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On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space

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Abstract. In this paper, we proved decay properties of solutions to the Stokes equations with surface tension and gravity in the half space $\mathbf{R}^N_+ = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N > 0\}$ $(N \geq 2)$. In order to prove the decay properties, we first show that the zero points $\lambda_\pm$ of Lopatinskii determinant for some resolvent problem associated with the Stokes equations have the asymptotics: $\lambda_\pm = \pm i c_g^{1/2}|\xi'|^{1/2} - 2|\xi'|^2 + O(|\xi'|^{5/2})$ as $|\xi'| \to 0$, where $c_g > 0$ is the gravitational acceleration and $\xi' \in \mathbf{R}^{N-1}$ is the tangential variable in the Fourier space. We next shift the integral path in the representation formula of the Stokes semi-group to the complex left half-plane by Cauchy's integral theorem, and then it is decomposed into closed curves enclosing $\lambda_\pm$ and the remainder part. We finally see, by the residue theorem, that the low frequency part of the solution to the Stokes equations behaves like the convolution of the $(N-1)$-dimensional heat kernel and $\mathcal{F}^{-1}_{\xi'}[e^{\pm i c_g^{1/2}|\xi'|^{1/2}t}](x')$ formally, where $\mathcal{F}^{-1}_{\xi'}$ is the inverse Fourier transform with respect to ξ'. However, main task in our approach is to show that the remainder part in the above decomposition decay faster than the residue part.

1. Introduction and main results.

Let $\mathbf{R}^N_+$ and $\mathbf{R}^N_0$ $(N \geq 2)$ be the half space and its boundary, that is,

$$\mathbf{R}^N_+ = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N > 0\}, \quad \mathbf{R}^N_0 = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N = 0\}.$$

In this paper, we consider the following Stokes equations with the surface tension and gravity in the half space $\mathbf{R}^N_+$:

$$\begin{cases} \partial_t U - \operatorname{Div} S(U, \Theta) = 0, \quad \operatorname{div} U = 0 & \text{in } \mathbf{R}^N_+,\ t > 0, \\ \partial_t H + U_N = 0 & \text{on } \mathbf{R}^N_0,\ t > 0, \\ S(U, \Theta)\nu + (c_g - c_\sigma \Delta')H\nu = 0 & \text{on } \mathbf{R}^N_0,\ t > 0, \\ U|_{t=0} = f \quad \text{in } \mathbf{R}^N_+, \quad H|_{t=0} = d & \text{on } \mathbf{R}^{N-1}. \end{cases} \tag{1.1}$$

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Here the unknowns $U = (U_1(x,t), \dots, U_N(x,t))^{T1}$ and $\Theta = \Theta(x,t)$ are the velocity field and the pressure at $(x,t) \in \mathbf{R}^N_+ \times (0,\infty)$, respectively, and also $H = H(x',t)$ is the height function at $(x',t) \in \mathbf{R}^{N-1} \times (0,\infty)$. The operators div and Δ' are defined by

$$\operatorname{div} U = \sum_{j=1}^{N} D_j U_j, \quad \Delta' H = \sum_{j=1}^{N-1} D_j^2 H \quad \left(D_j = \frac{\partial}{\partial x_j} \right)$$

for any N-component vector function U and scalar function H. $S(U,\Theta) = -\Theta I + D(U)$ is the stress tensor, where I is the $N \times N$ identity matrix and $D(U)$ is the doubled strain tensor whose (i,j) component is $D_{ij}(U) = D_i U_j + D_j U_i$. Moreover, $\operatorname{Div} S(U,\Theta)$ is the N-component vector function with the ith component:

$$\sum_{j=1}^{N} D_j (D_j U_i + D_i U_j - \delta_{ij}\Theta) = \Delta U_i + D_i \operatorname{div} U - D_i \Theta.$$

Let $\nu = (0, \dots, 0, -1)^T$ be the unit outer normal to $\mathbf{R}^N_0$, and then

$$i^{\text{th}} \text{ component of } S(U,\Theta)\nu = \begin{cases} -(D_N U_i + D_i U_N) & (i = 1, \dots, N-1), \\ -2D_N U_N + \Theta & (i = N). \end{cases}$$

The parameters $c_g > 0$ and $c_\sigma > 0$ describe the gravitational acceleration and the surface tension coefficient, respectively, and the functions $f = (f_1(x), \dots, f_N(x))^T$ and $d = d(x')$ are given initial data.

The equations (1.1) arise in the study of a free boundary problem for the incompressible Navier-Stokes equations. The free boundary problem is mathematically to find a N-component vector function $u = (u_1(x,t), \dots, u_N(x,t))^T$, a scalar function $\theta = \theta(x,t)$, and a free boundary $\Gamma(t) = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N = h(x',t)\}$ satisfying the following Navier-Stokes equations:

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) - \operatorname{Div} S(u,\theta) = -\rho c_g \nabla x_N, \quad \operatorname{div} u = 0 & \text{in } \Omega(t),\ t > 0, \\ \partial_t h + u' \cdot \nabla' h - u_N = 0 & \text{on } \Gamma(t),\ t > 0, \\ S(u,\theta)\nu_t = c_\sigma \kappa \nu_t & \text{on } \Gamma(t),\ t > 0, \\ u|_{t=0} = u_0 & \text{in } \Omega(0), \\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^{N-1}. \end{cases} \tag{1.2}$$

Here $\Omega(t) = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N < h(x',t)\}$, and $\Omega(0)$ is a given initial domain; ρ is a positive constant describing the density of the fluid; $\kappa = \kappa(x,t)$ is the mean curvature of $\Gamma(t)$, and ν_t is the unit outer normal to $\Gamma(t)$; $u \cdot \nabla u = \sum_{j=1}^{N} u_j D_j u$, and $u' \cdot \nabla' h = \sum_{j=1}^{N-1} u_j D_j h$.

A problem is called the finite depth one if the equations (1.2) is considered in $\Omega(t) =$

[1] M^T describes the transposed M.

$\{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ -b < x_N < h(x', t)\}$ for some constant $b > 0$ with Dirichlet boundary condition on the lower boundary: $\Gamma_b = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N = -b\}$. There are several results for the finite depth problem. In fact, Beale [**4**] proved the local well-posedness in the case of $c_\sigma = 0$ and $c_g > 0$, and also [**5**] proved the global well-posedness for small initial data when $c_\sigma > 0$ and $c_g > 0$. Beale and Nishida [**6**] proved decay properties of the solution obtained in [**5**], but the paper is just survey. We can find the detailed proof in Hataya [**9**]. Tani and Tanaka [**20**] also treated both case of $c_\sigma = 0$ and $c_\sigma > 0$ under the condition $c_g > 0$. Along with these results, we refer to Allain [**2**], Hataya and Kawashima [**8**], and Bae [**3**]. Note that they treated the problem in the L_2-L_2 framework, that is, their classes of solutions are contained in the space-time L_2 space, and their methods are based on the Hilbert space structure. Thus, their methods do not work in general Banach spaces. From this viewpoint, we need completely different techniques since our aim is to treat (1.2) in the L_p-L_q framework.

The study of free boundary problems with surface tension and gravity in the L_p-L_q maximal regularity class were started by Shibata and Shimizu [**16**]. We especially note that Abels [**1**] proved the local well-posedness of the finite depth problem with $p = q > N$, $c_\sigma = 0$, and $c_g > 0$. In the case of the L_p-L_q framework, Shibata [**19**] proved the local well-posedness of free boundary problems for the Navier-Stokes equations with $c_\sigma = c_g = 0$ in general unbounded domains containing the finite depth problem, where p and q are exponents satisfying the conditions: $1 < p, q < \infty$ and $2/p + N/q < 1$.

Concerning (1.2), under some smallness condition of initial data, Prüss and Simonett [**10**] showed the local well-posedness of the two-phase problem containing (1.2) with $c_\sigma > 0$ and $c_g = 0$, and also [**11**] and [**12**] proved the local well-posedness of the case where $c_g > 0$ and $c_\sigma > 0$. Recently, there are two papers due to Shibata and Shimizu [**15**], [**17**], which treat the linearized problem of (1.2) and some resolvent problem. But all the papers do not have any results about decay properties of solutions for the linearized problem of (1.2). In the present paper, we show decay properties of solutions to (1.1) as the first step to prove the global well-posedness of (1.2).

Now we shall state our main results. For this purpose, we introduce some symbols and function spaces. For any domain Ω in $\mathbf{R}^N$, positive integer m, and $1 \le q \le \infty$, $L_q(\Omega)$ and $W_q^m(\Omega)$ denote the usual Lebesgue and Sobolev spaces with $\|\cdot\|_{L_q(\Omega)}$ and $\|\cdot\|_{W_q^m(\Omega)}$, respectively, and we set $W_q^0(\Omega) = L_q(\Omega)$. Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and let $\mathbf{C}$ be the set of all complex numbers. For differentiations, we use the symbols D_x^α and $D_{\xi'}^{\beta'}$ defined by

$$D_x^\alpha f(x_1, \ldots, x_N) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} f(x_1, \ldots, x_N) = D_1^{\alpha_1} \cdots D_N^{\alpha_N} f(x_1, \ldots, x_N),$$

$$D_{\xi'}^{\beta'} g(\xi_1, \ldots, \xi_{N-1}) = \frac{\partial^{|\beta'|}}{\partial \xi_1^{\beta_1} \cdots \partial \xi_{N-1}^{\beta^{N-1}}} g(\xi_1, \ldots, \xi_{N-1}) = D_1^{\beta_1} \cdots D_{N-1}^{\beta_{N-1}} g(\xi_1, \ldots, \xi_{N-1}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}_0^N$ and $\beta' = (\beta_1, \ldots, \beta_{N-1}) \in \mathbf{N}_0^{N-1}$. In addition, for any vector functions $u(x) = (u_1(x), \ldots, u_N(x))^T$, $D_x^\alpha u(x)$ is given by $D_x^\alpha u(x) = (D_x^\alpha u_1(x), \ldots, D_x^\alpha u_N(x))^T$, and also

$$\nabla u = \{D_i u_j \mid i,j = 1,\dots,N\}, \quad \nabla^2 u = \{D_i D_j u_k \mid i,j,k = 1,\dots,N\}.$$

Let X and Y be Banach spaces with $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and then $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X to Y, and set $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $m \in \mathbf{N}_0$ and an interval I in $\mathbf{R}$, $C^m(I,X)$ is the set of all X-valued C^m-functions defined on I. Let X^m be the m-product space of X with $m \in \mathbf{N}$, while we use the symbol $\|\cdot\|_X$ to denote its norm for short, that is,

$$\|u\|_X = \sum_{j=1}^m \|u_j\|_X \quad \text{for } u = (u_1,\dots,u_m) \in X^m.$$

For $1 < q < \infty$, non-integer $s > 0$, and $m \in \mathbf{N}$, $W_q^s(\mathbf{R}^m)$ denotes the Slobodeckii spaces defined by

$$W_q^s(\mathbf{R}^m) = \bigl\{u \in W_q^{[s]}(\mathbf{R}^m) \mid \|u\|_{W_q^s(\mathbf{R}^m)} < \infty\bigr\},$$

$$\|u\|_{W_q^s(\mathbf{R}^m)} = \|u\|_{W_q^{[s]}(\mathbf{R}^m)} + \sum_{|\alpha|=[s]} \biggl(\int_{\mathbf{R}^m}\int_{\mathbf{R}^m} \frac{|D_x^\alpha u(x) - D_y^\alpha u(y)|^q}{|x-y|^{m+(s-[s])q}}\,dxdy\biggr)^{1/q},$$

where $[s]$ is the largest integer lower than s. For any vector function $u = (u_1,\dots,u_N)^T$ and $v = (v_1,\dots,v_N)^T$ defined on $\mathbf{R}^N_+$, we set

$$(u,v)_{\mathbf{R}^N_+} = \int_{\mathbf{R}^N_+} u(x)\cdot v(x)\,dx = \sum_{j=1}^N \int_{\mathbf{R}^N_+} u_j(x)v_j(x)\,dx.$$

The letter C denotes a generic constant and $C(a,b,c,\dots)$ a generic constant depending on the quantities $a,b,c,\dots$. The value of C and $C(a,b,c,\dots)$ may change from line to line.

Let $\widehat{W}_q^1(\mathbf{R}^N_+)$ be the homogeneous spaces of order 1 defined by $\widehat{W}_q^1(\mathbf{R}^N_+) = \{\theta \in L_{q,\mathrm{loc}}(\mathbf{R}^N_+) \mid \nabla\theta \in L_q(\mathbf{R}^N_+)^N\}$. In addition, we set $\widehat{W}_{q,0}^1(\mathbf{R}^N_+) = \{\theta \in \widehat{W}_q^1(\mathbf{R}^N_+) \mid \theta|_{\mathbf{R}^N_0} = 0\}$ and $W_{q,0}^1(\mathbf{R}^N_+) = \{\theta \in W_q^1(\mathbf{R}^N_+) \mid \theta|_{\mathbf{R}^N_0} = 0\}$. As was seen in [**18**, Theorem A.3], $W_{q,0}^1(\mathbf{R}^N_+)$ is dense in $\widehat{W}_{q,0}^1(\mathbf{R}^N_+)$ with the gradient norm $\|\nabla\cdot\|_{L_q(\mathbf{R}^N_+)}$. Then the second solenoidal space $J_q(\mathbf{R}^N_+)$ is defined by

$$J_q(\mathbf{R}^N_+) = \bigl\{f \in L_q(\mathbf{R}^N_+)^N \mid (f,\nabla\varphi)_{\mathbf{R}^N_+} = 0 \text{ for any } \varphi \in \widehat{W}_{q',0}^1(\mathbf{R}^N_+)\bigr\},$$

where $1/q + 1/q' = 1$. For simplicity, we set

$$X_q = J_q(\mathbf{R}^N_+) \times W_q^{2-1/q}(\mathbf{R}^{N-1}), \quad X_q^0 = L_q(\mathbf{R}^N_+) \times L_q(\mathbf{R}^{N-1}),$$
$$X_q^i = L_q(\mathbf{R}^N_+) \times W_q^{i-1/q}(\mathbf{R}^{N-1}) \quad (i = 1,2), \tag{1.3}$$

and let $\mathcal{E}H$ be the harmonic extension of H, that is,

$$\begin{cases} \Delta \mathcal{E} H = 0 & \text{in } \mathbf{R}^N_+, \\ \mathcal{E} H = H & \text{on } \mathbf{R}^N_0. \end{cases} \tag{1.4}$$

The main results of this paper then is stated as follows:

THEOREM 1.1. *Let $1 < p < \infty$, $c_g > 0$, and $c_\sigma > 0$.*

(1) *For every $t > 0$ there exists operators*

$$S(t) \in \mathcal{L}\big(X^2_p, W^2_p(\mathbf{R}^N_+)^N\big),\ \Pi(t) \in \mathcal{L}\big(X^2_p, \widehat{W}^1_p(\mathbf{R}^N_+)\big),\ T(t) \in \mathcal{L}\big(X^2_p, W^{3-1/p}_p(\mathbf{R}^{N-1})\big)$$

such that for $F = (f, d) \in X_p$

$$\begin{aligned} &S(\cdot)F \in C^1\big((0,\infty), J_p(\mathbf{R}^N_+)\big) \cap C^0\big((0,\infty), W^2_p(\mathbf{R}^N_+)^N\big), \\ &\Pi(\cdot)F \in C^0\big((0,\infty), \widehat{W}^1_p(\mathbf{R}^N_+)\big), \\ &T(\cdot)F \in C^1\big((0,\infty), W^{2-1/p}_p(\mathbf{R}^{N-1})\big) \cap C^0\big((0,\infty), W^{3-1/p}_p(\mathbf{R}^{N-1})\big), \end{aligned}$$

and that $(U, \Theta, H) = (S(t)F, \Pi(t)F, T(t)F)$ solves uniquely (1.1) *with*

$$\lim_{t\to 0} \|(U(t), H(t)) - (f, d)\|_{X_p} = 0.$$

(2) *Let $1 \le r \le 2 \le q \le \infty$ and $F = (f, d) \in X^0_r \cap X^2_p$. The operators, obtained in* (1), *then are decomposed into*

$$\begin{aligned} S(t)F &= S_0(t)F + S_\infty(t)F + R(t)f, \\ \Pi(t)F &= \Pi_0(t)F + \Pi_\infty(t)F + P(t)f, \\ T(t)F &= T_0(t)F + T_\infty(t)F, \end{aligned} \tag{1.5}$$

which satisfy the estimates as follows: *For $k = 1, 2$, $\ell = 0, 1, 2$, and $t \ge 1$*

$$\begin{aligned} \|(S_0(t)F, \partial_t \mathcal{E}(T_0(t)F))\|_{L_q(\mathbf{R}^N_+)} &\le C(t+1)^{-m(q,r)} \|F\|_{X^0_r} \quad \text{if } (q,r) \ne (2,2), \\ \|\nabla^k S_0(t)F\|_{L_q(\mathbf{R}^N_+)} &\le C(t+1)^{-n(q,r)-k/8} \|F\|_{X^0_r}, \\ \|(\partial_t S_0(t)F, \nabla \Pi_0(t)F)\|_{L_q(\mathbf{R}^N_+)} &\le C(t+1)^{-m(q,r)-1/4} \|F\|_{X^0_r}, \\ \|\nabla^k \partial_t \mathcal{E}(T_0(t)F)\|_{L_q(\mathbf{R}^N_+)} &\le C(t+1)^{-m(q,r)-k/2} \|F\|_{X^0_r}, \\ \|\nabla^{1+\ell} \mathcal{E}(T_0(t)F)\|_{L_q(\mathbf{R}^N_+)} &\le C(t+1)^{-m(q,r)-1/4-\ell/2} \|F\|_{X^0_r} \end{aligned} \tag{1.6}$$

with some positive constant C, where we have set

$$m(q,r) = \frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),$$
$$n(q,r) = \frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right) + \min\left\{\frac{1}{2}\left(\frac{1}{r}-\frac{1}{q}\right), \frac{1}{8}\left(2-\frac{1}{q}\right)\right\}.$$

In addition, there exist positive constants δ and C such that for $t \geq 1$

$$\|(\partial_t S_\infty(t)F, \nabla\Pi_\infty(t)F)\|_{L_p(\mathbf{R}^N_+)} + \|(S_\infty(t)F, \partial_t\mathcal{E}(T_\infty(t)F)), \nabla\mathcal{E}(T_\infty(t)F)))\|_{W^2_p(\mathbf{R}^N_+)} \leq Ce^{-\delta t}\|F\|_{X^2_p}. \tag{1.7}$$

Finally, for $t \geq 1$ and $\ell = 0, 1, 2$,

$$\|\nabla^\ell R(t)f\|_{L_p(\mathbf{R}^N_+)} \leq C(t+1)^{-\ell/2}\|f\|_{L_p(\mathbf{R}^N_+)},$$
$$\|(\partial_t R(t)f, \nabla P(t)f)\|_{L_p(\mathbf{R}^N_+)} \leq C(t+1)^{-1}\|f\|_{L_p(\mathbf{R}^N_+)}. \tag{1.8}$$

This paper consists of five sections. In the next section, we introduce some symbols and lemmas, and also consider some resolvent problem associated with (1.1) with $c_g = c_\sigma = 0$. In Section 3, we construct the operators $S(t), \Pi(t)$, and $T(t)$, and also give the decompositions (1.5). Finally, Theorem 1.1 (2) is proved in Section 4 and Section 5.

2. Preliminaries.

We first give some symbols used throughout this paper. Set

$$\Sigma_\varepsilon = \{\lambda \in \mathbf{C} \mid |\arg\lambda| < \pi - \varepsilon,\ \lambda \neq 0\}, \quad \Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0\}$$

for any $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. We then define

$$A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2} \quad (\mathrm{Re}\, B \geq 0), \quad \mathcal{M}(a) = \frac{e^{-Ba} - e^{-Aa}}{B - A},$$
$$D(A,B) = B^3 + AB^2 + 3A^2B - A^3,$$
$$L(A,B) = (B-A)D(A,B) + A(c_g + c_\sigma A^2) \tag{2.1}$$

for $\xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbf{R}^{N-1}$, $\lambda \in \Sigma_\varepsilon$, and $a > 0$. Especially, we have, for $\ell = 1, 2$,

$$\frac{\partial^\ell}{\partial a^\ell}\mathcal{M}(a) = (-1)^\ell\big((B+A)^{\ell-1}e^{-Ba} + A^\ell\mathcal{M}(a)\big),$$
$$\mathcal{M}(a) = -a\int_0^1 e^{-(B\theta + A(1-\theta))a}\, d\theta. \tag{2.2}$$

The following lemma was proved in [**17**, Lemma 5.2, Lemma 5.3, Lemma 7.2].

LEMMA 2.1. *Let $0 < \varepsilon < \pi/2$, $s \in \mathbf{R}$, $a > 0$, and $\alpha' \in \mathbf{N}_0^{N-1}$.*

(1) *There holds the estimate*

$$b_\varepsilon(|\lambda|^{1/2} + A) \le \operatorname{Re} B \le |B| \le (|\lambda|^{1/2} + A)$$

for any $(\xi', \lambda) \in \mathbf{R}^{N-1} \times \Sigma_\varepsilon$ with $b_\varepsilon = (1/\sqrt{2})\{\sin(\varepsilon/2)\}^{3/2}$.

(2) *There exist a positive constant $C = C(\varepsilon, s, \alpha')$ such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_\varepsilon$*

$$\begin{aligned}
\left|D_{\xi'}^{\alpha'} A^s\right| &\le C A^{s-|\alpha'|},\\
\left|D_{\xi'}^{\alpha'} e^{-Aa}\right| &\le C A^{-|\alpha'|} e^{-(A/2)a},\\
\left|D_{\xi'}^{\alpha'} B^s\right| &\le C(|\lambda|^{1/2} + A)^{s-|\alpha'|},\\
\left|D_{\xi'}^{\alpha'} e^{-Ba}\right| &\le C(|\lambda| + A)^{-|\alpha'|} e^{-(b_\varepsilon/8)(|\lambda|^{1/2}+A)a},\\
\left|D_{\xi'}^{\alpha'} D(A,B)^s\right| &\le C(|\lambda|^{1/2} + A)^{3s} A^{-|\alpha'|},\\
\left|D_{\xi'}^{\alpha'} \mathcal{M}(a)\right| &\le C A^{-1-|\alpha'|} e^{-(b_\varepsilon/8)Aa},\\
\left|D_{\xi'}^{\alpha'} \mathcal{M}(a)\right| &\le C|\lambda|^{-1/2} A^{-|\alpha'|} e^{-(b_\varepsilon/8)Aa}.
\end{aligned}$$

(3) *There exist positive constants $\lambda_0 = \lambda_0(\varepsilon) \ge 1$ and $C = C(\varepsilon, \lambda_0, \alpha')$ such that for any $(\xi', \lambda) \in (\mathbf{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\lambda_0}$*

$$\left|D_{\xi'}^{\alpha'} L(A,B)^{-1}\right| \le C\{|\lambda|(|\lambda|^{1/2} + A)^2 + A(c_g + c_\sigma A^2)\}^{-1} A^{-|\alpha'|}.$$

Let $f(x)$ and $g(\xi)$ be functions defined on $\mathbf{R}^N$, and then the Fourier transform of $f(x)$ and the inverse Fourier transform of $g(\xi)$ are defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbf{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

We also define the partial Fourier transform of $f(x)$ and the inverse partial Fourier transform of $g(\xi)$ with respect to tangential variables $x' = (x_1, \dots, x_{N-1})$ and its dual variable $\xi' = (\xi_1, \dots, \xi_{N-1})$ by

$$\begin{aligned}
\widehat{f}(\xi', x_N) &= \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\,dx',\\
\mathcal{F}_{\xi'}^{-1}[g](x', \xi_N) &= \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', \xi_N)\,d\xi'.
\end{aligned}$$

Next we consider the following resolvent problem:

$$\begin{cases} \lambda w - \mathrm{Div} S(w,p) = f, \quad \mathrm{div}\, w = 0 & \text{in } \mathbf{R}^N_+, \\ S(w,p)\nu = 0 & \text{on } \mathbf{R}^N_0. \end{cases} \tag{2.3}$$

LEMMA 2.2. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $\lambda \in \Sigma_\varepsilon$, and $f \in L_q(\mathbf{R}^N_+)^N$. Then the equations* (2.3) *admits a unique solution $(w,p) \in W^2_q(\mathbf{R}^N_+)^N \times \widehat{W}^1_q(\mathbf{R}^N_+)$ possessing the estimate*:

$$\|(\lambda w, \lambda^{1/2}\nabla w, \nabla^2 w, \nabla p)\|_{L_q(\mathbf{R}^N_+)} \le C \|f\|_{L_q(\mathbf{R}^N_+)}$$

with some positive constant $C = C(\varepsilon, q, N)$. In addition, $\widehat{w}_N(\xi', 0, \lambda)$ is given by

$$\begin{aligned} \widehat{w}_N(\xi', 0, \lambda) = & \sum_{k=1}^{N-1} \frac{i\xi_k (B - A)}{D(A,B)} \int_0^\infty e^{-By_N} \widehat{f}_k(\xi', y_N)\, dy_N \\ & + \frac{A(B+A)}{D(A,B)} \int_0^\infty e^{-By_N} \widehat{f}_N(\xi', y_N)\, dy_N \\ & - \sum_{k=1}^{N-1} \frac{i\xi_k (B^2 + A^2)}{D(A,B)} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N)\, dy_N \\ & - \frac{A(B^2 + A^2)}{D(A,B)} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N \qquad (2.4) \\ = & \sum_{k=1}^{N-1} \frac{i\xi_k (B - A)}{D(A,B)} \int_0^\infty e^{-Ay_N} \widehat{f}_k(\xi', y_N)\, dy_N \\ & + \frac{A(B+A)}{D(A,B)} \int_0^\infty e^{-Ay_N} \widehat{f}_N(\xi', y_N)\, dy_N \\ & - \sum_{k=1}^{N-1} \frac{2i\xi_k AB}{D(A,B)} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N)\, dy_N \\ & - \frac{2A^3}{D(A,B)} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N. \qquad (2.5) \end{aligned}$$

PROOF. The lemma was proved by Shibata and Shimizu [**14**, Theorem 4.1] except for (2.4) and (2.5), so that we prove (2.4) and (2.5) here.

Given functions $g(x)$ defined on $\mathbf{R}^N_+$, we set their even extensions $g^e(x)$ and odd extensions $g^o(x)$ as

$$g^e(x) = \begin{cases} g(x', x_N) & \text{in } \mathbf{R}^N_+, \\ g(x', -x_N) & \text{in } \mathbf{R}^N_-, \end{cases} \quad g^o(x) = \begin{cases} g(x', x_N) & \text{in } \mathbf{R}^N_+, \\ -g(x', -x_N) & \text{in } \mathbf{R}^N_-, \end{cases} \tag{2.6}$$

where $\mathbf{R}^N_- = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, x_N < 0\}$. In addition, given the right member $f = (f_1, \ldots, f_N)^T$ of (2.3), we set $Ef = (f^o_1, \ldots, f^o_{N-1}, f^e_N)^T$. Let (w^1, p^1) be the solution

to the following resolvent problem:

$$\lambda w^1 - \mathrm{Div} S(w^1, p^1) = Ef, \quad \mathrm{div}\, w^1 = 0 \quad \text{in } \mathbf{R}^N.$$

We then have the following solution formulas (cf. [**17**, Section 3]):

$$w_j^1(x,\lambda) = \mathcal{F}_\xi^{-1}\left[\frac{\widehat{(Ef)}_j(\xi)}{\lambda + |\xi|^2}\right](x) - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\xi_j\xi_k}{|\xi|^2(\lambda+|\xi|^2)}\widehat{(Ef)}_k(\xi)\right](x) \quad (j = 1, \dots, N),$$

$$p^1(x,\lambda) = -\mathcal{F}_\xi^{-1}\left[\frac{i\xi}{|\xi|^2}\cdot \widehat{Ef}(\xi)\right](x). \tag{2.7}$$

As was seen in [**14**, Section 4], we have, by the definition of the extension E,

$$D_N w_N^1(x', 0, \lambda) = 0, \quad p^1(x', 0, \lambda) = 0. \tag{2.8}$$

Next we give the exact formulas of $\widehat{w}_N^1(\xi', 0, \lambda)$ and $\widehat{D_N w_j^1}(\xi', 0, \lambda)$ for $j = 1, \dots, N-1$. To this end, we use the following lemma which is proved by the residue theorem.

LEMMA 2.3. *Let $a \in \mathbf{R} \setminus \{0\}$, and let $\xi = (\xi_1, \dots, \xi_N) \in \mathbf{R}^N$. Then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{|\xi|^2}\, d\xi_N = \frac{e^{-A|a|}}{2A}, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_N e^{ia\xi_N}}{|\xi|^2}\, d\xi_N = -\mathrm{sign}(a)\frac{e^{-A|a|}}{2},$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{\lambda+|\xi|^2}\, d\xi_N = \frac{e^{-B|a|}}{2B}, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi_N e^{ia\xi_N}}{\lambda+|\xi|^2} d\xi_N = \mathrm{sign}(a)\frac{i}{2}e^{-B|a|},$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi_N e^{ia\xi_N}}{|\xi|^2(\lambda+|\xi|^2)}\, d\xi_N = \mathrm{sign}(a)\frac{i}{2\lambda}(e^{-A|a|} - e^{-B|a|}),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi_N^2 e^{ia\xi_N}}{|\xi|^2(\lambda+|\xi|^2)}\, d\xi_N = -\frac{1}{2\lambda}(Ae^{-A|a|} - Be^{-B|a|}),$$

where $\mathrm{sign}(a)$ *defined by the formula*: $\mathrm{sign}(a) = 1$ *when* $a > 0$ *and* $\mathrm{sign}(a) = -1$ *when* $a < 0$.

In order to obtain

$$\widehat{w}_N^1(\xi', 0, \lambda) = \sum_{k=1}^{N-1} \frac{i\xi_k}{\lambda}\int_0^\infty (e^{-Ay_N} - e^{-By_N})\widehat{f}_k(\xi', y_N)\, dy_N + \int_0^\infty \frac{e^{-By_N}}{B}\widehat{f}_N(\xi', y_N)\, dy_N + \frac{1}{\lambda}\int_0^\infty (Ae^{-Ay_N} - Be^{-By_N})\widehat{f}_N(\xi', y_N)\, dy_N,$$

$$\widehat{D_N w_j^1}(\xi',0,\lambda) = -\sum_{k=1}^{N-1} \frac{\xi_j\xi_k}{\lambda} \int_0^\infty (e^{-Ay_N} - e^{-By_N}) \widehat{f}_k(\xi', y_N)\, dy_N + \int_0^\infty e^{-By_N} \widehat{f}_j(\xi', y_N)\, dy_N + \frac{i\xi_j}{\lambda} \int_0^\infty (Ae^{-Ay_N} - Be^{-By_N}) \widehat{f}_N(\xi', y_N)\, dy_N, \tag{2.9}$$

we apply the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ to (2.7), insert the identities in Lemma 2.3 into the resultant formula, and use the formulas:

$$\mathcal{F}[f_j^o](\xi) = \int_0^\infty \bigl(e^{-iy_N\xi_N} - e^{iy_N\xi_N}\bigr) \widehat{f}_j(\xi', y_N)\, dy_N \quad (j = 1, \ldots, N-1),$$
$$\mathcal{F}[f_N^e](\xi) = \int_0^\infty \bigl(e^{-iy_N\xi_N} + e^{iy_N\xi_N}\bigr) \widehat{f}_N(\xi', y_N)\, dy_N.$$

Here and in the following, j runs from 1 through $N-1$. By (2.9) and the fact that $\lambda = B^2 - A^2$ and $e^{-By_N} - e^{-Ay_N} = (B-A)\mathcal{M}(y_N)$, we have

$$\widehat{w}_N^1(\xi',0,\lambda) = \frac{A}{B(B+A)} \int_0^\infty e^{-Bx_N} \widehat{f}_N(\xi', y_N)\, dy_N - \sum_{k=1}^{N-1} \frac{i\xi_k}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N)\, dy_N - \frac{A}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N,$$
$$\widehat{D_N w_j^1}(\xi',0,\lambda) = \int_0^\infty e^{-By_N} \widehat{f}_j(\xi', y_N)\, dy_N - \frac{i\xi_j}{B+A} \int_0^\infty e^{-By_N} \widehat{f}_N(\xi', y_N)\, dy_N + \sum_{k=1}^{N-1} \frac{\xi_j\xi_k}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_k(\xi', y_N)\, dy_N - \frac{i\xi_j A}{B+A} \int_0^\infty \mathcal{M}(y_N) \widehat{f}_N(\xi', y_N)\, dy_N. \tag{2.10}$$

Next we give the exact formula of $\widehat{w}_N^2(\xi',0,\lambda)$. Setting $w = w^1 + w^2$ and $p = p^1 + p^2$ in (2.3) and noting (2.8), we achieve the equations:

$$\begin{cases} \lambda w^2 - \mathrm{Div} S(w^2, p^2) = 0, \quad \mathrm{div}\, w^2 = 0 & \text{in } \mathbf{R}_+^N, \\ D_j w_N^2 + D_N w_j^2 = -h_j & \text{on } \mathbf{R}_0^N, \\ -p^2 + 2D_N w_N^2 = 0 & \text{on } \mathbf{R}_0^N \end{cases}$$

with $h_j = D_j w_N^1 + D_N w_j^1$. We then obtain the formulas (cf. [**17**, Section 4]):

$$w_N^2(x',x_N,\lambda)=\mathcal{F}_{\xi'}^{-1}\big[\widehat{w}_N^2(\xi',x_N,\lambda)\big](x'),$$

$$\widehat{w}_N^2(\xi',x_N,\lambda)=\left(\frac{B-A}{D(A,B)}e^{-Bx_N}+\frac{2AB}{D(A,B)}\mathcal{M}(x_N)\right)\sum_{j=1}^{N-1}i\xi_j\widehat{h}_j(\xi',0,\lambda). \tag{2.11}$$

By (2.10) and (2.11),

$$\begin{aligned}
\widehat{w}_N^2(\xi',0,\lambda)&=\frac{B-A}{D(A,B)}\sum_{j=1}^{N-1}i\xi_j\widehat{h}_j(\xi',0,\lambda)\\
&=\sum_{j=1}^{N-1}\frac{i\xi_j(B-A)}{D(A,B)}\big(i\xi_j\widehat{w}_N^1(\xi',0,\lambda)+\widehat{D_Nw_j^1}(\xi',0,\lambda)\big)\\
&=\sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_k(\xi',y_N)\,dy_N\\
&\quad+\frac{A^2(B-A)^2}{B(B+A)D(A,B)}\int_0^\infty e^{-By_N}\widehat{f}_N(\xi',y_N)\,dy_N\\
&\quad+\sum_{k=1}^{N-1}\frac{2i\xi_kA^2(B-A)}{(B+A)D(A,B)}\int_0^\infty\mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N\\
&\quad+\frac{2A^3(B-A)}{(B+A)D(A,B)}\int_0^\infty\mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N,
\end{aligned}$$

which combined with (2.10) furnishes (2.4), because $\widehat{w}_N(\xi',0,\lambda)=\widehat{w}_N^1(\xi',0,\lambda)+\widehat{w}_N^2(\xi',0,\lambda)$.

Finally, using the relation: $e^{-By_N}=e^{-Ay_N}+(B-A)\mathcal{M}(y_N)$ in (2.4), we have (2.5). This completes the proof of the lemma. □

3. Decompositions of operators.

In this section, we construct the operators $S(t),\Pi(t)$, and $T(t)$ in Theorem 1.1, and also show the decompositions (1.5). For this purpose, we first give the exact formulas of the solution (u,θ,h) to

$$\begin{cases}\lambda u-\operatorname{Div}S(u,\theta)=f\quad \operatorname{div}u=0 & \text{in }\mathbf{R}^N_+,\\ \lambda h+u_N=d & \text{on }\mathbf{R}^N_0,\\ S(u,\theta)\nu+(c_g-c_\sigma\Delta')h\nu=0 & \text{on }\mathbf{R}^N_0.\end{cases} \tag{3.1}$$

Let (w,p) be the solution to (2.3) and (v,π,h) the solution to the equations:

$$\begin{cases} \lambda v - \Delta v + \nabla \pi = 0, \quad \operatorname{div} v = 0 & \text{in } \mathbf{R}^N_+, \\ \lambda h + v_N = -w_N + d & \text{on } \mathbf{R}^N_0, \\ S(v,\pi)\nu + (c_g - c_\sigma \Delta')h\nu = 0 & \text{on } \mathbf{R}^N_0. \end{cases} \tag{3.2}$$

Then, $u = v + w$, $\theta = \pi + p$, and h solve (3.1). Let j and k run from 1 through $N-1$ and J from 1 through N, respectively, in the present section. The exact formulas of (v, π, h) are given by

$$v_J(x,\lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{v}_J(\xi', x_N, \lambda)](x'), \quad \pi(x,\lambda) = \mathcal{F}^{-1}_{\xi'}[\widehat{\pi}(\xi', x_N, \lambda)](x')$$

$$h(x',\lambda) = \mathcal{F}^{-1}_{\xi'}\left[\frac{D(A,B)}{(B+A)L(A,B)}\big(- \widehat{w}_N(\xi', 0, \lambda) + \widehat{d}(\xi')\big)\right](x')$$

(cf. [**17**, Section 7]), where

$$\widehat{v}_j(\xi', x_N, \lambda) = \left(- \frac{i\xi_j(B-A)}{D(A,B)} e^{-Bx_N} + \frac{i\xi_j(B^2+A^2)}{D(A,B)} \mathcal{M}(x_N)\right)(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda),$$

$$\widehat{v}_N(\xi', x_N, \lambda) = \left(\frac{A(B+A)}{D(A,B)} e^{-Bx_N} - \frac{A(B^2+A^2)}{D(A,B)} \mathcal{M}(x_N)\right)(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda),$$

$$\widehat{\pi}(\xi', x_N, \lambda) = \frac{(B+A)(B^2+A^2)}{D(A,B)} e^{-Ax_N}(c_g + c_\sigma A^2)\widehat{h}(\xi', \lambda).$$

Inserting (2.4) into $h(x', \lambda)$, we have the decompositions:

$$\widehat{v}_J(\xi', x_N, \lambda) = \widehat{v}^f_J(\xi', x_N, \lambda) + \widehat{v}^d_J(\xi', x_N, \lambda), \quad \widehat{\pi}(\xi', x_N, \lambda) = \widehat{\pi}^f(\xi', x_N, \lambda) + \widehat{\pi}^d(\xi', x_N, \lambda),$$

where each term on the right-hand sides is given by

$$\begin{aligned} \widehat{v}^f_J(\xi', x_N, \lambda) = {} & \sum_{K=1}^N \frac{\mathcal{V}^{BB}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty e^{-B(x_N + y_N)} \widehat{f}_K(\xi', y_N)\, dy_N \\ & + \sum_{K=1}^N \frac{\mathcal{V}^{B\mathcal{M}}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_K(\xi', y_N)\, dy_N \\ & + \sum_{K=1}^N \frac{\mathcal{V}^{\mathcal{M}B}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty \mathcal{M}(x_N) e^{-By_N} \widehat{f}_K(\xi', y_N)\, dy_N \\ & + \sum_{K=1}^N \frac{\mathcal{V}^{\mathcal{M}\mathcal{M}}_{JK}(\xi', \lambda)(c_g + c_\sigma A^2)}{L(A,B)} \int_0^\infty \mathcal{M}(x_N)\mathcal{M}(y_N) \widehat{f}_K(\xi', y_N)\, dy_N \end{aligned}$$

$$\widehat{v}^d_j(\xi', x_N, \lambda) = \frac{i\xi_j(c_g + c_\sigma A^2)}{(B+A)L(A,B)}\big(- (B-A)e^{-Bx_N} + (B^2+A^2)\mathcal{M}(x_N)\big)\widehat{d}(\xi'),$$

$$\widehat{v}_N^d(\xi',x_N,\lambda)=\frac{A(c_g+c_\sigma A^2)}{(B+A)L(A,B)}\big((B+A)e^{-Bx_N}-(B^2+A^2)\mathcal{M}(x_N)\big)\widehat{d}(\xi'),$$

$$\widehat{\pi}^f(\xi',x_N,\lambda)=\sum_{K=1}^N\frac{\mathcal{P}_K^{AA}(\xi',\lambda)(c_g+c_\sigma A^2)}{L(A,B)}\int_0^\infty e^{-A(x_N+y_N)}\widehat{f}_K(\xi',y_N)\,dy_N$$
$$+\sum_{K=1}^N\frac{\mathcal{P}_K^{A\mathcal{M}}(\xi',\lambda)(c_g+c_\sigma A^2)}{L(A,B)}\int_0^\infty e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}_K(\xi',y_N)\,dy_N,$$

$$\widehat{\pi}^d(\xi',x_N,\lambda)=\frac{(B^2+A^2)(c_g+c_\sigma A^2)}{L(A,B)}e^{-Ax_N}\widehat{d}(\xi'),\tag{3.3}$$

where we have set

$$\mathcal{V}_{jk}^{BB}(\xi',\lambda)=-\frac{\xi_j\xi_k(B-A)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{BB}(\xi',\lambda)=\frac{i\xi_jA(B-A)}{D(A,B)},$$

$$\mathcal{V}_{Nk}^{BB}(\xi',\lambda)=-\frac{i\xi_kA(B-A)}{D(A,B)},\qquad \mathcal{V}_{NN}^{BB}(\xi',\lambda)=-\frac{A^2(B+A)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)=\frac{\xi_j\xi_k(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{B\mathcal{M}}(\xi',\lambda)=-\frac{i\xi_jA(B-A)(B^2+A^2)}{(B+A)D(A,B)},$$

$$\mathcal{V}_{Nk}^{B\mathcal{M}}(\xi',\lambda)=\frac{i\xi_kA(B^2+A^2)}{D(A,B)},\qquad \mathcal{V}_{NN}^{B\mathcal{M}}(\xi',\lambda)=\frac{A^2(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)=\frac{\xi_j\xi_k(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{\mathcal{M}B}(\xi',\lambda)=-\frac{i\xi_jA(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{Nk}^{\mathcal{M}B}(\xi',\lambda)=\frac{i\xi_kA(B-A)(B^2+A^2)}{(B+A)D(A,B)},\qquad \mathcal{V}_{NN}^{\mathcal{M}B}(\xi',\lambda)=\frac{A^2(B^2+A^2)}{D(A,B)},$$

$$\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{\xi_j\xi_k(B^2+A^2)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{jN}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=\frac{i\xi_jA(B^2+A^2)^2}{(B+A)D(A,B)},$$

$$\mathcal{V}_{Nk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{i\xi_kA(B^2+A^2)^2}{(B+A)D(A,B)},\qquad \mathcal{V}_{NN}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)=-\frac{A^2(B^2+A^2)^2}{(B+A)D(A,B)},$$

$$\mathcal{P}_k^{AA}(\xi',\lambda)=-\frac{i\xi_k(B-A)(B^2+A^2)}{D(A,B)},\qquad \mathcal{P}_N^{AA}(\xi',\lambda)=-\frac{A(B+A)(B^2+A^2)}{D(A,B)},$$

$$\mathcal{P}_k^{A\mathcal{M}}(\xi',\lambda)=\frac{2i\xi_kAB(B^2+A^2)}{D(A,B)},\qquad \mathcal{P}_N^{A\mathcal{M}}(\xi',\lambda)=\frac{2A^3(B^2+A^2)}{D(A,B)}.\tag{3.4}$$

In addition, we see, by inserting (2.5) into $\widehat{h}(\xi',\lambda)$, that $\widehat{h}(\xi',\lambda)=\widehat{h}^f(\xi',\lambda)+\widehat{h}^d(\xi',\lambda)$ with

$$\begin{aligned}
\widehat{h}^f(\xi',\lambda) &= -\sum_{k=1}^{N-1}\frac{i\xi_k(B-A)}{(B+A)L(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_k(\xi',y_N)\,dy_N \\
&\quad -\frac{A}{L(A,B)}\int_0^\infty e^{-Ay_N}\widehat{f}_N(\xi',y_N)\,dy_N \\
&\quad +\sum_{k=1}^{N-1}\frac{2i\xi_kAB}{(B+A)L(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_k(\xi',y_N)\,dy_N \\
&\quad +\frac{2A^3}{(B+A)L(A,B)}\int_0^\infty \mathcal{M}(y_N)\widehat{f}_N(\xi',y_N)\,dy_N, \\
\widehat{h}^d(\xi',\lambda) &= \frac{D(A,B)}{(B+A)L(A,B)}\widehat{d}(\xi'). \qquad (3.5)
\end{aligned}$$

Next we shall construct cut-off functions. Let $\varphi \in C_0^\infty(\mathbf{R}^{N-1})$ be a function such that $0 \le \varphi(\xi') \le 1$, $\varphi(\xi') = 1$ for $|\xi'| \le 1/3$, and $\varphi(\xi') = 0$ for $|\xi'| \ge 2/3$. Let A_0 be a number in $(0,1)$, which is determined in Section 4 below. We then define φ_0 and φ_∞ by

$$\varphi_0(\xi') = \varphi(\xi'/A_0), \quad \varphi_\infty(\xi') = 1 - \varphi(\xi'/A_0), \qquad (3.6)$$

and also set, for $a \in \{0,\infty\}$, $g \in \{f,d\}$, and $F = (f,d)$,

$$\begin{aligned}
S_a^g(t;A_0)F &= \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{v}^g(\xi',x_N,\lambda)](x')\,d\lambda, \\
\Pi_a^g(t;A_0)F &= \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{\pi}^g(\xi',x_N,\lambda)](x')\,d\lambda, \\
T_a^g(t;A_0)F &= \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\varphi_a(\xi')\widehat{h}^g(\xi',\lambda)](x')\,d\lambda, \\
R(t)f &= \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\widehat{w}(\xi',x_N,\lambda)](x')\,d\lambda, \\
P(t)f &= \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}_{\xi'}^{-1}[\widehat{p}(\xi',x_N,\lambda)](x')\,d\lambda \quad (t>0) \qquad (3.7)
\end{aligned}$$

with $\widehat{v}^g(\xi',x_N,\lambda) = (\widehat{v}_1^g(\xi',x_N,\lambda),\dots,\widehat{v}_N^g(\xi',x_N,\lambda))^T$. Here we have taken the integral path $\Gamma(\varepsilon)$ as follows:

$$\Gamma(\varepsilon) = \Gamma^+(\varepsilon)\cup\Gamma^-(\varepsilon), \quad \Gamma^\pm(\varepsilon) = \{\lambda\in\mathbf{C} \mid \lambda = \widetilde{\lambda}_0(\varepsilon) + se^{\pm i(\pi-\varepsilon)},\ s\in(0,\infty)\} \qquad (3.8)$$

for $\widetilde{\lambda}_0(\varepsilon) = 2\lambda_0(\varepsilon)/\sin\varepsilon$ with $\varepsilon \in (0,\pi/2)$, where $\lambda_0(\varepsilon)$ is the same number as in Lemma 2.1 (3).

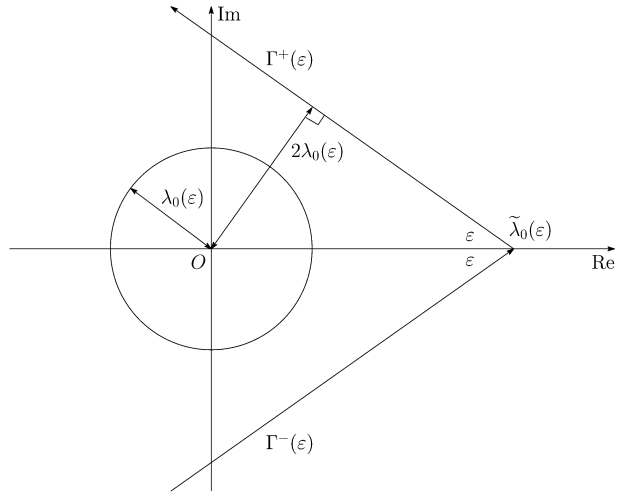

Figure 1. $\Gamma(\varepsilon) = \Gamma^+(\varepsilon) \cup \Gamma^-(\varepsilon)$.

REMARK 3.1. (1) If we set

$$S(t)F = \sum_{a\in\{0,\infty\}} \sum_{g\in\{f,d\}} S_a^g(t;A_0)F + R(t)f,$$

$$\Pi(t)F = \sum_{a\in\{0,\infty\}} \sum_{g\in\{f,d\}} \Pi_a^g(t;A_0)F + P(t)f,$$

$$T(t)F = \sum_{a\in\{0,\infty\}} \sum_{g\in\{f,d\}} T_a^g(t;A_0)F,$$

then $S(t)F$, $\Pi(t)F$, and $T(t)F$ are the requirements in Theorem 1.1 (1). Especially, let $\mathcal{S}(t) : F \mapsto (S(t)F, T(t)F)$ and $1 < p < \infty$, and then $\{\mathcal{S}(t)\}_{t\geq 0}$ is an analytic semi-group on X_p, defined in (1.3), as was seen in [**18**]. On the other hand, by Lemma 2.2, $\{R(t)\}_{t\geq 0}$ is an analytic semi-group on $J_p(\mathbf{R}^N_+)$, and also $R(t)$ and $P(t)$ satisfy

$$\|\nabla^\ell R(t)f\|_{L_p(\mathbf{R}^N_+)} \leq Ct^{-\ell/2}\|f\|_{L_p(\mathbf{R}^N_+)},$$

$$\|(\partial_t R(t)f, \nabla P(t)f)\|_{L_p(\mathbf{R}^N_+)} \leq Ct^{-1}\|f\|_{L_p(\mathbf{R}^N_+)}$$

for $f \in L_p(\mathbf{R}^N_+)^N$, $\ell = 0, 1, 2$, and $t > 0$. These estimates imply that (1.8) holds.

(2) For $a \in \{0, \infty\}$ and $g \in \{f, d\}$, the extension $\mathcal{E}(T_a^g(t;A_0)F)$ defined as (1.4) is decomposed into

$$\mathcal{E}(T_a^g(t;A_0)F) = \frac{1}{2\pi i}\int_{\Gamma(\varepsilon)} e^{\lambda t}\mathcal{F}^{-1}_{\xi'}[\varphi_a(\xi')e^{-Ax_N}\widehat{h}^g(\xi',\lambda)](x')\,d\lambda. \tag{3.9}$$

Figure 1 is reprinted from Ph.D. thesis of the first author.

(3) In the following sections, we show, for $a \in \{0, \infty\}$, that

$$S_a(t)F = \sum_{g \in \{f,d\}} S_a^g(t; A_0)F, \quad \Pi_a(t)F = \sum_{g \in \{f,d\}} \Pi_a^g(t; A_0)F,$$

$$T_a(t)F = \sum_{g \in \{f,d\}} T_a^g(t; A_0)F$$

satisfy the estimates (1.6) and (1.7), respectively.

We devote the last part of this section to the proof of the following lemma.

LEMMA 3.2. *Let $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$ and $\lambda \in \{z \in \mathbf{C} \mid \mathrm{Re} z \geq 0\}$. Then $L(A, B) \neq 0$.*

PROOF. Applying the partial Fourier transform with respect to tangential variable x' to the equations (3.1) with $f = 0$ and $d = 0$ yields that

$$\begin{aligned}
&\lambda \widehat{u}_j(x_N) - \sum_{k=1}^{N-1} i\xi_k (i\xi_j \widehat{u}_k(x_N) + i\xi_k \widehat{u}_j(x_N)) \\
&\qquad\qquad - D_N(D_N \widehat{u}_j(x_N) + i\xi_j \widehat{u}_N(x_N)) + i\xi_j \widehat{\theta}(x_N) = 0, \\
&\lambda \widehat{u}_N(x_N) - \sum_{k=1}^{N-1} i\xi_k (D_N \widehat{u}_k(x_N) + i\xi_k \widehat{u}_N(x_N)) - 2D_N^2 \widehat{u}_N(x_N) + D_N \widehat{\theta}(x_N) = 0, \\
&\sum_{k=1}^{N-1} i\xi_k \widehat{u}_k(x_N) + D_N \widehat{u}_N(x_N) = 0, \quad \lambda \widehat{h} + \widehat{u}_N(0) = 0, \\
&D_N \widehat{u}_j(0) + i\xi_j \widehat{u}_N(0) = 0, \quad -\widehat{\theta}(0) + 2D_N \widehat{u}_N(0) + (c_g + c_\sigma A^2)\widehat{h} = 0
\end{aligned} \tag{3.10}$$

for $x_N > 0$, where we have used the symbols:

$$\widehat{u}_J(x_N) = \widehat{u}_J(\xi', x_N), \quad \widehat{\theta}(x_N) = \widehat{\theta}(\xi', x_N), \quad \widehat{h} = \widehat{h}(\xi').$$

We here set

$$\widehat{u}(x_N) = (\widehat{u}_1(x_N), \ldots, \widehat{u}_N(x_N))^T, \quad \|f\|^2 = \int_0^\infty f(x_N)\overline{f(x_N)}\, dx_N,$$

and show that $L(A, B) \neq 0$ by contradiction. Suppose that $L(A, B) = 0$. We know that (3.10) admits a solution $(\widehat{u}(x_N), \widehat{\theta}(x_N), \widehat{h}) \neq 0$ that decays exponentially when $x_N \to \infty$ (see e.g. [**17**, Section 4]). To obtain

$$0 = \lambda \|\widehat{u}\|^2 + 2\|D_N \widehat{u}_N\|^2 + \sum_{j,k=1}^{N-1} \|i\xi_k \widehat{u}_j\|^2$$

$$+\|\sum_{j=1}^{N-1} i\xi_j \widehat{u}_j\|^2 + \sum_{j=1}^{N-1} \|D_N\widehat{u}_j + i\xi_j\widehat{u}_N\|^2 + \overline{\lambda}(c_g + c_\sigma A^2)|\widehat{h}|^2, \tag{3.11}$$

we multiply the first equation of (3.10) by $\overline{\widehat{u}_j(x_N)}$ and the second equation by $\overline{\widehat{u}_N(x_N)}$, and integrate the resultant formulas with respect to $x_N \in (0,\infty)$, and furthermore, after integration by parts, we use the third to sixth equations of (3.10). Taking the real part of (3.11), we have

$$D_N\widehat{u}_N(x_N) = 0, \quad D_N\widehat{u}_j(x_N) + i\xi_j\widehat{u}_N = 0 \quad \text{for } \mathrm{Re}\,\lambda \geq 0.$$

In particular, $\widehat{u}_N$ is a constant, but $\widehat{u}_N = 0$ since $\lim_{x_N\to\infty}\widehat{u}_N = 0$. We thus have $D_N\widehat{u}_j = 0$, which implies that $\widehat{u}_j = 0$ since $\lim_{x_N\to\infty}\widehat{u}_j = 0$. Combining $\widehat{u}_j = 0$ and the first equation of (3.10) yields that $i\xi_j\widehat{\theta} = 0$. This implies that $\widehat{\theta} = 0$ because $\xi' \neq 0$. In addition, by the sixth equation of (3.10), we have $(c_g + c_\sigma A^2)\widehat{h} = 0$. Since $c_g + c_\sigma A^2 \neq 0$, we see that $\widehat{h} = 0$. We thus have $\widehat{u} = 0$, $\widehat{\theta} = 0$, and $\widehat{h} = 0$, which leads to a contradiciton. This completes the proof of Lemma 3.2. □

4. Analysis of low frequency parts.

In this section, we show the estimates (1.6) in Theorem 1.1 (2). If we consider the Lopatinskii determinant $L(A,B)$ defined in (2.1) as a polynomial with respect to B, then it has four roots $B_j^{\pm}$ $(j = 1,2)$, which have the following asymptotics:

$$B_j^{\pm} = e^{\pm i(2j-1)(\pi/4)}c_g^{1/4}A^{1/4} - \frac{A^{7/4}}{2e^{\pm i(2j-1)(\pi/4)}c_g^{1/4}} - \frac{c_\sigma A^{9/4}}{e^{\pm i(2j-1)(3\pi/4)}c_g^{3/4}} + O(A^{10/4}) \tag{4.1}$$

as $A \to 0$. Set $\lambda_\pm = (B_1^{\pm})^2 - A^2$, and then

$$\lambda_\pm = \pm ic_g^{1/2}A^{1/2} - 2A^2 \mp \frac{2c_\sigma}{ic_g^{1/2}}A^{10/4} + O(A^{11/4}) \quad \text{as } A \to 0. \tag{4.2}$$

Remark 4.1. For $\lambda \in \Sigma_\epsilon$, we choose a branch such that $\mathrm{Re}\,B = \mathrm{Re}\sqrt{\lambda + A^2} > 0$. Note that $\lambda_\pm \in \Sigma_\epsilon$ and $\mathrm{Re}\,(\lambda_\pm + A^2) < 0$.

We define a positive number ε_0 by $\varepsilon_0 = \tan^{-1}\{(A^2/8)/A^2\} = \tan^{-1}(1/8)$, and furthermore, we set

$$\Gamma_0^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = \lambda_\pm + (c_g^{1/2}/4)A^{1/2}e^{\pm iu},\ u : 0 \to 2\pi\},$$
$$\Gamma_1^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -A^2 + (A^2/4)e^{\pm iu},\ u : 0 \to \pi/2\},$$
$$\Gamma_2^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -(A^2(1-u) + \gamma_0 u) \pm i((A^2/4)(1-u) + \widetilde{\gamma}_0 u),\ u : 0 \to 1\},$$
$$\Gamma_3^{\pm} = \{\lambda \in \mathbf{C} \mid \lambda = -(\gamma_0 \pm i\widetilde{\gamma}_0) + ue^{\pm i(\pi - \varepsilon_0)},\ u : 0 \to \infty\}$$

with $\gamma_0 = \lambda_0(\varepsilon_0)$ given by Lemma 2.1 (3) and

$$\widetilde{\gamma}_0 = \frac{1}{8}\big(\lambda_0(\varepsilon_0) + \widetilde{\lambda}_0(\varepsilon_0)\big) = \frac{1}{8}\left(1 + \frac{2}{\sin\varepsilon_0}\right)\lambda_0(\varepsilon_0) = \frac{(1+2\sqrt{65})\gamma_0}{8}, \tag{4.3}$$

where $\widetilde{\lambda}_0(\varepsilon_0)$ is the same constant as in (3.8) with $\varepsilon = \varepsilon_0$.

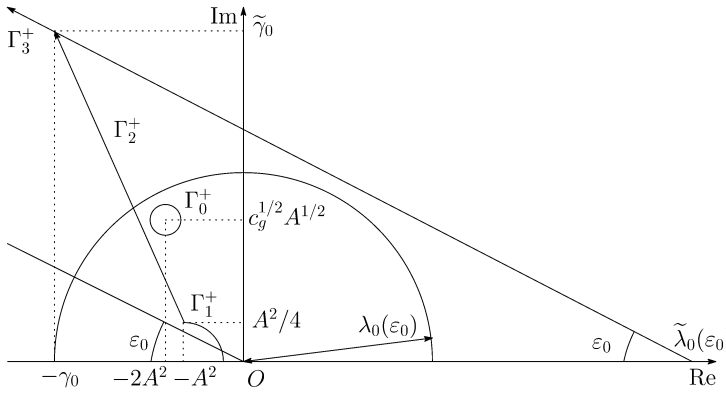

Figure 2. Γ_σ^+ $(\sigma = 0, 1, 2, 3)$.

Then, by Cauchy's integral theorem, we decompose $S_0^g(t; A_0)F$, $\Pi_0^g(t; A_0)F$, and $\mathcal{E}(T_0^g(t; A_0)F)$ given by (3.7) and (3.9) as follows: For $g \in \{f, d\}$

$$S_0^g(t; A_0)F = \sum_{\sigma=0}^{3} S_0^{g,\sigma}(t; A_0)F, \quad \Pi_0^g(t; A_0)F = \sum_{\sigma=0}^{3} \Pi_0^{g,\sigma}(t; A_0)F,$$

$$\mathcal{E}(T_0^g(t; A_0)F) = \sum_{\sigma=0}^{3} \mathcal{E}(T_0^{g,\sigma}(t; A_0)F) \tag{4.4}$$

with

$$S_0^{g,\sigma}(t; A_0)F = \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')\widehat{v}^g(\xi', x_N, \lambda)\,d\lambda\right](x'),$$

$$\Pi_0^{g,\sigma}(t; A_0)F = \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')\widehat{\pi}^g(\xi', x_N, \lambda)\,d\lambda\right](x'),$$

$$\mathcal{E}(T_0^{g,\sigma}(t; A_0)F) = \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^+\cup\Gamma_\sigma^-} e^{\lambda t}\varphi_0(\xi')e^{-Ax_N}\widehat{h}^g(\xi', \lambda)\,d\lambda\right](x'), \tag{4.5}$$

where $\varphi_0(\xi')$ is the cut-off function given in (3.6). In order to estimate each term in (4.5), we here introduce operators $K_n^{\pm,\sigma}(t; A_0)$ and $L_n^{\pm,\sigma}(t; A_0)$ defined by

Figure 2 is reprinted from Ph.D. thesis of the first author.

$$[K_n^{\pm,\sigma}(t;A_0)f](x)=\int_0^\infty \mathcal{F}_{\xi'}^{-1}\biggl[\int_{\Gamma_\sigma^\pm} e^{\lambda t}\varphi_0(\xi')k_n(\xi',\lambda)\mathcal{X}_n(x_N,y_N)\,d\lambda\,\widehat{f}(\xi',y_N)\biggr](x')\,dy_N,$$

$$[L_n^{\pm,\sigma}(t;A_0)d](x)=\mathcal{F}_{\xi'}^{-1}\biggl[\int_{\Gamma_\sigma^\pm} e^{\lambda t}\varphi_0(\xi')\ell_n(\xi',\lambda)\mathcal{Y}_n(x_N)\,d\lambda\,\widehat{d}(\xi')\biggr](x')\quad(\sigma=0,1,2,3) \tag{4.6}$$

with some multipliers $k_n(\xi',\lambda)$ and $\ell_n(\xi',\lambda)$, where $\mathcal{X}_n(x_N,y_N)$ and $\mathcal{Y}_n(x_N)$ are given by

$$\mathcal{X}_n(x_N,y_N)=\begin{cases} e^{-A(x_N+y_N)} & (n=1),\\ e^{-Ax_N}\mathcal{M}(y_N) & (n=2),\\ e^{-B(x_N+y_N)} & (n=3),\\ e^{-Bx_N}\mathcal{M}(y_N) & (n=4),\\ \mathcal{M}(x_N)e^{-By_N} & (n=5),\\ \mathcal{M}(x_N)\mathcal{M}(y_N) & (n=6),\end{cases}\qquad \mathcal{Y}_n(x_N)=\begin{cases} e^{-Ax_N} & (n=1),\\ e^{-Bx_N} & (n=2),\\ \mathcal{M}(x_N) & (n=3).\end{cases}$$

4.1. Analysis on $\boldsymbol{\Gamma_0^\pm}$.

Our aim here is to show the following theorem for the operators given in (4.5) with $\sigma=0$.

THEOREM 4.2. *Let $1\le r\le 2\le q\le\infty$ and $F=(f,d)\in L_r(\mathbf{R}^N_+)^N\times L_r(\mathbf{R}^{N-1})$. Then there exists an $A_0\in(0,1)$ such that the following assertions hold*:

(1) *Let $k=0,1$, $\ell=0,1,2$, and $\alpha'\in\mathbf{N}_0^{N-1}$. Then there exist a positive constant $C=C(\alpha')$ such that for any $t>0$*

$$\bigl\|\partial_t^k D_{x'}^{\alpha'}D_N^\ell S_0^{f,0}(t;A_0)F\bigr\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-(N/2)(1/r-1/q)-k/4-|\alpha'|/2-\ell/8}\|f\|_{L_r(\mathbf{R}^N_+)},$$

$$\begin{aligned}&\bigl\|\partial_t^k D_{x'}^{\alpha'}D_N^\ell S_0^{d,0}(t;A_0)F\bigr\|_{L_q(\mathbf{R}^N_+)}\\ &\quad\le C\begin{cases}(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-k/4-|\alpha'|/2}\|d\|_{L_r(\mathbf{R}^{N-1})} & (\ell=0),\\ (t+1)^{-((N-1)/2)(1/r-1/q)-(1/8)(2-1/q)-k/4-|\alpha'|/2-\ell/8}\|d\|_{L_r(\mathbf{R}^{N-1})} & (\ell=1,2).\end{cases}\end{aligned}$$

(2) *There exists a positive constant C such that for any $t>0$*

$$\bigl\|\nabla\Pi_0^{f,0}(t;A_0)F\bigr\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-(N/2)(1/r-1/q)-1/4}\|f\|_{L_r(\mathbf{R}^N_+)},$$

$$\begin{aligned}&\bigl\|\nabla\Pi_0^{d,0}(t;A_0)F\bigr\|_{L_q(\mathbf{R}^N_+)}\\ &\quad\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-1/4}\|d\|_{L_r(\mathbf{R}^{N-1})}.\end{aligned}$$

(3) *Let $\alpha\in\mathbf{N}_0^N$. Then there exists a positive constant $C=C(\alpha)$ such that for any $t>0$*

$$\left\|D_x^\alpha \nabla \mathcal{E}(T_0^{f,0}(t;A_0)F)\right\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-1/4-|\alpha|/2}\|f\|_{L_r(\mathbf{R}^N_+)},$$

$$\left\|D_x^\alpha \partial_t \mathcal{E}(T_0^{f,0}(t;A_0)F)\right\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-|\alpha|/2}\|f\|_{L_r(\mathbf{R}^N_+)},$$

$$\left\|D_x^\alpha \nabla \mathcal{E}(T_0^{d,0}(t;A_0)F)\right\|_{L_q(\mathbf{R}^N_+)}$$
$$\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-1/4-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

We here introduce some fundamental lemmas to show Theorem 4.2.

LEMMA 4.3. *Let $s_i \ge 0$ $(i = 0,1,2,3)$. Then there exists a positive constant $C = C(s_0,s_1,s_2,s_3)$ such that for any $\tau > 0$, $a \ge 0$, and $Z \ge 0$*

$$e^{-s_0(Z^2)\tau} Z^{s_1} e^{-s_2(Z^{s_3})a} \le C(\tau^{s_1/2} + a^{s_1/s_3})^{-1}.$$

LEMMA 4.4. *Let $1 \le q,r \le \infty$, $a > 0$, $b_1 > 0$, and $b_2 > 0$.*

(1) *Set $g(x_N,\tau) = (\tau^a + (x_N)^{b_1})^{-1}$ for $x_N > 0$ and $\tau > 0$. Then there exists a positive constant C such that for any $\tau > 0$*

$$\|g(\tau)\|_{L_q((0,\infty))} \le C\tau^{-a(1-1/b_1q)},$$

provided that $b_1 q > 1$.

(2) *Let $f \in L_r((0,\infty))$, and set, for $x_N > 0$ and $\tau > 0$,*

$$g(x_N,\tau) = \int_0^\infty \frac{f(y_N)}{\tau^a + (x_N)^{b_1} + (y_N)^{b_2}}\, dy_N.$$

Then there exists a positive constant C such that for any $\tau > 0$

$$\|g(\tau)\|_{L_q((0,\infty))} \le C\tau^{-a(1-1/b_1q-1/b_2+1/b_2r)}\|f\|_{L_r((0,\infty))},$$

provided that for $r' = r/(r-1)$

$$b_1 q > 1, \quad b_2\left(1 - \frac{1}{b_1 q}\right) r' > 1.$$

By using Lemma 4.3 and Lemma 4.4, we obtain the following lemma.

LEMMA 4.5. *Let $1 \le r \le 2 \le q \le \infty$, and let $f \in L_r(\mathbf{R}^N_+)^N$ and $d \in L_r(\mathbf{R}^{N-1})$. For multipliers $\kappa_n(\xi',\lambda)$ and $m_n(\xi',\lambda)$ given below, we set, in* (4.6),

$$k_n(\xi',\lambda) = \frac{\kappa_n(\xi',\lambda)}{L(A,B)}, \quad \ell_n(\xi',\lambda) = \frac{m_n(\xi',\lambda)}{L(A,B)}.$$

(1) *Let $s \ge 0$ and suppose that there exist constants $A_1 \in (0,1)$ and $C = C(s) > 0$ such that for any $A \in (0,A_1)$*

$$|\kappa_1(\xi',\lambda_\pm)| \le CA^{6/4+s}, \quad |\kappa_2(\xi',\lambda_\pm)| \le CA^{7/4+s}, \quad |\kappa_3(\xi',\lambda_\pm)| \le CA^{6/4+s},$$
$$|\kappa_4(\xi',\lambda_\pm)| \le CA^{7/4+s}, \quad |\kappa_5(\xi',\lambda_\pm)| \le CA^{7/4+s}, \quad |\kappa_6(\xi',\lambda_\pm)| \le CA^{8/4+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ *and* $C = C(s) > 0$ *such that for any* $t > 0$

$$\|K_n^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)} \quad (n=1,2,6),$$
$$\|K_3^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2+1/8)(1/r-1/q)-3/8-s/2}\|f\|_{L_r(\mathbf{R}^N_+)},$$
$$\|K_4^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2+1/8)(1/r-1/q)-3/8r-s/2}\|f\|_{L_r(\mathbf{R}^N_+)},$$
$$\|K_5^{\pm,0}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2+1/8)(1/r-1/q)-3/8(1-3/q)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)}.$$

(2) *Let* $s \ge 0$ *and suppose that there exist constants* $A_1 \in (0,1)$ *and* $C = C(s) > 0$ *such that for any* $A \in (0, A_1)$

$$|m_1(\xi',\lambda_\pm)| \le CA^{1+s}, \quad |m_2(\xi',\lambda_\pm)| \le CA^{1+s}, \quad |m_3(\xi',\lambda_\pm)| \le CA^{5/4+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ *and* $C = C(s) > 0$ *such that for any* $t > 0$

$$\|L_n^{\pm,0}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n=1,3),$$
$$\|L_2^{\pm,0}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/8)(2-1/q)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

Proof. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, and $\tilde{t} = t+1$ for $t > 0$ in this proof, and consider only the estimates on Γ_0^+ since the estimates on Γ_0^- can be shown similarly.

(1) We first show the inequality for $K_1^{+,0}(t;A_0)$. Noting that $B^2 - (B_1^+)^2 = \lambda - \lambda_+$, by the residue theorem, we have

$$\begin{aligned}
&[K_1^{+,0}(t;A_0)f](x)\\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma_0^+} e^{\lambda t}\frac{\varphi_0(\xi')\kappa_1(\xi',\lambda)(B+B_1^+)}{(\lambda-\lambda_+)(B-B_1^-)(B-B_2^+)(B-B_2^-)}\\
&\qquad\qquad \times e^{-A(x_N+y_N)}\,d\lambda\widehat{f}(y_N)\Big](x')\,dy_N\\
&= 4\pi i\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\frac{e^{\lambda_+ t}\varphi_0(\xi')\kappa_1(\xi',\lambda_+)B_1^+}{(B_1^+-B_1^-)(B_1^+-B_2^+)(B_1^+-B_2^-)}e^{-A(x_N+y_N)}\widehat{f}(y_N)\Big](x')\,dy_N. \quad (4.7)
\end{aligned}$$

In view of (4.1) and (4.2), we can choose $A_0 \in (0, A_1)$ in such a way that

$$|e^{\lambda_+ t}| \le Ce^{-A^2\tilde{t}}, \ |B_1^+ - B_1^-| \ge CA^{1/4}, \ |B_1^+ - B_2^+| \ge CA^{1/4}, \ |B_1^+ - B_2^-| \ge CA^{1/4} \quad (4.8)$$

for any $A \in (0, A_0)$ and $t > 0$ with some constant C. Thus, by L_q-L_r estimates of the $(N-1)$-dimensional heat kernel and Parseval's theorem, we have

$$\begin{aligned}
&\|[K_1^{+,0}(t; A_0)f](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)} \int_0^\infty \left\| \frac{e^{-(A^2/2)\tilde{t}}\varphi_0(\xi')A^{6/4+s}A^{1/4}}{A^{3/4}} e^{-A(x_N+y_N)}\widehat{f}(y_N) \right\|_2 dy_N \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2} \int_0^\infty \left\| e^{-(A^2/4)\tilde{t}} A e^{-A(x_N+y_N)}\widehat{f}(y_N) \right\|_2 dy_N \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2} \int_0^\infty \frac{\|e^{-(A^2/8)\tilde{t}}\,\widehat{f}(y_N)\|_{L_2(\mathbf{R}^{N-1})}}{\tilde{t}^{\,1/2} + x_N + y_N}\, dy_N \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2} \int_0^\infty \frac{\|f(\cdot\,, y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{\,1/2} + x_N + y_N}\, dy_N, \qquad (4.9)
\end{aligned}$$

where we have used Lemma 4.3 with $s_0 = 1/8$, $s_i = 1$ $(i = 1, 2, 3)$, $a = x_N + y_N$, and $Z = A$. If $q > 2$, then applying Lemma 4.4 (2) with $a = 1/2$ and $b_1 = b_2 = 1$ to (4.9) furnishes that

$$\|K_1^{+,0}(t; A_0)f\|_{L_q(\mathbf{R}^N_+)} \le C\tilde{t}^{-(N/2)(1/r-1/q)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)}.$$

In the case of $(q, r) = (2, 2)$, by (4.9)

$$\|[K_1^{+,0}(t; A_0)f](\cdot, x_N)\|_2 \le C\tilde{t}^{-s/2} \int_0^\infty \left\| \mathcal{F}_{\xi'}^{-1}\left[A e^{-A(x_N+y_N)}\widehat{f}(y_N) \right] \right\|_2 dy_N,$$

and then it follows from [**17**, Lemma 5.4] that

$$\|K_1^{+,0}(t; A_0)f\|_{L_2(\mathbf{R}^N_+)} \le C\tilde{t}^{-s/2}\|f\|_{L_2(\mathbf{R}^N_+)}.$$

On the other hand, in the case of $1 \le r < 2$ and $q = 2$, by the second inequality of (4.9), Lemma 4.3, and Hölder's inequality

$$\begin{aligned}
\|K_1^{+,0}(t; A_0)f\|_{L_2(\mathbf{R}^N_+)} &\le C\tilde{t}^{-s/2} \int_0^\infty \|e^{-(A^2/2)\tilde{t}}A^{1/2}e^{-Ay_N}\widehat{f}(y_N)\|_2\, dy_N \\
&\le C\tilde{t}^{-((N-1)/2)(1/r-1/2)-s/2} \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{1/4} + (y_N)^{1/2}}\, dy_N \\
&\le C\tilde{t}^{-(N/2)(1/r-1/2)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)},
\end{aligned}$$

which implies that the required inequality for $K_1^{+,0}(t; A_0)$ holds. Summing up the arguments above, we see that the following lemma holds.

LEMMA 4.6. *Let $1 \le r \le 2 \le q \le \infty$, $\tau > 0$, and $s_i > 0$ $(i = 1, 2)$. For $x_N > 0$*

and $f \in L_r(\mathbf{R}^N_+)$, *we set*

$$F(x_N,\tau) = \int_0^\infty \left\| e^{-s_1 A^2 \tau} A e^{-s_2 A(x_N+y_N)} \widehat{f}(\xi', y_N) \right\|_{L_2(\mathbf{R}^{N-1})} dy_N.$$

Then there exists a positive constant C *such that for any* $\tau > 0$

$$\|F(\tau)\|_{L_q((0,\infty))} \le C\tau^{-((N-1)/2)(1/r-1/2)-(1/2)(1/r-1/q)} \|f\|_{L_r(\mathbf{R}^N_+)}.$$

Secondly we show the inequality for $K_2^{+,0}(t;A_0)$. We here set

$$\mathcal{M}_\pm(a) = \frac{e^{-B_1^\pm a} - e^{-Aa}}{B_1^\pm - A} \quad \text{for } a > 0.$$

In view of (4.1) and (4.2), we can choose $A_0 \in (0, A_1)$ in such a way that for any $A \in (0, A_0)$ and $a > 0$

$$|\mathcal{M}_\pm(a)| = \frac{|e^{-B_1^\pm a} - e^{-Aa}|}{|B_1^\pm - A|} \le CA^{-1/4} e^{-Aa} \tag{4.10}$$

with some constant C. Thus, by the same calculations as in (4.7) and (4.9), we obtain

$$\begin{aligned}
&\|[K_2^{+,0}(t;A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2} \int_0^\infty \left\| e^{-(A^2/2)\tilde{t}} A e^{-A(x_N+y_N)} \widehat{f}(y_N) \right\|_2 dy_N,
\end{aligned}$$

which furnishes the required inequality of $K_2^{+,0}(t;A_0)$ by Lemma 4.6.

Thirdly we show the inequality for $K_3^{+,0}(t;A_0)$. In view of (4.1) and (4.2), we can choose $A_0 \in (0, A_1)$ such that

$$|e^{-B_1^+(x_N+y_N)}| \le e^{-CA^{1/4}(x_N+y_N)} \quad \text{for any } A \in (0, A_0)$$

with some constant C, so that we easily see that by Lemma 4.3

$$\begin{aligned}
&\|[K_3^{+,0}(t;A_0)f](\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2} \int_0^\infty \left\| e^{-(A^2/2)\tilde{t}} A e^{-CA^{1/4}(x_N+y_N)} \widehat{f}(y_N) \right\|_2 dy_N \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2} \int_0^\infty \frac{\|f(\cdot, y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{1/2} + (x_N)^4 + (y_N)^4} dy_N.
\end{aligned}$$

Combining the inequality above with Lemma 4.4 (2) with $a = 1/2$ and $b_1 = b_2 = 4$, we obtain the required inequality of $K_3^{+,0}(t;A_0)$.

Finally we show the inequalities for $K_n^{+,0}(t;A_0)$ $(n=4,5,6)$. Using similar argumentations to the above cases, we have for $n=4,5$

$$\begin{aligned}
&\|[K_4^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2}\int_0^\infty \frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{1/2}+(x_N)^4+y_N}\,dy_N,\\
&\|[K_5^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2}\int_0^\infty \frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{1/2}+x_N+(y_N)^4}\,dy_N,
\end{aligned}$$

which, combined with Lemma 4.4 (2), furnishes the required inequalities of $K_n^{+,0}(t;A_0)$ $(n=4,5)$. In addition, for $n=6$, we have

$$\begin{aligned}
&\|[K_6^{+,0}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\int_0^\infty \|e^{-(A^2/2)\tilde{t}}Ae^{-CA(x_N+y_N)}\widehat{f}(y_N)\|_2 dy_N
\end{aligned}$$

with a positive constant C, which yields the required inequality of $K_6^{+,0}(t;A_0)$ by Lemma 4.6.

(2) We consider the case of $n=1,3$. Noting that $B^2-(B_1^\pm)^2=\lambda-\lambda_\pm$, by the residue theorem, we have

$$[L_n^{+,0}(t;A_0)d](x)=4\pi i\mathcal{F}_{\xi'}^{-1}\left[\frac{e^{\lambda_+ t}\varphi_0(\xi')m_n(\xi',\lambda_+)B_1^+}{(B_1^+-B_1^-)(B_1^+-B_2^-)(B_1^+-B_2^-)}\mathcal{Y}_n(x_N)\widehat{d}(\xi')\right](x').$$

Thus, by (4.8), (4.10), Lemma 4.3, L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem, we have

$$\begin{aligned}
&\|[L_n^{+,0}(t;A_0)d](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\|e^{-(A^2/2)\tilde{t}}A^{1/2}e^{-Ax_N}\widehat{d}(\xi')\|_2\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\|e^{-(A^2/4)\tilde{t}}\ \widehat{d}(\xi')\|_2/(\tilde{t}^{1/4}+(x_N)^{1/2})\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}/(\tilde{t}^{1/4}+(x_N)^{1/2}).
\end{aligned}\tag{4.11}$$

If $q>2$, then by Lemma 4.4 (1) we obtain the required inequality of $L_n^{+,0}(t;A_0)$ $(n=1,3)$. In the case of $q=2$, we see that by the first inequality of (4.11)

$$\begin{aligned}
\|L_n^{+,0}(t;A_0)d\|_{L_2(\mathbf{R}^N_+)}&\le C\tilde{t}^{-s/2}\|e^{-(A^2/2)\tilde{t}}\widehat{d}(\xi')\|_2\\
&\le C\tilde{t}^{-((N-1)/2)(1/r-1/2)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

Analogously, we can obtain the required inequality of $L_2^{+,0}(t;A_0)$, which complete the proof of the lemma. □

Noting that for some $A_2 \in (0,1)$ and $C > 0$ there holds $|D(A, B_1^{\pm})| \geq CA^{3/4}$ for $A \in (0, A_2)$, we see that there exist positive numbers $A_1 \in (0, A_2)$ and C such that for any $A \in (0, A_1)$ and $j, k = 1, \ldots, N$

$$|\mathcal{V}_{jk}^{BB}(\xi', \lambda_\pm)| \leq CA^{6/4}, \quad |\mathcal{V}_{jk}^{B\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{7/4}, \quad |\mathcal{V}_{jk}^{\mathcal{M}B}(\xi', \lambda_\pm)| \leq CA^{7/4},$$
$$|\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{8/4}, \quad |\mathcal{P}_{j}^{AA}(\xi', \lambda_\pm)| \leq CA, \qquad |\mathcal{P}_{j}^{A\mathcal{M}}(\xi', \lambda_\pm)| \leq CA^{5/4}.$$

Therefore, recalling the formulas (3.3), (3.4), (3.5), and (4.5) with $\sigma = 0$ and using (2.2), we obtain the required inequalities of Theorem 4.2 by Lemma 4.5.

4.2. Analysis on $\Gamma_1^{\pm}$.

Our aim here is to show the following theorem for the operators defined in (4.5) with $\sigma = 1$.

THEOREM 4.7. *Let $1 \leq r \leq 2 \leq q \leq \infty$ and $F = (f, d) \in L_r(\mathbf{R}_+^N)^N \times L_r(\mathbf{R}^{N-1})$. Then, there exists an $A_0 \in (0,1)$ such that we have the following assertions*:

(1) *Let $k = 0, 1$, $\ell = 0, 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. Then there exists a positive constant $C = C(\alpha')$ such that for any $t > 0$*

$$\left\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell S_0^{f,1}(t;A_0)F\right\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(N/2)(1/r-1/q)-(2k+|\alpha'|+\ell)/2}\|f\|_{L_r(\mathbf{R}_+^N)},$$
$$\left\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell S_0^{d,1}(t;A_0)F\right\|_{L_q(\mathbf{R}_+^N)}$$
$$\leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-3/4-(2k+|\alpha'|+\ell)/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

(2) *There exists a positive constant C such that for any $t > 0$*

$$\left\|\nabla\Pi_0^{f,1}(t;A_0)F\right\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(N/2)(1/r-1/q)-1}\|f\|_{L_r(\mathbf{R}_+^N)},$$
$$\left\|\nabla\Pi_0^{d,1}(t;A_0)F\right\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-7/4}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

(3) *Let $\alpha \in \mathbf{N}_0^N$. Then there exists a positive constant $C = C(\alpha)$ such that for any $t > 0$*

$$\left\|D_x^\alpha \nabla\mathcal{E}(T_0^{f,1}(t;A_0)F)\right\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(N/2)(1/r-1/q)-1-|\alpha|/2}\|f\|_{L_r(\mathbf{R}_+^N)},$$
$$\left\|D_x^\alpha \partial_t\mathcal{E}(T_0^{f,1}(t;A_0)F)\right\|_{L_q(\mathbf{R}_+^N)} \leq C(t+1)^{-(N/2)(1/r-1/q)-3/2-|\alpha|/2}\|f\|_{L_r(\mathbf{R}_+^N)},$$
$$\left\|D_x^\alpha \nabla\mathcal{E}(T_0^{d,1}(t;A_0)F)\right\|_{L_q(\mathbf{R}_+^N)}$$
$$\leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-7/4-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

We start with the following lemmas in order to show Theorem 4.7.

LEMMA 4.8. *Let $f(z) = z^3 + 2z^2 + 12z - 8$. Then $f(z) \neq 0$ for $z \in \{\omega \in \mathbf{C} \mid \mathrm{Re}\,\omega \geq 0\} \setminus (0,1)$.*

PROOF. We note that $f(z)$ has only one real root α because $f(0) = -8$, $f(1) = 7$ and $f'(z) = 3z^2 + 4z + 12 > 0$ for $z \in \mathbf{R}$, and it is clear that α is in $(0,1)$. Let β and $\bar{\beta}$ be the other roots of $f(z)$. Since $\alpha + \beta + \bar{\beta} = -2$, we have $2\mathrm{Re}\beta = -2 - \alpha < 0$. This completes the proof. □

LEMMA 4.9. *Let $\lambda \in \Gamma_1^{\pm}$ and $\xi' \in \mathbf{R}^{N-1}$. Then*

$$\frac{A}{4} \leq \mathrm{Re}\, B \leq |B| \leq \frac{A}{2}, \quad |D(A,B)| \geq CA^3$$

for some positive constant C independent of ξ' and λ. In addition, there exist positive constants $A_1 \in (0,1)$ and C such that $|L(A,B)| \geq CA$ for any $A \in (0, A_1)$.

PROOF. We first show the inequalities for B and $D(A,B)$. Note that

$$B = \sqrt{\lambda + A^2} = (A/2)e^{\pm i(u/2)} \tag{4.12}$$

since $\lambda = -A^2 + (A^2/4)e^{\pm iu}$ for $u \in [0, \pi/2]$ on $\Gamma_1^{\pm}$. Therefore, it is clear that the required inequalities of B hold. We insert the identity (4.12) into $D(A,B)$ to obtain

$$D(A,B) = \frac{A^3}{8}\big((e^{\pm i(u/2)})^3 + 2(e^{\pm i(u/2)})^2 + 12(e^{\pm i(u/2)}) - 8\big),$$

which, combined with Lemma 4.8, furnishes that $|D(A,B)| \geq CA^3$ for some positive constant C independent of ξ' and λ.

We finally show the last inequality. By (4.2)

$$B^2 - (B_1^{\pm})^2 = \mp i c_g^{1/2} A^{1/2} + A^2\left(1 + \frac{e^{\pm iu}}{4}\right) + O(A^{10/4}) \quad \text{as } A \to 0,$$

so that there exist positive constants $A_1 \in (0,1)$ and C such that

$$|B^2 - (B_1^{\pm})^2| \geq CA^{1/2} \quad \text{for any } A \in (0, A_1). \tag{4.13}$$

On the other hand, we have $|B + B_1^{\pm}| \leq CA^{1/4}$ on $\Gamma_1^{\pm}$ when A is sufficiently small, which, combined with (4.13), furnishes that

$$|B - B_1^{\pm}| = \frac{|B^2 - (B_1^{\pm})^2|}{|B + B_1^{\pm}|} \geq CA^{1/4} \quad \text{for any } A \in (0, A_1).$$

Since $|B - B_1^{\pm}| \leq |B - B_2^{\pm}|$ as follows from $\mathrm{Re}\, B \geq 0$ and (4.1), we thus obtain

$$|L(A,B)| = |(B-B_1^+)(B-B_1^-)(B-B_2^+)(B-B_2^-)| \geq CA$$

for any $A \in (0, A_1)$, $\lambda \in \Gamma_1^{\pm}$, and a positive constant C independent of ξ' and λ. □

Next, we show some multiplier theorem on $\Gamma_1^{\pm}$.

LEMMA 4.10. *Let $1 \leq r \leq 2 \leq q \leq \infty$, and let $f \in L_r(\mathbf{R}^N_+)^N$ and $d \in L_r(\mathbf{R}^{N-1})$. We use the symbols defined in* (4.6).

(1) *Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0,1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_1^{\pm}$ and $A \in (0, A_1)$*

$$|k_n(\xi', \lambda)| \leq CA^{-1+s} \ (n = 1, 3), \quad |k_n(\xi', \lambda)| \leq CA^s \ (n = 2, 4, 5),$$
$$|k_6(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ we have the estimates:

$$\|K_n^{\pm,1}(t; A_0)f\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-(N/2)(1/r-1/q)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)} \quad (n = 1, 2, 3, 4, 5, 6).$$

(2) *Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0,1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_1^{\pm}$ and $A \in (0, A_1)$*

$$|\ell_n(\xi', \lambda)| \leq CA^s \ (n = 1, 2), \quad |\ell_3(\xi', \lambda)| \leq CA^{1+s}.$$

Then there exist constants $A_0 \in (0, A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ we have the estimates:

$$\|L_n^{\pm,1}(t; A_0)d\|_{L_q(\mathbf{R}^N_+)}$$
$$\leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-3/4-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n = 1, 2, 3).$$

PROOF. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi', y_N)$, and $\tilde{t} = t+1$ for $t > 0$ in this proof, and consider only the estimates on Γ_1^+ since the estimates on Γ_1^- can be shown similarly.

(1) Since $\lambda = -A^2 + (A^2/4)e^{iu}$ for $u \in [0, \pi/2]$ on Γ_1^+, we have

$$[K_1^{+,1}(t; A_0)f](x)$$
$$= \int_0^{\infty} \mathcal{F}_{\xi'}^{-1}\left[\int_0^{\pi/2} e^{(-A^2+(A^2/4)e^{iu})t}\varphi_0(\xi')k_1(\xi', \lambda)e^{-A(x_N+y_N)}\frac{iA^2}{4}e^{iu}\,du\widehat{f}(y_N)\right](x')\,dy_N.$$

Noting that $|e^{(-A^2+(A^2/4)e^{iu})t}| \leq Ce^{-(3/4)A^2\tilde{t}}$ for some positive constant C independent of ξ', u, and t, we see that by Lemma 4.3, L_q-L_r estimates of the $(N-1)$-dimensional heat kernel, and Parseval's theorem

$$\|[K_1^{+,1}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\int_0^\infty \|e^{-(A^2/2)\tilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\|_2 dy_N,$$

and furthermore, for $n=2,3,4,5,6$

$$\|[K_n^{+,1}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\int_0^\infty \|e^{-(A^2/2)\tilde{t}}Ae^{-CA(x_N+y_N)}\widehat{f}(y_N)\|_2 dy_N$$

with some positive constant C analogously, where we have used the fact that for $a>0$ and $\lambda\in\Gamma_1^{\pm}$

$$|\mathcal{M}(a)| \le a\int_0^1 |e^{-(B\theta+A(1-\theta))y_N}|\,d\theta \le ae^{-(A/4)a} \le 8A^{-1}e^{-(A/8)a}. \tag{4.14}$$

We thus obtain the required inequality for $K_n^{+,1}(t;A_0)$ $(n=1,\dots,6)$ by Lemma 4.6.

(2) Since $\lambda=-A^2+(A^2/4)e^{iu}$ for $u\in[0,\pi/2]$ on Γ_1^+, we have

$$[L_1^{+,1}(t;A_0)d](x)$$
$$=\mathcal{F}_{\xi'}^{-1}\left[\int_0^{\pi/2} e^{(-A^2+(A^2/4)e^{iu})t}\varphi_0(\xi')\ell_1(\xi',\lambda)e^{-Ax_N}\frac{iA^2}{4}e^{iu}\,du\,\widehat{d}(\xi')\right](x').$$

By calculations similar to the case of $K_1^{+,1}(t)$ and Lemma 4.3,

$$\|[L_1^{+,1}(t;A_0)d](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-1/2-s/2}\int_0^{\pi/2}\|e^{-(A^2/2)\tilde{t}}Ae^{-Ax_N}\widehat{d}(\xi')\|_2\,du$$
$$\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-1/2-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}/(\tilde{t}^{\,1/2}+x_N),$$

and also for $n=2,3$ we have by (4.14)

$$\|[L_n^{+,1}(t;A_0)d](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})} \le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-1/2-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}/(\tilde{t}^{\,1/2}+x_N).$$

We thus obtain the required inequality for $L_n^{+,1}(t;A_0)$ $(n=1,2,3)$ by Lemma 4.4 (1). □

By Lemma 4.9 we see that there exist constants $A_1\in(0,1)$ and $C>0$ such that for any $\lambda\in\Gamma_1^{\pm}$, $A\in(0,A_1)$, and $j,k=1,\dots,N$

$$|\mathcal{V}_{jk}^{BB}(\xi',\lambda)/L(A,B)| \le CA^{-1}, \quad |\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)/L(A,B)| \le C,$$
$$|\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)/L(A,B)| \le C, \qquad |\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)/L(A,B)| \le CA,$$

$$|\mathcal{P}_j^{AA}(\xi',\lambda)/L(A,B)| \le C, \qquad |\mathcal{P}_j^{A\mathcal{M}}(\xi',\lambda)/L(A,B)| \le CA.$$

Therefore, recalling (3.3)–(3.5) and (4.5) with $\sigma = 1$ and using Lemma 4.10, we have Theorem 4.7.

4.3. Analysis on $\mathbf{\Gamma_2^{\pm}}$.

Our aim here is to show the following theorem for the operators defined in (4.5) with $\sigma = 2$.

THEOREM 4.11. *Let $1 \le r \le 2 \le q \le \infty$ and $F = (f,d) \in L_r(\mathbf{R}^N_+)^N \times L_r(\mathbf{R}^{N-1})$. Then there exists an $A_0 \in (0,1)$ such that the following assertions hold*:

(1) *Let $k = 0,1$, $\ell = 0,1,2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. Then there exists a positive constant $C = C(\alpha')$ such that for any $t > 0$*

$$\begin{aligned}
&\big\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell S_0^{f,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-k-(|\alpha'|+\ell)/2}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\big\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell S_0^{d,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)}\\
&\quad \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-k-(|\alpha'|+\ell)/2}\|d\|_{L_r(\mathbf{R}^{N-1})},
\end{aligned}$$

provided that $k + \ell + |\alpha'| \ne 0$. In addition, if $(q,r) \ne (2,2)$, then

$$\begin{aligned}
&\big\|S_0^{f,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\big\|S_0^{d,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

(2) *There exists a positive constant C such that for any $t > 0$*

$$\begin{aligned}
&\big\|\nabla\Pi_0^{f,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-1/4}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\big\|\nabla\Pi_0^{d,2}(t;A_0)F\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-1}\|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

(3) *Let $\alpha \in \mathbf{N}_0^N$. Then there exists a positive constant $C = C(\alpha)$ such that for any $t > 0$*

$$\begin{aligned}
&\big\|D_x^\alpha \nabla\mathcal{E}(T_0^{f,2}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-1/4-|\alpha|/2}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\big\|D_x^\alpha \partial_t\mathcal{E}(T_0^{f,2}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-|\alpha|/2}\|f\|_{L_r(\mathbf{R}^N_+)} \quad \text{if } |\alpha| \ne 0,\\
&\big\|D_x^\alpha \nabla\mathcal{E}(T_0^{d,2}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)}\\
&\quad \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-1-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

In addition, if $(q,r) \ne (2,2)$, then

$$\big\|\partial_t \mathcal{E}(T_0^{f,2}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)}\|f\|_{L_r(\mathbf{R}^N_+)}.$$

We start with the following lemma in order to show Theorem 4.11.

LEMMA 4.12. *There exist positive constants $A_1 \in (0,1)$, $b_0 \ge 1$, and C such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$b_0^{-1}(A\sqrt{1-u}+\sqrt{u}+A) \le \operatorname{Re} B \le |B| \le b_0(A\sqrt{1-u}+\sqrt{u}+A),$$
$$|D(A,B)| \ge C(A\sqrt{1-u}+\sqrt{u}+A)^3,$$
$$|L(A,B)| \ge C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^4.$$

PROOF. We fist show the inequalities for B. Set $\sigma = \lambda + A^2$ and $\theta = \arg\sigma$. Noting that

$$\lambda = -(A^2(1-u)+\gamma_0 u) + \pm i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)$$

for $u \in [0,1]$ on $\Gamma_2^{\pm}$, we have

$$\begin{aligned}|\sigma| + A^2(1-u)+\gamma_0 u - A^2 &\le 2(A^2(1-u)+\gamma_0 u + A^2) + \frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\\ &\le 3\max(\gamma_0,\widetilde{\gamma}_0)(A^2(1-u)+u+A^2)\\ &\le 3\max(\gamma_0,\widetilde{\gamma}_0)(A\sqrt{1-u}+\sqrt{u}+A)^2,\end{aligned}$$

which is used to obtain

$$\begin{aligned}\operatorname{Re} B &= |\sigma|^{1/2}\cos\frac{\theta}{2} = \frac{|\sigma|^{1/2}}{\sqrt{2}}(1+\cos\theta)^{1/2}\\ &= \frac{|\sigma|^{1/2}}{\sqrt{2}}\left(\frac{|\sigma|+\operatorname{Re}\sigma}{|\sigma|}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\frac{|\sigma|^2-(\operatorname{Re}\sigma)^2}{|\sigma|-\operatorname{Re}\sigma}\right)^{1/2}\\ &= \frac{(A^2/4)(1-u)+\gamma_0 u + (A^2/8) - (A^2/8)(1-u) - (A^2/8)u}{\sqrt{2}(|\sigma|+A^2(1-u)+\gamma_0 u - A^2)^{1/2}}\\ &\ge \frac{(A^2/8)(1-u)+\gamma_0 u + (A^2/8) - (A_0^2/8)u}{\sqrt{6}\max(\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2})(A\sqrt{1-u}+\sqrt{u}+A)}\\ &\ge \frac{(1/8)\{A^2(1-u)+\gamma_0 u + A^2\}}{\sqrt{6}\max(\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2})(A\sqrt{1-u}+\sqrt{u}+A)} \ge \frac{A\sqrt{1-u}+u+A}{24\sqrt{6}\max(\gamma_0^{1/2},\widetilde{\gamma}_0^{1/2})}\end{aligned}$$

for any $A \in (0, A_1)$ provided that $A_1^2 \le 7\gamma_0$. It is clear that the other inequalities concerning B hold.

Next we consider $D(A,B)$. Noting that $\lambda \in \Gamma_2^{\pm} \subset \Sigma_{\varepsilon_0}$ and using Lemma 2.1 (2), we obtain

$$|D(A,B)| \geq C(\varepsilon_0)(|\lambda|^{1/2}+A)^3 \geq C(\varepsilon_0)(A\sqrt{1-u}+\sqrt{u}+A)^3.$$

Finally, we show the inequality for $L(A,B)$. By (4.2)

$$B^2-(B_1^{\pm})^2 = -(A^2(1-u)+\gamma_0 u) \pm i\biggl(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u - c_g^{1/2}A^{1/2}\biggr) + 2A^2 + O(A^{10/4})$$

as $A \to 0$, and also we have

$$\begin{aligned}
&\biggl| -(A^2(1-u)+\gamma_0 u) \pm i\biggl(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u - c_g^{1/2}A^{1/2}\biggr)\biggr|^2 \\
&\quad = (A^2(1-u)+\gamma_0 u)^2 + \biggl(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\biggr)^2 + c_g A - 2c_g^{1/2}A^{1/2}\biggl(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\biggr) \\
&\quad \geq \biggl(A^2(1-u)+\frac{\widetilde{\gamma}_0}{3}u\biggr)^2 + \frac{1}{11}c_g A - \frac{1}{10}\biggl(\frac{A^2}{4}(1-u)+\widetilde{\gamma}_0 u\biggr)^2 \\
&\quad \geq \frac{1}{90}(A^2(1-u)+\widetilde{\gamma}_0 u)^2 + \frac{1}{11}c_g A \geq C\bigl(A\sqrt{1-u}+\sqrt{u}+A^{1/4}\bigr)^4.
\end{aligned}$$

We thus see that there exist positive constants A_1 and C such that for any $A \in (0,A_1)$ and $\lambda \in \Gamma_2^{\pm}$

$$\begin{aligned}
|B-B_1^{\pm}| &= \frac{|B^2-(B_1^{\pm})^2|}{|B+B_1^{\pm}|} \\
&\geq \frac{C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}{b_0(A\sqrt{1-u}+\sqrt{u}+A)+c_g^{1/4}A^{1/4}} \geq C\bigl(A\sqrt{1-u}+\sqrt{u}+A^{1/4}\bigr).
\end{aligned}$$

Since $|B-B_1^{\pm}| \leq |B-B_2^{\pm}|$ as follows from $\operatorname{Re} B \geq 0$ and (4.1), we have the required inequality for $L(A,B)$, which completes the proof of Lemma 4.12. □

LEMMA 4.13. *Let $1 \leq r \leq 2 \leq q \leq \infty$, and let $f \in L_r(\mathbf{R}^N_+)^N$ and $d \in L_r(\mathbf{R}^{N-1})$. We use the symbols defined in* (4.6).

(1) *Let $s \geq 0$ and suppose that there exist constants $A_1 \in (0,1)$ and $C = C(s) > 0$ such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0,A_1)$*

$$\begin{aligned}
&|k_n(\xi',\lambda)| \leq C(A\sqrt{1-u}+\sqrt{u}+A)^{-2}A|B|^s \quad (n=1,3), \\
&|k_2(\xi',\lambda)| \leq C(A\sqrt{1-u}+\sqrt{u}+A)^{-2}A^2|B|^s, \\
&|k_n(\xi',\lambda)| \leq C(A\sqrt{1-u}+\sqrt{u}+A)^{-1}A|B|^s \quad (n=4,5), \\
&|k_6(\xi',\lambda)| \leq CA|B|^s.
\end{aligned}$$

Then there exist constants $A_0 \in (0,A_1)$ and $C = C(s) > 0$ such that for any $t > 0$ and $n = 1,\dots,6$ we have the estimates:

$$\big\|K_n^{\pm,2}(t;A_0)f\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)-s/2}\|f\|_{L_r(\mathbf{R}^N_+)},$$

provided that $s>0$. *In the case of* $s=0$, *we have*

$$\big\|K_n^{\pm,2}(t;A_0)f\big\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-(N/2)(1/r-1/q)}\|f\|_{L_r(\mathbf{R}^N_+)} \quad \text{if } (q,r)\ne(2,2).$$

(2) *Let* $s\ge 0$ *and suppose that there exist constants* $A_1\in(0,1)$ *and* $C=C(s)>0$ *such that for any* $\lambda\in\Gamma_2^{\pm}$ *and* $A\in(0,A_1)$

$$|\ell_n(\xi',\lambda)| \le C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-4}A|B|^s \quad (n=1,2),$$
$$|\ell_3(\xi',\lambda)| \le C(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-3}A|B|^s.$$

Then there exist constants $A_0\in(0,A_1)$ *and* $C=C(s)>0$ *such that for any* $t>0$ *we have the estimates*:

$$\|L_n^{\pm,2}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)}$$
$$\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n=1,3,\ s>0),$$
$$\|L_2^{\pm,2}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)}$$
$$\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-3/4-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (s\ge 0).$$

In the case of $s=0$, *for* $n=1,3$, *we have*

$$\|L_n^{\pm,2}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)}$$
$$\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad \text{if } (q,r)\ne(2,2).$$

PROOF. We use the abbreviations: $\|\cdot\|_2=\|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N)=\widehat{f}(\xi',y_N)$ and $\tilde{t}=t+1$ for $t>0$ in this proof, and consider only the estimates on Γ_2^+ since the estimates on Γ_2^- can be shown similarly.

(1) We first show the inequality for $K_1^{+,2}(t;A_0)$. Recalling that $\lambda=-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)$ for $u\in[0,1]$ on Γ_2^+, we have, by (4.6),

$$[K_1^{+,2}(t;A_0)f](x)$$
$$= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigg[\int_0^1 e^{\{-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}\varphi_0(\xi')k_1(\xi',\lambda)e^{-A(x_N+y_N)}$$
$$\times\bigg\{-(\gamma_0-A^2)+i\bigg(\widetilde{\gamma}_0-\frac{A^2}{4}\bigg)\bigg\}\,du\,\widehat{f}(y_N)\bigg](x')\,dy_N.$$

Since it follows from Lemma 4.12 that

$$\begin{aligned}&\left|e^{\{-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}\right|\\&\quad\le e^{-(3/4)A^2 t}e^{-(1/4)(A^2(1-u)+\gamma_0 u)t}\le Ce^{-(3/4)A^2\tilde{t}}e^{-C|B|^2\tilde{t}}\end{aligned}\tag{4.15}$$

with some positive constant C, independent of ξ', λ, and t, for any $A\in(0,A_0)$ by choosing suitable $A_0\in(0,A_1)$, we have, by L_q-L_r estimates of the $(N-1)$-dimensional heat kernel and Parseval's theorem,

$$\begin{aligned}&\left\|[K_1^{+,2}(t;A_0)f](\cdot,x_N)\right\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\\&\qquad\times\int_0^\infty\left\|\int_0^1\frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')A|B|^s e^{-A(x_N+y_N)}}{(A\sqrt{1-u}+\sqrt{u}+A)^2}\,du\,\widehat{f}(y_N)\right\|_2 dy_N\\&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\\&\qquad\times\int_0^\infty\left\|\int_0^1\frac{e^{-C|B|^2\tilde{t}}|B|^{s-\delta}\varphi_0(\xi')}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N\end{aligned}\tag{4.16}$$

for a sufficiently small $\delta>0$. If $s>0$, then we have, by Lemma 4.12 and Lemma 4.3 with $Z=|B|$ and $a=0$,

$$\int_0^1\frac{e^{-C|B|^2\tilde{t}}|B|^{s-\delta}\varphi_0(\xi')}{(\sqrt{u})^{2-\delta}}\,du\le C\widetilde{t}^{-(s-\delta)/2}\int_0^1\frac{e^{-Cu\tilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\le C\widetilde{t}^{-s/2}.\tag{4.17}$$

We thus obtain

$$\begin{aligned}&\left\|[K_1^{+,2}(t;A_0)f](\cdot,x_N)\right\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\int_0^\infty\left\|e^{-(A^2/2)\tilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\right\|_2 dy_N,\end{aligned}$$

which furnishes the required inequality by Lemma 4.6. In the case of $s=0$, by Lemma 4.3 and (4.16)

$$\begin{aligned}&\|K_1^{+,2}(t;A_0)f\|_{L_q(\mathbf{R}^{N-1})}\\&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\\&\qquad\times\int_0^\infty\left\|\int_0^1\frac{e^{-C|B|^2\tilde{t}}\varphi_0(\xi')}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{1-\delta}e^{-A(x_N+y_N)}\,\widehat{f}(y_N)\right\|_2 dy_N\\&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-\delta/2}\int_0^\infty\frac{\|f(\cdot,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\widetilde{t}^{(1-\delta)/2}+(x_N)^{1-\delta}+(y_N)^{1-\delta}}\,dy_N,\end{aligned}$$

which implies that the required inequality holds by Lemma 4.4 (2) if we choose a sufficiently small $\delta>0$ and $(q,r)\ne(2,2)$. Analogously, for $K_2^{+,2}(t;A_0)$, we see that the

required inequality holds, noting that there holds, by (2.2) and Lemma 4.12,

$$|\mathcal{M}(a)| \le a\int_0^1 e^{-\{(\mathrm{Re}\,B)\theta+A(1-\theta)\}a}\,d\theta \le ae^{-b_0^{-1}Aa} \le 2b_0A^{-1}e^{-(b_0^{-1}/2)Aa}$$

for $a>0$ and any $A\in(0,A_0)$ by choosing suitable $A_0\in(0,A_1)$.

Next we show the inequalities for $K_n^{+,2}(t;A_0)$ $(n=3,4,5)$. Note that by Lemma 4.12 we have

$$|\mathcal{M}(a)| = \frac{|e^{-Ba}-e^{-Aa}|}{|B-A|} \le C|B|^{-1}e^{-CAa} \le C\frac{e^{-CAa}}{\sqrt{u}} \tag{4.18}$$

with $a>0$ and some positive constant C for any $A\in(0,A_0)$ by choosing suitable $A_0\in(0,A_1)$. Then, by (4.15) and Lemma 4.12, there holds

$$\begin{aligned}
&\|[K_n^{+,2}(t;A_0)f](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\left\|\int_0^1 \frac{e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}A|B|^s e^{-CA(x_N+y_N)}\varphi_0(\xi')}{(A\sqrt{1-u}+\sqrt{u}+A)\sqrt{u}}\,du\,\widehat{f}(y_N)\right\|_2 dy_N,
\end{aligned}$$

which furnishes that the required inequalities of $K_n^{+,2}(t;A_0)$ $(n=3,4,5)$ hold in the same manner as we have obtained the inequality of $K_1^{+,2}(t;A_0)$ from (4.16).

Finally, we consider $K_6^{+,2}(t;A_0)$. By (4.17) and (4.18), we have for $s>0$

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\left\|\int_0^1 e^{-(A^2/2)\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0(\xi')A|B|^s\mathcal{M}(x_N)\mathcal{M}(y_N)\,du\,\widehat{f}(y_N)\right\|_2 dy_N\\
&\quad\le C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\left\|\int_0^1 e^{-\frac{A^2}{2}\widetilde{t}}e^{-C|B|^2\widetilde{t}}\varphi_0(\xi')A|B|^{s-2}e^{-CA(x_N+y_N)}\,du\,\widehat{f}(y_N)\right\|_2 dy_N\\
&\quad\le C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\left\|\int_0^1 \frac{e^{-C|B|^2\widetilde{t}}|B|^{s-\delta}\varphi_0(\xi')}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\widetilde{t}}Ae^{-CA(x_N+y_N)}\,\widehat{f}(y_N)\right\|_2 dy_N\\
&\quad\le C\widetilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\int_0^\infty\|e^{-(A^2/2)\widetilde{t}}Ae^{-CA(x_N+y_N)}\,\widehat{f}(y_N)\|_2 dy_N
\end{aligned}$$

by choosing sufficiently small $\delta>0$. We thus obtain the required inequality of $K_6^{+,2}(t;A_0)$ by Lemma 4.6 if $s>0$. In the case of $s=0$, since it follows from Lemma 4.12 that

$$|\mathcal{M}(a)| \le a\int_0^1 e^{-\{(\mathrm{Re}\,B)\theta+A(1-\theta)\}a}\,d\theta \le a\int_0^1 e^{-\{(b_0^{-1}(\sqrt{u}+A))\theta+A(1-\theta)\}x_N}\,d\theta$$
$$\le ae^{-b_0^{-1}Aa}\int_0^1 e^{-b_0^{-1}\sqrt{u}\theta a}\,d\theta \quad (a>0,\ \lambda\in\Gamma_2^+)$$

for any $A\in(0,A_0)$ by choosing some $A_0\in(0,A_1)$, we easily obtain by Lemma 4.3

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\Big\|\int_0^1 e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')A\mathcal{M}(x_N)\mathcal{M}(y_N)\,du\,\widehat{f}(y_N)\Big\|_2\,dy_N\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\int_0^\infty x_Ny_N\big\|\,e^{-(A^2/2)\tilde{t}}Ae^{-CA(x_N+y_N)}\,\widehat{f}(y_N)\big\|_2\\
&\qquad\times\iiint_{[0,1]^3}e^{-Cu\tilde{t}}e^{-C\sqrt{u}\varphi x_N}e^{-C\sqrt{u}\psi y_N}\,dud\varphi d\psi dy_N\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)}\\
&\qquad\times\int_0^\infty\frac{x_Ny_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{\,1/2}+x_N+y_N}\iint_{[0,1]^2}\frac{d\varphi d\psi}{\tilde{t}+(\varphi x_N)^2+(\psi y_N)^2}\,dy_N
\end{aligned}$$

for some positive constant C. The change of variable: $\psi y_N=\{\tilde{t}+(\varphi x_N)^2\}^{1/2}\ell$ yields that

$$\begin{aligned}
&\int_0^1\frac{d\psi}{\tilde{t}+(\varphi x_N)^2+(\psi y_N)^2}\\
&\quad\le\frac{1}{\tilde{t}+(\varphi x_N)^2}\int_0^\infty\frac{1}{1+\ell^2}\frac{\{\tilde{t}+(\varphi x_N)^2\}^{1/2}}{y_N}\,d\ell\le\frac{C}{y_N(\tilde{t}^{\,1/2}+\varphi x_N)}
\end{aligned}$$

for a positive constant C, so that

$$\begin{aligned}
&\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)}\int_0^\infty\frac{x_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{1/2}+x_N+y_N}\int_0^1\frac{d\varphi}{\tilde{t}^{1/2}+\varphi x_N}\,dy_N\\
&\quad=C\tilde{t}^{-((N-1)/2)(1/r-1/q)}\\
&\qquad\times\int_0^\infty\frac{x_N\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{(\tilde{t}^{\,1/2}+x_N+y_N)^{1-\delta}}\int_0^1\frac{d\varphi dy_N}{(\tilde{t}^{\,1/2}+x_N+y_N)^{\delta}(\tilde{t}^{\,1/2}+\varphi x_N)}
\end{aligned}$$

for any $0<\delta<1$. By the change of variable: $\varphi x_N=\tilde{t}^{\,1/2}\ell$, we then have

$$\int_0^1 \frac{d\varphi}{(\tilde{t}^{1/2}+x_N+y_N)^\delta(\tilde{t}^{1/2}+\varphi x_N)} \le \int_0^1 \frac{d\varphi}{(\tilde{t}^{1/2}+\varphi x_N)^\delta(\tilde{t}^{1/2}+\varphi x_N)}$$
$$\le C\int_0^1 \frac{d\varphi}{\tilde{t}^{(1+\delta)/2}+(\varphi x_N)^{1+\delta}}$$
$$\le \frac{C}{\tilde{t}^{(1+\delta)/2}}\int_0^\infty \frac{1}{1+\ell^{1+\delta}}\frac{\tilde{t}^{1/2}}{x_N}\,d\ell \le \frac{C}{x_N\tilde{t}^{\delta/2}}$$

with a positive constant C, which furnishes that

$$\|[K_6^{+,2}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-\delta/2}\int_0^\infty \frac{\|f(\cdot\,,y_N)\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t}^{(1-\delta)/2}+x_N^{1-\delta}+y_N^{1-\delta}}\,dy_N.$$

We therefore obtain the required inequality by Lemma 4.4 (2) with choosing a sufficiently small $\delta>0$ when $(q,r)\neq(2,2)$.

(2) First, we show the inequality for $L_1^{+,2}(t;A_0)$. Noting that $\lambda=-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\gamma_0 u)$ for $u\in[0,1]$ on Γ_2^+, we have, by (4.6),

$$[L_1^{+,2}(t;A_0)d](x)$$
$$=\mathcal{F}_{\xi'}^{-1}\Bigg[\int_0^1 e^{\{-(A^2(1-u)+\gamma_0 u)+i((A^2/4)(1-u)+\widetilde{\gamma}_0 u)\}t}\varphi_0(\xi')\ell_1(\xi',\lambda)e^{-A(x_N+y_N)}$$
$$\times\Bigg\{-(\gamma_0-A^2)+i\Big(\widetilde{\gamma}_0-\frac{A^2}{4}\Big)\Bigg\}\widehat{d}(\xi')\Bigg](x').$$

In a similar way to the case of $K_1^{+,2}(t;A_0)$, we have by (4.17) and Lemma 4.3

$$\|[L_1^{+,2}(t;A_0)d](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\Bigg\|\int_0^1 \frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')A^{1/2}|B|^s e^{-Ax_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\,du\,\widehat{d}(\xi')\Bigg\|_2 \tag{4.19}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\Bigg\|\int_0^1 \frac{e^{-C|B|^2\tilde{t}}|B|^{s-\delta}\varphi_0(\xi')}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{1/2}e^{-Ax_N}\widehat{d}(\xi')\Bigg\|_2 \tag{4.20}$$
$$\le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{1/4}+(x_N)^{1/2}).$$

We thus obtain the required inequality by Lemma 4.4 (1) if $s>0$ and $q>2$. In the case of $s>0$ and $q=2$, by (4.20) and using (4.17) again, we have

$$\|L_1^{+,2}(t;A_0)d\|_{L_2(\mathbf{R}^N_+)} \le C\Bigg\|\int_0^1 \frac{e^{-C|B|^2\tilde{t}}|B|^{s-\delta}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}\widehat{d}(\xi')\Bigg\|_2$$

$$\leq C\widetilde{t}^{-((N-1)/2)(1/r-1/2)-s/2}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

If $s=0$, then we have by Lemma 4.3 and Lemma 4.12

$$\begin{aligned}
&\|[L_1^{+,2}(t;A_0)d](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\leq C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\left\|\int_0^1 \frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')A^{1/2}e^{-Ax_N}}{(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^2}\,du\,\widehat{d}(\xi')\right\|_2\\
&\quad\leq C\widetilde{t}^{-((N-1)/2)(1/2-1/q)}\left\|\int_0^1 \frac{e^{-Cu\tilde{t}}}{(\sqrt{u})^{2-\delta}}\,du\,e^{-(A^2/2)\tilde{t}}A^{(1/2)-(\delta/4)}e^{-Ax_N}\,\widehat{d}(\xi')\right\|_2 \qquad (4.21)\\
&\quad\leq C\widetilde{t}^{-((N-1)/2)(1/r-1/q)-\delta/2}\|d\|_{L_r(\mathbf{R}^{N-1})}/(\widetilde{t}^{(1/4)-(\delta/8)}+(x_N)^{(1/2)-(\delta/4)}),
\end{aligned}$$

which, combined with Lemma 4.4 (1), furnishes that the required inequality holds for $q>2$ by choosing a sufficiently small $\delta>0$. In the case of $s=0$ and $q=2$, by (4.21) and Young's inequality with $1+(1/2)=(1/p)+(1/r)$ for $1\leq r<2$, we have

$$\|L_1^{+,2}(t;A_0)d\|_{L_2(\mathbf{R}^N_+)}\leq C\widetilde{t}^{-\delta/2}\|\mathcal{F}_{\xi'}^{-1}[e^{-(A^2/2)\widetilde{t}}A^{-\delta/4}]\|_{L_p(\mathbf{R}^{N-1})}\|d\|_{L_r(\mathbf{R}^{N-1})}. \qquad (4.22)$$

We use the following proposition proved by [**13**, Theorem 2.3] to calculate the right-hand side of (4.22).

PROPOSITION 4.14. *Let X be a Banach space and $\|\cdot\|_X$ its norm. Suppose that L and n be a non-negative integer and positive integer, respectively. Let $0<\sigma\leq 1$ and $s=L+\sigma-n$. Let $f(\xi)$ be a C^∞-function, defined on $\mathbf{R}^n\setminus\{0\}$ with value X, which satisfies the following two conditions*:

(1) *$D_\xi^\alpha f\in L_1(\mathbf{R}^n,X)$ for any multi-index $\alpha\in\mathbf{N}_0^n$ with $|\alpha|\leq L$.*
(2) *For any multi-index $\alpha\in\mathbf{N}_0^n$, there exists a positive constant $C(\alpha)$ such that*

$$\|D_\xi^\alpha f(\xi)\|_X\leq C(\alpha)|\xi|^{s-|\alpha|}\quad(\xi\in\mathbf{R}^n\setminus\{0\}).$$

Then there exists a positive constant $C(n,s)$ such that

$$\|\mathcal{F}_\xi^{-1}[f](x)\|_X\leq C(n,s)\left(\max_{|\alpha|\leq L+2}C(\alpha)\right)|x|^{-(n+s)}\quad(x\in\mathbf{R}^n\setminus\{0\}).$$

By Proposition 4.14 with $n=N-1$, $L=N-2$, and $\sigma=1-\delta/4$, we have

$$|\mathcal{F}_{\xi'}^{-1}[e^{-(A^2/2)\widetilde{t}}A^{-\delta/4}](x')|\leq C|x'|^{-(N-1-\delta/4)}$$

for a positive constant C, and furthermore, by direct calculations

$$|\mathcal{F}_{\xi'}^{-1}[e^{-(A^2/2)\widetilde{t}}A^{-\delta/4}](x')|\leq C\widetilde{t}^{-(1/2)(N-1-\delta/4)}.$$

We thus obtain

$$|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}](x')| \le \frac{C}{\tilde{t}^{(1/2)(N-1-\delta/4)} + |x'|^{(N-1-\delta/4)}}$$

for some positive constant C. Therefore, by choosing a sufficiently small $\delta > 0$, we see that

$$\|\mathcal{F}^{-1}_{\xi'}[e^{-(A^2/2)\tilde{t}}A^{-\delta/4}]\|_{L_p(\mathbf{R}^{N-1})} \le C\tilde{t}^{-((N-1)/2)(1-1/p)+\delta/8} = C\tilde{t}^{-((N-1)/2)(1/r-1/2)+\delta/8}$$

since $p > 1$ by $1 \le r < 2$, which, combined with (4.22), furnishes that the required inequality holds. Summing up in the case of $s = 0$, we have obtained

$$\|L_1^{+,2}(t; A_0)d\|_{L_q(\mathbf{R}^N_+)} \le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})}$$

for some positive constant C and $1 \le r \le 2 \le q \le \infty$ when $(q, r) \ne (2, 2)$.

Concerning $L_2^{+,2}(t; A_0)$, we see, by Lemma 4.3, that

$$\begin{aligned}
&\|[L_2^{+,2}(t; A_0)d](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\left\| \int_0^1 e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')e^{-(\operatorname{Re} B)x_N}\, du\, \widehat{d}(\xi')\right\|_2 \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\left\| \int_0^1 e^{-Cu\tilde{t}}e^{-C\sqrt{u}x_N}\, du\, e^{-(A^2/2)\tilde{t}}\widehat{d}(\xi')\right\|_2 \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)-s/2}\frac{\| e^{-(A^2/2)\tilde{t}}\widehat{d}(\xi')\|_2}{\tilde{t} + (x_N)^2} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/r-1/q)-s/2}\frac{\|d\|_{L_r(\mathbf{R}^{N-1})}}{\tilde{t} + (x_N)^2},
\end{aligned}$$

which, combined with Lemma 4.4 (1), furnishes the required inequality for $L_2^{+,2}(t; A_0)$.

Finally, we show the inequality for $L_3^{+,2}(t; A_0)$. We easily have by (4.18) and Lemma 4.12

$$\begin{aligned}
&\|[L_3^{+,2}(t)d](\cdot\,, x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\quad \le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\left\| \int_0^1 \frac{e^{-(A^2/2)\tilde{t}}e^{-C|B|^2\tilde{t}}\varphi_0(\xi')A^{1/2}|B|^s e^{-CAx_N}}{(A\sqrt{1-u} + \sqrt{u} + A^{1/4})\sqrt{u}}\, du\, \widehat{d}(\xi')\right\|_2
\end{aligned}$$

for a positive constant C. We thus obtain the required inequality in the same manner as we have obtained the inequality of $L_1^{+,2}(t; A_0)$ from (4.19). □

COROLLARY 4.15. *Let $1 \le r \le 2 \le q \le \infty$, and let $f \in L_r(\mathbf{R}^N_+)^N$ and $d \in L_r(\mathbf{R}^{N-1})$. We use the symbols defined in* (4.6).

(1) *Let $\alpha \in \mathbf{N}_0^N$ and we assume that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$
\begin{aligned}
|k_1(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A, \\
|k_2(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A^2, \\
|\ell_1(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} |B|^2.
\end{aligned}
$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(\alpha)$ such that for any $t > 0$ and $n = 1, 2$

$$
\begin{aligned}
&\|D_x^\alpha \nabla K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-(N/2)(1/r-1/q)-1/4-|\alpha|/2} \|f\|_{L_r(\mathbf{R}_+^N)}, \\
&\|D_x^\alpha \partial_t K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-(N/2)(1/r-1/q)-|\alpha|/2} \|f\|_{L_r(\mathbf{R}_+^N)} \quad \text{if } |\alpha| \ne 0, \\
&\|D_x^\alpha \nabla L_1^{\pm,2}(t; A_0) d\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-1-|\alpha|/2} \|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}
$$

In addition, if $(q, r) \ne (2, 2)$, then we have for any $t > 0$ and $n = 1, 2$

$$
\|\partial_t K_n^{\pm,2}(t; A_0) f\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-(N/2)(1/r-1/q)} \|f\|_{L_r(\mathbf{R}_+^N)}.
$$

(2) *Let $k = 0, 1$, $\ell = 0, 1, 2$, and $\alpha' \in \mathbf{N}_0^{N-1}$. We assume that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_1)$*

$$
\begin{aligned}
|k_3(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A, \\
|k_n(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B| \quad (n = 4, 5), \\
|k_6(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A|B|^2 \\
|\ell_2(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A, \\
|\ell_3(A, B)| &\le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A.
\end{aligned}
$$

Then there exist positive constants $A_0 \in (0, A_1)$ and $C = C(\alpha')$ such that for any $t > 0$

$$
\begin{aligned}
&\left\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell K_n^{\pm,2}(t; A_0) f\right\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \le C(t+1)^{-(N/2)(1/r-1/q)-k-(|\alpha'|+\ell)/2} \|f\|_{L_r(\mathbf{R}_+^N)} \quad (n = 3, 4, 5, 6), \\
&\left\|\partial_t^k D_{x'}^{\alpha'} D_N^\ell L_n^{\pm,2}(t; A_0) d\right\|_{L_q(\mathbf{R}_+^N)} \\
&\quad \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-k-(|\alpha'|+\ell)/2} \|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n = 2, 3),
\end{aligned}
$$

provided that $k+\ell+|\alpha'|\neq 0$. In addition, if $(q,r)\neq(2,2)$, then there hold for any $t>0$

$$\|K_n^{\pm,2}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-(N/2)(1/r-1/q)}\|f\|_{L_r(\mathbf{R}^N_+)}\quad(n=3,4,5,6),$$
$$\|L_n^{\pm,2}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})}\quad(n=2,3).$$

PROOF. We only show the inequalities for $K_5^{\pm,2}(t)$, $K_6^{\pm,2}(t)$, and $L_3^{\pm,2}(t)$. The other inequalities can be proved by Lemma 4.13 directly. By (4.6)

$$\partial_t^k D_{x'}^{\alpha'}[K_n^{\pm,2}(t;A_0)f](x)$$
$$=\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0(\xi')\lambda^k(i\xi')^{\alpha'}k_n(\xi',\lambda)\mathcal{X}_n(x_N,y_N)\,d\lambda\,\widehat{f}(\xi',y_N)\Big](x'),$$
$$\partial_t^k D_{x'}^{\alpha'}[L_3^{\pm,2}(t;A_0)d](x)=\mathcal{F}_{\xi'}^{-1}\Big[\int_{\Gamma_2^\pm} e^{\lambda t}\varphi_0(\xi')\lambda^k(i\xi')^{\alpha'}\ell_3(\xi',\lambda)\mathcal{M}(x_N)\,d\lambda\,\widehat{d}(\xi')\Big](x')$$

for $n=5,6$. Since by Lemma 4.12

$$|\lambda^k(i\xi')^{\alpha'}k_n(\xi',\lambda)|\le C\begin{cases}(A\sqrt{1-u}+\sqrt{u}+A)^{-1}A|B|^{2k+|\alpha'|} & (n=5),\\ A|B|^{2k+|\alpha'|} & (n=6),\end{cases}$$
$$|\lambda^k(i\xi')^{\alpha'}\ell_3(\xi',\lambda)|\le(A\sqrt{1-u}+\sqrt{u}+A^{1/4})^{-3}A|B|^{2k+|\alpha'|}$$

for $\lambda\in\Gamma_2^\pm$ and $A\in(0,A_0)$ by choosing some $A_0\in(0,A_1)$, we obtain by Lemma 4.13

$$\big\|\partial_t^k D_{x'}^{\alpha'}K_n^{\pm,2}(t)f\big\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-(N/2)(1/r-1/q)-k-|\alpha'|/2}\|f\|_{L_r(\mathbf{R}^N_+)}\quad(n=5,6),$$
$$\big\|\partial_t^k D_{x'}^{\alpha'}L_3^{\pm,2}(t)d\big\|_{L_q(\mathbf{R}^N_+)}$$
$$\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-k-|\alpha'|/2}\|d\|_{L_r(\mathbf{R}^{N-1})}\tag{4.23}$$

for any $t>0$, provided that $k+|\alpha'|\neq 0$. In the case of $k+|\alpha'|=0$, we have by Lemma 4.13

$$\|K_n^{\pm,2}(t)f\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-(N/2)(1/r-1/q)}\|f\|_{L_r(\mathbf{R}^N_+)}\quad(n=5,6),$$
$$\|L_3^{\pm,2}(t)d\|_{L_q(\mathbf{R}^N_+)}\le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})}\tag{4.24}$$

when $(q,r)\neq(2,2)$. On the other hand, by (2.2)

$$
\begin{aligned}
&\partial_t^k D_{x'}^{\alpha'} D_N^\ell [K_5^{\pm,2}(t) f](x) \\
&\quad = (-1)^\ell \Big\{ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} \\
&\qquad\qquad \times k_5(\xi', \lambda) e^{-B(x_N+y_N)} d\lambda\, \widehat{f}(\xi', y_N) \Big] (x')\, dy_N \\
&\qquad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} A^\ell \\
&\qquad\qquad \times k_5(\xi', \lambda) \mathcal{M}(x_N) e^{-By_N} d\lambda\, \widehat{f}(\xi', y_N) \Big] (x')\, dy_N \Big\}, \\
&\partial_t^k D_{x'}^{\alpha'} D_N^\ell [K_6^{\pm,2}(t) f](x) \\
&\quad = (-1)^\ell \Big\{ \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} \\
&\qquad\qquad \times k_6(\xi', \lambda) e^{-Bx_N} \mathcal{M}(y_N) d\lambda\, \widehat{f}(\xi', y_N) \Big] (x')\, dy_N \\
&\qquad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} A^\ell \\
&\qquad\qquad \times k_6(\xi', \lambda) \mathcal{M}(x_N) \mathcal{M}(y_N) d\lambda\, \widehat{f}(\xi', y_N) \Big] (x')\, dy_N \Big\}, \\
&\partial_t^k D_{x'}^{\alpha'} D_N^\ell [L_3^{\pm,2}(t) d](x) \\
&\quad = (-1)^\ell \Big\{ \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} \ell_3(\xi', \lambda) e^{-Bx_N} d\lambda\, \widehat{f}(\xi', y_N) \Big] (x') \\
&\qquad + \mathcal{F}_{\xi'}^{-1} \Big[\int_{\Gamma_2^\pm} e^{\lambda t} \varphi_0(\xi') \lambda^k (i\xi')^{\alpha'} A^\ell \ell_3(\xi', \lambda) \mathcal{M}(x_N) d\lambda\, \widehat{f}(\xi', y_N) \Big] (x') \Big\}
\end{aligned}
$$

for $\ell = 1, 2$. Since by Lemma 4.12

$$
\begin{aligned}
|\lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} k_5(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-2} A |B|^{2k+|\alpha'|+\ell} \\
|\lambda^k (i\xi')^{\alpha'} A^\ell k_5(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-1} A |B|^{2k+|\alpha'|+\ell}, \\
|\lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} k_6(\xi', \lambda)| &\le C(A\sqrt{1-u} + \sqrt{u} + A)^{-1} A |B|^{2k+|\alpha'|+\ell}, \\
|\lambda^k (i\xi')^{\alpha'} A^\ell k_6(\xi', \lambda)| &\le C A |B|^{2k+|\alpha'|+\ell}
\end{aligned}
$$

for any $\lambda \in \Gamma_2^\pm$ and $A \in (0, A_0)$ by choosing suitable $A_0 \in (0, A_1)$, we have by Lemma 4.13

$$
\big\| \partial_t^k D_{x'}^{\alpha'} D_N^\ell K_n^{\pm,2}(t) f \big\|_{L_q(\mathbf{R}_+^N)} \le C(t+1)^{-(N/2)(1/r-1/q)-k-(|\alpha'|+\ell)/2} \|f\|_{L_r(\mathbf{R}_+^N)} \quad (n = 5, 6) \tag{4.25}
$$

for $\ell = 1, 2$. In addition,

$$|\lambda^k (i\xi')^{\alpha'} (B+A)^{\ell-1} \ell_3(\xi', \lambda)| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A|B|^{2k+|\alpha'|+\ell}$$
$$|\lambda^k (i\xi')^{\alpha'} A^{\ell} \ell_3(\xi', \lambda)| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A|B|^{2k+|\alpha'|+\ell}$$

for any $\lambda \in \Gamma_2^{\pm}$ and $A \in (0, A_0)$, and therefore by Lemma 4.13

$$\begin{aligned}&\left\|\partial_t^k D_{x'}^{\alpha'} D_N^{\ell} L_3^{\pm,2}(t) d\right\|_{L_q(\mathbf{R}^N_+)} \\ &\quad \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-k-(|\alpha'|+\ell)/2} \|d\|_{L_r(\mathbf{R}^{N-1})}\end{aligned}$$

for $\ell = 1, 2$, which, combined with (4.23), (4.24), and (4.25), furnishes the required estimates for $K_5^{\pm,2}(t)$, $K_6^{\pm,2}(t)$, and $L_3^{\pm,2}(t)$. This completes the proof of Corollary 4.15. □

By Lemma 4.12 there exist a positive number $A_1 \in (0, 1)$ and a positive constant C such that for $j, k = 1, \dots, N$, $\lambda \in \Gamma_2^{\pm}$, and $A \in (0, A_1)$ we have

$$\left|\frac{\mathcal{V}_{jk}^{BB}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA}{(A\sqrt{1-u} + \sqrt{u} + A)^2}, \qquad \left|\frac{\mathcal{V}_{jk}^{B\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA|B|}{(A\sqrt{1-u} + \sqrt{u} + A)^2},$$
$$\left|\frac{\mathcal{V}_{jk}^{\mathcal{M}B}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA|B|}{(A\sqrt{1-u} + \sqrt{u} + A)^2}, \qquad \left|\frac{\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA|B|^2}{(A\sqrt{1-u} + \sqrt{u} + A)^2},$$
$$\left|\frac{\mathcal{P}_{j}^{AA}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA}{(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^4}, \qquad \left|\frac{\mathcal{P}_{j}^{A\mathcal{M}}(\xi', \lambda)}{L(A, B)}\right| \le \frac{CA^2}{(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^4},$$

and furthermore,

$$|A/L(A, B)| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} A,$$
$$|\{A(B^2 + A^2)\}/\{(B+A)L_(A, B)\}| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-3} A,$$
$$|D(A, B)/\{(B+A)L(A, B)\}| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4} |B|^2.$$

Therefore, remembering (3.3)–(3.5), and (4.5) with $\sigma = 2$, and using Corollary 4.15, we have Theorem 4.11. This completes the proof of Theorem 4.11.

4.4. Analysis on $\Gamma_3^{\pm}$.

Our aim here is to show the following theorem for the operators defined in (4.5) with $\sigma = 3$.

THEOREM 4.16. *Let $1 \le r \le 2 \le q \le \infty$, $(\alpha', \alpha) \in \mathbf{N}_0^{N-1} \times \mathbf{N}_0^N$, and $F = (f, d) \in L_r(\mathbf{R}^N_+)^N \times L_r(\mathbf{R}^{N-1})$. Then there exist positive constants δ_0, A_0, and C such that for any $t \ge 1$*

$$\begin{aligned}
&\big\|(\partial_t S_0^{f,3}(t;A_0)F, \nabla\Pi_0^{f,3}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)}\\
&\quad + \big\|(D_{x'}^{\alpha'} S_0^{f,3}(t;A_0)F, D_x^\alpha \partial_t \mathcal{E}(T_0^{f,3}(t;A_0)F), D_x^\alpha \nabla\mathcal{E}(T_0^{f,3}(t;A_0)F))\big\|_{W_q^2(\mathbf{R}^N_+)}\\
&\quad \le Ce^{-\delta_0 t}\|f\|_{L_r(\mathbf{R}^N_+)},\\
&\big\|(\partial_t S_0^{d,3}(t;A_0)F, \nabla\Pi_0^{d,3}(t;A_0)F)\big\|_{L_q(\mathbf{R}^N_+)}\\
&\quad + \big\|(D_{x'}^{\alpha'} S_0^{d,3}(t;A_0)F, D_x^\alpha \nabla\mathcal{E}(T_0^{d,3}(t;A_0)F))\big\|_{W_q^2(\mathbf{R}^N_+)} \le Ce^{-\delta_0 t}\|d\|_{L_r(\mathbf{R}^{N-1})}.
\end{aligned}$$

In order to show Theorem 4.16, we start with the following lemma.

LEMMA 4.17. *Let $1 \le r \le 2 \le q \le \infty$, and let $f \in L_r(\mathbf{R}^N_+)^N$ and $d \in L_r(\mathbf{R}^{N-1})$. We use the operators defined in* (4.6) *with the forms*:

$$k_n(\xi',\lambda) = \kappa_n(\xi',\lambda)/L(A,B), \quad \ell_n(\xi',\lambda) = m_n(\xi',\lambda)/L(A,B).$$

(1) *Let $s \ge 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_3^\pm$ and $A \in (0,A_1)$*

$$|\kappa_n(\xi',\lambda)| \le C(|\lambda|^{1/2}+A)^2A^{1+s} \ \ (n=1,2,4,5,6), \quad |\kappa_3(\xi',\lambda)| \le C(|\lambda|^{1/2}+A)^2A^s.$$

Then there exist positive constants δ_0, $A_0 \in (0,A_1)$, and C such that for any $t \ge 1$

$$\|K_n^{\pm,3}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le Ce^{-\delta_0 t}\|f\|_{L_r(\mathbf{R}^N_+)} \quad (n=1,\dots,6).$$

(2) *Let $s \ge 0$ and suppose that there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_3^\pm$ and $A \in (0,A_1)$*

$$|m_1(\xi',\lambda)| \le C(|\lambda|^{1/2}+A)^2A^{1+s}, \quad |m_2(\xi',\lambda)| \le C(|\lambda|^{1/2}+A)^2A^s,$$
$$|m_3(\xi',\lambda)| \le C|\lambda|^{1/2}(|\lambda|^{1/2}+A)^2A^{1+s}.$$

Then there exist positive constants $A_0 \in (0,A_1)$, δ_0, and C such that for any $t \ge 1$

$$\|L_n^{\pm,3}(t;A_0)d\|_{L_q(\mathbf{R}^N_+)} \le Ce^{-\delta_0 t}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n=1,2,3).$$

PROOF. We use the abbreviations: $\|\cdot\|_2 = \|\cdot\|_{L_2(\mathbf{R}^{N-1})}$, $\widehat{f}(y_N) = \widehat{f}(\xi',y_N)$, and $\tilde{t} = t+1$ for $t > 0$ in this proof, and consider only the estimates on Γ_3^+, because the estimates on Γ_3^- can be shown similarly.

(1) First, we show the inequality for $K_1^{+,3}(t)$. Noting that $\lambda = -\gamma_0 + i\widetilde{\gamma}_0 + ue^{i(\pi-\varepsilon_0)}$ for $u \in [0,\infty)$ on Γ_3^+, we have, by (4.6),

$$[K_1^{+,3}(t)f](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Bigg[\int_0^\infty e^{\{-\gamma_0+i\tilde{\gamma}_0+ue^{i(\pi-\varepsilon_0)}\}t} \times \varphi_0(\xi')\frac{\kappa_1(\xi',\lambda)}{L(A,B)}e^{-A(x_N+y_N)}e^{i(\pi-\varepsilon_0)}\,du\widehat{f}(y_N)\Bigg](x')\,dy_N.$$

Since $e^{-(\gamma_0/2)t}e^{A^2\tilde{t}} \le Ce^{-A^2\tilde{t}}$ for any $A \in (0, A_0)$ by choosing some $A_0 \in (0, A_1)$, we obtain by Lemma 2.1 (3), L_q-L_r estimates of the $(N-1)$ dimensional heat kernel, and Parseval's theorem

$$\begin{aligned}
&\|[K_1^{+,3}(t;A_0)f](\cdot\,,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}\\
&\qquad\times\int_0^\infty\Big\|\int_0^\infty \varphi_0(\xi')e^{A^2\tilde{t}}e^{-(\gamma_0+u\cos\varepsilon_0)t}\frac{A^{1+s}}{|\lambda|}e^{-A(x_N+y_N)}\,du\widehat{f}(y_N)\Big\|_2\,dy_N\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}e^{-(\gamma_0/2)t}\\
&\qquad\times\int_0^\infty\Big\|\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}\,du\,e^{-A^2\tilde{t}}Ae^{-A(x_N+y_N)}\,\widehat{f}(y_N)\Big\|_2\,dy_N\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}e^{-(\gamma_0/2)t}\int_0^\infty\big\|e^{-A^2\tilde{t}}Ae^{-A(x_N+y_N)}\widehat{f}(y_N)\big\|_2 dy_N
\end{aligned}$$

for any $t \ge 1$ with some positive constant C, where we note that $|\lambda| \ge \gamma_\infty$ on Γ_3^+ and $A^s \le C$ on $\operatorname{supp}\varphi_0$. We thus obtain the required inequality of $K_1^{+,3}(t;A_0)$ by Lemma 4.6. Analogously, we can show the case of $n = 2, 4, 5, 6$ by using the fact that

$$|e^{-Ba}| \le Ce^{-Ca}, \quad |\mathcal{M}(a)| \le C|\lambda|^{-1}e^{-Ca} \le Ce^{-Ca} \tag{4.26}$$

for any $a > 0$ and $\lambda \in \Gamma_3^+$ with some positive constant C by Lemma 2.1 (1) and (2.2).

We finally show the inequality for $K_3^{+,3}(t;A_0)$. By Hölder's inequality and (4.26), we easily have for $r' = r/(r-1)$

$$\begin{aligned}
&\|[K_3^{+,3}(t;A_0)f](\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\quad\le C\tilde{t}^{-((N-1)/2)(1/2-1/q)}e^{-(\gamma_0/2)t}\\
&\qquad\times\int_0^\infty\Big\|\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|}e^{-C|\lambda|^{1/2}(x_N+y_N)}\,du\,e^{-A^2\tilde{t}}\widehat{f}(y_n)\Big\|_2\,dy_N\\
&\quad\le Ce^{-(\gamma_0/2)t}\int_0^\infty\frac{e^{-u(\cos\varepsilon_0)t}e^{-C|\lambda|^{1/2}x_N}}{|\lambda|}\Big(\int_0^\infty e^{-Cr'|\lambda|^{1/2}y_N}\,dy_N\Big)^{1/r'}du\|f\|_{L_r(\mathbf{R}^N_+)}.
\end{aligned}$$

Therefore, we see that

$$\|K_3^{+,3}(t;A_0)f\|_{L_q(\mathbf{R}^N_+)} \le Ce^{-(\gamma_0/2)t}\int_0^\infty \frac{e^{-u(\cos\varepsilon_0)t}}{|\lambda|^{1+1/(2q)+1/(2r')}}\,du\,\|f\|_{L_r(\mathbf{R}^N_+)}$$
$$\le Ce^{-(\gamma_0/2)t}\|f\|_{L_r(\mathbf{R}^N_+)}$$

for any $t \ge 1$ with some positive constant C.

(2) Employing an argumentation similar to (1) and using (4.26) for $L_3^{+,3}(t;A_0)$, we can prove (2), so that we may omit the detailed proof of (2). This completes the proof of Lemma 4.17. □

We see that by Lemma 2.1 there exist positive constants $A_1 \in (0,1)$ and C such that for any $A \in (0,A_1)$ and $\lambda \in \Gamma_3^+$ we have

$$|\mathcal{V}_{jk}^{BB}(\xi',\lambda)| \le C, \quad |\mathcal{V}_{jk}^{B\mathcal{M}}(\xi',\lambda)| \le CA, \quad |\mathcal{V}_{jk}^{\mathcal{M}B}(\xi',\lambda)| \le CA,$$
$$|\mathcal{V}_{jk}^{\mathcal{M}\mathcal{M}}(\xi',\lambda)| \le CA, \quad |\mathcal{P}_j^{AA}(\xi',\lambda)| \le CA, \quad |\mathcal{P}_j^{A\mathcal{M}}(\xi',\lambda)| \le CA$$

for $j,k = 1,\dots,N$. Therefore, remembering (3.3)–(3.5), and (4.5) with $\sigma = 3$, and using Lemma 4.17, we have Theorem 4.16. This completes the proof of Theorem 4.16.

We finally consider the term $\partial_t\mathcal{E}(T_0^d(t;A_0)F)$ given by

$$\partial_t\mathcal{E}(T_0^d(t;A_0)F) = \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{\varphi_0(\xi')\lambda D(A,B)}{(B+A)L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{d}(\xi')\right](x')$$
$$= \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,d\lambda\,\varphi_0(\xi')e^{-Ax_N}\widehat{d}(\xi')\right](x')$$
$$- \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{\varphi_0(\xi')A(c_g+c_\sigma A^2)}{L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{d}(\xi')\right](x'),$$

where we have used the relations: $D(A,B) = (B-A)^{-1}\{L(A,B) - A(c_g + c_\sigma A^2)\}$ and $B^2 - A^2 = \lambda$. Note that the first term vanishes by Cauchy's integral theorem, so that it suffices to consider the second term only. Set

$$I_\sigma^\pm(t;A_0) = -\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_\sigma^\pm} e^{\lambda t}\frac{\varphi_0(\xi')A(c_g+c_\sigma A^2)}{L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{d}(\xi')\right](x') \quad (\sigma = 0,1,2,3).$$

Since by Lemma 4.12 there exist positive constants $A_1 \in (0,1)$ and C such that for any $\lambda \in \Gamma_2^+$ and $A \in (0,A_1)$

$$|A(c_g + c_\sigma A^2)/L(A,B)| \le C(A\sqrt{1-u} + \sqrt{u} + A^{1/4})^{-4}A,$$

by Lemma 4.13 we have for $t > 0$, $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \ne 0$, and $1 \le r \le 2 \le q \le \infty$

$$\|D_x^\alpha I_2^\pm(t;A_0)\|_{L_q(\mathbf{R}^N_+)} \le C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})}$$

with some positive constant C. If $(q,r) \neq (2,2)$, then we also have

$$\|I_2^{\pm}(t;A_0)\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})}.$$

In addition, by Lemma 4.5, 4.10, and 4.17, we have

$$\|D_x^{\alpha} I_n^{\pm}(t;A_0)\|_{L_q(\mathbf{R}^N_+)} \leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \quad (n=0,1),$$

$$\|D_x^{\alpha} I_3^{\pm}(t;A_0)\|_{L_q(\mathbf{R}^N_+)} \leq Ce^{-\delta_0 t}\|d\|_{L_r(\mathbf{R}^{N-1})}$$

for any $t \geq 1$, $\alpha \in \mathbf{N}_0^N$, and $1 \leq r \leq 2 \leq q \leq \infty$ with some positive constant C. Thus, we have

$$\begin{aligned}
&\|D_x^{\alpha}\partial_t \mathcal{E}(T_0^d(t;A_0)F)\|_{L_q(\mathbf{R}^N_+)} \\
&\quad \leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)-|\alpha|/2}\|d\|_{L_r(\mathbf{R}^{N-1})} \\
&\qquad\qquad (1 \leq r \leq 2 \leq q \leq \infty,\ |\alpha| \neq 0), \\
&\|\partial_t \mathcal{E}(T_0^d(t;A_0)F)\|_{L_q(\mathbf{R}^N_+)} \\
&\quad \leq C(t+1)^{-((N-1)/2)(1/r-1/q)-(1/2)(1/2-1/q)}\|d\|_{L_r(\mathbf{R}^{N-1})} \\
&\qquad\qquad (1 \leq r \leq 2 \leq q \leq \infty \text{ and } (q,r) \neq (2,2))
\end{aligned}$$

for any $t \geq 1$ with some positive constant C, which, combined with Theorem 4.2, 4.7, 4.11, and 4.16, completes the proof of (1.6) in Theorem 1.1 (2), because

$$S_0(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=0}^{3} S_0^{g,\sigma}(t;A_0)F, \quad \Pi_0(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=0}^{3} \Pi_0^{g,\sigma}(t;A_0)F,$$

$$T_0(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=0}^{3} T_0^{g,\sigma}(t;A_0)F.$$

5. Analysis of high frequency part.

In this section, we show the estimate (1.7) in Theorem 1.1 (2). If we consider the Lopatinskii determinant $L(A,B)$ defined by (2.1) as a polynomial with respect to B, it has the following four roots:

$$B_j = a_j A + \frac{c_\sigma}{4(1-a_j-a_j^3)} + \frac{(1+3a_j^2)c_\sigma^2}{32(1-a_j-a_j^3)^3}\frac{1}{A} + O\left(\frac{1}{A^2}\right) \quad \text{as } A \to \infty, \tag{5.1}$$

where a_j $(j=1,\ldots,4)$ are the solutions to the equation: $x^4+2x^2-4x+1=0$. We

have the following informations about a_j: a_1 and a_2 are real numbers such that $a_1 = 1$ and $0 < a_2 < 1/2$, and a_3 and a_4 are complex numbers satisfying $\operatorname{Re} a_j < 0$ for $j = 3, 4$. We define λ_j by $\lambda_j = B_j^2 - A^2$ for $j = 1, 2$, and then

$$\lambda_1 = -\frac{c_\sigma}{2}A - \frac{3}{16}c_\sigma^2 + O\left(\frac{1}{A}\right), \quad \lambda_2 = -(1-a_2^2)A^2 + \frac{a_2 c_\sigma}{2(1-a_2-a_2^3)}A + O(1) \quad \text{as } A \to \infty. \tag{5.2}$$

Let $L_0 = \{\lambda \in \mathbf{C} \mid L(A,B) = 0,\ \operatorname{Re} B \geq 0,\ A \in \operatorname{supp}\varphi_\infty\}$, where φ_∞ is defined in (3.6), and then we see, by the expansion formulas (4.2), (5.2), and Lemma 3.2, that there exist positive numbers $0 < \varepsilon_\infty < \pi/2$ and $\lambda_\infty > 0$ such that $L_0 \subset \Sigma_{\varepsilon_\infty} \cap \{z \in \mathbf{C} \mid \operatorname{Re} z < -\lambda_\infty\}$. Set $\gamma_\infty = \min\{\lambda_\infty, 4^{-1} \times (A_0/6)^2\}$ for A_0 defined in (3.6), and set, for (3.7) and $g \in \{f, d\}$,

$$S_\infty^g(t) = S_\infty^g(t; A_0), \quad \Pi_\infty^g(t) = \Pi_\infty^g(t; A_0), \quad T_\infty^g(t) = T_\infty^g(t; A_0).$$

In order to estimate each term above, we use the integral paths:

$$\Gamma_4^\pm = \{\lambda \in \mathbf{C} \mid \lambda = -\gamma_\infty \pm iu,\ u : 0 \to \widetilde{\gamma}_\infty\},$$
$$\Gamma_5^\pm = \{\lambda \in \mathbf{C} \mid \lambda = -\gamma_\infty \pm i\widetilde{\gamma}_\infty + ue^{\pm i(\pi - \varepsilon_\infty)},\ u : 0 \to \infty\},$$

where $\widetilde{\gamma}_\infty = (\tan\varepsilon_\infty)(\widetilde{\lambda}_0(\varepsilon_\infty) + \gamma_\infty)$ and $\widetilde{\lambda}_0(\varepsilon_\infty)$ is the same constant as in (3.8) with $\varepsilon = \varepsilon_\infty$. Furthermore, for $g \in \{f, d\}$, setting $v_\infty^g(x,\lambda) = (v_{1,\infty}^g(x,\lambda), \ldots, v_{N,\infty}^g(x,\lambda))^T$ and

$$v_{j,\infty}^g(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\widehat{v}_j^g(\xi', x_N, \lambda)](x') \quad (j = 1, \ldots, N),$$
$$\pi_\infty^g(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\widehat{\pi}^g(\xi', x_N, \lambda)](x'),$$
$$h_{A,\infty}^g(x,\lambda) = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')e^{-Ax_N}\widehat{h}^g(\xi', \lambda)](x')$$

by (3.3)–(3.6), we have, by Cauchy's integral theorem, the following decompositions:

$$S_\infty^g(t)F = \sum_{\sigma=4}^5 S_\infty^{g,\sigma}(t)F, \quad \Pi_\infty^g(t)F = \sum_{\sigma=4}^5 \Pi_\infty^{g,\sigma}(t)F, \quad \mathcal{E}(T_\infty^g(t)F) = \sum_{\sigma=4}^5 \mathcal{E}(T_\infty^{g,\sigma}(t)F),$$

where the right-hand sides are given by

$$S_\infty^{g,\sigma}(t)F = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} e^{\lambda t} v_\infty^g(x,\lambda)\, d\lambda, \quad \Pi_\infty^{g,\sigma}(t)F = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} e^{\lambda t} \pi_\infty^g(x,\lambda)\, d\lambda,$$
$$\mathcal{E}(T_\infty^{g,\sigma}(t)F) = \frac{1}{2\pi i}\int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} e^{\lambda t} h_{A,\infty}^g(x,\lambda)\, d\lambda. \tag{5.3}$$

By the relation $1 = B^2/B^2 = (\lambda + A^2)/B^2$, we write v_∞^f, π_∞^f, and $h_{A,\infty}^f$ as follows: For

$j = 1, \dots, N$, $\widehat{f}_j(y_N) = \widehat{f}_j(\xi', y_N)$, and $\varphi_\infty = \varphi_\infty(\xi')$,

$$
\begin{aligned}
v^f_{j,\infty}(x,\lambda) &= \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{V}^{BB}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{AL(A,B)} A e^{-B(x_N+y_N)} \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\lambda |\lambda|^{-1/2} \mathcal{V}^{B\mathcal{M}}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2 L(A,B)} \\
&\qquad\qquad \times A|\lambda|^{1/2} e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{V}^{B\mathcal{M}}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{B^2 L(A,B)} A^2 e^{-Bx_N} \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\lambda |\lambda|^{-1/2} \mathcal{V}^{\mathcal{M}B}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2 L(A,B)} \\
&\qquad\qquad \times A|\lambda|^{1/2} \mathcal{M}(x_N) e^{-By_N} \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{V}^{\mathcal{M}B}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{B^2 L(A,B)} A^2 \mathcal{M}(x_N) e^{-By_N} \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{V}^{\mathcal{M}\mathcal{M}}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2 L(A,B)} A\lambda \mathcal{M}(x_N) \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{V}^{\mathcal{M}\mathcal{M}}_{jk}(\xi',\lambda)(c_g + c_\sigma A^2)}{AB^2 L(A,B)} A^3 \mathcal{M}(x_N) \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N, \\
\pi^f_\infty(x,\lambda) &= \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{P}^{AA}_{k}(\xi',\lambda)(c_g + c_\sigma A^2)}{AL(A,B)} A e^{-A(x_N+y_N)} \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{\mathcal{P}^{A\mathcal{M}}_{k}(\xi',\lambda)(c_g + c_\sigma A^2)}{A^2 L(A,B)} A^2 e^{-Ax_N} \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N, \\
h^f_{A,\infty}(x,\lambda) &= -\sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{i\xi_k (B-A)}{A(B+A)L(A,B)} A e^{-A(x_N+y_N)} \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad - \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{1}{L(A,B)} A e^{-A(x_N+y_N)} \widehat{f}_N(y_N) \biggr](x')\, dy_N \\
&\quad + \sum_{k=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{2i\xi_k B}{A(B+A)L(A,B)} A^2 e^{-Ax_N} \mathcal{M}(y_N) \widehat{f}_k(y_N) \biggr](x')\, dy_N \\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'} \biggl[\varphi_\infty \frac{2A}{(B+A)L(A,B)} A^2 e^{-Ax_N} \mathcal{M}(y_N) \widehat{f}_N(y_N) \biggr](x')\, dy_N. \qquad (5.4)
\end{aligned}
$$

Moreover, using the relations:

$$
\begin{aligned}
e^{-Bx_N}\widehat{g}(0) &= \int_0^\infty Be^{-B(x_N+y_N)}\widehat{g}(y_N)\,dy_N - \int_0^\infty e^{-B(x_N+y_N)}\widehat{D_N g}(y_N)\,dy_N,\\
\mathcal{M}(x_N)\widehat{g}(0) &= \int_0^\infty \big(e^{-B(x_N+y_N)} + A\mathcal{M}(x_N+y_N)\big)\widehat{g}(y_N)\,dy_N\\
&\quad + \int_0^\infty \mathcal{M}(x_N+y_N)\widehat{D_N g}(y_N)\,dy_N, \qquad (5.5)
\end{aligned}
$$

where $\widehat{g}(y_N) = \widehat{g}(\xi', y_N)$, and using the identity: $1 = A^2/A^2 = -\sum_{k=1}^{N-1}(i\xi_k)^2/A^2$, we write v^d_∞, π^d_∞, and $h^d_{A,\infty}$ as follows: For $j = 1, \dots, N-1$,

$$
\begin{aligned}
v^d_{j,\infty}(x,\lambda) &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_j(c_g + c_\sigma A^2)}{A^2 L(A,B)} Ae^{-B(x_N+y_N)}\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad + \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{\xi_j\xi_k(B-A)(c_g + c_\sigma A^2)}{A^3(B+A)L(A,B)} Ae^{-B(x_N+y_N)}\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N\\
&\quad - \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_j(B^2+A^2)(c_g + c_\sigma A^2)}{A^3(B+A)L(A,B)} A^2\mathcal{M}(x_N+y_N)\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad - \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{\xi_j\xi_k(B^2+A^2)(c_g + c_\sigma A^2)}{A^4(B+A)L(A,B)}\\
&\qquad\qquad \times A^2\mathcal{M}(x_N+y_N)\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N,\\
v^d_{N,\infty}(x,\lambda) &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{(B-A)(c_g + c_\sigma A^2)}{A(B+A)L(A,B)} Ae^{-B(x_N+y_N)}\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad + \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_k(c_g + c_\sigma A^2)}{A^2 L(A,B)} Ae^{-B(x_N+y_N)}\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N\\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{(B^2+A^2)(c_g + c_\sigma A^2)}{A^2(B+A)L(A,B)} A^2\mathcal{M}(x_N+y_N)\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad - \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_k(B^2+A^2)(c_g + c_\sigma A^2)}{A^3(B+A)L(A,B)} A^2\mathcal{M}(x_N+y_N)\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N,\\
\pi^d_\infty(x,\lambda) &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{(B^2+A^2)(c_g + c_\sigma A^2)}{A^2 L(A,B)} Ae^{-A(x_N+y_N)}\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad + \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_k(B^2+A^2)(c_g + c_\sigma A^2)}{A^3 L(A,B)} Ae^{-A(x_N+y_N)}\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N,\\
h^d_{A,\infty}(x,\lambda) &= -\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{D(A,B)}{A^2(B+A)L(A,B)} Ae^{-A(x_N+y_N)}\widehat{\Delta' d}(y_N)\Big](x')\,dy_N\\
&\quad + \sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\varphi_\infty \frac{i\xi_k D(A,B)}{A^3(B+A)L(A,B)} Ae^{-A(x_N+y_N)}\widehat{D_k D_N d}(y_N)\Big](x')\,dy_N. \qquad (5.6)
\end{aligned}
$$

REMARK 5.1. We extend $d \in W_p^{2-1/p}(\mathbf{R}^{N-1})$ to a function $\widetilde{d}$, which is defined on $\mathbf{R}^N_+$ and satisfies $\|\widetilde{d}\|_{W_p^2(\mathbf{R}^N_+)} \le C\|d\|_{W_p^{2-1/p}(\mathbf{R}^{N-1})}$ for a positive constant C independent of d and $\widetilde{d}$. For simplicity, such a $\widetilde{d}$ is denoted by d again in the present section.

To estimates all the terms given in (5.4) and (5.6), we introduce the following operators:

$$
\begin{aligned}
[K_1(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_1(\xi',\lambda)Ae^{-A(x_N+y_N)}\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_2(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_2(\xi',\lambda)A^2e^{-Ax_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_3(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_3(\xi',\lambda)Ae^{-B(x_N+y_N)}\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_4(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_4(\xi',\lambda)A^2e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_5(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_5(\xi',\lambda)A|\lambda|^{1/2}e^{-Bx_N}\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_6(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_6(\xi',\lambda)A^2\mathcal{M}(x_N)e^{-By_N}\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_7(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_7(\xi',\lambda)A|\lambda|^{1/2}\mathcal{M}(x_N)e^{-By_N}\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_8(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_8(\xi',\lambda)A^2\mathcal{M}(x_N+y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_9(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_9(\xi',\lambda)A^3\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N,\\
[K_{10}(\lambda)f](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\bigl[\varphi_\infty(\xi')k_{10}(\xi',\lambda)A\lambda\mathcal{M}(x_N)\mathcal{M}(y_N)\widehat{f}(\xi',y_N)\bigr](x')\,dy_N. \qquad (5.7)
\end{aligned}
$$

We know the following proposition (cf. [**17**, Lemma 5.4]).

PROPOSITION 5.2. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$, and $f \in L_p(\mathbf{R}^N_+)$, and let Λ_ε be a subset of Σ_ε. Suppose that for every $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C = C(\alpha')$ such that for any $\lambda \in \Lambda_\varepsilon$ and $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$*

$$
\bigl|D_{\xi'}^{\alpha'}\{\varphi_\infty(\xi')k_n(\xi',\lambda)\}\bigr| \le CA^{-|\alpha'|} \quad (n = 1,\dots,10).
$$

Then there exists a positive constant C such that for any $\lambda \in \Lambda_\varepsilon$

$$
\|K_n(\lambda)f\|_{L_p(\mathbf{R}^N_+)} \le C\|f\|_{L_p(\mathbf{R}^N_+)} \quad (n = 1,\dots,10).
$$

5.1. Analysis on $\Gamma_4^{\pm}$.

We first show the following lemma concerning estimates of the symbols defined in (2.1).

LEMMA 5.3. (1) *There exists a positive constant A_∞ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^{\pm}$*

$$2^{-1}A \leq \operatorname{Re} B \leq |B| \leq 2A, \quad |D(A,B)| \geq A^3, \quad |L(A,B)| \geq (c_\sigma/16)(8^{-1}A)^3.$$

(2) *There exist positive constants C_1, C_2, and C such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^{\pm}$,*

$$C_1 A \leq \operatorname{Re} B \leq |B| \leq C_2 A, \quad |D(A,B)| \geq CA^3, \quad |L(A,B)| \geq CA^3.$$

where A_∞ and A_0 are the same constants as in (1) *and in* (3.6)*, respectively.*

(3) *Let $\alpha' \in \mathbf{N}_0^{N-1}$, $s \in \mathbf{R}$, and $a > 0$. Then there exist constants $C > 0$ and $b_\infty \geq 1$, independent of a, such that for any $\lambda \in \Gamma_4^{\pm}$ and $A \geq A_0/6$ with A_0 defined as in* (3.6)

$$|D_{\xi'}^{\alpha'} B^s| \leq CA^{s-|\alpha'|}, \quad |D_{\xi'}^{\alpha'} D(A,B)^s| \leq CA^{3s-|\alpha'|}, \quad |D_{\xi'}^{\alpha'} e^{-Ba}| \leq CA^{-|\alpha'|} e^{-b_\infty^{-1}Aa},$$
$$|D_{\xi'}^{\alpha'} L(A,B)^{-1}| \leq CA^{-3-|\alpha'|}, \quad |D_{\xi'}^{\alpha'} \mathcal{M}(a)| \leq CA^{-1-|\alpha'|} e^{-b_\infty^{-1}Aa}.$$

PROOF. (1) We first consider the estimates of B. For $\lambda \in \Gamma_4^{\pm}$, set $\sigma = \lambda + A^2 = -\gamma_\infty + A^2 \pm iu$ ($u \in [0, \widetilde{\gamma}_\infty]$) and $\theta = \arg\sigma$. Then we have

$$\operatorname{Re} B = |\sigma|^{1/2} \cos\frac{\theta}{2} = \frac{|\sigma|^{1/2}}{\sqrt{2}}(1+\cos\theta)^{1/2} = \frac{1}{\sqrt{2}}(|\sigma| + A^2 - \gamma_\infty)^{1/2},$$

so that for any $A \geq A_\infty$

$$\operatorname{Re} B \geq \frac{1}{\sqrt{2}}(2A^2 - 2\gamma_\infty - \widetilde{\gamma}_\infty)^{1/2} \geq \frac{A}{\sqrt{2}},$$

provided that A_∞ satisifes $A_\infty^2 \geq 2\gamma_\infty + \widetilde{\gamma}_\infty$. On the other hand, it is clear that $|B| \leq 2A$.

Next, we show the inequality for $D(A,B)$. Since

$$D(A,B) = B(B^2+3A^2) + A(B^2-A^2) = B(\lambda+4A^2) + \lambda A = 4A^2B + (B+A)(-\gamma_\infty \pm iu),$$

we see, by the inequality of B obtained above, that

$$\begin{aligned} |D(A,B)| &\geq 4A^2|B| - |B+A||-\gamma_\infty \pm iu| \geq 4A^2(\operatorname{Re} B) - (|B|+A)(\gamma_\infty + \widetilde{\gamma}_\infty) \\ &\geq 2A^3 - 3(\gamma_\infty + \widetilde{\gamma}_\infty)A \geq A^3 \end{aligned}$$

for any $A \geq A_\infty$, provided that A_∞ satisfies $A_\infty^2 \geq 3(\gamma_\infty + \widetilde{\gamma}_\infty)$.

Finally, we show the inequality for $L(A,B)$. Since

$$B_1^2 - B^2 = -\frac{c_\sigma}{2}A - \frac{3}{16}c_\sigma^2 - (-\gamma_\infty \pm iu) + O\left(\frac{1}{A}\right) \quad \text{as } A \to \infty,$$

there exist positive constants A_∞ and C such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^\pm$ we have $|B_1^2 - B^2| \geq (c_\sigma/4)A$, which, combined with the inequality of B obtained above and (5.1), furnishes that

$$|B_1 - B| \geq \frac{|B_1^2 - B^2|}{|B_1 + B|} \geq \frac{(c_\sigma/4)A}{4A} \geq \frac{c_\sigma}{16} \quad (A \geq A_\infty \text{ and } \lambda \in \Gamma_4^\pm).$$

On the other hand, we have

$$B_2^2 - B^2 = -(1 - a_2^2)A^2 + O(A) \quad \text{as } A \to \infty,$$

so that there exists a positive number A_∞ such that for any $A \geq A_\infty$ and $\lambda \in \Gamma_4^\pm$ we have $|B_2^2 - B^2| \geq (A^2/2)$, from which it follows that

$$|B_2 - B| = \frac{|B_2^2 - B^2|}{|B_2 + B|} \geq \frac{(A^2/2)}{4A} = \frac{A}{8}.$$

Since $|B - B_2| \leq |B - B_j|$ $(j = 3, 4)$, we thus obtain

$$|L(A,B)| \geq (c_\sigma/16)(8^{-1}A)^3 \quad (A \geq A_\infty \text{ and } \lambda \in \Gamma_4^\pm).$$

(2) It is sufficient to show the existence of positive constants C_1, C_2, and C such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^\pm$

$$C_1 \leq \operatorname{Re} B \leq |B| \leq C_2, \quad |D(A,B)| \geq C, \quad |L(A,B)| \geq C.$$

It is obvious that the inequalities for B holds, so that we here consider $D(A,B)$ and $L(A,B)$ only.

First, we show the inequality for $D(A,B)$. Set

$$\widetilde{A} = \frac{A}{2}, \quad \widetilde{\lambda} = -\gamma_\infty + 3\widetilde{A}^2 \pm iu \quad \text{for } u \in [0, \widetilde{\gamma}_\infty],$$

and note that $B = (\widetilde{\lambda} + \widetilde{A}^2)^{1/2}$. We then see that

$$\{B/\widetilde{A} \in \mathbf{C} \mid \lambda \in \Gamma_4^\pm \text{ and } A \in [A_0/6, 2A_\infty]\} \subset \{z \in \mathbf{C} \mid 1 \leq \operatorname{Re} z\}.$$

In fact, setting $\sigma = 1 - (\gamma_\infty/A^2) \pm i(u/A^2)$ and $\theta = \arg\sigma$, we have

$$\operatorname{Re}\frac{B}{\widetilde{A}} = 2|\sigma|^{1/2}\cos\frac{\theta}{2} = 2|\sigma|^{1/2}\biggl(\frac{1+\cos\theta}{2}\biggr)^{1/2} = \sqrt{2}(|\sigma|+\operatorname{Re}\sigma)^{1/2} \geq 2(\operatorname{Re}\sigma)^{1/2}$$
$$= 2\biggl(1-\frac{\gamma_\infty}{A^2}\biggr)^{1/2} \geq 2\biggl(1-\frac{4^{-1}\times(A_0/6)^2}{(A_0/6)^2}\biggr)^{1/2} = \sqrt{3}.$$

This combined with Lemma 4.8 and the formula:

$$D(A,B) = B^3 + 2\widetilde{A}B^2 + 12\widetilde{A}^2B - 8\widetilde{A}^3 = \widetilde{A}^3\biggl\{\biggl(\frac{B}{\widetilde{A}}\biggr)^3 + 2\biggl(\frac{B}{\widetilde{A}}\biggr)^2 + 12\biggl(\frac{B}{\widetilde{A}}\biggr) - 8\biggr\},$$

furnishes the existence of a positive constant C such that for any $A \in [A_0/6, 2A_\infty]$ and $\lambda \in \Gamma_4^\pm$ we have $|D(A,B)| \geq C$. The inequality for $L(A,B)$ follows clearly from the definition of the integral path $\Gamma_4^\pm$.

(3) We see, by Lemma 5.3 (1) and (2), that there exist positive constants C_1, C_2, and C such that for any $\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$

$$C_1A \leq \operatorname{Re}B \leq |B| \leq C_2A, \quad |D(A,B)| \geq CA^3, \quad |L(A,B)| \geq CA^3. \tag{5.8}$$

We thus obtain the required inequalities by using Leibniz's rule and Bell's formula, and noting

$$\bigl|D_{\xi'}^{\alpha'}D(A,B)\bigr| = \bigl|D_{\xi'}^{\alpha'}(B^3+AB^2+3A^2B-A^3)\bigr| \leq CA^3,$$
$$\bigl|D_{\xi'}^{\alpha'}L(A,B)\bigr| = \biggl|D_{\xi'}^{\alpha'}\biggl(\frac{\lambda}{B+A}D(A,B)+A(c_g+c_\sigma A^2)\biggr)\biggr| \leq CA^3$$

for any $\alpha' \in \mathbf{N}_0^{N-1}$, $\lambda \in \Gamma_4^\pm$, and $A \geq A_0/6$ by (5.8) (cf. [**17**, Lemma 5.2, Lemma 5.3, Lemma 7.2]). □

Now, we have a multiplier theorems on $\Gamma_4^\pm$.

Lemma 5.4. *Let $1 < p < \infty$, $n = 1, \ldots, 10$, and $f \in L_p(\mathbf{R}_+^N)$. We use the symbols defined in* (5.7) *and assume that for any $\alpha' \in \mathbf{N}_0^{N-1}$ there exists a positive constant $C = C(\alpha')$ such that $|D_{\xi'}^{\alpha'}k_n(\xi',\lambda)| \leq CA^{-|\alpha'|}$ for any $\lambda \in \Gamma_4^\pm$ and $A \geq A_0/6$ with A_0 defined as in* (3.6). *Then there exists a positive constant C such that for any $\lambda \in \Gamma_4^\pm$*

$$\|K_n(\lambda)f\|_{L_p(\mathbf{R}_+^N)} \leq C\|f\|_{L_p(\mathbf{R}_+^N)} \quad (n = 1, \ldots, 10).$$

Proof. Employing the similar argumentation to the proof of [**17**, Lemma 5.4] and using Lemma 5.3, we can prove the lemma. □

By (3.4), (5.4), (5.6), Lemma 5.3, and Lemma 5.4, we have the following lemma.

Lemma 5.5. *Let $1 < p < \infty$, $f \in L_p(\mathbf{R}_+^N)^N$, and $d \in W_p^2(\mathbf{R}_+^N)$. Then there exists a positive constant C such that for any $\lambda \in \Gamma_4^\pm$*

$$\|v^f_\infty\|_{W^2_p(\mathbf{R}^N_+)} + \|\pi^f_\infty\|_{W^1_p(\mathbf{R}^N_+)} + \|h^f_{A,\infty}\|_{W^3_p(\mathbf{R}^N_+)} \le C\|f\|_{L_p(\mathbf{R}^N_+)},$$
$$\|v^d_\infty\|_{W^2_p(\mathbf{R}^N_+)} + \|\pi^d_\infty\|_{W^1_p(\mathbf{R}^N_+)} + \|h^d_{A,\infty}\|_{W^3_p(\mathbf{R}^N_+)} \le C\|d\|_{W^2_p(\mathbf{R}^N_+)}.$$

Applying Lemma 5.5 to the terms in (5.3), we have

$$\begin{aligned}
&\big\|(\partial_t S^{f,4}_\infty(t)F, \nabla\Pi^{f,4}_\infty(t)F)\big\|_{L_p(\mathbf{R}^N_+)} \\
&\quad + \big\|(S^{f,4}_\infty(t)F, \partial_t\mathcal{E}(T^{f,4}_\infty(t)F), \nabla\mathcal{E}(T^{f,4}_\infty(t)F))\big\|_{W^2_p(\mathbf{R}^N_+)} \le Ce^{-\gamma_\infty t}\|f\|_{L_p(\mathbf{R}^N_+)}, \\
&\big\|(\partial_t S^{d,4}_\infty(t)F, \nabla\Pi^{d,4}_\infty(t)F)\big\|_{L_p(\mathbf{R}^N_+)} \\
&\quad + \big\|(S^{d,4}_\infty(t)F, \partial_t\mathcal{E}(T^{d,4}_\infty(t)F), \nabla\mathcal{E}(T^{d,4}_\infty(t)F))\big\|_{W^2_p(\mathbf{R}^N_+)} \le Ce^{-\gamma_\infty t}\|d\|_{W^2_p(\mathbf{R}^N_+)} \qquad (5.9)
\end{aligned}$$

for any $t > 0$ with some positive constant C.

5.2. Analysis on $\mathbf{\Gamma^{\pm}_5}$.

By Lemma 2.1, (3.4), (5.4), (5.6), and Proposition 5.2, we easily see that the following lemma holds.

LEMMA 5.6. *Let $1 < p < \infty$, $f \in L_p(\mathbf{R}^N_+)^N$, and $d \in W^2_p(\mathbf{R}^N_+)$. Then there exists a positive constant C such that for any $\lambda \in \Gamma^{\pm}_5$*

$$\begin{aligned}
\big\|(\lambda^{3/2}v^f_\infty, \lambda\nabla v^f_\infty, \nabla^2 v^f_\infty, \nabla\pi^f_\infty)\big\|_{L_p(\mathbf{R}^N_+)} &\le C\|f\|_{L_p(\mathbf{R}^N_+)}, \\
\big\|(\lambda^2 h^f_{A,\infty}, \lambda^{3/2}\nabla h^f_{A,\infty}, \lambda\nabla^2 h^f_{A,\infty}, \nabla^3 h^f_{A,\infty})\big\|_{L_p(\mathbf{R}^N_+)} &\le C\|f\|_{L_p(\mathbf{R}^N_+)}, \\
\big\|(\lambda^{3/2}v^d_\infty, \lambda\nabla v^d_\infty, \nabla^2 v^d_\infty, \nabla\pi^d_\infty)\big\|_{L_p(\mathbf{R}^N_+)} &\le C\|d\|_{W^2_p(\mathbf{R}^N_+)}, \\
\big\|\lambda h^d_{A,\infty}\big\|_{W^2_p(\mathbf{R}^N_+)} + \|h^d_{A,\infty}\|_{W^3_p(\mathbf{R}^N_+)} &\le C\|d\|_{W^2_p(\mathbf{R}^N_+)}.
\end{aligned}$$

Applying Lemma 5.6 to the terms in (5.3), we have for $t \ge 1$

$$\begin{aligned}
&\big\|(\partial_t S^{f,5}_\infty(t)F, \nabla\Pi^{f,5}_\infty(t)F)\big\|_{L_p(\mathbf{R}^N_+)} \\
&\quad + \big\|(S^{f,5}_\infty(t)F, \partial_t\mathcal{E}(T^{5,f}_\infty(t)F), \nabla\mathcal{E}(T^{5,f}_\infty(t)F))\big\|_{W^2_p(\mathbf{R}^N_+)} \le Ce^{-\gamma_\infty t}\|f\|_{L_p(\mathbf{R}^N_+)}, \\
&\big\|(\partial_t S^{d,5}_\infty(t)F, \nabla\Pi^{d,5}_\infty(t)F)\big\|_{L_p(\mathbf{R}^N_+)} \\
&\quad + \big\|(S^{d,5}_\infty(t)F, \partial_t\mathcal{E}(T^{5,d}_\infty(t)F), \nabla\mathcal{E}(T^{5,d}_\infty(t)F))\big\|_{W^2_p(\mathbf{R}^N_+)} \le Ce^{-\gamma_\infty t}\|d\|_{W^2_p(\mathbf{R}^N_+)} \qquad (5.10)
\end{aligned}$$

with some positive constant C.

Summing up (5.9) and (5.10), we have obtained the estimate (1.7) in Theorem 1.1 (2), since

$$S_\infty(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=4}^{5} S_\infty^{g,\sigma}(t)F, \quad \Pi_\infty(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=4}^{5} \Pi_\infty^{g,\sigma}(t)F,$$

$$T_\infty(t)F = \sum_{g\in\{f,d\}}\sum_{\sigma=4}^{5} T_\infty^{g,\sigma}(t)F.$$

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