Hitting times of Bessel processes, volume of the Wiener sausages and zeros of Macdonald functions

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Abstract. We derive formulae for some ratios of the Macdonald functions by using their zeros, which are simpler and easier to treat than known formulae. The result gives two applications in probability theory and one in classical analysis. We show a formula for the Lévy measure of the distribution of the first hitting time of a Bessel process and an explicit form for the expected volume of the Wiener sausage for an even dimensional Brownian motion. In addition, we show that the complex zeros of the Macdonald functions are the roots of some algebraic equations with real coefficients.

1. Introduction.

The (modified) Bessel functions appear in various kinds of situations. In probability theory, for example, I_ν's appear in the explicit form for the transition probability densities of the Bessel processes. In this article we are concerned with the ratio of the modified Bessel functions. It is known that such functions represent the Laplace transforms of the first hitting times of the Bessel processes (cf. [2], [13]) and of the expected volume of the Wiener sausage (cf. [5]). From an analytical point of view Ismail et al. [10], [11] have studied when such functions are completely monotone.

We are mainly concerned with the ratios of the modified Bessel function of the second kind K_ν, so-called the Macdonald functions. Ismail [10] has shown that $(K_{\nu+1}/K_\nu)(\sqrt{z})$ is completely monotone by expressing it as a Stieltjes transform of some function. From his expression we can invert the Laplace transform, but the resulting formula seems complicated. The purpose of this article is to rewrite the ratio in a simpler form by means of the zeros of K_ν and to invert the Laplace transform. The result is applied to two questions in probability theory and a study on the zeros of K_ν.

Recently, in connection with the first hitting times of Bessel processes, the authors [8] have studied another type of the ratios of the Macdonald functions and decomposed it into a sum of several functions which are easy to treat. A similar method via some contour integrals is effective in this article.

The purpose of [8] is to show an explicit form of the distribution function for the first hitting time $\tau_{a,b}^{(\nu)}$ to b of the Bessel process with index ν starting at a. The density for $\tau_{a,b}^{(\nu)}$ and its asymptotics have been discussed in [7]. The infinite divisibility of the distribution was first investigated by Kent [13]. To be accurate, the conditional distribution of $\tau_{a,b}^{(\nu)}$

under the condition that it is finite is infinitely divisible (cf. [11]). General theory in the infinite divisibility of the distributions of the first hitting times of one-dimensional diffusion processes is given by Yamazato [22].

As referred in [8], the function $K_{\nu+1}/K_\nu$ appears when we give an expression for the Lévy measure. We may apply our result on the ratio of the Macdonald functions to obtain an explicit expression for the Lévy measure of the distribution of $\tau_{a,b}^{(\nu)}$.

Moreover, the function $(K_{d/2}/K_{d/2-1})(r\sqrt{2\lambda})$ in $\lambda > 0$ represents the Laplace transform of the expectation of the Wiener sausage for the d-dimensional Brownian motion associated with a close ball with radius r (cf. [5]). In the case when d is odd, Hamana [6] divided the function into a sum of several functions of which the inverse Laplace transforms can be easily obtained and he deduced an exact form of the mean volume of the Wiener sausage by means of zeros of $K_{d/2-1}$. By using our result we can show that, also in the even dimensional case, the expectation is represented in a similar form. However, if we consider the large time asymptotics, we see a difference between the odd and the even dimensional cases. The logarithmic terms appear in the even dimensional cases, while the asymptotic expansions are given only by the powers of t in the odd cases.

We should remark that the Wiener sausage for a Brownian motion associated with a general compact set is investigated in [16], [17], [20] and so on, and that the same problem for a stable process is discussed in [1], [18]. These articles deal with the large time asymptotics of their expectation and the first several terms are given. In the Brownian cases it is shown that the asymptotic expansion in powers of $1/\log t$ is admitted if $d = 2$ and that the logarithmic term appears if $d = 4$. In higher dimensional cases we do not find the logarithmic terms in the literature.

In the results mentioned so far we have express several quantities by using the zeros of K_ν. If we consider the asymptotic behavior of the ratio $(K_{\nu+1}/K_\nu)(w)$ as $w \to \infty$ or $w \to 0$, we may conversely obtain information on the zeros of K_ν. In particular, we see that the zeros are the roots of some algebraic equations with real coefficients and show the way to obtain the coefficients.

This article is organized as follows. Section 2 is devoted to a decomposition of the functions $K_{\nu+1}/K_\nu$. Section 3 is devoted to a representation of the Lévy measure of the first hitting time of the Bessel process. We calculate the expected volume of the Wiener sausage for the even dimensional Brownian motion in Section 4 and discuss its large time asymptotics in Section 5. In the final Section 6 we study the complex zeros of the Macdonald functions.

2. Ratios of Macdonald functions.

For $\nu \in \mathbb{C}$ the notation I_ν and K_ν are used to denote the modified Bessel functions of the first kind and the second kind, respectively. We call ν the order of the functions. Especially, K_ν is also called the Macdonald function. In this article we treat only the case when the all orders of modified Bessel functions are real.

Before giving the result, we recall several facts on the zeros of the Macdonald function. For $\nu \in \mathbb{R}$ let $N(\nu)$ be the number of zeros of K_ν. It is known that $N(\nu)$ is equal to $|\nu| - 1/2$ if $\nu - 1/2$ is an integer and that $N(\nu)$ is the even number closest to $|\nu| - 1/2$ otherwise. Thus we have $N(\nu) = 0$ if $|\nu| < 3/2$. If $|\nu| = 2n + 3/2$ for some integer

n, K_ν has a real (negative) zero. Otherwise, K_ν does not have real zeros. Each zero, if exists, lies in the half plain $\mathbb{C}^- = \{z \in \mathbb{C}\,;\, \mathrm{Re}(z) < 0\}$. When $N(\nu) \geq 1$, we write $z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,N(\nu)}$ for the zeros. It is known that all zeros of K_ν are distinct. For details, see [**8**, p. 5240] and [**21**, pp. 511–513] for example.

Let $D = \{z \in \mathbb{C} \setminus \{0\}\,;\, |\arg z| < \pi\}$ and $D_\nu = \{z \in D\,;\, K_\nu(z) \neq 0\}$ for $\nu \in \mathbb{R}$. The purpose of this section is to show the following theorem.

THEOREM 2.1. *Let $w \in D_\nu$ and $\nu^+ = \max\{\nu, 0\}$. In addition, we put, for $\mu \geq 0$,*

$$G_\mu(x) = K_\mu(x)^2 + \pi^2 I_\mu(x)^2 + 2\pi \sin(\pi\mu) K_\mu(x) I_\mu(x), \quad x > 0. \tag{2.1}$$

(1) *When $\nu - 1/2$ is an integer, we have that, if $|\nu| = 1/2$,*

$$\frac{K_{\nu+1}(w)}{K_\nu(w)} = 1 + \frac{2\nu^+}{w} \tag{2.2}$$

and that, if $|\nu| \geq 3/2$,

$$\frac{K_{\nu+1}(w)}{K_\nu(w)} = 1 + \frac{2\nu^+}{w} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - w}. \tag{2.3}$$

(2) *When $\nu - 1/2$ is not an integer, we have that, if $|\nu| < 3/2$,*

$$\frac{K_{\nu+1}(w)}{K_\nu(w)} = 1 + \frac{2\nu^+}{w} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+w)G_{|\nu|}(x)} \tag{2.4}$$

and that, if $|\nu| > 3/2$,

$$\frac{K_{\nu+1}(w)}{K_\nu(w)} = 1 + \frac{2\nu^+}{w} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - w} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+w)G_{|\nu|}(x)}. \tag{2.5}$$

For a proof of Theorem 2.1, it is sufficient to consider the case of $\nu \geq 0$ because of the formula $K_\mu = K_{-\mu}$ and the recurrence relation

$$K_{\mu+1}(z) - K_{\mu-1}(z) = \frac{2\mu}{z} K_\mu(z) \tag{2.6}$$

(cf. [**21**, p. 79]). In fact, we see for $\nu < 0$

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{K_{|\nu|+1}(z)}{K_{|\nu|}(z)} + \frac{2\nu}{z}, \quad z \in D_\nu = D_{|\nu|}.$$

Formula (2.2) is easily obtained from

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad K_{3/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \frac{z+1}{z}.$$

Moreover (2.3) has been already established in [**6**]. Therefore we concentrate on the case when $\nu - 1/2$ is not an integer.

To prove (2.4) and (2.5), we need two lemmas. One is an uniform estimate of the Macdonald function, which has been proved in [**8**].

LEMMA 2.2.   *Let $\delta \in (0, 3\pi/2)$ be given. For $\mu \geq 0$ we have*

$$K_\mu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \{1 + E_\mu(z)\} \tag{2.7}$$

if $|\arg z| \leq 3\pi/2 - \delta$. Here $|E_\mu(z)| \leq A_\mu/|z|$ for a constant A_μ which is independent of z.

The other gives asymptotic behavior on the real line of the functions involving the modified Bessel functions. It is easily shown by the formula

$$I_\mu(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 - \frac{4\mu^2 - 1}{8x} + o\left(\frac{1}{x}\right) \right\} = \frac{e^x}{\sqrt{2\pi x}} \{1 + o(1)\}, \quad x \to \infty \tag{2.8}$$

for $\mu \geq 0$ (cf. [**21**, p. 203]) and we omit the detailed proofs.

LEMMA 2.3.   *Let $\zeta, \eta, \xi \geq 0$. It follows that, as $x \to \infty$,*

$$G_\zeta(x) = \frac{\pi e^{2x}}{2x} \{1 + o(1)\}, \quad \frac{K_\eta(x) K_\xi(x)}{G_\zeta(x)} = e^{-4x} \{1 + o(1)\},$$

$$\frac{I_\eta(x) I_\xi(x)}{G_\zeta(x)} = \frac{1}{\pi^2} \{1 + o(1)\}, \quad \frac{I_\eta(x) K_\xi(x)}{G_\zeta(x)} = \frac{e^{-2x}}{\pi} \{1 + o(1)\}.$$

We are now ready to show Theorem 2.1. We only consider the case when $\nu - 1/2$ is not an integer. Let $\alpha \in (0, 1)$ and $w \in D_\nu$. For $z \in D_\nu$ with $z \neq w$, we set

$$f_{\nu,\alpha}^w(z) = \frac{1}{z^\alpha(z - w)} \frac{K_{\nu+1}(z)}{K_\nu(z)}.$$

Letting ε and R be positive numbers with $\varepsilon < 1$ and $\varepsilon < R$ and setting

$$\theta_{R,\varepsilon} = \text{Arcsin} \frac{\varepsilon}{R} \in \left(0, \frac{\pi}{2}\right),$$

we consider the same piecewise C^1-curve $\gamma = \gamma_0 + \gamma_1 - \gamma_3 - \gamma_2$ as in [**8**], where

$$\gamma_0 : z = Re^{i\theta}, \quad -\pi + \theta_{R,\varepsilon} \leq \theta \leq \pi - \theta_{R,\varepsilon},$$

$$\gamma_1 \; : \; z = x + i\varepsilon, \quad -R\cos\theta_{R,\varepsilon} \le x \le 0,$$

$$\gamma_2 \; : \; z = x - i\varepsilon, \quad -R\cos\theta_{R,\varepsilon} \le x \le 0,$$

$$\gamma_3 \; : \; z = \varepsilon e^{i\theta}, \quad -\pi/2 \le \theta \le \pi/2.$$

We take R so large and ε so small that w and all zeros of K_ν are inside γ. Then, setting

$$\Psi(R,\alpha,\varepsilon) = \frac{1}{2\pi i}\int_\gamma f^w_{\nu,\alpha}(z)dz, \quad \Psi_k(R,\alpha,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma_k} f^w_{\nu,\alpha}(z)dz$$

for $k = 0, 1, 2$ and

$$\Psi_3(\alpha,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma_3} f^w_{\nu,\alpha}(z)dz,$$

we have $\Psi(R,\alpha,\varepsilon) = \Psi_0(R,\alpha,\varepsilon) + \Psi_1(R,\alpha,\varepsilon) - \Psi_2(R,\alpha,\varepsilon) - \Psi_3(\alpha,\varepsilon)$.

The singular points of $f^w_{\nu,\alpha}$ inside γ are w and the zeros of K_ν, which are all poles of order one. Hence the residue theorem yields

$$\Psi(R,\alpha,\varepsilon) = \begin{cases} \mathrm{Res}(w; f^w_{\nu,\alpha}) & \text{if } N(\nu) = 0, \\ \mathrm{Res}(w; f^w_{\nu,\alpha}) + \displaystyle\sum_{j=1}^{N(\nu)} \mathrm{Res}(z_{\nu,j}; f^w_{\nu,\alpha}) & \text{if } N(\nu) \ge 1. \end{cases} \tag{2.9}$$

Here $\mathrm{Res}(v; f)$ is the residue of a function f at a pole v. It is obvious that

$$\mathrm{Res}(w; f^w_{\nu,\alpha}) = \frac{1}{w^\alpha}\frac{K_{\nu+1}(w)}{K_\nu(w)}.$$

When $N(\nu) \ge 1$, by the formula $zK'_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z)$ (cf. [**21**, p. 29]), we have

$$\mathrm{Res}(z_{\nu,j}; f^w_{\nu,\alpha}) = \frac{1}{z^\alpha_{\nu,j}(z_{\nu,j} - w)}\frac{K_{\nu+1}(z_{\nu,j})}{K'_\nu(z_{\nu,j})} = -\frac{1}{z^\alpha_{\nu,j}(z_{\nu,j} - w)}.$$

Hence we obtain from (2.9)

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}\lim_{R\to\infty}\Psi(R,\alpha,\varepsilon) = \begin{cases} \dfrac{K_{\nu+1}(w)}{K_\nu(w)} & \text{if } N(\nu) = 0, \\ \dfrac{K_{\nu+1}(w)}{K_\nu(w)} - \displaystyle\sum_{j=1}^{N(\nu)}\frac{1}{z_{\nu,j} - w} & \text{if } N(\nu) \ge 1 \end{cases} \tag{2.10}$$

if we show that the limit on the left hand side exists.

We fix $\varepsilon > 0$ and $\alpha > 0$. By (2.7) we have for sufficiently large R

$$|\Psi_0(R,\alpha,\varepsilon)| \leq \frac{R}{2\pi} \int_{-\pi+\theta_{R,\varepsilon}}^{\pi-\theta_{R,\varepsilon}} |f_{\nu,\alpha}^w(Re^{i\theta})|d\theta \leq \frac{1}{R^\alpha}\frac{R}{R-|w|}\frac{1+A_{\nu+1}/R}{1-A_\nu/R},$$

which immediately yields that $\Psi_0(R,\alpha,\varepsilon)$ converges to 0 as $R \to \infty$.

For the integral $\Psi_1(R,\alpha,\varepsilon)$, we have

$$\Psi_1(R,\alpha,\varepsilon) = \frac{1}{2\pi i}\int_0^{R\cos\theta_{R,\varepsilon}} \frac{1}{(-x+i\varepsilon)^\alpha(-x+i\varepsilon-w)}\frac{K_{\nu+1}(-x+i\varepsilon)}{K_\nu(-x+i\varepsilon)}dx.$$

Note that $K_\mu(e^{i\pi}z) = e^{-i\pi\mu}K_\mu(z) - i\pi I_\mu(z)$ for $z \in D$ and $\mu \geq 0$ (cf. [21, p. 80]). Then, setting

$$g_{\nu,1}(z) = \frac{K_{\nu+1}(z) + i\pi e^{i\pi\nu}I_{\nu+1}(z)}{K_\nu(z) - i\pi e^{i\pi\nu}I_\nu(z)},$$

we have

$$\frac{K_{\nu+1}(e^{i\pi}z)}{K_\nu(e^{i\pi}z)} = -g_{\nu,1}(z), \tag{2.11}$$

$$\Psi_1(R,\alpha,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma_1^0} \frac{g_{\nu,1}(z)}{e^{i\pi\alpha}z^\alpha(z+w)}dz, \tag{2.12}$$

where γ_1^0 is the line in D defined by $\gamma_1^0 : z = x - i\varepsilon$, $0 \leq x \leq R\cos\theta_{R,\varepsilon}$.

We now define three paths as follows:

$$\gamma_1^1 : z = \varepsilon e^{i\theta}, \quad -\pi/2 \leq \theta \leq 0, \qquad \gamma_1^2 : z = x, \ \varepsilon \leq x \leq R,$$

$$\gamma_1^3 : z = Re^{i\theta}, \quad -\theta_{R,\varepsilon} \leq \theta \leq 0.$$

Since w is inside γ, we have that $|\operatorname{Im}(w)| > \varepsilon$ if $\operatorname{Re}(w) < 0$. There is no zero of K_ν on the real axis and the integrand of the right hand side of (2.12) is holomorphic inside and on the contour consisting of γ_1^0, γ_1^1, γ_1^2 and γ_1^3. Then the Cauchy integral theorem yields that $\Psi_1(R,\alpha,\varepsilon) = \Psi_1^1(\alpha,\varepsilon) + \Psi_1^2(R,\alpha,\varepsilon) - \Psi_1^3(R,\alpha,\varepsilon)$, where

$$\Psi_1^1(\alpha,\varepsilon) = \frac{1}{2\pi}\int_{-\pi/2}^0 \frac{\varepsilon e^{i\theta}g_{\nu,1}(\varepsilon e^{i\theta})}{e^{i(\theta+\pi)\alpha}\varepsilon^\alpha(\varepsilon e^{i\theta}+w)}d\theta,$$

$$\Psi_1^2(R,\alpha,\varepsilon) = \frac{1}{2\pi i}\int_\varepsilon^R \frac{g_{\nu,1}(x)}{e^{i\pi\alpha}x^\alpha(x+w)}dx,$$

$$\Psi_1^3(R,\alpha,\varepsilon) = \frac{1}{2\pi}\int_{-\theta_{R,\varepsilon}}^0 \frac{Re^{i\theta}g_{\nu,1}(Re^{i\theta})}{e^{i(\theta+\pi)\alpha}R^\alpha(Re^{i\theta}+w)}d\theta.$$

If $-\pi/2 \leq \theta \leq 0$, we deduce from (2.7) and (2.11)

$$|g_{\nu,1}(Re^{i\theta})| = \left| \frac{K_{\nu+1}(Re^{i(\theta+\pi)})}{K_{\nu}(Re^{i(\theta+\pi)})} \right| \le \frac{1 + A_{\nu+1}/R}{1 - A_{\nu}/R} \tag{2.13}$$

for sufficiently large R, which immediately implies that $\Psi_1^3(R,\alpha,\varepsilon) \to 0$ as $R \to \infty$. Moreover, from (2.13) for $R = x$ and $\theta = 0$, it follows that

$$\lim_{R\to\infty} \Psi_1^2(R,\alpha,\varepsilon) = \frac{1}{2\pi i} \int_{\varepsilon}^{\infty} \frac{g_{\nu,1}(x)}{e^{i\pi\alpha}x^{\alpha}(x+w)}dx,$$

for which we write $\Psi_1^2(\alpha,\varepsilon)$.

For the integral $\Psi_2(R,\alpha,\varepsilon)$, we use $K_{\mu}(e^{-i\pi}z) = e^{i\pi\mu}K_{\mu}(z) + i\pi I_{\mu}(z)$ for $z \in D$ and $\mu \ge 0$ (cf. [**21**, p. 80]) instead of the formula for $K_{\mu}(e^{i\pi}z)$. In the same fashion as $\Psi_1(R,\alpha,\varepsilon)$, we can easily see that $\Psi_2(R,\alpha,\varepsilon)$ converges to $-\Psi_2^1(\alpha,\varepsilon) + \Psi_2^2(\alpha,\varepsilon)$ as $R \to \infty$, where

$$\Psi_2^1(\alpha,\varepsilon) = \frac{1}{2\pi} \int_0^{\pi/2} \frac{\varepsilon e^{i\theta} g_{\nu,2}(\varepsilon e^{i\theta})}{e^{i(\theta-\pi)\alpha}\varepsilon^{\alpha}(\varepsilon e^{i\theta}+w)}d\theta,$$

$$\Psi_2^2(\alpha,\varepsilon) = \frac{1}{2\pi i} \int_{\varepsilon}^{\infty} \frac{g_{\nu,2}(x)}{e^{-i\pi\alpha}x^{\alpha}(x+w)}dx,$$

$$g_{\nu,2}(z) = \frac{K_{\nu+1}(z) - i\pi e^{-i\pi\nu}I_{\nu+1}(z)}{K_{\nu}(z) + i\pi e^{-i\pi\nu}I_{\nu}(z)}.$$

Therefore we conclude that

$$\lim_{R\to\infty} \Psi(R,\alpha,\varepsilon) = \Psi_1^1(\alpha,\varepsilon) + \Psi_1^2(\alpha,\varepsilon) + \Psi_2^1(\alpha,\varepsilon) - \Psi_2^2(\alpha,\varepsilon) - \Psi_3(\alpha,\varepsilon). \tag{2.14}$$

We next consider the limiting behavior of the integral on (2.14) as $\alpha,\varepsilon \to 0$. We first let $\alpha \to 0$. Then it is easy to see

$$\lim_{\alpha\to 0} \Psi_1^1(\alpha,\varepsilon) = \frac{1}{2\pi} \int_{-\pi/2}^{0} \frac{\varepsilon e^{i\theta} g_{\nu,1}(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}+w}d\theta,$$

$$\lim_{\alpha\to 0} \Psi_2^1(\alpha,\varepsilon) = \frac{1}{2\pi} \int_0^{\pi/2} \frac{\varepsilon e^{i\theta} g_{\nu,2}(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}+w}d\theta,$$

$$\lim_{\alpha\to 0} \Psi_3(\alpha,\varepsilon) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}-w} \frac{K_{\nu+1}(\varepsilon e^{i\theta})}{K_{\nu}(\varepsilon e^{i\theta})}d\theta.$$

It is known that

$$K_{\mu}(z) = \begin{cases} \left(\log\dfrac{2}{z}\right)\{1+o(1)\} & \text{if } \mu = 0, \\[2mm] \dfrac{\Gamma(\mu)}{2}\left(\dfrac{2}{z}\right)^{\mu}\{1+o(1)\} & \text{if } \mu > 0 \end{cases} \tag{2.15}$$

as $|z| \to 0$ in D (cf. [**21**, p. 512]). Moreover, it is easy to see

$$I_\mu(z) = \begin{cases} 1 + o(1) & \text{if } \mu = 0, \\ \dfrac{1}{\Gamma(\mu+1)} \left(\dfrac{z}{2}\right)^\mu \{1 + o(1)\} & \text{if } \mu > 0 \end{cases} \tag{2.16}$$

as $|z| \to 0$ in D by the series expression. With the help of (2.15) and (2.16), we obtain that $zg_{\nu,1}(z)/(z+w)$ and $zg_{\nu,2}(z)/(z+w)$ tend to 0 if $\nu = 0$ and to $2\nu/w$ if $\nu > 0$ as $|z| \to 0$ in D. Hence we have

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \Psi_1^1(\alpha, \varepsilon) = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \Psi_2^1(\alpha, \varepsilon) = \frac{\nu}{2w}. \tag{2.17}$$

Moreover, since $zK_{\nu+1}(z)/(z-w)K_\nu(z)$ tends to $-2\nu/w$ as $|z| \to 0$ in D, we obtain

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \Psi_3(\alpha, \varepsilon) = -\frac{\nu}{w}$$

and

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \{\Psi_1^1(\alpha, \varepsilon) + \Psi_1^2(\alpha, \varepsilon) - \Psi_3(\alpha, \varepsilon)\} = \frac{2\nu}{w}. \tag{2.18}$$

Next we set $\Phi(\alpha, \varepsilon) = \Psi_1^2(\alpha, \varepsilon) - \Psi_2^2(\alpha, \varepsilon)$. Then, putting

$$F_\nu(x) = -2i\sin(\pi\alpha)\{K_{\nu+1}(x)K_\nu(x) - \pi^2 I_{\nu+1}(x)I_\nu(x)\}$$
$$+ i\pi e^{-i\pi\alpha}\{e^{i\pi\nu}I_{\nu+1}(x)K_\nu(x) + e^{-i\pi\nu}K_{\nu+1}(x)I_\nu(x)\}$$
$$+ i\pi e^{i\pi\alpha}\{e^{-i\pi\nu}I_{\nu+1}(x)K_\nu(x) + e^{i\pi\nu}K_{\nu+1}(x)I_\nu(x)\}$$

and

$$h_\nu(x) = \frac{F_\nu(x)}{G_\nu(x)} = \frac{g_{\nu,1}(x)}{e^{i\pi\alpha}} - \frac{g_{\nu,2}(x)}{e^{-i\pi\alpha}},$$

where $G_\nu(x)$ is given by (2.1), we have

$$\Phi(\alpha, \varepsilon) = \frac{1}{2\pi i} \int_\varepsilon^\infty \frac{h_\nu(x)}{x^\alpha(x+w)} dx.$$

The formula $K_{\nu+1}(x)I_\nu(x) + I_{\nu+1}(x)K_\nu(x) = 1/x$ (cf. [**21**, p. 80]) yields that

$$e^{-i\pi\alpha}\{e^{i\pi\nu}I_{\nu+1}(x)K_\nu(x) + e^{-i\pi\nu}K_{\nu+1}(x)I_\nu(x)\}$$
$$+ e^{i\pi\alpha}\{e^{-i\pi\nu}I_{\nu+1}(x)K_\nu(x) + e^{i\pi\nu}K_{\nu+1}(x)I_\nu(x)\}$$

$$= e^{-i\pi\alpha}\left\{\frac{e^{-i\pi\nu}}{x} + 2i\sin(\pi\nu)I_{\nu+1}(x)K_\nu(x)\right\} + e^{i\pi\alpha}\left\{\frac{e^{i\pi\nu}}{x} - 2i\sin(\pi\nu)I_{\nu+1}(x)K_\nu(x)\right\}$$

$$= \frac{2\cos\pi(\alpha+\nu)}{x} + 4\sin(\pi\nu)\sin(\pi\alpha)I_{\nu+1}(x)K_\nu(x).$$

Hence, letting

$$F_{\nu,1}(x) = 2i\sin(\pi\alpha)K_{\nu+1}(x)K_\nu(x), \quad F_{\nu,2}(x) = 2i\pi^2\sin(\pi\alpha)I_{\nu+1}(x)I_\nu(x),$$

$$F_{\nu,3}(x) = \frac{2i\pi\cos\pi(\alpha+\nu)}{x}, \qquad\qquad F_{\nu,4}(x) = 4i\pi\sin(\pi\nu)\sin(\pi\alpha)I_{\nu+1}(x)K_\nu(x),$$

we have

$$h_\nu(x) = \frac{-F_{\nu,1}(x) + F_{\nu,2}(x) + F_{\nu,3}(x) + F_{\nu,4}(x)}{G_\nu(x)}.$$

We set

$$\Phi_k(\alpha,\varepsilon) = \frac{1}{2\pi i}\int_\varepsilon^\infty \frac{1}{x^\alpha(x+w)}\frac{F_{\nu,k}(x)}{G_\nu(x)}\,dx, \quad 1 \le k \le 4.$$

By virtue of Lemma 2.3, the functions

$$\frac{1}{x+w}\frac{K_{\nu+1}(x)K_\nu(x)}{G_\nu(x)}, \quad \frac{1}{x+w}\frac{I_{\nu+1}(x)K_\nu(x)}{G_\nu(x)}$$

are integrable on (ε,∞). Hence we get

$$\lim_{\alpha\to 0}\Phi_1(\alpha,\varepsilon) = \lim_{\alpha\to 0}\Phi_4(\alpha,\varepsilon) = 0. \tag{2.19}$$

The integral $\Phi_3(\alpha,\varepsilon)$ is written as

$$\Phi_3(\alpha,\varepsilon) = \cos\pi(\alpha+\nu)\int_\varepsilon^\infty \frac{dx}{x^{1+\alpha}(x+w)G_\nu(x)}$$

and, with the help of Lemma 2.3, we can derive

$$\lim_{\alpha\to 0}\Phi_3(\alpha,\varepsilon) = \cos(\pi\nu)\int_\varepsilon^\infty \frac{dx}{x(x+w)G_\nu(x)}. \tag{2.20}$$

It follows from (2.15) and (2.16) that

$$G_\nu(x) = \begin{cases} \left(\log\dfrac{1}{x}\right)^2\{1+o(1)\} & \text{if } \nu = 0, \\[2mm] \dfrac{1}{\kappa_\nu x^{2\nu}}\{1+o(1)\} & \text{if } \nu > 0 \end{cases} \tag{2.21}$$

as $x \to 0$, where $\kappa_\nu = 1/4^{\nu-1}\{\Gamma(\nu)\}^2$. This implies the convergence of the right hand side of (2.20) as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \Phi_3(\alpha, \varepsilon) = \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+w)G_\nu(x)}. \tag{2.22}$$

It remains to consider $\Phi_2(\alpha, \varepsilon)$,

$$\Phi_2(\alpha, \varepsilon) = \pi \sin(\pi\alpha) \int_\varepsilon^\infty \frac{1}{x^\alpha(x+w)} \frac{I_{\nu+1}(x)I_\nu(x)}{G_\nu(x)} dx.$$

We should remark that the function $I_{\nu+1}(x)I_\nu(x)/(x+w)G_\nu(x)$ is not integrable on (ε, ∞). We write $\Phi_2(\alpha, \varepsilon) = -\Phi_2^1(\alpha, \varepsilon) - \Phi_2^2(\alpha, \varepsilon) + \Phi_2^3(\alpha, \varepsilon) + \Phi_2^4(\alpha, \varepsilon)$, where

$$\Phi_2^1(\alpha, \varepsilon) = \frac{\sin(\pi\alpha)}{\pi} \int_\varepsilon^\infty \frac{1}{x^\alpha(x+w)} \frac{K_\nu(x)^2}{G_\nu(x)} dx,$$

$$\Phi_2^2(\alpha, \varepsilon) = 2\sin(\pi\nu)\sin(\pi\alpha) \int_\varepsilon^\infty \frac{1}{x^\alpha(x+w)} \frac{I_{\nu+1}(x)K_\nu(x)}{G_\nu(x)} dx,$$

$$\Phi_2^3(\alpha, \varepsilon) = \pi \sin(\pi\alpha) \int_\varepsilon^\infty \frac{1}{x^\alpha(x+w)} \frac{I_\nu(x)\{I_\nu(x) - I_{\nu+1}(x)\}}{G_\nu(x)} dx,$$

$$\Phi_2^4(\alpha, \varepsilon) = \frac{\sin(\pi\alpha)}{\pi} \int_\varepsilon^\infty \frac{dx}{x^\alpha(x+w)}.$$

We can easily deduce from Lemma 2.3

$$\lim_{\alpha \to 0} \Phi_2^1(\alpha, \varepsilon) = \lim_{\alpha \to 0} \Phi_2^2(\alpha, \varepsilon) = 0$$

in the same way as (2.19).

To compute $\Phi_2^3(\alpha, \varepsilon)$, we note the following:

$$\frac{I_\nu(x)\{I_\nu(x) - I_{\nu+1}(x)\}}{G_\nu(x)} = \frac{2\nu+1}{\pi^2 x}\{1 + o(1)\}, \quad x \to \infty,$$

which is obtained from (2.8) and Lemma 2.3. Since

$$\frac{1}{x+w} \frac{I_\nu(x)\{I_\nu(x) - I_{\nu+1}(x)\}}{G_\nu(x)}$$

is integrable on (ε, ∞), we have that $\Phi_2^3(\alpha, \varepsilon)$ tends to 0 as $\alpha \to 0$.

The calculation of $\Phi_2^4(\alpha, \varepsilon)$ is easy. In fact we have

$$\Phi_2^4(\alpha, \varepsilon) = \frac{\sin(\pi\alpha)}{\pi} \int_\varepsilon^\infty \frac{dx}{x^{1+\alpha}} - \frac{w\sin(\pi\alpha)}{\pi} \int_\varepsilon^\infty \frac{dx}{x^{1+\alpha}(x+w)}$$

$$= \frac{\sin(\pi\alpha)}{\pi\alpha\varepsilon^\alpha} - \frac{w\sin(\pi\alpha)}{\pi} \int_\varepsilon^\infty \frac{dx}{x^{1+\alpha}(x+w)} \to 1, \quad \alpha \to 0$$

for any $\varepsilon > 0$. Hence we conclude from (2.19) and (2.22)

$$\lim_{\varepsilon\to0}\lim_{\alpha\to0} \Phi(\alpha,\varepsilon) = 1 + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+w)G_\nu(x)}. \tag{2.23}$$

Combining (2.10), (2.14), (2.17), (2.18) and (2.23), we complete our proof of (2.4) and (2.5) in the case of $\nu \geq 0$. $\square$

3. The first hitting time of the Bessel process.

For $\nu \in \mathbb{R}$ the one-dimensional diffusion process with infinitesimal generator

$$\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx} = \frac{1}{2x^{2\nu+1}}\frac{d}{dx}\left(x^{2\nu+1}\frac{d}{dx}\right)$$

is called the Bessel process with index ν. The classification of boundary points gives the following information. The endpoint ∞ is a natural boundary for any $\nu \in \mathbb{R}$. For $\nu \geq 0$, 0 is an entrance and not exit boundary. For $-1 < \nu < 0$, 0 is a regular boundary, which is instantly reflecting. For $\nu \leq -1$, 0 is an exit but not entrance boundary. For more details, see [**12**] and [**19**] for example.

For $a, b \in \mathbb{R}$ let $\tau_{a,b}^{(\nu)}$ be the first hitting time to b of the Bessel process with index ν starting at a. The conditional distribution of $\tau_{a,b}^{(\nu)}$ under $\tau_{a,b}^{(\nu)} < \infty$ is infinitely divisible. The purpose of this section is to give the exact form of the Lévy measure $m_{a,b}^{(\nu)}$ when $0 \leq b < a$ by applying Theorem 2.1.

It is known that, when $0 \leq a < b$, the distribution of $\tau_{a,b}^{(\nu)}$ is a mixture of exponential distributions. Let us recall the results in [**2**]. See also [**12**]. In this case, the Laplace transforms of the conditional distributions are given by the following. For $\lambda > 0$ we have

$$E\left[e^{-\lambda\tau_{a,b}^{(\nu)}} \,\middle|\, \tau_{a,b}^{(\nu)} < \infty\right] = \begin{cases} \dfrac{(b\sqrt{2\lambda})^\nu}{2^\nu \Gamma(\nu+1)} \dfrac{1}{I_\nu(b\sqrt{2\lambda})} & \text{if } a = 0 < b,\ \nu > -1, \\[3mm] \left(\dfrac{b}{a}\right)^\nu \dfrac{I_\nu(a\sqrt{2\lambda})}{I_\nu(b\sqrt{2\lambda})} & \text{if } 0 < a \leq b,\ \nu > -1, \\[3mm] \left(\dfrac{a}{b}\right)^\nu \dfrac{I_{-\nu}(a\sqrt{2\lambda})}{I_{-\nu}(b\sqrt{2\lambda})} & \text{if } 0 < a \leq b,\ \nu \leq -1. \end{cases}$$

Combining these results with the formula

$$I_\mu(x) = \left(\frac{x}{2}\right)^\mu \frac{1}{\Gamma(\mu+1)} \prod_{n=1}^\infty \left(1 + \frac{x^2}{j_{\mu,n}^2}\right)$$

for $\mu > -1$ and $x > 0$, where $\{j_{\mu,n}\}_{n=1}^\infty$ is an increasing sequence of positive zeros of the

Bessel function J_μ of the first kind of order μ, we obtain the following expressions for the Lévy measures:

$$
\frac{dm_{a,b}^{(\nu)}(x)}{dx} =
\begin{cases}
\dfrac{1_{(0,\infty)}(x)}{x} \displaystyle\sum_{n=1}^{\infty} e^{-(j_{\nu,n}^2/2b^2)x} & \text{if } a = 0 < b, \ \nu > -1, \\[3mm]
\dfrac{1_{(0,\infty)}(x)}{x} \displaystyle\sum_{n=1}^{\infty} \left(e^{-(j_{\nu,n}^2/2b^2)x} - e^{-(j_{\nu,n}^2/2a^2)x} \right) & \text{if } 0 < a \le b, \ \nu > -1, \\[3mm]
\dfrac{1_{(0,\infty)}(x)}{x} \displaystyle\sum_{n=1}^{\infty} \left(e^{-(j_{-\nu,n}^2/2b^2)x} - e^{-(j_{-\nu,n}^2/2a^2)x} \right) & \text{if } 0 < a \le b, \ \nu \le -1,
\end{cases}
$$

where 1_A is the indicator function of a set A.

The following is the main result in this section.

THEOREM 3.1. *For $0 \le b < a$ the support of the Lévy measure $m_{a,b}^{(\nu)}$ is $[0,\infty)$ and it is absolutely continuous with respect to the Lebesgue measure. We have the following expressions for the density $p_{a,b}^{(\nu)}$, $x > 0$ of $m_{a,b}^{(\nu)}$.*

(1) *If $a > 0$,*

$$
p_{a,0}^{(-1/2)}(x) = \frac{a}{\sqrt{2\pi x^3}}.
$$

(2) *If $a > 0$, $\nu - 1/2 \in \mathbb{Z}$ and $\nu \le -3/2$,*

$$
p_{a,0}^{(\nu)}(x) = \frac{a}{\sqrt{2\pi x^3}} - \frac{1}{2\sqrt{\pi x^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\xi^2/4x + z_{\nu,j}\xi/\sqrt{2}a}\,d\xi.
$$

(3) *If $a > 0$, $-3/2 < \nu < 0$ and $\nu \neq -1/2$,*

$$
p_{a,0}^{(\nu)}(x) = \frac{a}{\sqrt{2\pi x^3}} + \frac{\cos(\pi\nu)}{2\sqrt{\pi x^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_{|\nu|}(\eta)} e^{-\xi^2/4x + \xi\eta/\sqrt{2}a}\,d\xi d\eta.
$$

(4) *If $a > 0$, $\nu - 1/2 \notin \mathbb{Z}$ and $\nu < -3/2$,*

$$
p_{a,0}^{(\nu)}(x) = \frac{a}{\sqrt{2\pi x^3}} - \frac{1}{2\sqrt{\pi x^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\xi^2/4x + z_{\nu,j}\xi/\sqrt{2}a}\,d\xi
$$

$$
+ \frac{\cos(\pi\nu)}{2\sqrt{\pi x^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_{|\nu|}(\eta)} e^{-\xi^2/4x + \xi\eta/\sqrt{2}a}\,d\xi d\eta.
$$

(5) *If $0 < b < a$ and $\nu = \pm 1/2$,*

$$
p_{a,b}^{(\nu)}(x) = \frac{a - b}{\sqrt{2\pi x^3}}.
$$

(6) If $0 < b < a$, $\nu - 1/2 \in \mathbb{Z}$ and $|\nu| \geq 3/2$,

$$p_{a,b}^{(\nu)}(x) = \frac{a-b}{\sqrt{2\pi x^3}} - \frac{1}{2\sqrt{\pi x^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\xi^2/4x} \big(e^{z_{\nu,j}\xi/\sqrt{2a}} - e^{z_{\nu,j}\xi/\sqrt{2b}}\big) d\xi.$$

(7) If $0 < b < a$, $0 \leq |\nu| < 3/2$ and $\nu \neq \pm 1/2$,

$$p_{a,b}^{(\nu)}(x) = \frac{a-b}{\sqrt{2\pi x^3}} + \frac{\cos(\pi\nu)}{2\sqrt{\pi x^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_{|\nu|}(\eta)} e^{-\xi^2/4x} \big(e^{-\xi\eta/\sqrt{2a}} - e^{-\xi\eta/\sqrt{2b}}\big) d\xi d\eta.$$

(8) If $0 < b < a$, $\nu - 1/2 \notin \mathbb{Z}$ and $|\nu| > 3/2$,

$$p_{a,b}^{(\nu)}(x) = \frac{a-b}{\sqrt{2\pi x^3}} - \frac{1}{2\sqrt{\pi x^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\xi^2/4x} \big(e^{z_{\nu,j}\xi/\sqrt{2a}} - e^{z_{\nu,j}\xi/\sqrt{2b}}\big) d\xi$$

$$+ \frac{\cos(\pi\nu)}{2\sqrt{\pi x^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_{|\nu|}(\eta)} e^{-\xi^2/4x} \big(e^{-\xi\eta/\sqrt{2a}} - e^{-\xi\eta/\sqrt{2b}}\big) d\xi d\eta.$$

The rest of this section is devoted to the proof of Theorem 3.1. By the formulae for the Laplace transforms of $\tau_{a,b}^{(\nu)}$, we have, for $\lambda > 0$, if $a > 0$ and $\nu < 0$,

$$E\big[e^{-\lambda\tau_{a,0}^{(\nu)}} \,\big|\, \tau_{a,0}^{(\nu)} < \infty\big] = \frac{2^{\nu+1}}{\Gamma(|\nu|)(a\sqrt{2\lambda})^\nu} K_\nu(a\sqrt{2\lambda}), \tag{3.1}$$

if $0 < b \leq a$ and $\nu \in \mathbb{R}$,

$$E\big[e^{-\lambda\tau_{a,b}^{(\nu)}} \,\big|\, \tau_{a,b}^{(\nu)} < \infty\big] = \Big(\frac{a}{b}\Big)^{|\nu|} \frac{K_{|\nu|}(a\sqrt{2\lambda})}{K_{|\nu|}(b\sqrt{2\lambda})}. \tag{3.2}$$

We represent $K_\nu(x)$ in terms of G_ν and zeros of K_ν.

PROPOSITION 3.2. *For $x > 0$ we have the following formulae.*

(1) If $\mu = 1/2$,

$$\log\{x^{1/2} K_{1/2}(x)\} = \frac{1}{2} \log \frac{\pi}{2} - x. \tag{3.3}$$

(2) If $\mu - 1/2 \in \mathbb{Z}$ and $\mu \geq 3/2$,

$$\log\{x^\mu K_\mu(x)\} = \log\{2^{\mu-1}\Gamma(\mu)\} - x - \sum_{j=1}^{N(\mu)} \log \frac{z_{\mu,j}}{z_{\mu,j} - x}. \tag{3.4}$$

(3) If $0 < \mu < 3/2$ and $\mu \neq 1/2$,

$$\log\{x^\mu K_\mu(x)\} = \log\{2^{\mu-1}\Gamma(\mu)\} - x - \cos(\pi\mu)\int_0^\infty \frac{1}{yG_\mu(y)}\log\frac{y+x}{y}dy. \quad (3.5)$$

(4) If $\mu - 1/2 \notin \mathbb{Z}$ and $\mu > 3/2$,

$$\log\{x^\mu K_\mu(x)\} = \log\{2^{\mu-1}\Gamma(\mu)\} - x - \sum_{j=1}^{N(\mu)}\log\frac{z_{\mu,j}}{z_{\mu,j}-x}$$

$$- \cos(\pi\mu)\int_0^\infty \frac{1}{yG_\mu(y)}\log\frac{y+x}{y}dy. \quad (3.6)$$

Proof. Formula (3.3) is obtained from the explicit expression for $K_{1/2}(x)$. In order to show the others, we note the following formula:

$$\frac{d}{dx}\log\{x^\nu K_\nu(x)\} = -\frac{K_{\nu+1}(x)}{K_\nu(x)} + \frac{2\nu}{x}, \quad (3.7)$$

which can be derived by (2.6) and $(x^\nu K_\nu(x))' = -x^\nu K_{\nu-1}(x)$ (cf. [**21**, p. 79]).

If $\nu - 1/2 \in \mathbb{Z}$ and $\nu \geq 3/2$, it follows from (2.3) and (3.7) that

$$\frac{d}{dx}\log\{x^\nu K_\nu(x)\} = -1 - \sum_{j=1}^{N(\nu)}\frac{1}{z_{\nu,j}-x}.$$

Then we obtain that, for any $\varepsilon > 0$,

$$\log\{x^\nu K_\nu(x)\} - \log\{\varepsilon^\nu K_\nu(\varepsilon)\} = -(x-\varepsilon) + \sum_{j=1}^{N(\nu)}\int_\varepsilon^x \frac{d\xi}{\xi - z_{\nu,j}}.$$

Hence, letting $\varepsilon \to 0$, we get (3.4) with the help of

$$\lim_{x\to 0} x^\nu K_\nu(x) = 2^{\nu-1}\Gamma(\nu). \quad (3.8)$$

If $0 < \nu < 3/2$ and $\nu \neq 1/2$, it follows from (2.4) and (3.7) that

$$\frac{d}{dx}\log\{x^\nu K_\nu(x)\} = -1 - \cos(\pi\nu)\int_0^\infty \frac{dy}{y(y+x)G_\nu(y)}.$$

Hence we have

$$\log\{x^\nu K_\nu(x)\} - \log\{\varepsilon^\nu K_\nu(\varepsilon)\} = -(x-\varepsilon) - \cos(\pi\nu)\int_\varepsilon^x d\xi \int_0^\infty \frac{dy}{y(y+\xi)G_\nu(y)}.$$

Note that $G_\nu(x)$ is positive for $x > 0$ unless ν is $2n + 3/2$ for any integer n. Thus it follows from (3.8) that

$$\log\{x^\nu K_\nu(x)\} - \log\{2^{\nu-1}\Gamma(\nu)\} = -x - \cos(\pi\nu)\int_0^x d\xi \int_0^\infty \frac{dy}{y(y+\xi)G_\nu(y)}.$$

By the Fubini theorem, the right hand side is equal to

$$-x - \cos(\pi\nu)\int_0^\infty \frac{dy}{yG_\nu(y)}\int_0^x \frac{d\xi}{y+\xi},$$

which yields (3.5).

We can show (3.6) in the same way. □

In order to see Theorem 3.1, we only have to check

$$\phi_{a,b}^{(\nu)}(\lambda) := \log E\big[e^{-\lambda\tau_{a,b}^{(\nu)}}\,\big|\,\tau_{a,b}^{(\nu)} < \infty\big] = \int_0^\infty (e^{-\lambda x} - 1)p_{a,b}^{(\nu)}(x)dx$$

for each case. It follows from (3.1) and (3.2) that, if $a > 0$ and $\nu < 0$,

$$\phi_{a,0}^{(\nu)}(\lambda) = \log\big\{(a\sqrt{2\lambda})^{|\nu|}K_{|\nu|}(a\sqrt{2\lambda})\big\} - \log\{2^{|\nu|-1}\Gamma(|\nu|)\},$$

and that, if $0 < b < a$ and $\nu \in \mathbb{R}$,

$$\phi_{a,b}^{(\nu)}(\lambda) = \log\big\{(a\sqrt{2\lambda})^{|\nu|}K_{|\nu|}(a\sqrt{2\lambda})\big\} - \log\{(b\sqrt{2\lambda})^{|\nu|}K_{|\nu|}(b\sqrt{2\lambda})\}. \tag{3.9}$$

The following lemma and Proposition 3.2 prove Theorem 3.1 for $\nu \neq 0$.

LEMMA 3.3.   *Let $c > 0$, $\nu > 0$ and $z \in \mathbb{C}^-$. For $\lambda > 0$ it follows that*

$$\sqrt{2\lambda} = -\int_0^\infty \frac{e^{-\lambda x} - 1}{\sqrt{2\pi x^3}}dx, \tag{3.10}$$

$$\log\frac{z}{z - c\sqrt{2\lambda}} = \int_0^\infty \frac{e^{-\lambda x} - 1}{2\sqrt{\pi x^3}}dx\int_0^\infty e^{-\xi^2/4x + z\xi/\sqrt{2}c}d\xi \tag{3.11}$$

and

$$\int_0^\infty \log\frac{\eta + c\sqrt{2\lambda}}{\eta}\frac{1}{\eta G_\nu(\eta)}d\eta$$

$$= -\int_0^\infty \frac{e^{-\lambda x} - 1}{2\sqrt{\pi x^3}}dx\int_0^\infty\int_0^\infty \frac{1}{\eta G_\nu(\eta)}e^{-\xi^2/4x - \xi\eta/\sqrt{2}c}d\xi d\eta. \tag{3.12}$$

PROOF.   We recall the following formula (cf. [**3**, p. 361]): for $-1 < p < 0$,

$$\int_0^\infty y^{p-1}(e^{-y} - 1)dy = \Gamma(p).$$

Setting $p = -1/2$ and noting $\Gamma(-1/2) = -2\sqrt{\pi}$, we obtain (3.10).

Recall the formulae

$$\int_0^\infty e^{-\lambda x - \xi^2/4x} x^{-3/2} dx = \frac{2\sqrt{\pi}}{\xi} e^{-\xi\sqrt{\lambda}} \tag{3.13}$$

for $\lambda \geq 0$, $\xi > 0$ and

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx = \log \frac{\beta}{\alpha}$$

for $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$ (cf. [**3**, p. 361]). Then we obtain

$$\int_0^\infty \int_0^\infty \frac{1 - e^{-\lambda x}}{x^{3/2}} e^{-\xi^2/4x - \alpha\xi} dx d\xi = 2\sqrt{\pi} \log \frac{\sqrt{\lambda} + \alpha}{\alpha}.$$

This yields (3.11). Since $G_\nu(x)$ is positive for $x > 0$, we may apply the Fubini theorem and obtain (3.12) in a similar way to (3.11). $\qquad\square$

REMARK 3.4. If $\nu = 0$, it follows from (2.21) that the left hand side of (3.12) diverges.

We finally consider the case of $\nu = 0$. Since $K_0'(x) = -K_1(x)$, we have

$$\frac{d}{dx} \log K_0(x) = -\frac{K_1(x)}{K_0(x)} = -1 - \int_0^\infty \frac{d\eta}{\eta(\eta + x)G_0(\eta)}$$

by Theorem 2.1. Note that $1/xG_0(x)$ is integrable on $(0, \infty)$. Hence, by (3.9), we get

$$\phi_{a,b}^{(0)}(\lambda) = -(a - b)\sqrt{2\lambda} - \int_{b\sqrt{2\lambda}}^{a\sqrt{2\lambda}} d\xi \int_0^\infty \frac{d\eta}{\eta(\eta + x)G_0(\eta)}$$

$$= -(a - b)\sqrt{2\lambda} - \int_0^\infty \frac{1}{\eta G_0(\eta)} \log \frac{\eta + a\sqrt{2\lambda}}{\eta + b\sqrt{2\lambda}} d\eta.$$

For $c > 0$ and $\varepsilon > 0$, we write

$$\int_\varepsilon^\infty \log \frac{\eta + c\sqrt{2\lambda}}{\eta} \frac{1}{\eta G_0(\eta)} d\eta = -\int_0^\infty \frac{e^{-\lambda x} - 1}{2\sqrt{\pi}x^3} dx \int_\varepsilon^\infty \frac{d\eta}{\eta G_0(\eta)} \int_0^\infty e^{-\xi^2/4x - \xi\eta/\sqrt{2}c} d\xi.$$

This formula immediately implies

$$\int_\varepsilon^\infty \frac{1}{\eta G_0(\eta)} \log \frac{\eta + a\sqrt{2\lambda}}{\eta + b\sqrt{2\lambda}} d\eta$$

$$= \int_0^\infty \frac{e^{-\lambda x} - 1}{2\sqrt{\pi}x^3} dx \int_\varepsilon^\infty \frac{d\eta}{\eta G_0(\eta)} \int_0^\infty e^{-\xi^2/4x} \left(e^{-\xi\eta/\sqrt{2}b} - e^{-\xi\eta/\sqrt{2}a} \right) d\xi. \tag{3.14}$$

It is easy to see from (3.12) and (3.13) that the right hand side of (3.14) converges as $\varepsilon \to 0$ and thus we obtain Theorem 3.1 in the case of $\nu = 0$.

4. The expected volume of the Wiener sausage.

Let $r > 0$ be given. The Wiener sausage $\{W(t)\}_{t \geq 0}$ for the Brownian motion with radius r is defined by

$$W(t) = \{x \in \mathbb{R}^d \,;\, x + B(s) \in U \text{ for some } s \in [0, t]\}$$

for $t \geq 0$, where $\{B(t)\}_{t \geq 0}$ is a Brownian motion on $\mathbb{R}^d$ and U is the closed ball with center 0 and radius r. For $t > 0$ let

$$L(t) = \int_{\mathbb{R}^d \setminus U} P_x[\tau \leq t] dx,$$

where $\tau = \inf\{t \geq 0 \,;\, B(t) \in U\}$ and P_x is the law of the Brownian motion starting from $x \in \mathbb{R}^d$. It is easy to see that the expectation of the volume of $W(t)$ coincides with the sum of $L(t)$ and the volume of U.

In the case when d is odd, the explicit form of $L(t)$ has already given. One and three dimensional cases are easy. Indeed, we have that $L(t)$ is equal to $2\sqrt{2t/\pi}$ if $d = 1$ and $2\pi rt + 4r^2\sqrt{2\pi t}$ if $d = 3$. These formulae can be obtained directly from the well-known formula for $P_x[\tau \leq t]$. For details, see [5], [12], [16]. In the higher dimensional cases, the authors [8] recently obtained an explicit form of $P_x[\tau \leq t]$. However it is not of a convenient form for the integration on x.

We here consider the Laplace transform of L given by

$$\int_0^\infty e^{-\lambda t} L(t) dt = \frac{S_{d-1} r^{d-1}}{\sqrt{2\lambda^3}} \frac{K_{d/2}(r\sqrt{2\lambda})}{K_{d/2-1}(r\sqrt{2\lambda})}, \quad \lambda > 0, \tag{4.1}$$

where S_{d-1} is the surface area of $d - 1$ dimensional unit sphere (cf. [5]). When d is odd, since $K_{d/2}(x)/K_{d/2-1}(x)$ may be expressed by the ratio of polynomials for $x > 0$, and the right hand side of (4.1) may be represented by the linear combination of rational functions of the following four types: $\lambda^{-1/2}$, λ^{-1}, $\lambda^{-3/2}$, $(\sqrt{\lambda} - z)^{-1}$. Hence the Laplace transform on (4.1) can be inverted. When d is odd and more than or equal to five, Theorem 1.1 in [6] shows that, for $t > 0$

$$L(t) = S_{d-1} r^{d-2} \left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\sqrt{2} r^3}{\sqrt{\pi t}} \sum_{j=1}^{N_d} \frac{1}{(z_j^{(d)})^2} \int_0^\infty e^{-r^2 x^2/2t + z_j^{(d)} x} dx \right].$$

Here we have used $z_j^{(d)}$ and N_d instead of $z_{d/2-1, j}$ and $N(d/2 - 1)$, respectively.

Our goal in this section is to give similar results in even dimensional cases by applying the results in Theorem 2.1. For $t > 0$ and $x \in \mathbb{R}$ let

$$q(t,x) = \sqrt{\frac{2}{\pi t}}\, e^{-x^2/2t}.$$

THEOREM 4.1.   *For $x > 0$ we set $G^{(d)}(x) = G_{d/2-1}(x)$.*

(1) *If $d = 2$, we have*

$$L(t) = 2\pi r\left[\sqrt{\frac{2t}{\pi}} + r^2\int_0^\infty\int_0^\infty\frac{xy - 1 + e^{-xy}}{y^3 G^{(2)}(y)}q(t,rx)dxdy\right].$$

(2) *If $d = 4$, we have*

$$L(t) = 2\pi^2 r^2\left[t + r^3\int_0^\infty\int_0^\infty\frac{1 - e^{-xy}}{y^3 G^{(4)}(y)}q(t,rx)dxdy\right].$$

(3) *If $d \geq 6$ and d is even, we have*

$$L(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - r^3\sum_{j=1}^{N_d}\frac{1}{(z_j^{(d)})^2}\int_0^\infty e^{z_j^{(d)}x}q(t,rx)dx\right.$$

$$\left. + (-1)^{d/2-1}r^3\int_0^\infty\int_0^\infty\frac{e^{-xy}}{y^3 G^{(d)}(y)}q(t,rx)dxdy\right].$$

We set $T(t) = L(2r^2 t)$ for $t > 0$ and then deduce from (4.1) that, for $\lambda > 0$

$$\int_0^\infty e^{-\lambda t}T(t)dt = \frac{S_{d-1}r^d}{\sqrt{\lambda^3}}\frac{K_{d/2}(\sqrt{\lambda})}{K_{d/2-1}(\sqrt{\lambda})}.$$

For a proof of Theorem 4.1 we consider the function:

$$\Sigma_\nu(\lambda) = \frac{1}{\sqrt{\lambda^3}}\frac{K_{\nu+1}(\sqrt{\lambda})}{K_\nu(\sqrt{\lambda})}, \quad \lambda > 0.$$

Let T_ν be the inverse Laplace transform of Σ_ν. Then we have the following, which immediately yields Theorem 4.1.

THEOREM 4.2.   *Setting $\nu^+ = \max\{\nu, 0\}$, we have the following.*

(1) *If $|\nu| < 1/2$,*

$$T_\nu(t) = 2\nu^+ t + 2\sqrt{\frac{t}{\pi}} + \cos(\pi\nu)\int_0^\infty\int_0^\infty\frac{xy - 1 + e^{-xy}}{y^3 G_{|\nu|}(y)}q(2t,x)dxdy. \qquad (4.2)$$

(2) *If $1/2 < |\nu| \leq 1$,*

$$T_\nu(t) = 2\nu^+ t - \cos(\pi\nu) \int_0^\infty \int_0^\infty \frac{1 - e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy. \tag{4.3}$$

(3) If $1 < |\nu| < 3/2$,

$$T_\nu(t) = 2\nu^+ t + \frac{1}{2(|\nu| - 1)} + \cos(\pi\nu) \int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy. \tag{4.4}$$

(4) If $|\nu| > 3/2$ and $\nu - 1/2$ is not an integer,

$$T_\nu(t) = 2\nu^+ t + \frac{1}{2(|\nu| - 1)} - \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}^2} \int_0^\infty e^{z_{\nu,j}x} q(2t, x) dx$$

$$+ \cos(\pi\nu) \int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy. \tag{4.5}$$

From now on, we shall treat $K_{\nu+1}/K_\nu$ as the function on the half real line $(0, \infty)$. For our purpose we need three lemmas.

LEMMA 4.3. *If $n + 1/2 < \mu < n + 3/2$ for an integer $n \geq 1$, we have*

$$K_\mu(x) = \frac{\Gamma(\mu)}{2} \left(\frac{2}{x}\right)^\mu e^{-x} \left\{ 1 + x + \sum_{k=2}^{2n+1} \binom{\mu - 1/2}{k} \frac{\Gamma(2\mu - k)}{\Gamma(2\mu)} (2x)^k + o(x^{2n+1}) \right\}$$

as $x \downarrow 0$. Especially, we have

$$K_\mu(x) = \frac{\Gamma(\mu)}{2} \left(\frac{2}{x}\right)^\mu e^{-x} \left\{ 1 + x + \frac{1}{2} \frac{2\mu - 3}{2\mu - 2} x^2 + \frac{1}{6} \frac{2\mu - 5}{2\mu - 2} x^3 + o(x^3) \right\}.$$

Moreover we have that, if $1 < \mu < 3/2$,

$$K_\mu(x) = \begin{cases} \dfrac{\Gamma(\mu)}{2} \left(\dfrac{2}{x}\right)^\mu e^{-x} \left\{ 1 + x + \dfrac{1}{2} \dfrac{2\mu - 3}{2\mu - 2} x^2 + o(x^2) \right\} & \text{if } 1 < \mu < 3/2, \\ \dfrac{\Gamma(\mu)}{2} \left(\dfrac{2}{x}\right)^\mu e^{-x} \{ 1 + x + o(x) \} & \text{if } 1/2 < \mu \leq 1. \end{cases}$$

PROOF. It is well-known that

$$K_\mu(x) = \frac{\sqrt{\pi}}{(2x)^\mu} \frac{e^{-x}}{\Gamma(\mu + 1/2)} \int_0^\infty e^{-y} y^{2\mu - 1} \left(1 + \frac{2x}{y} \right)^{\mu - 1/2} dy$$

(cf. [**15**, p. 140], [**21**, p. 206]). The Taylor formula yields that

$$\int_0^\infty e^{-y} y^{2\mu-1} \left(1 + \frac{2x}{y}\right)^{\mu-1/2} dy$$

$$= \int_0^\infty e^{-y} y^{2\mu-1} \left\{ 1 + \sum_{k=1}^{2n} \binom{\mu-1/2}{k} \left(\frac{2x}{y}\right)^k \right.$$

$$\left. + \binom{\mu-1/2}{2n+1} \left(\frac{2x}{y}\right)^{2n+1} \left(1 + \frac{2\xi x}{y}\right)^{\mu-2n-3/2} \right\} dy$$

for some $\xi \in [0,1]$. Since $2\mu - 2n - 2 > -1$ and $\mu - 2n - 3/2 < 0$, we have

$$\int_0^\infty e^{-y} y^{2\mu-2n-2} \left(1 + \frac{2x}{y}\right)^{\mu-2n-3/2} dy \leq \int_0^\infty e^{-y} y^{2\mu-2n-2} \left(1 + \frac{2\xi x}{y}\right)^{\mu-2n-3/2} dy$$

$$\leq \int_0^\infty e^{-y} y^{2\mu-2n-2} dy = \Gamma(2\mu - 2n - 1).$$

Hence the dominated convergence theorem yields that

$$\lim_{x\downarrow 0} \int_0^\infty e^{-y} y^{2\mu-2n-2} \left(1 + \frac{2x}{y}\right)^{\mu-2n-3/2} dy = \Gamma(2\mu - 2n - 1).$$

Therefore we obtain

$$K_\mu(x) = \frac{\sqrt{\pi}}{(2x)^\mu} \frac{e^{-x}}{\Gamma(\mu+1/2)} \left[\Gamma(2\mu) + (2\mu - 1)\Gamma(2\mu - 1)x \right.$$

$$+ \sum_{k=2}^{2n} \binom{\mu-1/2}{k} \Gamma(2\mu - k)(2x)^k$$

$$\left. + \binom{\mu-1/2}{2n+1} \Gamma(2\mu - 2n - 1)(2x)^{2n+1}\{1 + o(1)\} \right]$$

as $x \downarrow 0$. With the help of the formula

$$2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi} \Gamma(2z) \tag{4.6}$$

(cf. [**15**, p. 3]), we easily obtain the first assertion. The remainders can be seen similarly and the calculation is left to the reader. □

The following lemma is a straightforward consequence of Lemma 4.3.

LEMMA 4.4.   *We have that, as* $x \downarrow 0$,

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \begin{cases} \dfrac{2\nu^+}{x} + o(1) & \text{if } \dfrac{1}{2} < |\nu| \le 1, \\[3mm] \dfrac{2\nu^+}{x} + \dfrac{x}{2(|\nu|-1)} + o(x) & \text{if } 1 < |\nu| < \dfrac{3}{2}, \\[3mm] \dfrac{2\nu^+}{x} + \dfrac{x}{2(|\nu|-1)} + o(x^2) & \text{if } |\nu| > \dfrac{3}{2},\ \nu - \dfrac{1}{2} \notin \mathbb{Z}. \end{cases}$$

For an integer $k \ge 1$ we set

$$\zeta_{\nu,k} = \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}^k}, \qquad \varrho_{\nu,k} = \int_0^\infty \frac{dy}{y^{k+1}G_{|\nu|}(y)}.$$

By virtue of Theorem 2.1 and Lemma 4.4, we can derive the connection between $\zeta_{\nu,k}$ and $\varrho_{\nu,k}$.

LEMMA 4.5. (1) *If* $1/2 < |\nu| \le 1$, *we have*

$$1 + \varrho_{\nu,1}\cos(\pi\nu) = 0. \tag{4.7}$$

(2) *If* $1 < |\nu| < 3/2$, *we have*

$$1 + \varrho_{\nu,1}\cos(\pi\nu) = 0, \quad -\varrho_{\nu,2}\cos(\pi\nu) = \frac{1}{2(|\nu|-1)}. \tag{4.8}$$

(3) *If* $|\nu| > 3/2$ *and* $\nu - 1/2$ *is not an integer, we have*

$$1 + \zeta_{\nu,1} + \varrho_{\nu,1}\cos(\pi\nu) = 0, \quad \zeta_{\nu,2} - \varrho_{\nu,2}\cos(\pi\nu) = \frac{1}{2(|\nu|-1)}. \tag{4.9}$$

PROOF. Recall that, for $\nu \ne 0$

$$G_{|\nu|}(x) = \begin{cases} \dfrac{\pi}{2x}e^{2x}\{1 + o(1)\} & \text{as } x \to \infty, \\[3mm] \dfrac{1}{\kappa_{|\nu|}x^{2|\nu|}}\{1 + o(1)\} & \text{as } x \downarrow 0. \end{cases} \tag{4.10}$$

If $1/2 < |\nu| \le 1$, we obtain by (4.10) that $1/y^2G_{|\nu|}(y)$ is asymptotically equal to $\kappa_{|\nu|}y^{2|\nu|-2}$ as $y \downarrow 0$. This implies the convergence of $\varrho_{\nu,1}$. The dominated convergence theorem shows that, as $x \downarrow 0$,

$$\int_0^\infty \frac{dy}{(y+x)yG_{|\nu|}(y)} = \varrho_{\nu,1} + o(1).$$

Hence we deduce from (2.4) that

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = 1 + \frac{2\nu^+}{x} + \cos(\pi\nu)\varrho_{\nu,1} + o(1).$$

With the help of Lemma 4.4, we conclude (4.7).

In the case of $1 < |\nu| < 3/2$, $1/y^3 G_{|\nu|}(y)$ is integrable by (4.10) and we have

$$\int_0^\infty \frac{dy}{(y+x)yG_{|\nu|}(y)} = \int_0^\infty \left\{ \frac{1}{y} - \frac{x}{y(y+x)} \right\} \frac{dy}{yG_{|\nu|}(y)}$$
$$= \varrho_{\nu,1} - x\varrho_{\nu,2} + o(x) \tag{4.11}$$

by the dominated convergence theorem. Then we get by (2.4) that

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \frac{2\nu^+}{x} + 1 + \varrho_{\nu,1}\cos(\pi\nu) - x\varrho_{\nu,2}\cos(\pi\nu) + o(x),$$

and hence, combining with Lemma 4.4, we get (4.8).

If $|\nu| > 3/2$ and $\nu - 1/2$ is not an integer, we deduce

$$\sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - x} = \sum_{j=1}^{N(\nu)} \left\{ \frac{1}{z_{\nu,j}} + \frac{x}{z_{\nu,j}(z_{\nu,j} - x)} \right\} = \zeta_{\nu,1} + x\zeta_{\nu,2} + o(x).$$

Since $1/y^3 G_{|\nu|}(y)$ is integrable on $(0,\infty)$ by (4.10), (4.11) is valid for this case. Therefore it follows from (2.5) that

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \frac{2\nu^+}{x} + 1 + \zeta_{\nu,1} + \varrho_{\nu,1}\cos(\pi\nu) + x\{\zeta_{\nu,2} - \varrho_{\nu,2}\cos(\pi\nu)\} + o(x).$$

We immediately obtain (4.9) by Lemma 4.4. □

REMARK 4.6. In the case when $|\nu| > 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$, we can derive that $\zeta_{\nu,3} + \varrho_{\nu,3}\cos(\pi\nu) = 0$. It is not necessary for the proof of Theorem 4.2.

By virtue of Theorem 2.1, we have that, if $|\nu| < 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$,

$$\Sigma_\nu(\lambda) = \frac{2\nu^+}{\lambda^2} + \frac{1}{\sqrt{\lambda^3}} + \cos(\pi\nu)\int_0^\infty \frac{dy}{\sqrt{\lambda^3}(\sqrt{\lambda}+y)yG_{|\nu|}(y)}$$

and that, if $|\nu| > 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$,

$$\Sigma_\nu(\lambda) = \frac{2\nu^+}{\lambda^2} + \frac{1}{\sqrt{\lambda^3}} + \sum_{j=1}^{N(\nu)} \frac{1}{\sqrt{\lambda^3}(z_{\nu,j} - \sqrt{\lambda})}$$
$$+ \cos(\pi\nu)\int_0^\infty \frac{dy}{\sqrt{\lambda^3}(\sqrt{\lambda}+y)yG_{|\nu|}(y)}. \tag{4.12}$$

LEMMA 4.7.   *For $t > 0$ let*

$$q_\nu(t) = \int_0^\infty \int_0^\infty \frac{xy - 1 + e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy.$$

Then, we have, for $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} q_\nu(t) dt = \int_0^\infty \frac{dy}{\sqrt{\lambda^3}(\sqrt{\lambda} + y) y G_{|\nu|}(y)}. \tag{4.13}$$

PROOF.   We first recall the elementary formula

$$\int_0^\infty e^{-\lambda t} q(2t, x) dt = \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda} x}, \quad \lambda > 0. \tag{4.14}$$

Then, with the help of the Fubini theorem, we deduce

$$\int_0^\infty e^{-\lambda t} q_\nu(t) dt = \int_0^\infty dy \int_0^\infty \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda} x} \frac{xy - 1 + e^{-xy}}{y^3 G_{|\nu|}(y)} dx.$$

Carrying out the elementary integral in x, we obtain (4.13).   □

We now complete our proof of Theorem 4.2. If $|\nu| < 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$,

$$T_\nu(t) = 2\nu^+ t + 2\sqrt{\frac{t}{\pi}} + \cos(\pi\nu) q_\nu(t). \tag{4.15}$$

When $|\nu| < 1/2$, (4.15) immediately implies (4.2).

When $|\nu| > 1/2$ and $\nu - 1/2 \notin \mathbb{Z}$, we have by (4.10) that $1/y^2 G_{|\nu|}(y)$ is integrable on $(0, \infty)$. Since

$$\left| \frac{1 - e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) \right| \leq \frac{xq(2t, x)}{y^2 G_{|\nu|}(y)},$$

we have

$$q_\nu(t) = 2\sqrt{\frac{t}{\pi}} \varrho_{\nu,1} - \int_0^\infty \int_0^\infty \frac{1 - e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy. \tag{4.16}$$

Combining this formula with (4.7) and (4.15), we conclude (4.3).

When $|\nu| > 1$ and $\nu - 1/2 \notin \mathbb{Z}$, we can further improve (4.16). Indeed, since $1/y^3 G_{|\nu|}(y)$ is integrable on $(0, \infty)$, we get that

$$\int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx$$

converges and we get

$$q_\nu(t) = 2\sqrt{\frac{t}{\pi}}\varrho_{\nu,1} - \varrho_{\nu,2} + \int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G_{|\nu|}(y)} q(2t, x) dx dy. \qquad (4.17)$$

Hence, we get (4.4) by (4.15), (4.17) and Lemma 4.4.

To invert Σ_ν in the case of $|\nu| > 3/2$, we note

$$\sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - \sqrt{\lambda}} = \zeta_{\nu,1} + \sqrt{\lambda}\zeta_{\nu,2} + \sum_{j=1}^{N(\nu)} \frac{\lambda}{z_{\nu,j}^2(z_{\nu,j} - \sqrt{\lambda})}.$$

Hence (4.12) is equivalent to

$$\Sigma_\nu(\lambda) = \frac{2\nu^+}{\lambda^2} + \frac{1 + \zeta_{\nu,1}}{\sqrt{\lambda^3}} + \frac{\zeta_{\nu,2}}{\lambda} - \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}^2 \sqrt{\lambda}(\sqrt{\lambda} - z_{\nu,j})}$$

$$+ \cos(\pi\nu) \int_0^\infty \frac{dy}{\sqrt{\lambda^3}(\sqrt{\lambda} + y)y G_{|\nu|}(y)}.$$

With the help of (4.14), we easily see that, for $z \in \mathbb{C}^-$

$$\int_0^\infty e^{-\lambda t} dt \int_0^\infty e^{zx} q(2t, x) dx = \frac{1}{\sqrt{\lambda}(\sqrt{\lambda} - z)}.$$

Hence, we deduce (4.5) from (4.17) and Lemma 4.4.   □

5. Large time asymptotics for the Wiener sausage.

This section is devoted to show an asymptotic behavior of $L(t)$ for large t in even dimensional cases. Le Gall [16] considered the Wiener sausage associated with a general compact set and proved

$$L(t) = \begin{cases} c_1^{(4)}t + c_2^{(4)}\log t + c_3^{(4)} + c_4^{(4)}\dfrac{\log t}{t} + o\left(\dfrac{\log t}{t}\right) & \text{if } d = 4, \\ c_1^{(d)}t + c_2^{(d)} + c_3^{(d)}t^{2-d/2} + O(t^{1-d/2}) & \text{if } d \geq 5 \end{cases} \qquad (5.1)$$

and gave the explicit expression of each constant $c_j^{(d)}$. If $d = 2$, Le Gall [17] also showed that $L(t)$ admits the asymptotic expansion in powers of $1/\log t$.

When d is odd, Hamana [6] gave the asymptotic expansion of $L(t)$. The purpose in this section is to improve the asymptotic behavior of $L(t)$ if d is even and not less than 6. Throughout this section, we use C_i's for positive constants independent of the variable.

THEOREM 5.1.   *If d is even and not less than 6, there is a family of constants* $\{\alpha_n^{(d)}\}_{n=0}^{d-5}$ *such that*

$$L(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\alpha_d\,r^{d-2}}{2^{d/2-2}(d-4)\Gamma(d/2-1)}\frac{1}{t^{d/2-2}}\right.$$

$$\left. + \frac{1}{t^{d/2-1}}\sum_{n=0}^{d-5}\frac{\alpha_n^{(d)}}{t^{n/2}} + \frac{\Gamma((d-3)/2)r^{2d-4}}{\sqrt{\pi}(d-2)\Gamma(d/2-1)^3}\frac{\log t}{t^{d-3}} + O\left(\frac{1}{t^{d-3}}\right)\right],$$

where

$$\alpha_d = \int_0^\infty \frac{1}{y^d}\left\{\frac{1}{G^{(d)}(y)} - \frac{y^{d-2}}{2^{d-4}\Gamma(d/2-1)^2}\right\}dy.$$

Recalling the notation $G^{(d)} = G_{d/2-1}$, we set

$$L_1(t) = r^3\sum_{j=1}^{N_d}\frac{1}{(z_j^{(d)})^2}\int_0^\infty e^{z_j^{(d)}x}q(t,rx)dx,$$

$$L_2(t) = r^3\int_0^\infty\int_0^\infty \frac{e^{-xy}}{y^3 G^{(d)}(y)}q(t,rx)dxdy.$$

Then we have from Theorem 4.1 that, if d is even and not less than 6,

$$L(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - L_1(t) + (-1)^{d/2-1}L_2(t)\right]. \qquad (5.2)$$

The calculation of $L_1(t)$ is easy since $\mathrm{Re}(z_j^{(d)}) < 0$ for each $j = 1, 2, \ldots, N_d$. For $x \geq 0$ and an integer $n \geq 0$ we put

$$R_n(x) = e^{-x} - \sum_{k=0}^n \frac{(-1)^k}{k!}x^k.$$

and let M be a positive integer. Then it follows that

$$\int_0^\infty e^{-r^2x^2/2t+z_j^{(d)}x}dx = \sum_{n=0}^M \frac{(-1)^n r^{2n}}{n!\,(2t)^n}\int_0^\infty x^{2n}e^{z_j^{(d)}x}dx + \int_0^\infty R_M\left(\frac{r^2x^2}{2t}\right)e^{z_j^{(d)}x}dx$$

$$= \sum_{n=0}^M \frac{(-1)^n r^{2n}(2n)!}{2^n n!\,(z_j^{(d)})^{2n+1}}\frac{1}{t^n} + O\left(\frac{1}{t^{M+1}}\right).$$

We use m instead of $d/2 - 1$ for simplicity and hence obtain

$$L_1(t) = \sqrt{\frac{2}{\pi}}\sum_{n=0}^M \frac{(-1)^n r^{2n+3}(2n)!\,\zeta_{m,2n+3}}{2^n n!}\frac{1}{t^{n+1/2}} + O\left(\frac{1}{t^{M+3/2}}\right) \qquad (5.3)$$

as $t \to \infty$.

For a proof of Theorem 5.1 we need to give an asymptotic behavior of $L_2(t)$ for large t. We set $L_2^0(t) = L_2(2r^2t)/r^2$ and then have

$$L_2^0(t) = \int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G_m(y)} q(2t, x) dx dy. \tag{5.4}$$

We note that $G_m(x) = K_m(x)^2 + \pi^2 I_m(x)^2$.

LEMMA 5.2. *For an integer $n \geq 1$ we have*

$$\int_0^\infty \frac{R_{n-1}(x^2)}{x^{2n}} dx = \frac{(-1)^n \pi}{2\Gamma(n + 1/2)}. \tag{5.5}$$

PROOF. From a change of variables from x to y given by $y = x^2$ we deduce that the left hand side of (5.5) is equal to

$$\frac{1}{2} \int_0^\infty \frac{1}{y^{n+1/2}} \left\{ e^{-y} - \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} y^k \right\} dy = \frac{1}{2} \Gamma\left(\frac{1}{2} - n\right)$$

(cf. [3, p. 361]). The formula $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$, $z \in \mathbb{C} \setminus \mathbb{Z}$ (cf. [15, p. 3]) yields (5.5). □

We give several constants which we need to describe the asymptotic behavior of $1/G_m(x)$ as $x \downarrow 0$. For integers h, k with $1 \leq h \leq k \leq m - 1$ we put

$$b_{k,h} = \sum_{\substack{k_1 + k_2 + \cdots + k_h = k \\ k_1, k_2, \ldots, k_h \geq 1}} b_{k_1} b_{k_2} \cdots b_{k_h},$$

where

$$b_k = \frac{(-1)^{k+1}}{4^k \Gamma(m)^2} \sum_{h=0}^k \frac{\Gamma(m - h)\Gamma(m - k + h)}{\Gamma(h + 1)\Gamma(k - h + 1)}.$$

We set

$$a_k = \sum_{h=1}^k b_{k,h} \ (k = 1, 2, \ldots, m - 1), \quad a_m = \frac{(-1)^{m+1}}{4^{m-1} m \Gamma(m)^2}.$$

We note that $a_1 = b_1$. Moreover recall that $\kappa_m = 1/4^{m-1}\Gamma(m)^2$. See (2.21) for the definition of κ_ν. The second lemma gives the asymptotic behavior of $1/G_m(x)$.

LEMMA 5.3. *We have that, as $x \downarrow 0$,*

$$\frac{1}{G_m(x)} = \kappa_m x^{2m} \left\{ 1 + \sum_{k=1}^{m-1} a_k x^{2k} + a_m x^{2m} \log \frac{1}{x} + O(x^{2m}) \right\}. \tag{5.6}$$

PROOF. It is known that

$$K_m(x) = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(-1)^k (m-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-m}$$

$$+ (-1)^{m+1} \log \frac{x}{2} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+m)!} \left(\frac{x}{2}\right)^{2k+m}$$

$$- \frac{(-1)^{m+1}}{2} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+m)!} \left(\frac{x}{2}\right)^{2k+m} \{\psi(k+1) + \psi(k+m+1)\},$$

where ψ is the logarithmic derivative of the gamma function (cf. [**21**, p. 80]). This formula immediately yields that the leading term of $K_m(x)$ for small x is

$$\frac{\Gamma(m)}{2} \left(\frac{2}{x}\right)^m \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m-k)}{\Gamma(k+1)\Gamma(m)} \left(\frac{x}{2}\right)^{2k} + \frac{(-1)^m}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m \log \frac{1}{x}$$

and its error term is of order x^m. Hence we have that, as $x \downarrow 0$

$$K_m(x)^2 = \frac{\Gamma(m)^2}{4} \left(\frac{2}{x}\right)^{2m} \left\{ \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m-k)}{\Gamma(k+1)\Gamma(m)} \left(\frac{x}{2}\right)^{2k} \right\}^2$$

$$+ \frac{(-1)^m}{m} \log \frac{1}{x} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m-k)}{\Gamma(k+1)\Gamma(m)} \left(\frac{x}{2}\right)^{2k} + O(1).$$

Since $I_m(x)^2 = O(x^{2m})$, we conclude that, as $x \downarrow 0$,

$$G_m(x) = \frac{1}{\kappa_m x^{2m}} \left\{ 1 - \sum_{k=1}^{m-1} b_k x^{2k} - a_m x^{2m} \log \frac{1}{x} + O(x^{2m}) \right\}.$$

We set $G_m^0(x) = \kappa_m x^{2m} G_m(x)$ for simplicity. It is sufficient to obtain the asymptotic behavior of $1/G_m^0(x)$. We can easily derive

$$\frac{1}{G_m^0(x)} = 1 + \sum_{h=1}^{m-1} \left(\sum_{k=1}^{m-1} b_k x^{2k} + a_m x^{2m} \log \frac{1}{x} \right)^h + O(x^{2m}). \tag{5.7}$$

In the case of $m = 2$, (5.7) immediately implies (5.6). We concentrate on considering the case of $m \geq 3$. In this case, the summation on the right hand side of (5.7) is

$$1 + \sum_{h=1}^{m-1} \left(\sum_{k=1}^{m-1} b_k x^{2k} \right)^h + a_m x^{2m} \log \frac{1}{x} + O(x^{2m}). \tag{5.8}$$

A simple calculation shows that the double sum in the right hand side of (5.8) is equal to

$$\sum_{h=1}^{m-1} \sum_{k=h}^{m-1} b_{k,h} x^{2k} + O(x^{2m}).$$

Hence we deduce

$$\frac{1}{G_m^0(x)} = 1 + \sum_{k=1}^{m-1} a_k x^{2k} + a_m x^{2m} \log\frac{1}{x} + O(x^{2m}),$$

which implies (5.6) for $m \geq 3$. $\qquad\square$

We now proceed to a proof of Theorem 5.1. From (5.4) it follows that

$$L_2^0(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{1}{y^3 G_m(y)} e^{-2\sqrt{t}uy} e^{-u^2} du\,dy,$$

which is the sum of

$$L_2^1(t) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{m-2} \frac{(-1)^n}{n!} \int_0^\infty \int_0^\infty \frac{1}{y^3 G_m(y)} u^{2n} e^{-2\sqrt{t}uy} du\,dy,$$

$$L_2^2(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{1}{y^3 G_m(y)} e^{-2\sqrt{t}uy} R_{m-2}(u^2) du\,dy.$$

We easily derive

$$L_2^1(t) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{m-2} \frac{(-1)^n (2n)!}{4^n n!} \varrho_{m,2n+3} \frac{1}{t^{n+1/2}}.$$

For $x > 0$ and an integer $k = 0, 1, 2, \ldots, m-1$ let

$$Q_k(x) = \frac{1}{G_m(x)} - \kappa_m x^{2m} \sum_{n=0}^k a_n x^{2n},$$

where we have put $a_0 = 1$ for convenience. We need the following lemma to derive the large time asymptotics of $L_2^2(t)$.

LEMMA 5.4.   *We have that $L_2^2(t)$ is the sum of the following three integrals;*

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \kappa_m y^{2m} \sum_{k=0}^{m-1} a_k y^{2k} R_{m+k-2}(u^2) du\,dy, \qquad (5.9)$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1}Q_k(y)}{(m+k-1)!} u^{2(m+k-1)} dudy, \tag{5.10}$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} Q_{m-1}(y) R_{2m-2}(u^2) dudy. \tag{5.11}$$

PROOF. By the definition of Q_0 and R_k, we have

$$L_2^2(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \kappa_m y^{2m} R_{m-2}(u^2) dudy$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \frac{(-1)^{m-1}Q_0(y)}{(m-1)!} u^{2(m-1)} dudy$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} Q_0(y) R_{m-1}(u^2) dudy. \tag{5.12}$$

Moreover, it follows that, for $y > 0$, $u > 0$ and $k \geq 1$, $Q_{k-1}(y)R_{m+k-2}(u^2)$ is

$$Q_k(y)R_{m+k-1}(u^2) + \kappa_m a_k y^{2(m+k)} R_{m+k-2}(u^2) + \frac{(-1)^{m+k-1}}{(m+k-1)!} Q_k(y) u^{2(m+k-1)}.$$

Taking the sum on k over $[1, m-1]$, we deduce that the third term of the right hand side of (5.12) is equal to

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} Q_{m-1}(y) R_{2m-2}(u^2) dudy$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \kappa_m y^{2m} \sum_{k=1}^{m-1} a_k y^{2k} R_{m+k-2}(u^2) dudy$$

$$+ \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{e^{-2\sqrt{t}uy}}{y^3} \sum_{k=1}^{m-1} \frac{(-1)^{m+k-1}Q_k(y)}{(m+k-1)!} u^{2(m+k-1)} dudy.$$

Hence we conclude that $L_2^2(t)$ is the sum of (5.9), (5.10) and (5.11). □

For (5.9) we first carry out the integral in y and use (5.5). Then we see that (5.9) is equal to

$$\frac{2\kappa_m}{\sqrt{\pi}} \sum_{k=0}^{m-1} \frac{a_k(2m+2k-3)!}{2^{2m+2k-2}t^{m+k-1}} \int_0^\infty \frac{R_{m+k-2}(u^2)}{u^{2(m+k-1)}} du$$

$$= \sqrt{\pi} \kappa_m \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1} a_k \Gamma(2m+2k-2)}{2^{2m+2k-2} \Gamma(m+k-1/2)} \frac{1}{t^{m+k-1}}. \tag{5.13}$$

The formula (4.6) yields that the right hand side of (5.13) and so (5.9) are equal to

$$\frac{\kappa_m}{2} \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1}\Gamma(m+k-1)a_k}{t^{m+k-1}}.$$

Carrying out the integral on u in (5.10), we see that (5.10) is equal to

$$\frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1}\Gamma(m+k-1/2)}{t^{m+k-1/2}} \int_0^\infty \frac{Q_k(y)}{y^{2m+2k+2}} dy. \tag{5.14}$$

Here we have applied

$$\frac{\Gamma(2z-1)}{\Gamma(z)} = \frac{1}{2z-1}\frac{\Gamma(2z)}{\Gamma(z)} = \frac{2^{2z-2}}{\sqrt{\pi}}\Gamma\left(z-\frac{1}{2}\right). \tag{5.15}$$

We should note that the integral in the right hand side of (5.14) converges for each integer $k = 0, 1, 2, \ldots, m-1$. It is easy to see that (4.10) yields that, as $x \to \infty$,

$$Q_k(x) = \kappa_m x^{2m+2k} + o(x^{2m+2k}). \tag{5.16}$$

Moreover we deduce from (5.6) that, as $x \downarrow 0$,

$$Q_k(x) = \begin{cases} \kappa_m a_{k+1} x^{2m+2k+2} + O(x^{2m+2k+4}) & \text{if } 0 \le k \le m-2, \\ \kappa_m a_m x^{4m} \log\dfrac{1}{x} + O(x^{4m}) & \text{if } k = m-1. \end{cases} \tag{5.17}$$

Hence $Q_k(y)/y^{2m+2k+2}$ is integrable on $(0, \infty)$ for each $k = 0, 1, 2, \ldots, m-1$.

For $t > 0$ we let

$$P_1(t) = \frac{2}{\sqrt{\pi}} \int_1^\infty \frac{Q_{m-1}(y)}{y^3} dy \int_0^\infty e^{-2\sqrt{t}uy} R_{2m-2}(u^2) du,$$

$$P_2(t) = \frac{2}{\sqrt{\pi}} \int_0^1 \frac{Q_m(y)}{y^3} dy \int_0^\infty e^{-2\sqrt{t}uy} R_{2m-2}(u^2) du,$$

$$P_3(t) = \frac{2\kappa_m a_m}{\sqrt{\pi}} \int_0^1 y^{4m-3} \log\frac{1}{y} dy \int_0^\infty e^{-2\sqrt{t}uy} R_{2m-2}(u^2) du,$$

where we put $Q_m(x) = Q_{m-1}(x) - \kappa_m a_m x^{4m} \log(1/x)$. Then (5.11) is the sum of $P_1(t)$, $P_2(t)$ and $P_3(t)$.

By virtue of (5.16), we obtain that $|Q_{m-1}(y)| \le C_3 y^{4m-2}$ for $y \ge 1$. Combining this estimate with $|R_{2m-2}(u^2)| \le C_4 u^{4m-2}$, we deduce

$$|P_1(t)| \le C_5 \int_1^\infty y^{4m-5} dy \int_0^\infty e^{-2\sqrt{t}uy} u^{4m-2} du = \frac{C_6}{t^{2m-1/2}} \int_1^\infty \frac{dy}{y^4}.$$

This means that $P_1(t)$ is of order $1/t^{2m-1/2}$ and is negligible.

We next show $P_2(t) = O(1/t^{2m-1})$. It follows from (5.17) that $|Q_m(y)| \leq C_7 y^{4m}$ for $y < 1$. Noting $R_{2m-2}(x) \leq 0$ for $x \geq 0$, we obtain by the Fubini theorem that

$$|P_2(t)| \leq C_8 \int_0^\infty -R_{2m-2}(u^2)du \int_0^\infty e^{-2\sqrt{t}uy} y^{4m-3} dy.$$

The formula (5.5) implies that

$$|P_2(t)| \leq \frac{C_9}{t^{2m-1}} \int_0^\infty \frac{-R_{2m-2}(u^2)}{u^{4m-2}} du = \frac{C_9 \pi}{2\Gamma(2m-1/2)} \frac{1}{t^{2m-1}}.$$

The calculation of $P_3(t)$ is slightly complicated but not difficult. A change of variables from y to v given by $2\sqrt{t}y = v$ yields

$$P_3(t) = \frac{\kappa_m a_m}{2^{4m-3}\sqrt{\pi}t^{2m-1}} \int_0^{2\sqrt{t}} v^{4m-3} \log \frac{2\sqrt{t}}{v} dv \int_0^\infty e^{-uv} R_{2m-2}(u^2)du.$$

For $t > 1/4$ we set

$$P_3^1(t) = \log(2\sqrt{t}) \int_0^\infty v^{4m-3} dv \int_0^\infty e^{-uv} R_{2m-2}(u^2)du,$$

$$P_3^2(t) = \log(2\sqrt{t}) \int_{2\sqrt{t}}^\infty v^{4m-3} dv \int_0^\infty e^{-uv} R_{2m-2}(u^2)du,$$

$$P_3^3(t) = \int_1^{2\sqrt{t}} v^{4m-3} \log v \, dv \int_0^\infty e^{-uv} R_{2m-2}(u^2)du,$$

$$P_3^4(t) = \int_0^1 v^{4m-3} \log \frac{1}{v} dv \int_0^1 e^{-uv} R_{2m-2}(u^2)du,$$

$$P_3^5(t) = \int_0^1 v^{4m-3} \log \frac{1}{v} dv \int_1^\infty e^{-uv} R_{2m-2}(u^2)du.$$

Then we have

$$P_3(t) = \frac{\kappa_m a_m}{2^{4m-3}\sqrt{\pi}t^{2m-1}} \{P_3^1(t) - P_3^2(t) - P_3^3(t) + P_3^4(t) + P_3^5(t)\}.$$

We show that $P_3^1(t)$ is the leading part of $P_3(t)$ and the others are all negligible. Recall that $R_{2m-2}(u^2) \leq 0$. It follows from (4.6) for $z = 2m - 1$ and (5.5) that

$$P_3^1(t) = \left(\frac{\log t}{2} + \log 2\right)\Gamma(4m-2) \int_0^\infty \frac{R_{2m-2}(u^2)}{u^{4m-2}} du$$

$$= -2^{4m-5}\sqrt{\pi}\Gamma(2m-1)\log t + O(1).$$

The estimate of $P_3^2(t)$ is easy. Indeed, we have

$$|P_3^2(t)| \leq C_{10} \log t \int_{2\sqrt{t}}^\infty v^{4m-3} dv \int_0^\infty e^{-uv} u^{4m-2} du \leq C_{11} \log t \int_{2\sqrt{t}}^\infty \frac{dv}{v^2},$$

which is of order $\log t / \sqrt{t}$. The way of estimates of the remaining integrals is similar to that of $P_3^2(t)$. We deduce

$$|P_3^3(t)| \leq C_{12} \int_1^\infty v^{4m-3} \log v \, dv \int_0^\infty e^{-uv} u^{4m-2} du \leq C_{13} \int_1^\infty \frac{\log v}{v^2} dv$$

and that, by $0 \leq e^{-uv} u^{4m-2} \leq 1$ for $u, v \in [0, 1]$,

$$|P_3^4(t)| \leq C_{14} \int_0^1 v^{4m-3} \log \frac{1}{v} dv \int_0^1 e^{-uv} u^{4m-2} du \leq C_{14} \int_0^1 v^{4m-3} \log \frac{1}{v} dv.$$

These immediately imply that $P_3^3(t)$ and $P_3^4(t)$ are bounded. Note that $R_n(x)$ is asymptotically equal to $(-1)^{n+1} x^n / n!$ as $x \to \infty$ for an integer $n \geq 0$. This yields that $|R_{2m-2}(u^2)| \leq C_{15} u^{4m-4}$ for $u \geq 1$ and then we deduce

$$|P_3^5(t)| \leq C_{15} \int_0^1 v^{4m-3} \log \frac{1}{v} dv \int_1^\infty e^{-uv} u^{4m-4} du \leq C_{16} \int_0^1 \log \frac{1}{v} dv = C_{16}.$$

Therefore, by virtue of Lemma 5.4, we accordingly obtain

$$
\begin{aligned}
L_2^2(t) = {} & \frac{\kappa_m}{2} \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1} \Gamma(m+k-1) \, a_k}{t^{m+k-1}} \\
& + \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(-1)^{m+k-1} \Gamma(m+k-1/2)}{t^{m+k-1/2}} \int_0^\infty \frac{Q_k(y)}{y^{2m+2k+2}} dy \\
& - \frac{\kappa_m a_m \Gamma(2m-1)}{4} \frac{\log t}{t^{2m-1}} + O\left(\frac{1}{t^{2m-1}}\right),
\end{aligned}
$$

which implies that we finished to give the asymptotic behavior of $L_2^0(t)$.

Recall the definition of κ_m and a_m. We deduce from (5.15) that

$$-\frac{\kappa_m a_m \Gamma(2m-1)}{4} = \frac{(-1)^m}{4^m \sqrt{\pi} \, m \Gamma(m)^3} \Gamma\left(m - \frac{1}{2}\right).$$

Since $L_2(t) = r^2 L_2^0(t)(t/2r^2)$, we obtain, by (5.3) for $M = d - 4$, that

$$
\begin{aligned}
& - L_1(t) + (-1)^{d/2-1} L_2(t) \\
& = -\sqrt{\frac{2}{\pi}} \sum_{n=0}^{d-4} \frac{(-1)^n (2n)! \, r^{2n+3} \zeta_{m,2n+3}}{2^n n!} \frac{1}{t^{n+1/2}}
\end{aligned}
$$

$$-\sqrt{\frac{2}{\pi}}\sum_{n=0}^{d/2-3}\frac{(-1)^{d/2+n}(2n)!\,r^{2n+3}\varrho_{m,2n+3}}{2^n n!}\frac{1}{t^{n+1/2}}$$

$$-\kappa_{d/2-1}\sum_{n=0}^{d/2-2}\frac{(-1)^n\Gamma(d/2+n-2)a_n 2^{d/2+n-3}r^{d+2n-2}\xi_n^{(d)}}{t^{d/2+n-2}}$$

$$+\frac{\Gamma((d-3)/2)r^{2d-4}}{\sqrt{\pi}(d-2)\Gamma(d/2-1)^3}\frac{\log t}{t^{d-3}}+O\left(\frac{1}{t^{d-3}}\right),$$

where

$$\xi_k^{(d)}=\int_0^\infty\frac{Q_k(y)}{y^{d+2k}}\,dy.$$

(5.1) and (5.2) imply that $\zeta_{m,d-1}=0$ and

$$\zeta_{m,2n+3}+(-1)^{d/2}\varrho_{m,2n+3}=0$$

for $n=0,1,2,\ldots,d/2-3$. Therefore we can conclude that

$$-L_1(t)+(-1)^{d/2-1}L_2(t)=-\frac{r^{d-2}\xi_0^{(d)}}{2^{d/2-2}(d-4)\Gamma(d/2-1)}\frac{1}{t^{d/2-2}}+\frac{1}{t^{d/2-1}}\sum_{n=0}^{d-6}\frac{\alpha_n^{(d)}}{t^{n/2}}$$

$$+\frac{\Gamma((d-3)/2)r^{2d-4}}{\sqrt{\pi}(d-2)\Gamma(d/2-1)^3}\frac{\log t}{t^{d-3}}+O\left(\frac{1}{t^{d-3}}\right).$$

Our proof of Theorem 5.1 is completed. □

6. Zeros of Macdonald functions.

We can find several properties concerning the zeros of J_ν, Y_ν and I_ν (cf. [15], [21]). However there is less information on the zeros of K_ν. Some numerical evaluations have only been given. See [14] for example.

Our purpose in this section is to represent all zeros of K_ν as the root of a polynomial of order $N(\nu)$. Since $K_\nu=K_{-\nu}$ and $N(\nu)\geq 1$ if $|\nu|\geq 3/2$, it is sufficient to consider the case of $\nu\geq 3/2$. Moreover, if $\nu=n+1/2$ for an integer $n\geq 1$, the formula

$$K_\nu(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^n\frac{(\nu,k)}{(2z)^k}$$

yields that all zeros of K_ν are the solutions of the equation

$$\sum_{k=0}^n\frac{(\nu,n-k)}{2^{n-k}}z^k=0.$$

Here we have used the notation

$$(\nu, k) = \frac{\Gamma(\nu + k + 1/2)}{k!\Gamma(\nu - k + 1/2)}.$$

From now on, we discuss the cases when $\nu > 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$. By virtue of (2.5), we have that, for $x > 0$

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = 1 + \frac{2\nu}{x} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - x} + \cos(\pi\nu) \int_0^\infty \frac{dy}{y(y+x)G_\nu(y)}. \qquad (6.1)$$

We can derive the power sum of $z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,N(\nu)}$ by (6.1). The Newton formula (cf. [4, p. 276]) gives the polynomial whose roots are $z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,N(\nu)}$.

LEMMA 6.1.   *Let*

$$p_n = \sum_{j=1}^N z_j^n$$

for positive integers n, N and $z_1, z_2, \ldots, z_N \in \mathbb{C}$. We define a sequence $\{\xi_n\}_{n=0}^N$ of complex numbers by $\xi_0 = 1$ and

$$\xi_n = -\frac{1}{n} \sum_{k=1}^n \xi_{n-k} p_k$$

for $n = 1, 2, \ldots, N$. Then we have that, for $z \in \mathbb{C}$

$$\prod_{j=1}^N (z - z_j) = \sum_{n=0}^N \xi_{N-n} z^n. \qquad (6.2)$$

PROOF.   Let s_n be the elementary symmetric polynomial of degree n, that is,

$$s_n = \sum_{1 \le j_1 < j_2 < \cdots < j_n \le N} z_{j_1} z_{j_2} \cdots z_{j_n}.$$

The Newton formula yields that $p_1 = -s_1$ and

$$p_n = \sum_{k=1}^{n-1} (-1)^{n-k+1} s_{n-k} p_k + (-1)^{n+1} n s_n$$

for $n = 2, 3, \ldots, N$. Therefore we easily deduce (6.2) from the formula

$$\prod_{j=1}^N (z - z_j) = \sum_{n=0}^N (-1)^{N-k} s_{N-k} z^n.$$

This completes the proof of the lemma. □

We first consider the asymptotic expansion of (6.1) for large x. Recall that, for $\mu \geq 0$ and a given integer $M \geq 0$

$$K_\mu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{n=0}^{M+1} \frac{(\mu, n)}{(2x)^n} + O\left(\frac{1}{x^{M+2}}\right)$$

as $x \to \infty$ (cf. [**15**, p. 123], [**21**, p. 202]) and we have

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \sum_{n=0}^{M+1} \frac{a_n}{x^n} + O\left(\frac{1}{x^{M+2}}\right), \tag{6.3}$$

where $\{a_n\}_{n=0}^{M+1}$ is the sequence of real numbers defined by

$$\frac{(\nu+1, n)}{2^n} = \sum_{k=0}^{n} \frac{(\nu, n-k)}{2^{n-k}} a_k$$

for $n = 0, 1, 2, \ldots, M+1$. A simple calculation shows

$$a_0 = 1, \quad a_1 = \nu + \frac{1}{2}, \quad a_2 = \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right), \quad a_3 = -\frac{1}{2}\left(\nu^2 - \frac{1}{4}\right),$$

$$a_4 = -\frac{1}{8}\left(\nu^2 - \frac{1}{4}\right)\left(\nu^2 - \frac{25}{4}\right), \quad a_5 = \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right)\left(\nu^2 - \frac{13}{4}\right).$$

The remaining constants $a_6, a_7, \ldots$ have complicated forms. It is easy to give the asymptotic expansion of the right hand side of (6.1) and then we have that, for $M \geq N(\nu)$

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = 1 + \frac{2\nu}{x} - \sum_{n=0}^{M} \frac{1}{x^{n+1}} \sum_{j=1}^{N(\nu)} z_{\nu,j}^n$$

$$+ \cos(\pi\nu) \sum_{n=0}^{M} \frac{(-1)^n}{x^{n+1}} \int_0^\infty \frac{y^{n-1}}{G_\nu(y)} dy + O\left(\frac{1}{x^{M+2}}\right). \tag{6.4}$$

Here we should note that the integral of $y^{m-1}/G_\nu(y)$ over $(0, \infty)$ converges for each integer $m \geq 0$, which can be shown by Lemma 2.3 and (2.21). Comparing the corresponding coefficients in (6.3) and (6.4), we obtain

$$N(\nu) = \nu - \frac{1}{2} + \cos(\pi\nu) \int_0^\infty \frac{dy}{y G_\nu(y)}, \tag{6.5}$$

$$\sum_{j=1}^{N(\nu)} z_{\nu,j}^n = -a_{n+1} + (-1)^n \cos(\pi\nu) \int_0^\infty \frac{y^{n-1}}{G_\nu(y)} dy \tag{6.6}$$

for $n = 1, 2, \ldots, M$. We define a sequence $\{\alpha_n^\nu\}_{n=0}^{N(\nu)}$ of complex numbers by $\alpha_0^\nu = 1$ and

$$\alpha_n^\nu = \frac{1}{n} \sum_{k=1}^{n} \alpha_{n-k}^\nu \left\{ a_{k+1} - (-1)^k \cos(\pi\nu) \int_0^\infty \frac{y^{k-1}}{G_\nu(y)} dy \right\}$$

for $n = 1, 2, \ldots, N(\nu)$. Therefore, by (6.6) and Lemma 6.1, we have the following theorem.

THEOREM 6.2. *For $|\nu| > 3/2$ the zeros of K_ν are the solutions of*

$$\sum_{n=0}^{N(\nu)} \alpha_{N(\nu)-n}^\nu z^n = 0.$$

We obtain another polynomial whose roots are $z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,N(\nu)}$ by considering the asymptotic behavior of (6.1) for small x, which is an improvement of Lemma 4.4. Let $N \geq 1$ be an integer with $N + 1/2 < \nu < N + 3/2$. It follows from Lemma 4.3 that, as $x \downarrow 0$,

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \frac{2\nu}{x} \sum_{n=0}^{2N+1} b_n x^n + o(x^{2N}), \tag{6.7}$$

where $\{b_n\}_{n=0}^{2N+1}$ is a sequence of real numbers defined by

$$\binom{\nu + 1/2}{n} \frac{2^n \Gamma(2\nu - n + 2)}{\Gamma(2\nu + 2)} = \sum_{k=0}^{n} \binom{\nu - 1/2}{n - k} \frac{2^{n-k} \Gamma(2\nu - n + k)}{\Gamma(2\nu)} b_k$$

for $n = 0, 1, 2, \ldots, 2N + 1$. A simple calculation shows

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = \frac{1}{4\nu(\nu - 1)}, \quad b_3 = 0, \quad b_4 = -\frac{1}{16\nu(\nu - 1)^2(\nu - 2)},$$

$$b_5 = 0, \quad b_6 = \frac{1}{32\nu(\nu - 1)^3(\nu - 2)(\nu - 3)}, \quad b_7 = 0.$$

The remaining coefficients have complicated forms. We easily get

$$\frac{K_{\nu+1}(x)}{K_\nu(x)} = \frac{2\nu}{x} + 1 + \sum_{n=0}^{2N} x^n \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}^{n+1}}$$

$$+ \cos(\pi\nu) \sum_{n=0}^{2N} (-1)^n x^n \int_0^\infty \frac{dy}{y^{n+2} G_\nu(y)} + o(x^{2N}). \tag{6.8}$$

We have to remark that we can not derive the higher term in (6.8) while M in (6.4) is arbitrary. This deference is caused from the integrability of $1/y^m G_\nu(y)$. Comparing the

corresponding coefficients in (6.7) and (6.8), we obtain

$$\sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}} = -1 - \cos(\pi\nu) \int_0^\infty \frac{dy}{y^2 G_\nu(y)}, \tag{6.9}$$

$$\sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j}^n} = 2\nu b_n + (-1)^n \cos(\pi\nu) \int_0^\infty \frac{dy}{y^{n+1} G_\nu(y)} \tag{6.10}$$

for $n = 2, 3, \ldots, 2N$. We define a sequence $\{\beta_n^\nu\}_{n=0}^{N(\nu)}$ of complex numbers by $\beta_0^\nu = 1$,

$$\beta_1^\nu = 1 + \cos(\pi\nu) \int_0^\infty \frac{dy}{y^2 G_\nu(y)},$$

$$\beta_n^\nu = -\frac{1}{n} \sum_{k=2}^n \beta_{n-k}^\nu \left\{ 2\nu b_k + (-1)^k \cos(\pi\nu) \int_0^\infty \frac{dy}{y^{k+1} G_\nu(y)} \right\}$$

$$+ \frac{1}{n} \beta_{n-1}^\nu \left\{ 1 + \cos(\pi\nu) \int_0^\infty \frac{dy}{y^2 G_\nu(y)} \right\}$$

for $n = 2, 3, \ldots, N(\nu)$. By (6.9), (6.10) and Lemma 6.1, we have the following theorem.

THEOREM 6.3. *For $|\nu| > 3/2$ the zeros of K_ν are the solutions of*

$$\sum_{n=0}^{N(\nu)} \beta_n^\nu z^n = 0.$$

REMARK 6.4. It is known that, if $\nu - 1/2$ is not an odd integer,

$$N(\nu) = \nu - \frac{1}{2} + \frac{\theta_\nu}{\pi}$$

(cf. [**21**, p. 512]). Here θ_ν is the unique number determined by $|\theta_\nu| < \pi$, $\cos\theta_\nu = \sin\pi\nu$, $\sin\theta_\nu = \cos\pi\nu$. Hence we deduce from (6.5) that, if $\nu > 3/2$ and $\nu - 1/2 \notin \mathbb{Z}$,

$$\int_0^\infty \frac{dy}{y G_\nu(y)} = \frac{\theta_\nu}{\pi \cos(\pi\nu)}. \tag{6.11}$$

Similarly, when $0 \le \nu < 3/2$ and $\nu \ne 1/2$, we can easily derive (6.11) by virtue of (2.4) and (6.3). In particular, we obtain that, for an integer $n \ge 0$,

$$\int_0^\infty \frac{dy}{y\{K_n(y)^2 + \pi^2 I_n(y)^2\}} = \frac{1}{2}.$$

REMARK 6.5. When $3/2 < \nu < 7/2$, $N(\nu) = 2$ and the zeros of K_ν satisfy some quadratic equations. In particular, when $\nu = 2$, the zeros $z_{2,1}$ and $z_{2,2}$ satisfy

$$\begin{cases} z_{2,1} + z_{2,2} = -\dfrac{15}{8} - \displaystyle\int_0^\infty \dfrac{dy}{G_2(y)}, \\[3mm] z_{2,1}^2 + z_{2,2}^2 = \dfrac{15}{8} + \displaystyle\int_0^\infty \dfrac{y\,dy}{G_2(y)}, \end{cases} \qquad \begin{cases} \dfrac{1}{z_{2,1}} + \dfrac{1}{z_{2,2}} = -1 - \displaystyle\int_0^\infty \dfrac{dy}{y^2 G_2(y)}, \\[3mm] \dfrac{1}{z_{2,1}^2} + \dfrac{1}{z_{2,2}^2} = \dfrac{1}{2} + \displaystyle\int_0^\infty \dfrac{dy}{y^3 G_2(y)}. \end{cases}$$

By using Mathematica we obtain from each system of equations that the zeros are close to $-1.28137 \pm 0.4294851\,i$, and check the comment "The two zeros of $K_2(z)$ are not very far from the points $-1.29 \pm 0.44\,i$." in [**21**, p. 512]. A table of the zeros is given in [**9**], where the continuity of each zero as a function of the index is shown and its graph is also given.

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