Se] F. Semin, Darboux lines, Rev. Fac. Sci. Univ. Istanbul (A), **17** (1952), 351–383.

[So] J. Sotomayor, Lines of curvature and an integral form of Mainardi-Codazzi equations, An. Acad. Brasil. Ciênc., **68** (1996), 133–137.

[Sp] M. Spivak, A Comprehensive Introduction to Differential Geometry, **III**, Publish or Perish Berkeley, 1979.

[St] D. Struik, Lectures on Classical Differential Geometry, Addison Wesley, 1950, Reprinted by Dover Collections, 1988.

[Tho] G. Thomsen, Über konforme Geometrie, II: Über Kreisscharen und Kurven in der Ebene und über Kugelscharen und Kurven im Raum, Abhandlungen Hamburg, **4** (1925), 117–147.

[Tr] M. A. Tresse, Sur les invariants différentiels d'une surface par rapport aux transformations conformes de l'espace, Note Comptes Rendus Acad. Sci. Paris, **192** (1892), 948–950.

[Ve] E. Vessiot, Contributions à la géométrie conforme. Cercles et surfaces cerclées, J. Math. Pures Appliqués, **2** (1923), 99–165.

Ronaldo Garcia
Instituto de Matemática e Estatística
Universidade Federal de Goiás
CEP 74001-970
Caixa Postal 131
Goiânia, GO, Brazil
E-mail: ragarcia@ufg.br

Rémi Langevin
Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne Franche-Comté
F-21000 Dijon, France
E-mail: Remi.Langevin@u-bourgogne.fr

Paweł Walczak
Katedra Geometrii
Wydział Matematyki i Informatyki
Uniwersytet Łódzki
ul. Banacha 22
90-238 Łódź, Poland
E-mail: pawelwal@math.uni.lodz.pl