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Local maximal operators on fractional Sobolev spaces

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Abstract. In this note we establish the boundedness properties of local maximal operators M_G on the fractional Sobolev spaces $W^{s,p}(G)$ whenever G is an open set in $\mathbb{R}^n$, $0 < s < 1$ and $1 < p < \infty$. As an application, we characterize the fractional (s,p)-Hardy inequality on a bounded open set by a Maz'ya-type testing condition localized to Whitney cubes.

1. Introduction.

The local Hardy–Littlewood maximal operator $M_G\colon f \mapsto M_G f$ is defined for an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ and a function $f \in L^p(G)$ by

$$M_G f(x) = \sup_r \fint_{B(x,r)} |f(y)|\, dy, \qquad x \in G,$$

where the supremum ranges over all radii $0 < r < \operatorname{dist}(x, \partial G)$. Whereas the (local) Hardy–Littlewood maximal operator is often used to estimate the absolute size, its Sobolev mapping properties are perhaps less known. The classical Sobolev regularity of M_G is established by Kinnunen and Lindqvist in [**12**]; we also refer to [**7**], [**11**], [**13**], [**14**], [**16**]. Concerning smoothness of fractional order, the first author established in [**17**] the boundedness and continuity properties of M_G on the Triebel–Lizorkin spaces $F^s_{pq}(G)$ whenever G is an open set in $\mathbb{R}^n$, $0 < s < 1$ and $1 < p, q < \infty$.

Our main focus lies in the mapping properties of M_G on a fractional Sobolev space $W^{s,p}(G)$ with $0 < s < 1$ and $1 < p < \infty$, see Section 2 for the definition or [**3**] for a survey of this space. The intrinsically defined function space $W^{s,p}(G)$ on a given domain G coincides with the trace space $F^s_{pp}(G)$ if and only if G is regular, i.e., $|B(x,r) \cap G| \simeq r^n$ whenever $x \in G$ and $0 < r < 1$, see [**22**, Theorem 1.1] and [**21**, pp. 6–7]. As a consequence, if G is a regular domain then M_G is bounded on $W^{s,p}(G)$. Moreover, the following question arises: is M_G a bounded operator on $W^{s,p}(G)$ even if G is not regular, e.g., if G has an exterior cusp? Our main result is an affirmative answer to the last question:

Theorem 1.1. *Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set, $0 < s < 1$ and $1 < p < \infty$. Then, there is a constant $C = C(n,p,s) > 0$ such that inequality*

$$\int_G \int_G \frac{|M_G f(x) - M_G f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx \le C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx \tag{1}$$

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holds for every $f \in L^p(G)$. In particular, the local Hardy–Littlewood maximal operator M_G is bounded on the fractional Sobolev space $W^{s,p}(G)$.

The simple proof of Theorem 1.1 is based on a pointwise inequality in $\mathbb{R}^{2n}$, see Proposition 3.1. That is, for $f \in L^p(G)$ we define an auxiliary function $S(f) : \mathbb{R}^{2n} \to \mathbb{R}$

$$S(f)(x,y) = \frac{\chi_G(x)\chi_G(y)|f(x) - f(y)|}{|x-y|^{n/p+s}}, \qquad \text{a.e. } (x,y) \in \mathbb{R}^{2n}.$$

Observe that the $L^p(\mathbb{R}^{2n})$-norm of $S(f)$ coincides with $|f|_{W^{s,p}(G)}$, see definition (5). The key step is to show that $S(M_G f)(x,y)$ is almost everywhere dominated by

$$C(n,p,s) \sum_{i,j,k,l \in \{0,1\}} \big(M_{ij}(M_{kl}(Sf))(x,y) + M_{ij}(M_{kl}(Sf))(y,x)\big),$$

where each M_{ij} and M_{kl} is either $F \mapsto |F|$ or a V-directional maximal operator in $\mathbb{R}^{2n}$ that is defined in terms of a fixed n-dimensional subspace $V \subset \mathbb{R}^{2n}$, see definition (8). The geometry of the open set G does not have a pivotal role, hence, we are able to prove the pointwise domination without imposing additional restrictions on G. Theorem 1.1 is then a consequence of the fact that the compositions $M_{ij}M_{kl}$ are bounded on $L^p(\mathbb{R}^{2n})$ if $1 < p < \infty$. The transference to the $2n$-dimensional Euclidean space is a typical step when dealing with norm estimates for the spaces $W^{s,p}(G)$, we refer to [**6**], [**8**], [**22**] for other examples. We plan to adapt the transference method to norm estimates on intrinsically defined Triebel–Lizorkin and Besov function spaces on open sets, [**21**].

As an application of our main result, Theorem 1.1, we study certain fractional Hardy inequalities. Let us recall that an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s,p)-Hardy inequality, for $0 < s < 1$ and $1 < p < \infty$, if there exists a constant $C > 0$ such that inequality

$$\int_G \frac{|f(x)|^p}{\operatorname{dist}(x,\partial G)^{sp}}\,dx \le C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx \tag{2}$$

holds for all functions $f \in C_c(G)$. These inequalities have attracted interest recently, we refer to [**4**], [**5**], [**6**], [**8**], [**9**], [**10**] and the references therein.

Theorem 4.5 answers a question from [**4**] by characterizing bounded open sets which admit an (s,p)-Hardy inequality. The characterization is given in terms of a localized Maz'ya-type testing condition, where a lower bound $\ell(Q)^{n-sp} \lesssim \operatorname{cap}_{s,p}(Q,G)$ for the fractional (s,p)-capacities of Whitney cubes $Q \in \mathcal{W}(G)$ is required and a quasiadditivity property of the same capacity is assumed with respect to all finite families of Whitney cubes. Let us mention in passing that the quasiadditivity property of certain capacities with respect to Whitney cubes was first considered by Aikawa [**1**], [**2**]. Aside from (1), an important ingredient in the proof of Theorem 4.5 is the estimate

$$\fint_Q f\,dx \le C \inf_Q M_G f, \tag{3}$$

which holds for a constant $C > 0$ that is independent of both $Q \in \mathcal{W}(G)$ and $f \in C_c(G)$. Inequality (3) allows us to omit the (apparently unknown) weak Harnack inequalities for the minimizers that are associated with (s,p)-capacities. We remark that the weak Harnack based approach is taken up in [**15**]; therein the counterpart of Theorem 4.5 is obtained in case of the classical Hardy inequality, i.e., for the gradient instead of the fractional Sobolev seminorm.

The structure of this paper is as follows. In Section 2 we present the notation and recall various maximal operators. The proof of Theorem 1.1 is taken up in Section 3. Finally, in Section 4, we give an application of our main result by characterizing fractional (s,p)-Hardy inequalities on bounded open sets.

2. Notation and maximal operators.

Notation. The open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is written as $B(x,r)$. The Euclidean distance from $x \in \mathbb{R}^n$ to a set E in $\mathbb{R}^n$ is written as $\operatorname{dist}(x,E)$. The Euclidean diameter of E is $\operatorname{diam}(E)$. The Lebesgue n-measure of a measurable set E is denoted by $|E|$. The characteristic function of a set E is written as χ_E. We write $f \in C_c(G)$ if $f : G \to \mathbb{R}$ is a continuous function with compact support in an open set G. We let $C(\star,\dots,\star)$ denote a positive constant which depends on the quantities appearing in the parentheses only.

For an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ in $\mathbb{R}^n$, we let $\mathcal{W}(G)$ be its Whitney decomposition. For the properties of Whitney cubes $Q \in \mathcal{W}(G)$ we refer to [**20**, VI.1]. In particular, we need the inequalities

$$\operatorname{diam}(Q) \le \operatorname{dist}(Q,\partial G) \le 4\operatorname{diam}(Q), \quad Q \in \mathcal{W}(G). \tag{4}$$

The center of a cube $Q \in \mathcal{W}(G)$ is written as x_Q and $\ell(Q)$ is its side length. By tQ, $t > 0$, we mean a cube whose sides are parallel to those of Q and that is centered at x_Q and whose side length is $t\ell(Q)$.

Let G be an open set in $\mathbb{R}^n$. Let $1 < p < \infty$ and $0 < s < 1$ be given. We write

$$|f|_{W^{s,p}(G)} = \left(\int_G \int_G \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx \right)^{1/p} \tag{5}$$

for measurable functions f on G that are finite a.e. By $W^{s,p}(G)$ we mean the fractional Sobolev space of functions f in $L^p(G)$ with $\|f\|_{W^{s,p}(G)} = \|f\|_{L^p(G)} + |f|_{W^{s,p}(G)} < \infty$.

Maximal operators. Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set. The local Hardy–Littlewood maximal function of $f \in L^p(G)$ $(1 < p < \infty)$ is defined as follows. For every $x \in G$, we write

$$M_G f(x) = \sup_r \fint_{B(x,r)} |f(y)|\, dy, \tag{6}$$

where the supremum ranges over $0 < r < \operatorname{dist}(x,\partial G)$. For notational convenience, we write

$$\fint_{B(x,0)} |f(y)|\, dy = |f(x)| \tag{7}$$

whenever $x \in G$ is a Lebesgue point of $|f|$. It is clear that, at the Lebesgue points of $|f|$, the supremum in (6) can equivalently be taken over $0 \le r \le \operatorname{dist}(x, \partial G)$.

The following lemma is [**4**, Lemma 2.3].

Lemma 2.1. *Suppose that $\emptyset \neq G \subsetneq \mathbb{R}^n$ is an open set and $f \in C_c(G)$. Then $M_G f$ is continuous on G.*

Let us fix $i, j \in \{0, 1\}$ and $1 < p < \infty$. For a function $F \in L^p(\mathbb{R}^{2n})$ we write

$$M_{ij}(F)(x, y) = \sup_{r>0} \fint_{B(0,r)} |F(x + iz, y + jz)|\, dz \tag{8}$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Observe that $M_{00}(F) = |F|$. By applying Fubini's theorem in suitable coordinates and boundedness of the centred Hardy–Littlewood maximal operator in $L^p(\mathbb{R}^n)$ we find that $M_{ij} \colon F \mapsto M_{ij}(F)$ is a bounded operator on $L^p(\mathbb{R}^{2n})$; let us remark that the measurability of $M_{ij}(F)$ for a given $F \in L^p(\mathbb{R}^{2n})$ can be checked by first noting that the supremum in (8) can be restricted to the rational numbers $r > 0$ and then adapting the proof of [**19**, Theorem 8.14] with each r separately.

3. The proof of Theorem 1.1.

We prove our main result, namely Theorem 1.1 that is stated in the Introduction. Let us first recall a convenient notation. For $f \in L^p(G)$ we write

$$S(f)(x, y) = S_{G,n,s,p}(f)(x, y) = \frac{\chi_G(x)\chi_G(y)|f(x) - f(y)|}{|x - y|^{(n/p)+s}}$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. The main tool for proving Theorem 1.1 is a pointwise inequality, stated in Proposition 3.1, which might be of independent interest.

Proposition 3.1. *Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set, $0 < s < 1$ and $1 < p < \infty$. Then there exists a constant $C = C(n, p, s) > 0$ such that, for almost every $(x, y) \in \mathbb{R}^{2n}$, inequality*

$$S(M_G f)(x, y) \le C \sum_{i,j,k,l \in \{0,1\}} \big(M_{ij}(M_{kl}(Sf))(x, y) + M_{ij}(M_{kl}(Sf))(y, x)\big) \tag{9}$$

holds whenever $f \in L^p(G)$ and $Sf \in L^p(\mathbb{R}^{2n})$.

By postponing the proof of Proposition 3.1 for a while, we can prove Theorem 1.1.

Proof of Theorem 1.1. Fix $f \in L^p(G)$. Without loss of generality, we may assume that the right hand side of inequality (1) is finite. Hence $Sf \in L^p(\mathbb{R}^{2n})$ and inequality (1) is a consequence of Proposition 3.1 and the boundedness of operators M_{ij}

on $L^p(\mathbb{R}^{2n})$. $\square$

We proceed to the postponed proof that is motivated by that of [**17**, Theorem 3.2].

Proof of Proposition 3.1. By replacing the function f with $|f|$ we may assume that $f \geq 0$. Since $f \in L^p(G)$ and, hence, $M_G f \in L^p(G)$ we may restrict ourselves to $(x,y) \in G \times G$ for which both x and y are Lebesgue points of f and both $M_G f(x)$ and $M_G f(y)$ are finite. Moreover, by symmetry, we may further assume that $M_G f(x) > M_G f(y)$. These reductions allow us to find

$$0 \leq r(x) \leq \operatorname{dist}(x, \partial G) \quad \text{and} \quad 0 \leq r(y) \leq \operatorname{dist}(y, \partial G)$$

such that the estimate

$$\begin{aligned} S(M_G f)(x,y) &= \frac{|M_G f(x) - M_G f(y)|}{|x-y|^{(n/p)+s}} \\ &= \frac{|\fint_{B(x,r(x))} f - \fint_{B(y,r(y))} f\,|}{|x-y|^{(n/p)+s}} \leq \frac{|\fint_{B(x,r(x))} f - \fint_{B(y,r_2)} f\,|}{|x-y|^{(n/p)+s}} \end{aligned}$$

is valid for any given number

$$0 \leq r_2 \leq \operatorname{dist}(y, \partial G);$$

this number will be chosen in a convenient manner in the two case studies below.

Case: $r(x) \leq |x-y|$. Let us denote $r_1 = r(x)$ and choose

$$r_2 = 0. \tag{10}$$

If $r_1 = 0$, then we get from (10) and our notational convention (7) that

$$S(M_G f)(x,y) \leq S(f)(x,y).$$

Suppose then that $r_1 > 0$. Now

$$\begin{aligned} S(M_G f)(x,y) &\leq \frac{1}{|x-y|^{(n/p)+s}} \left| \fint_{B(x,r_1)} f(z)\,dz - \fint_{B(y,r_2)} f(z)\,dz \right| \\ &= \frac{1}{|x-y|^{(n/p)+s}} \left| \fint_{B(x,r_1)} f(z) - f(y)\,dz \right| \\ &\lesssim \fint_{B(0,r_1)} \frac{\chi_G(x+z)\chi_G(y)|f(x+z) - f(y)|}{|x+z-y|^{(n/p)+s}}\,dz \leq M_{10}(Sf)(x,y). \end{aligned}$$

We have shown that

$$S(M_G f)(x,y) \lesssim S(f)(x,y) + M_{10}(Sf)(x,y)$$

and it is clear that inequality (9) follows (recall that M_{00} is the identity operator when restricted to non-negative functions).

Case: $r(x) > |x-y|$. Let us denote $r_1 = r(x) > 0$ and choose

$$r_2 = r(x) - |x-y| > 0.$$

We then have

$$\begin{aligned}
&\left| \fint_{B(x,r_1)} f(z)\,dz - \fint_{B(y,r_2)} f(z)\,dz \right| \\
&\quad = \left| \fint_{B(0,r_1)} \left(f(x+z) - f\left(y + \frac{r_2}{r_1} z \right) \right) dz \right| \\
&\quad = \left| \fint_{B(0,r_1)} \left(f(x+z) - \fint_{B(y+(r_2/r_1)z,2|x-y|)\cap G} f(a)\,da \right.\right. \\
&\qquad\qquad \left.\left. + \fint_{B(y+(r_2/r_1)z,2|x-y|)\cap G} f(a)\,da - f\left(y + \frac{r_2}{r_1} z \right) \right) dz \right| \\
&\quad \le A_1 + A_2,
\end{aligned}$$

where we have written

$$\begin{aligned}
A_1 &= \fint_{B(0,r_1)} \left(\fint_{B(y+(r_2/r_1)z,2|x-y|)\cap G} |f(x+z) - f(a)|da \right) dz, \\
A_2 &= \fint_{B(0,r_1)} \left(\fint_{B(y+(r_2/r_1)z,2|x-y|)\cap G} \left| f\left(y + \frac{r_2}{r_1} z \right) - f(a) \right| da \right) dz.
\end{aligned}$$

We estimate both of these terms separately, but first we need certain auxiliary estimates.

Recall that $r_2 = r_1 - |x-y|$. Hence, for every $z \in B(0,r_1)$,

$$\left| y + \frac{r_2}{r_1} z - (x+z) \right| = \left| y - x + \frac{(r_2 - r_1)}{r_1} z \right| \le |y-x| + \frac{|x-y|}{r_1}|z| \le 2|y-x|.$$

This, in turn, implies that

$$B\left(y + \frac{r_2}{r_1} z, 2|x-y| \right) \subset B(x+z, 4|x-y|) \tag{11}$$

whenever $z \in B(0,r_1)$. Moreover, since $r_1 > |x-y|$ and $\{y + (r_2/r_1)z, x+z\} \subset B(x,r_1) \subset G$ if $|z| < r_1$, we obtain the two equivalences

$$\left| B\left(y + \frac{r_2}{r_1} z, 2|x-y| \right) \cap G \right| \simeq |x-y|^n \simeq |B(x+z, 4|x-y|) \cap G| \tag{12}$$

for every $z \in B(0, r_1)$. Here the implied constants depend only on n.

An estimate for A_1. The inclusion (11) and inequalities (12) show that, in the definition of A_1, we can replace the domain of integration in the inner integral by $B(x+z, 4|x-y|) \cap G$ and, at the same time, control the error term while integrating on average. That is,

$$A_1 \lesssim \fint_{B(0,r_1)} \left(\fint_{B(x+z,4|x-y|)\cap G} |f(x+z) - f(a)| da \right) dz.$$

By observing that both $x + z$ and a in the last double integral belong to G and using (12) again, we can continue as follows:

$$\begin{aligned} \frac{A_1}{|x-y|^{(n/p)+s}} &\lesssim \fint_{B(0,r_1)} \left(\fint_{B(x+z,4|x-y|)} \frac{\chi_G(x+z)\chi_G(a)|f(x+z) - f(a)|}{|x+z-a|^{(n/p)+s}} da \right) dz \\ &\lesssim \fint_{B(0,r_1)} \left(\fint_{B(y+z,5|x-y|)} S(f)(x+z, a) da \right) dz. \end{aligned}$$

Applying the maximal operators defined in Section 2 we find that

$$\frac{A_1}{|x-y|^{(n/p)+s}} \lesssim \fint_{B(0,r_1)} M_{01}(Sf)(x+z, y+z)\, dz \le M_{11}(M_{01}(Sf))(x, y).$$

An estimate for A_2. We use the inclusion $y + (r_2/r_1)z \in G$ for all $z \in B(0, r_1)$ and then apply the first equivalence in (12) to obtain

$$\begin{aligned} A_2 &= \fint_{B(0,r_1)} \left(\fint_{B(y+(r_2/r_1)z,2|x-y|)\cap G} \chi_G\left(y + \frac{r_2}{r_1}z\right) \chi_G(a) \left| f\left(y + \frac{r_2}{r_1}z\right) - f(a) \right| da \right) dz \\ &\lesssim \fint_{B(0,r_1)} \left(\fint_{B(y+(r_2/r_1)z,2|x-y|)} \chi_G\left(y + \frac{r_2}{r_1}z\right) \chi_G(a) \left| f\left(y + \frac{r_2}{r_1}z\right) - f(a) \right| da \right) dz. \end{aligned}$$

Hence, a change of variables yields

$$\begin{aligned} \frac{A_2}{|x-y|^{(n/p)+s}} &\lesssim \fint_{B(0,r_2)} \left(\fint_{B(y+z,2|x-y|)} \frac{\chi_G(y+z)\chi_G(a)|f(y+z) - f(a)|}{|y+z-a|^{(n/p)+s}} da \right) dz \\ &\lesssim \fint_{B(0,r_2)} \left(\fint_{B(x+z,3|x-y|)} S(f)(y+z, a) da \right) dz. \end{aligned}$$

Applying operators M_{01} and M_{11} from Section 2, we can proceed as follows

$$\frac{A_2}{|x-y|^{(n/p)+s}} \lesssim \fint_{B(0,r_2)} M_{01}(Sf)(y+z, x+z)\, dz \le M_{11}(M_{01}(Sf))(y, x).$$

Combining the above estimates for A_1 and A_2 we end up with

$$S(M_G f)(x,y) \le \frac{A_1 + A_2}{|x-y|^{(n/p)+s}} \lesssim M_{11}(M_{01}(Sf))(x,y) + M_{11}(M_{01}(Sf))(y,x)$$

and inequality (9) follows. □

4. Application to fractional Hardy inequalities.

We apply Theorem 1.1 by solving a certain localization problem for (s,p)-Hardy inequalities and our result is formulated in Theorem 4.5. This generalizes an earlier result from [**4**] that is formulated as Theorem 4.4 for the sake of comparison. Recall that an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s,p)-Hardy inequality, for $0 < s < 1$ and $1 < p < \infty$, if there is a constant $C > 0$ such that inequality

$$\int_G \frac{|f(x)|^p}{\operatorname{dist}(x,\partial G)^{sp}}\,dx \le C \int_G \int_G \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx \tag{13}$$

holds for all functions $f \in C_c(G)$. We need a characterization of (s,p)-Hardy inequality in terms of the following (s,p)-capacities of compact sets $K \subset G$; we write

$$\operatorname{cap}_{s,p}(K,G) = \inf_u |u|^p_{W^{s,p}(G)},$$

where the infimum is taken over all real-valued functions $u \in C_c(G)$ such that $u(x) \ge 1$ for every point $x \in K$. The 'Maz'ya-type characterization' in Theorem 4.1 can be found in [**4**, Theorem 1.1] (this result, in fact, applies when $0 < p < \infty$); we also refer to [**18**, Section 11.11.2; Remark 3].

Theorem 4.1. *Let $0 < s < 1$ and $1 < p < \infty$. Then an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s,p)-Hardy inequality if and only if there is a constant $C > 0$ such that*

$$\int_K \operatorname{dist}(x,\partial G)^{-sp}\,dx \le C \operatorname{cap}_{s,p}(K,G) \tag{14}$$

for every compact set $K \subset G$.

We solve a 'localization problem of the testing condition (14)'. Roughly speaking, we prove that if $\operatorname{cap}_{s,p}(\cdot,G)$ satisfies a weak quasiadditivity property, see Definition 4.2, then G admits an (s,p)-Hardy inequality if and only if inequality (14) holds for all Whitney cubes $K = Q \in \mathcal{W}(G)$.

Definition 4.2. The (s,p)-capacity $\operatorname{cap}_{s,p}(\cdot,G)$ is weakly $\mathcal{W}(G)$-quasiadditive, if there exists a constant $N > 0$ such that

$$\sum_{Q \in \mathcal{W}(G)} \operatorname{cap}_{s,p}(K \cap Q, G) \le N \operatorname{cap}_{s,p}(K,G) \tag{15}$$

whenever $K = \bigcup_{Q\in\mathcal{E}} Q$ and $\mathcal{E} \subset \mathcal{W}(G)$ is a finite family of Whitney cubes. If there is a constant $N > 0$ such that inequality (15) holds for all compact sets $K \subset G$, then we say that the (s,p)-capacity $\operatorname{cap}_{s,p}(\cdot, G)$ is $\mathcal{W}(G)$-quasiadditive.

We remark that the quasiadditivity property of (both Riesz and Green) capacity with respect to Whitney cubes of an open set G was first considered by Aikawa [**1**], [**2**]. In order to formulate an earlier localization result from [**4**] we also need the following definition.

DEFINITION 4.3. An open set G is said to admit an (s,p)-zero extension, if there is a constant $C > 0$ such that $|E_G u|_{W^{s,p}(\mathbb{R}^n)} \le C|u|_{W^{s,p}(G)}$ for every function $u \in C_c(G)$. Here $E_G u(x) = u(x)$ if $x \in G$ and $E_G u(x) = 0$ otherwise.

Let us emphasise that only continuous functions with compact support need to have a bounded zero extension, and not all open sets admit an (s,p)-zero extension, [**4**]. We aim to improve on the following earlier result, which (essentially) is [**4**, Theorem 1.2].

THEOREM 4.4. *Let $0 < s < 1$ and $1 < p < \infty$ satisfy $sp < n$. Suppose $G \neq \emptyset$ is a bounded open set in $\mathbb{R}^n$. Then the following conditions are equivalent.*

(1) *G admits an (s,p)-Hardy inequality*;
(2) $\operatorname{cap}_{s,p}(\cdot, G)$ *is $\mathcal{W}(G)$-quasiadditive and G admits an (s,p)-zero extension*;
(3) $\operatorname{cap}_{s,p}(\cdot, G)$ *is weakly $\mathcal{W}(G)$-quasiadditive and G admits an (s,p)-zero extension.*

Our main result in this section is Theorem 4.5 which answers a question in [**4**, p. 2]. That is, we generalize Theorem 4.4 by adding one more condition that is equivalent with G admitting an (s,p)-Hardy inequality.

THEOREM 4.5. *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp < n$. Suppose that $G \neq \emptyset$ is a bounded open set in $\mathbb{R}^n$. Then the conditions* (A) *and* (B) *are equivalent.*

(A) *G admits an (s,p)-Hardy inequality*;
(B) $\operatorname{cap}_{s,p}(\cdot, G)$ *is weakly $\mathcal{W}(G)$-quasiadditive and there exists $c > 0$ such that*

$$\ell(Q)^{n-sp} \le c \operatorname{cap}_{s,p}(Q, G) \tag{16}$$

for every $Q \in \mathcal{W}(G)$.

REMARK 4.6. The counterexamples that are given in [**4**, Section 6] show that neither one of the two conditions (i.e., weak $\mathcal{W}(G)$-quasiadditivity of the capacity and the lower bound (16) for the capacities of Whitney cubes) appearing in Theorem 4.5(B) is implied by the other one. Accordingly, both of these conditions are needed for the characterization.

Whereas in Theorem 4.4(3) it is assumed that G admits an (s,p)-zero extension, in Theorem 4.5(B) the lower bound (16) for the capacity is assumed instead. This lower bound and the boundedness inequality $|M_G u|_{W^{s,p}(G)} \lesssim |u|_{W^{s,p}(G)}$ if $u \in C_c(G)$ are, in fact, both consequences of the assumption that G admits an (s,p)-zero extension; we refer to [**4**, Lemma 2.1 and Lemma 2.2]. These two consequences, together with the weak quasiadditivity, are the key facts that are needed for the proof of the implication from (3) to (1) in Theorem 4.4. Since by Theorem 1.1 we now know the boundedness inequality for the local maximal operator, in Theorem 4.5(B) we only need to assume that the lower bound (16) for the capacity holds (and that the capacity is weakly quasiadditive).

Proof of Theorem 4.5. The implication from (A) to (B) follows from [**4**, Proposition 4.1] in combination with [**4**, Lemma 2.1]. For convenience of the reader, we give the proof of the implication from (B) to (A) by closely following the proof of [**4**, Proposition 5.1]. The last proposition is a reformulation of the implication from (3) to (1) in Theorem 4.4.

By Theorem 4.1, it suffices to show that

$$\int_K \operatorname{dist}(x,\partial G)^{-sp}\,dx \lesssim \operatorname{cap}_{s,p}(K,G), \tag{17}$$

whenever $K\subset G$ is compact. Let us fix a compact set $K\subset G$ and an admissible test function u for $\operatorname{cap}_{s,p}(K,G)$. We partition $\mathcal{W}(G)$ as $\mathcal{W}_1\cup\mathcal{W}_2$, where

$$\mathcal{W}_1=\biggl\{Q\in\mathcal{W}(G):\langle u\rangle_{2^{-1}Q}:=\fint_{2^{-1}Q}u<1/2\biggr\},\quad \mathcal{W}_2=\mathcal{W}(G)\setminus\mathcal{W}_1.$$

As opposed to the proof of [**4**, Proposition 5.1], here we use the integral average $\langle u\rangle_{2^{-1}Q}$ instead of $\langle u\rangle_Q$.

Write the left-hand side of (17) as

$$\biggl\{\sum_{Q\in\mathcal{W}_1}+\sum_{Q\in\mathcal{W}_2}\biggr\}\int_{K\cap Q}\operatorname{dist}(x,\partial G)^{-sp}\,dx. \tag{18}$$

The first series is estimated as in [**4**, Proposition 5.1]; in particular, condition (B) is not needed here. Instead we first observe that, for every $Q\in\mathcal{W}_1$ and every $x\in K\cap Q$, $(1/2)=1-(1/2)<u(x)-\langle u\rangle_{2^{-1}Q}=|u(x)-\langle u\rangle_{2^{-1}Q}|$. Thus, by Jensen's inequality and (4),

$$\begin{aligned}\sum_{Q\in\mathcal{W}_1}\int_{K\cap Q}\operatorname{dist}(x,\partial G)^{-sp}\,dx &\lesssim \sum_{Q\in\mathcal{W}_1}\ell(Q)^{-sp}\int_Q|u(x)-\langle u\rangle_{2^{-1}Q}|^p\,dx\\ &\lesssim \sum_{Q\in\mathcal{W}_1}\ell(Q)^{-n-sp}\int_Q\int_Q|u(x)-u(y)|^p\,dy\,dx\\ &\lesssim \sum_{Q\in\mathcal{W}_1}\int_Q\int_Q\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx.\end{aligned}$$

Since G is the union of its Whitney cubes and the interiors of Whitney cubes are pairwise disjoint, the last term is dominated by $|u|^p_{W^{s,p}(G)}$.

Let us then focus on the remaining series in (18), whose analysis depends both on Theorem 1.1 for the local maximal operator and the full generality of condition (B). Let us consider $Q \in \mathcal{W}_2$ and $x \in Q$. Observe that $2^{-1}Q \subset B(x, (4/5)\mathrm{diam}(Q))$. Hence, by inequalities (4),

$$M_G u(x) \gtrsim \fint_{2^{-1}Q} u(y)\, dy \geq \frac{1}{2}. \tag{19}$$

The support of $M_G u$ is a compact set in G by the boundedness of G and the fact that $u \in C_c(G)$. By Lemma 2.1, we find that $M_G u$ is continuous. Concluding from these remarks we find that there is $\rho > 0$, depending only on n, such that $\rho M_G u$ is an admissible test function for $\mathrm{cap}_{s,p}(\cup_{Q\in\mathcal{W}_2} Q, G)$. The family $\mathcal{W}_2$ is finite, as $u \in C_c(G)$. Hence, by condition (B) and the inequality (19),

$$\begin{aligned}
&\sum_{Q\in\mathcal{W}_2} \int_{K\cap Q} \mathrm{dist}(x, \partial G)^{-sp}\, dx \\
&\quad \lesssim \sum_{Q\in\mathcal{W}_2} \ell(Q)^{n-sp} \leq c \sum_{Q\in\mathcal{W}_2} \mathrm{cap}_{s,p}(Q, G) \\
&\quad \leq cN \mathrm{cap}_{s,p}\Big(\bigcup_{Q\in\mathcal{W}_2} Q, G\Big) \leq cN\rho^p \int_G \int_G \frac{|M_G u(x) - M_G u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx.
\end{aligned}$$

By Theorem 1.1, the last term is dominated by $C(n, s, p, N, c, \rho)|u|^p_{W^{s,p}(G)}$. The desired inequality (17) follows from the considerations above. □

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References

[1] H. Aikawa, Quasiadditivity of Riesz capacity, Math. Scand., **69** (1991), 15–30.
[2] H. Aikawa, Quasiadditivity of capacity and minimal thinness, Ann. Acad. Sci. Fenn. Ser. A I Math., **18** (1993), 65–75.
[3] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math., **136** (2012), 521–573.
[4] B. Dyda and A. V. Vähäkangas, Characterizations for fractional Hardy inequality, Adv. Calc. Var., **8** (2015), 173–182.
[5] B. Dyda and A. V. Vähäkangas, A framework for fractional Hardy inequalities, Ann. Acad. Sci. Fenn. Math., **39** (2014), 675–689.
[6] D. Edmunds, R. Hurri-Syrjänen and A. V. Vähäkangas, Fractional Hardy-type inequalities in domains with uniformly fat complement, Proc. Amer. Math. Soc., **142** (2014), 897–907.
[7] P. Hajłasz and J. Onninen, On boundedness of maximal functions in Sobolev spaces, Ann. Acad. Sci. Fenn. Math., **29** (2004), 167–176.
[8] L. Ihnatsyeva, J. Lehrbäck, H. Tuominen and A. V. Vähäkangas, Fractional Hardy inequalities and visibility of the boundary, Studia Math., **224** (2014), 47–80.
[9] L. Ihnatsyeva and A. V. Vähäkangas, Hardy inequalities in Triebel–Lizorkin spaces II. Aikawa dimension, Ann. Mat. Pura Appl. (4), **194** (2015), 479–493.

[10] L. Ihnatsyeva and A. V. Vähäkangas, Hardy inequalities in Triebel–Lizorkin spaces, Indiana Univ. Math. J., **62** (2013), 1785–1807.

[11] J. Kinnunen, The Hardy–Littlewood maximal function of a Sobolev function, Israel J. Math., **100** (1997), 117–124.

[12] J. Kinnunen and P. Lindqvist, The derivative of the maximal function, J. Reine Angew. Math., **503** (1998), 161–167.

[13] J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, Bull. London Math. Soc., **35** (2003), 529–535.

[14] S. Korry, Boundedness of Hardy–Littlewood maximal operator in the framework of Lizorkin–Triebel spaces, Rev. Mat. Complut., **15** (2002), 401–416.

[15] J. Lehrbäck and N. Shanmugalingam, Quasiadditivity of Variational Capacity, Potential Anal., **40** (2014), 247–265.

[16] H. Luiro, Continuity of the maximal operator in Sobolev spaces, Proc. Amer. Math. Soc., **135** (2007), 243–251.

[17] H. Luiro, On the regularity of the Hardy–Littlewood maximal operator on subdomains of $\mathbb{R}^n$, Proc. Edinb. Math. Soc. (2), **53** (2010), 211–237.

[18] V. Maz'ya, Sobolev Spaces with Applications to Elliptic Partial Differential Equations. 2nd, revised and augmented Edition. A Series of Comprehensive Studies in Mathematics, **342**, Springer-Verlag, 2011.

[19] W. Rudin, Real and complex analysis, McGraw-Hill Book Co., New York, third edition, 1987.

[20] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[21] H. Triebel, Theory of function spaces. II, Monographs in Mathematics, **84**, Birkhäuser Verlag, Basel, 1992.

[22] Y. Zhou, Fractional Sobolev extension and imbedding, Trans. Amer. Math. Soc., **367** (2015), 959–979.

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