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Parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter

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Abstract. Stokes phenomena with respect to parameters are investigated for the Gauss hypergeometric differential equation with a large parameter. For this purpose, the notion of the Voros coefficient is introduced for the equation. The explicit forms of the Voros coefficients are given as well as their Borel sums. By using them, formulas which describe the Stokes phenomena are obtained.

1. Introduction.

The purpose of this article is to describe parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter from a viewpoint of the exact WKB analysis. Parametric Stokes phenomena mean Stokes phenomena associated with a change of parameters contained in the equation. The classical Gauss hypergeometric differential equation has three complex parameters. We introduce a large parameter η so that the difference of two characteristic exponents at every regular singular point is proportional to η. We consider formal series solutions in η^{-1} of the Riccati equation associated with the hypergeometric equation and corresponding formal solutions of the hypergeometric equation which are called WKB solutions. We will describe the parametric Stokes phenomena of the hypergeometric equation in terms of the WKB solutions. The WKB solutions are Borel summable in a region surrounded by Stokes curves if the Stokes geometry is non-degenerate. If it is the case, we can obtain analytic solutions by taking the Borel sums of the WKB solutions. These analytic solutions can be analytically continued with respect to the parameters. If the parameters have moved and the Stokes geometry changes via a degeneration, the Borel sums of the WKB solutions are, in general, not equal to the analytic continuation of the original Borel sums even if the region where we take the Borel resummation has been deformed consistently with the analytic continuation. The discrepancy between the latter Borel sums and the continuation of the former Borel sums is called a parametric Stokes phenomenon. To analyze the phenomena, we will make use of Voros coefficients. The notion of Voros coefficients was introduced by Voros [**17**] for the Weber equation and for quartic oscillator. It plays a role in the analysis of the Stokes phenomena of WKB solutions with respect to parameters in equations (see [**6**], [**7**] also). Concrete forms of the Voros coefficients have been obtained

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by Shen and Silverstone [**13**] and Takei [**14**] for the Weber equation and by Koike and Takei [**12**] for the Whittaker equation of a degenerated type. We note that the Jost function for the Weber equation was computed in [**17**] first and as an asymptotic expansion of it, the explicit form of the Voros coefficient was obtained there. On the other hand, [**13**], [**14**] and [**12**] defined the Voros coefficients directly by using formal solutions of the Riccati equations associated with their equations and computed them as a formal series.

In this paper, we firstly show that we can define the Voros coefficients for the Gauss hypergeometric equation. Our definition follows that of [**14**] and [**12**] with suitable modifications. Secondly we compute the explicit forms of them. We use an extension of the method developed by [**14**] and [**12**], that is, we derive systems of difference equations that characterize the Voros coefficients and solve them. In our case, the method used in [**14**], [**12**] cannot be applied directly because the number of the difference equations for a Voros coefficient are three, which is the number of the parameters of the hypergeometric equation. To solve the systems, we employ formal differential operators of infinite order used by Candelpergher, Coppo and Delabaere [**5**]. Thirdly we see that the Voros coefficients are Borel summable in suitable regions in the space of parameters and compute the explicit forms of the Borel sums. Using the Borel sums, we describe the parametric Stokes phenomena for the WKB solutions. We restrict ourselves to discuss parametric Stokes phenomena associated with the Weber-type degeneration of Stokes geometry, namely, the case where two distinct turning points are connected by a Stokes segment (cf. [**6**], [**7**], [**14**]). There is another type of degeneration where a Stokes curve forms a loop. The parametric Stokes phenomena associated with this type of degeneration will be discussed in our forthcoming paper.

The relation between the Borel sums of the WKB solutions to the hypergeometric equation and the hypergeometric function is given by the second author [**16**] up to multiplicative constants. Hence we can obtain, in principle, parametric Stokes phenomena for asymptotic expansions of the hypergeometric function with respect to the inverse of the large parameter. This subject will be discussed also in our forthcoming article as well as determining the multiplicative constants mentioned above.

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2. Voros coefficients for the Gauss hypergeometric differential equation with a large parameter.

2.1. The Gauss hypergeometric differential equation with a large parameter.

Let us consider the following Schrödinger-type equation with a large parameter $\eta > 0$ and complex parameters α, β, γ:

$$\Big(-\frac{d^2}{dx^2} + \eta^2 Q(x)\Big)\psi = 0, \tag{1}$$

where we set $Q(x) = Q_0(x) + \eta^{-2} Q_1(x)$ with

$$Q_0(x) = Q_0(\alpha, \beta, \gamma; x) = \frac{(\alpha - \beta)^2 x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} \tag{2}$$

and

$$Q_1(x) = -\frac{x^2 - x + 1}{4x^2(x-1)^2}. \tag{3}$$

Equation (1) comes from the Gauss hypergeometric differential equation with complex parameters a, b and c:

$$x(1-x)\frac{d^2 w}{dx^2} + (c - (a+b+1)x)\frac{dw}{dx} - abw = 0. \tag{4}$$

If we introduce a large parameter η by setting

$$a = \frac{1}{2} + \alpha\eta, \tag{5}$$

$$b = \frac{1}{2} + \beta\eta, \tag{6}$$

$$c = 1 + \gamma\eta, \tag{7}$$

we have

$$x(1-x)\frac{d^2 w}{dx^2} + (1 + \gamma\eta - ((\alpha+\beta)\eta + 2)x)\frac{dw}{dx} - \left(\frac{1}{2} + \alpha\eta\right)\left(\frac{1}{2} + \beta\eta\right)w = 0. \tag{8}$$

Next we eliminate the first-order term of (8) by taking

$$\psi = x^{(1+\gamma\eta)/2}(1-x)^{(1+(\alpha+\beta-\gamma)\eta)/2} w, \tag{9}$$

as an unknown function. Then we have (1). In this paper, (1) is also called the Gauss hypergeometric differential equation. Equation (1) has the formal power series solutions which are called the WKB solutions

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{x_0}^{x} S_{\mathrm{odd}}\, dx\right), \tag{10}$$

where x_0 is a fixed point and S_{odd} denotes the odd-order part of formal solution

$$S(x) = S_{\mathrm{odd}} + S_{\mathrm{even}} = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \eta^{-2} S_2(x) + \cdots \tag{11}$$

in η^{-1} of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q(x) \tag{12}$$

associated with (1). (See also [**3**], [**12**], [**10**], [**16**] for the notation.) Here we have taken a branch of $S_{-1} = \sqrt{Q_0}$ suitably. Equation (1) has singular points $b_0 = 0$, $b_1 = 1$ and $b_2 = \infty$. A turning point of (1) is, by definition, a zero point or of a simple pole of Q_0. Let a be a turning point. A Stokes curve emanating from the turning point a is a curve defined by

$$\operatorname{Im} \int_a^x \sqrt{Q_0}\, dx = 0. \tag{13}$$

A Stokes curve flows into a singular point or a turning point. We assume that (α, β, γ) is not contained in the following set E_0:

$$E_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) = 0\}. \tag{14}$$

This implies that there are two distinct turning points a_0 and a_1 which do not coincide with 0, 1, ∞. The Stokes graph (cf. [**1**]) of (1) is, by definition, a two-colored sphere graph consisting of all Stokes curves (emanating from a_0 and a_1) as edges, $\{a_0, a_1\}$ as vertices of the first color and $\{b_0, b_1, b_2\}$ as vertices of the second color. The Stokes graph of (4) is, by definition, that of (1). We set

$$E_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \operatorname{Re}\alpha \cdot \operatorname{Re}\beta \cdot \operatorname{Re}(\gamma - \alpha) \cdot \operatorname{Re}(\gamma - \beta) = 0\}, \tag{15}$$

$$E_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \operatorname{Re}(\alpha - \beta) \cdot \operatorname{Re}(\alpha + \beta - \gamma) \cdot \operatorname{Re}\gamma = 0\}. \tag{16}$$

If one of Stokes curves flows into a turning point, (α, β, γ) is contained in the set $E_1 \cup E_2$ (cf. [**16**, Theorem 3.1]). In this case, we say that the Stokes geometry is degenerate. The sets ω_h $(h = 1, 2, 3, 4)$ of the parameters (α, β, γ) are defined by

$$\omega_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}\beta\}, \tag{17}$$

$$\omega_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\alpha < \operatorname{Re}\beta < \operatorname{Re}\gamma < \operatorname{Re}\alpha + \operatorname{Re}\beta\}, \tag{18}$$

$$\omega_3 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\gamma < \operatorname{Re}\alpha < \operatorname{Re}\beta\}, \tag{19}$$

$$\omega_4 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\gamma < \operatorname{Re}\alpha + \operatorname{Re}\beta < \operatorname{Re}\beta\}. \tag{20}$$

Let G denote the group generated by the involutions ι_j $(j = 0, 1, 2)$ which are defined in the space $\mathbb{C}^3$ of parameters (α, β, γ) as follows:

$$\iota_0 : (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma), \tag{21}$$

$$\iota_1 : (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma), \tag{22}$$

$$\iota_2 : (\alpha, \beta, \gamma) \mapsto (\beta, \alpha, \gamma). \tag{23}$$

Moreover, we define open subsets Π_h $(h = 1, 2, 3, 4)$ in $\mathbb{C}^3$ by

$$\Pi_h = \bigcup_{r \in G} r(\omega_h). \tag{24}$$

We assume that (α, β, γ) is not contained in the sets $E_0 \cup E_1 \cup E_2$. A Stokes graph can be classified by its order sequence $\hat{n} = (n_0, n_1, n_2)$, where n_0, n_1 and n_2 are numbers of Stokes curves that flow into 0, 1 and ∞, respectively (cf. [**3**]).

THEOREM 2.1 ([**3**, Theorem 3.2]). *Let $\hat{n} = (n_0, n_1, n_2)$ denote the order sequences of the Stokes graph with parameters (α, β, γ).*

(1) *If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.*
(2) *If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.*
(3) *If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$.*
(4) *If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.*

We introduce the following notations:

$$\iota_3 = \iota_1 \iota_2 \quad : (\alpha, \beta, \gamma) \mapsto (\gamma - \alpha, \gamma - \beta, \gamma), \tag{25}$$

$$\iota_4 = \iota_0 \iota_2 \quad : (\alpha, \beta, \gamma) \mapsto (-\beta, -\alpha, -\gamma), \tag{26}$$

$$\iota_5 = \iota_0 \iota_1 \quad : (\alpha, \beta, \gamma) \mapsto (\beta - \gamma, \alpha - \gamma, -\gamma), \tag{27}$$

$$\iota_6 = \iota_0 \iota_1 \iota_2 : (\alpha, \beta, \gamma) \mapsto (\alpha - \gamma, \beta - \gamma, -\gamma). \tag{28}$$

Then we have $G = \{\mathrm{id}, \iota_0, \ldots, \iota_6\}$. When τ runs over G and h on $\{1, 2, 3, 4\}$, $\iota_m(\omega_h)$ $(m = 0, 1, \ldots, 6;\ h = 1, \ldots, 4)$ covers most of $\mathbb{C}^3$:

$$\bigcup_{\tau \in G} \bigcup_{h=1}^{4} \tau(\omega_h) = \mathbb{C}^3 - \{(\alpha, \beta, \gamma) \mid \operatorname{Re}\alpha \operatorname{Re}\beta \operatorname{Re}\gamma \operatorname{Re}(\gamma - \alpha) \operatorname{Re}(\gamma - \beta) \cdot \operatorname{Re}(\alpha - \beta) \operatorname{Re}(\alpha + \beta - \gamma) = 0\}.$$

We denote $\iota_m(\omega_h)$ by $\omega_{hm} (m = 0, 1, \ldots, 6;\ h = 1, \ldots, 4)$. For a fixed $\operatorname{Re}\gamma > 0$ (resp. $\operatorname{Re}\gamma < 0$), the configurations of ω_h and ω_{hm} in the $\operatorname{Re}\alpha$-$\operatorname{Re}\beta$ plane are as follows:

2.2. Voros coefficients.

In this section, we assume that (α, β, γ) is not contained in the set $E_0 \cup E_1 \cup E_2$. Let a be a turning point of (1). Let C_j $(j = 0, 1, 2)$ be a closed path going around a with the base point b_j in a counterclockwise manner. We can take C_j so that the inside of C_j does not contain another turning point or singular points.

DEFINITION 2.2 ([**2**]). We define the following integrals:

$$V_0 = V_0(\alpha, \beta, \gamma; \eta) := \frac{1}{2} \int_{C_0} (S_{\mathrm{odd}} - \eta S_{-1})\, dx, \tag{29}$$

$$V_1 = V_1(\alpha, \beta, \gamma; \eta) := \frac{1}{2} \int_{C_1} (S_{\mathrm{odd}} - \eta S_{-1})\, dx, \tag{30}$$

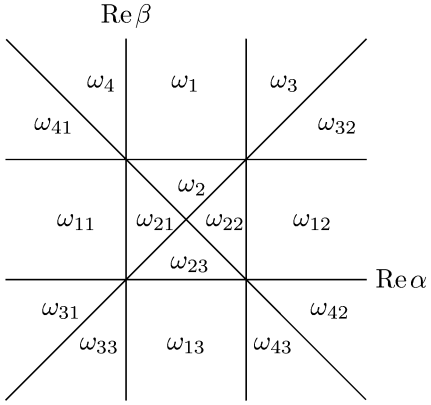

Figure 1.1. $\mathrm{Re}\,\gamma > 0$.

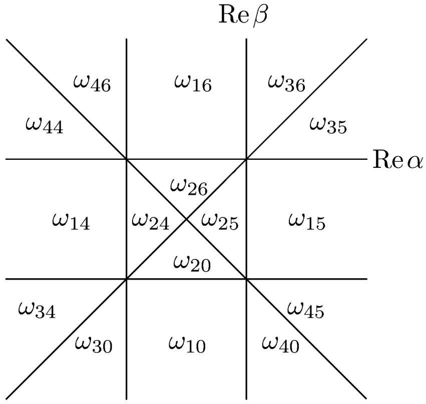

Figure 1.2. $\mathrm{Re}\,\gamma < 0$.

$$V_2 = V_2(\alpha, \beta, \gamma; \eta) := \frac{1}{2} \int_{C_2} (S_{\mathrm{odd}} - \eta S_{-1})\, dx. \tag{31}$$

Then V_j $(j = 0, 1, 2)$ are formal power series in η^{-1}. We call V_j the Voros coefficients of (1) with respect to b_j.

Since the residues of S_{odd} and ηS_{-1} at the singular points coincide (see [**10**] for the computation of residues of S_{odd}), these integrals are well-defined for every homotopy class of the path of integration and they do not depend on the choice of the turning point a. Thanks to the square root character of S_{odd} at $x = a$, we can rewrite the right-hand side of (29), (30) and (31) as

$$\frac{1}{2} \int_{C_j} (S_{\mathrm{odd}} - \eta S_{-1})\, dx = \int_{b_j}^{a} (S_{\mathrm{odd}} - \eta S_{-1})\, dx. \tag{32}$$

Hereafter, $\psi_\pm$ denote the WKB solutions normalized a turning point a ($a = a_0$ or $a = a_1$):

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}}\, dx \right). \tag{33}$$

Let

$$\psi_\pm^{(j)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{b_j}^x (S_{\mathrm{odd}} - \eta S_{-1})\, dx \pm \eta \int_a^x S_{-1}\, dx \right) \tag{34}$$

be the WKB solutions normalized at the singular point b_j (cf. [**7**]). For $j = 0, 1$ and 2, $V_j(\alpha, \beta, \gamma; \eta)$ describe the discrepancy between WKB solutions $\psi_\pm$ and $\psi_\pm^{(j)}$, that is, we factorize $\psi_\pm$ as

$$\psi_\pm = \exp(\mp V_j) \psi_\pm^{(j)}. \tag{35}$$

Here the paths of integration should be chosen suitably.

To give the explicit form of V_j, we specify the branch of $S_{-1}(x) = \sqrt{Q_0(x)}$ precisely. For this purpose we consider the case where (α, β, γ) is contained in ω_2. Firstly we take a point $(\alpha, \beta, \gamma) = (0.5 + \delta' i, 1 - \epsilon - \delta i, 1)$ in ω_2. Here δ', ϵ and δ are sufficiently small positive numbers. We show a configuration of the Stokes curves for this case in Figure 2.1. Here bullets and white bullets designate turning points a_0, a_1 and singular points $0, 1$, respectively. We take a segment connecting two turning points as a branch cut for $\sqrt{Q_0}$. This is shown by the wavy line in Figure 2.1.

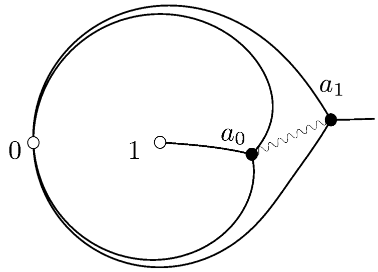

Figure 2.1.

We specify the branch of $\sqrt{Q_0}$ on the first sheet of the Riemann surface of $\sqrt{Q_0}$ so that

$$\sqrt{Q_0} \sim \frac{\beta - \alpha}{2x} \tag{36}$$

holds near $x = \infty$. In this case, the behavior of $\sqrt{Q_0}$ near 0 and 1 are

$$\sqrt{Q_0} \sim \frac{\gamma}{2x}, \tag{37}$$

$$\sqrt{Q_0} \sim -\frac{\alpha + \beta - \gamma}{2(x-1)}, \tag{38}$$

respectively. This can be observed by the following discussion: We consider that $Q_0(x) = Q_0(0.5, 1 - \epsilon - \delta i, x)$ is a perturbation of

$$Q_0(0.5, 1, 1; x) = \frac{(x-2)^2}{(4x(x-1))^2}. \tag{39}$$

These values of parameters are located on the boundary between ω_1 and ω_2. The Stokes curves of the equation

$$\left(-\frac{d^2}{dx^2} + \eta^2 \frac{(x-2)^2}{16x^2(x-1)^2} + Q_1 \right) \psi = 0 \tag{40}$$

can be described explicitly, namely,

$$\{u+iv \mid 1<u, v=0\} \cup \{u+iv \mid (u-1)^2+v^2=1, 0<u\}, \tag{41}$$

where $x=u+iv$ $(u,v\in\mathbb{R})$ (Figure 2.2).

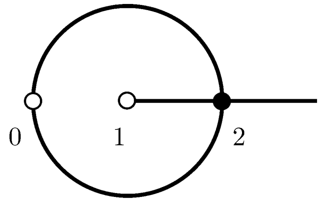

Figure 2.2.

Here $x=2$ is a double turning point. We take the branch

$$\sqrt{Q_0(0.5,1,1;x)} = \frac{x-2}{4x(x-1)}. \tag{42}$$

We have

$$\operatorname{Re}\int_2^x \sqrt{Q_0(0.5,1,1;x)}\,dx \geq 0 \tag{43}$$

on the Stokes curve $\{x \mid x\geq 2\}$ (see Figure 2.2). Hence

$$\operatorname{Re}\int_2^x \sqrt{Q_0(0.5,1-\epsilon-\delta i,1;x)}\,dx \geq 0 \tag{44}$$

on the Stokes curve emanating from a_1 and going to the infinity (see Figure 2.1). This is consistent with the choice of the branch satisfying (36). This implies ψ_+ is dominant to ψ_- on the Stokes curve. Similarly, we can see that ψ_- (resp. ψ_+) is dominant to ψ_+ (resp. ψ_-) on the Stokes curve(s) flowing into $b_0=0$ (resp. $b_1=1$). Thus we have (37) and (38).

Under the above choice of the branch of $\sqrt{Q_0}$, the explicit forms of V_j are given in the following theorem which has been announced in [**2**] (up to the multiplicative factor ± 1 coming from the choice of the branch).

Theorem 2.3. *The Voros coefficients V_j have the following forms*:

$$\begin{aligned} &V_0(\alpha,\beta,\gamma;\eta) \\ &\quad = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\}, \end{aligned} \tag{45}$$

$$
\begin{aligned}
V_1(\alpha,\beta,\gamma;\eta) &= -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)} \\
&\times\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right\},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
V_2(\alpha,\beta,\gamma;\eta) &= \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)} \\
&\times\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)-\frac{2}{(\beta-\alpha)^{n-1}}\right\}.
\end{aligned}
\tag{47}
$$

Here B_n are the Bernoulli numbers defined by

$$
\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.
$$

Proof. We only prove (46). Others can be proved similarly. A key of the proof is to use the method developed by Takei [**14**], which employs the ladder operator for the Weber equation. For the Gauss equation, we have the following three operators which play the role of the ladder operator for the parameters a, b and c in (4) (cf. [**9**]):

$$
H_1(a,b,c)=x\frac{d}{dx}+a:\mathcal{S}(a,b,c)\to\mathcal{S}(a+1,b,c), \tag{48}
$$

$$
H_2(a,b,c)=x\frac{d}{dx}+b:\mathcal{S}(a,b,c)\to\mathcal{S}(a,b+1,c), \tag{49}
$$

$$
B_3(a,b,c)=x\frac{d}{dx}+c:\mathcal{S}(a,b,c+1)\to\mathcal{S}(a,b,c). \tag{50}
$$

Here $\mathcal{S}(a,b,c)$ denotes the solution space of (4). Using (48), (49) and (50), we can prove the following lemma.

Lemma 2.4. *The solution $S(x)=S(\alpha,\beta,\gamma;x,\eta)$ of* (12) *satisfies the following system of difference equations*:

$$
\begin{aligned}
&S(\alpha+\eta^{-1},\beta,\gamma;x,\eta)-S(\alpha,\beta,\gamma;x,\eta) \\
&\quad=-\frac{1}{2(1-x)}+\frac{d}{dx}\log\left\{-\frac{1}{2}\gamma\eta+\frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)\eta)+xS(\alpha,\beta,\gamma;x,\eta)+\alpha\eta\right\},
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
&S(\alpha,\beta+\eta^{-1},\gamma;x,\eta)-S(\alpha,\beta,\gamma;x,\eta) \\
&\quad=-\frac{1}{2(1-x)}+\frac{d}{dx}\log\left\{-\frac{1}{2}\gamma\eta+\frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)\eta)+xS(\alpha,\beta,\gamma;x,\eta)+\beta\eta\right\},
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&S(\alpha,\beta,\gamma+\eta^{-1};x,\eta)-S(\alpha,\beta,\gamma;x,\eta)\\
&\quad=\frac{1}{2(1-x)}+\frac{1}{2x}-\frac{d}{dx}\log\left\{\frac{1}{2}\gamma\eta+\frac{x}{2(1-x)}(\alpha+\beta-\gamma)\eta+xS(\alpha,\beta,\gamma+\eta^{-1};x,\eta)\right\}.
\end{aligned}
\tag{53}
$$

Proof. To prove (51), we use the operator (48). As we have introduced the large parameter η by setting (5), (6) and (7), we have the following operator:

$$
\begin{aligned}
H_1\left(\frac{1}{2}+\alpha\eta,\frac{1}{2}+\beta\eta,1+\gamma\eta;\eta\right)&=x\frac{d}{dx}+\frac{1}{2}+\alpha\eta:\\
&\qquad\mathcal{S}(\alpha,\beta,\gamma;\eta)\to\mathcal{S}(\alpha+\eta^{-1},\beta,\gamma;\eta).
\end{aligned}
\tag{54}
$$

Here we have abbreviated $\mathcal{S}(1/2+\alpha\eta,\ 1/2+\beta\eta,\ 1+\gamma\eta;\eta)$ to $\mathcal{S}(\alpha,\beta,\gamma;\eta)$. Let $T(\alpha,\beta,\gamma;x,\eta)$ be a solution of the Riccati equation

$$
x(1-x)\left(\frac{dT}{dx}+T^2\right)+(1+\gamma\eta-((\alpha+\beta)\eta+2)x)T-\left(\frac{1}{2}+\alpha\eta\right)\left(\frac{1}{2}+\beta\eta\right)w=0 \tag{55}
$$

associated with (8) and $\hat{T}$ the logarithmic derivative of

$$
\left(x\frac{d}{dx}+\alpha\eta+\frac{1}{2}\right)\exp\int T\,dx=\left(xT+\alpha\eta+\frac{1}{2}\right)\exp\int T\,dx, \tag{56}
$$

namely,

$$
\hat{T}=T+\frac{d}{dx}\log\left(xT+\alpha\eta+\frac{1}{2}\right). \tag{57}
$$

We can confirm that $\hat{T}$ satisfies the equation obtained from (55) by replacing α by $\alpha+\eta^{-1}$. If S is a formal solution of (12), then

$$
T=S-\frac{1+\gamma\eta}{2x}+\frac{1+(\alpha+\beta-\gamma)\eta}{2(1-x)} \tag{58}
$$

becomes a formal solution of (55) and

$$
\hat{S}=\hat{T}+\frac{1+\gamma\eta}{2x}-\frac{1+(\alpha+\eta^{-1}+\beta-\gamma)\eta}{2(1-x)} \tag{59}
$$

is a formal solution of the equation obtained from (12) by replacing α by $\alpha+\eta^{-1}$. Hence we have

$$
\hat{S}=S(\alpha+\eta^{-1},\beta,\gamma;x,\eta). \tag{60}
$$

Combining (57), (58), (59) and (60), we obtain

$$\begin{aligned}
&S(\alpha+\eta^{-1},\beta,\gamma;x,\eta)-S(\alpha,\beta,\gamma;x,\eta)\\
&=-\frac{1}{2(1-x)}+\frac{d}{dx}\log\left(x\left(S(\alpha,\beta,\gamma;x,\eta)-\frac{1+\gamma\eta}{2x}+\frac{1+(\alpha+\beta-\gamma)\eta}{2(1-x)}\right)+\alpha\eta+\frac{1}{2}\right),
\end{aligned} \tag{61}$$

namely, (51). Similarly, we have (53). We obtain (52) by exchanging α for β in (51). $\square$

Since each coefficient of $S_{\text{even}}=S-S_{\text{odd}}$ is single valued at $x=a$ and

$$\operatorname*{Res}_{x=a} S_{\text{even}}=\operatorname*{Res}_{x=a} S_0=-\frac{1}{4} \tag{62}$$

hold in view of (12), we have

$$\frac{1}{2}\int_{C_1}(S_{\text{odd}}-\eta S_{-1})\,dx=\frac{1}{2}\int_{C_1}(S-\eta S_{-1}-S_0)\,dx. \tag{63}$$

Let x_0 be a point sufficiently close to $b_1=1$. To specify the value of $\log(x_0-1)$, we assume that $-\pi<\arg(x_0-b_1)\le\pi$. Let C_{x_0} be a path that runs from x_0, encircles a in a counterclockwise manner and returns to x_0. We can take C_{x_0} so that another turning point and singular points are not contained inside C_{x_0}. Note that the branch of S_{-1} at the starting point x_0 is different from that of S_{-1} at the final point x_0. To distinguish these two different branches, we use the notation $\hat{x}_0$ to specify x_0 on the second sheet (cf. Figure 2.3).

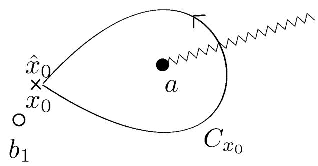

Figure 2.3.

We set

$$I(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S(\alpha+\eta^{-1},\beta,\gamma;x,\eta)-S(\alpha,\beta,\gamma;x,\eta))\,dx, \tag{64}$$

$$J(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S(\alpha,\beta+\eta^{-1},\gamma;x,\eta)-S(\alpha,\beta,\gamma;x,\eta))\,dx, \tag{65}$$

$$K(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S(\alpha,\beta,\gamma+\eta^{-1};x,\eta)-S(\alpha,\beta,\gamma;x,\eta))\,dx, \tag{66}$$

$$I_{-1}(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S_{-1}(\alpha+\eta^{-1},\beta,\gamma;x,\eta)-S_{-1}(\alpha,\beta,\gamma;x,\eta))\,dx, \tag{67}$$

$$J_{-1}(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S_{-1}(\alpha,\beta+\eta^{-1},\gamma;x,\eta)-S_{-1}(\alpha,\beta,\gamma;x,\eta))\,dx, \tag{68}$$

$$K_{-1}(\alpha,\beta,\gamma;x_0,\eta)=\frac{1}{2}\int_{C_{x_0}}(S_{-1}(\alpha,\beta,\gamma+\eta^{-1};x,\eta)-S_{-1}(\alpha,\beta,\gamma;x,\eta))\,dx. \tag{69}$$

By using Lemma 2.4, we obtain

$$\begin{aligned}
&I(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{1}{2}\log\Big\{-\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)\eta)+x_0S(\alpha,\beta,\gamma;\hat{x}_0,\eta)+\alpha\eta\Big\}\\
&\qquad-\frac{1}{2}\log\Big\{-\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)\eta)+x_0S(\alpha,\beta,\gamma;x_0,\eta)+\alpha\eta\Big\},
\end{aligned} \tag{70}$$

$$\begin{aligned}
&J(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{1}{2}\log\Big\{-\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)\eta)+x_0S(\alpha,\beta,\gamma;\hat{x}_0,\eta)+\beta\eta\Big\}\\
&\qquad-\frac{1}{2}\log\Big\{-\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)\eta)+x_0S(\alpha,\beta,\gamma;x_0,\eta)+\beta\eta\Big\},
\end{aligned} \tag{71}$$

$$\begin{aligned}
&K(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=-\frac{1}{2}\log\Big\{\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(\alpha+\beta-\gamma)\eta+x_0S(\alpha,\beta,\gamma+\eta^{-1};\hat{x}_0,\eta)\Big\}\\
&\qquad+\frac{1}{2}\log\Big\{\frac{1}{2}\gamma\eta+\frac{x_0}{2(1-x_0)}(\alpha+\beta-\gamma)\eta+x_0S(\alpha,\beta,\gamma+\eta^{-1};x_0,\eta)\Big\}.
\end{aligned} \tag{72}$$

We chose the semiaxis $\mathrm{Re}(x-1)<0$ as a branch cut of the logarithmic function. We fix the arguments of α, β, γ, $\gamma-\alpha$, $\gamma-\beta$, $\alpha+\beta-\gamma$ and $\beta-\alpha$. In the following computation, we use the conventions

$$-\alpha=e^{-\pi i}\alpha, \tag{73}$$

$$-\beta=e^{\pi i}\beta, \tag{74}$$

$$-\gamma=e^{\pi i}\gamma, \tag{75}$$

$$\alpha-\gamma=e^{\pi i}(\gamma-\alpha), \tag{76}$$

$$\beta-\gamma=e^{-\pi i}(\gamma-\beta), \tag{77}$$

$$\gamma-\alpha-\beta=e^{\pi i}(\alpha+\beta-\gamma), \tag{78}$$

$$\alpha-\beta=e^{\pi i}(\beta-\alpha) \tag{79}$$

which are corresponding to the conditions

$$0 < \arg \alpha \leq \pi, \tag{80}$$

$$-\pi < \arg \beta \leq 0, \tag{81}$$

$$-\pi < \arg \gamma \leq 0, \tag{82}$$

$$-\pi < \arg(\gamma - \alpha) \leq 0, \tag{83}$$

$$0 < \arg(\gamma - \beta) \leq \pi, \tag{84}$$

$$-\pi < \arg(\alpha + \beta - \gamma) \leq 0, \tag{85}$$

$$-\pi < \arg(\beta - \alpha) \leq 0, \tag{86}$$

respectively. By the definition of the Voros coefficients, V_j ($j = 0, 1, 2$) are determined once the branch of $\sqrt{Q_0}$ is fixed and they do not depend on the arguments of parameters α, β, γ. Hence we can compute the concrete form of V_j under the above conventions without loss of generality.

To compute the leading terms and subleading terms of the Laurent expansion of S at $x = b_1$ on the Riemann surface of $\sqrt{Q_0}$, we use a method given in [**10**] (see Section 3.1). On the first sheet, we have near $x = 1$:

$$x_0 S(\alpha, \beta, \gamma; x_0, \eta) = x_0 \left\{ \frac{1 - (\alpha + \beta - \gamma)\eta}{2(x_0 - 1)} - \frac{(2\gamma^2 + p)\eta^2 - 1}{4(1 - (\alpha + \beta - \gamma)\eta)} + O(x_0 - 1) \right\}, \tag{87}$$

where the branch of S_{-1} is chosen as (38) and where p denotes $2(2\alpha\beta - \alpha\gamma - \beta\gamma)$. On the second sheet, we have

$$x_0 S(\alpha, \beta, \gamma; \hat{x}_0, \eta) = x_0 \left\{ \frac{1 + (\alpha + \beta - \gamma)\eta}{2(x_0 - 1)} - \frac{(2\gamma^2 + p)\eta^2 - 1}{4(1 + (\alpha + \beta - \gamma)\eta)} + O(x_0 - 1) \right\} \tag{88}$$

near $x = 1$. Using (87) and (88), we obtain as $x_0 \to 1$

$$\begin{aligned} I(\alpha, \beta, \gamma; x_0, \eta) &= \frac{1}{2} \log \frac{(\alpha + (\eta^{-1}/2))(\gamma - \alpha - (\eta^{-1}/2))}{(\alpha + \beta - \gamma)(\alpha + \beta - \gamma + \eta^{-1})} \\ &\quad + \frac{1}{2} \log(x_0 - 1) + O(x_0 - 1), \end{aligned} \tag{89}$$

$$\begin{aligned} J(\alpha, \beta, \gamma; x_0, \eta) &= \frac{1}{2} \log \frac{(\beta + (\eta^{-1}/2))(\gamma - \beta - (\eta^{-1}/2))}{(\alpha + \beta - \gamma)(\alpha + \beta - \gamma + \eta^{-1})} \\ &\quad + \frac{1}{2} \log(x_0 - 1) + O(x_0 - 1), \end{aligned} \tag{90}$$

$$\begin{aligned} K(\alpha, \beta, \gamma; x_0, \eta) &= \frac{1}{2} \log \frac{(\alpha + \beta - \gamma)(\alpha + \beta - \gamma - \eta^{-1})}{(\gamma - \alpha + (\eta^{-1}/2))(\gamma - \beta + (\eta^{-1}/2))} \\ &\quad - \frac{1}{2} \log(x_0 - 1) + O(x_0 - 1), \end{aligned} \tag{91}$$

and the indefinite integral of S_{-1} can be computed explicitly as follows:

$$\begin{aligned}
&\int^x S_{-1}(\alpha,\beta,\gamma;x,\eta)\,dx \\
&\quad= \frac{1}{2}\Big\{\gamma\log\frac{2\gamma^2+px+2\gamma\sqrt{\delta^2x^2+px+\gamma^2}}{x} \\
&\qquad -\sqrt{\delta^2+p+\gamma^2}\log\frac{p+px+2\gamma^2+2\delta^2x+2\sqrt{\delta^2+p+\gamma^2}\sqrt{\delta^2x^2+px+\gamma^2}}{x-1} \\
&\qquad +\delta\log\big(p+2\delta^2x+2\delta\sqrt{\delta^2x^2+px+\gamma^2}\big)\Big\}. \qquad (92)
\end{aligned}$$

Here $\delta=\beta-\alpha$. Hence we can evaluate the contour integral of S_{-1} on C_{x_0}:

$$\begin{aligned}
&\frac{1}{2}\int_{C_{x_0}} S_{-1}(\alpha,\beta,\gamma;x,\eta)\,dx \\
&\quad= \frac{\gamma}{4}\log\frac{2\gamma^2+px_0+e^{\pi i}2\gamma\sqrt{\delta^2x_0^2+px_0+\gamma^2}}{2\gamma^2+px_0+2\gamma\sqrt{\delta^2x_0^2+px_0+\gamma^2}} \\
&\qquad -\frac{\sqrt{\delta^2+p+\gamma^2}}{4}\log\frac{p+px_0+2\gamma^2+2\delta^2x_0+e^{\pi i}2\sqrt{\delta^2+p+\gamma^2}\sqrt{\delta^2x_0^2+px_0+\gamma^2}}{p+px_0+2\gamma^2+2\delta^2x_0+2\sqrt{\delta^2+p+\gamma^2}\sqrt{\delta^2x_0^2+px_0+\gamma^2}} \\
&\qquad +\frac{\delta}{4}\log\frac{p+2\delta^2x_0+e^{\pi i}2\delta\sqrt{\delta^2x_0^2+px_0+\gamma^2}}{p+2\delta^2x_0+2\delta\sqrt{\delta^2x_0^2+px_0+\gamma^2}}. \qquad (93)
\end{aligned}$$

Let $M_1(\alpha,\beta,\gamma;x_0,\eta)$, $M_2(\alpha,\beta,\gamma;x_0,\eta)$ and $M_3(\alpha,\beta,\gamma;x_0,\eta)$ be the first term, the second term and the third term of right-hand side of (93), respectively. We investigate the asymptotic behavior for $I_{-1}(\alpha,\beta,\gamma;x_0,\eta)$ as $x_0\to 1$. We compute the difference of M_1 in α variable to obtain

$$\begin{aligned}
&M_1(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_1(\alpha,\beta,\gamma;x_0,\eta) \\
&\quad= \frac{\gamma}{4}\log\frac{(2\gamma^2+\hat{p}+e^{\pi i}2\gamma\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2})(2\gamma^2+p+2\gamma\sqrt{\delta^2+p+\gamma^2})}{(2\gamma^2+p+e^{\pi i}2\gamma\sqrt{\delta^2+p+\gamma^2})(2\gamma^2+\hat{p}+2\gamma\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2})}+O(x_0-1), \qquad (94)
\end{aligned}$$

where we set $\hat{p}=2(2(\alpha+\eta^{-1})\beta-(\alpha+\eta^{-1})\gamma-\beta\gamma)$, $\hat{\delta}=\beta-\alpha-\eta^{-1}$. Similarly, we have

$$\begin{aligned}
&M_2(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_2(\alpha,\beta,\gamma;x_0,\eta) \\
&\quad= -\frac{\sqrt{\delta^2+p+\gamma^2}}{4}\log\frac{\big(4\hat{\delta}^2(\hat{\delta}^2+\hat{p}+\gamma^2)-(2\hat{\delta}^2+\hat{p})^2\big)(\delta^2+p+\gamma^2)^2}{\big(4\delta^2(\delta^2+p+\gamma^2)-(2\delta^2+p)^2\big)\big(\hat{\delta}^2+\hat{p}+\gamma^2\big)^2} \\
&\qquad +\frac{\eta^{-1}}{4}\log\frac{e^{\pi i}\big(4\hat{\delta}^2(\hat{\delta}^2+\hat{p}+\gamma^2)-(2\hat{\delta}^2+\hat{p})^2\big)}{16\big(\hat{\delta}^2+\hat{p}+\gamma^2\big)^2}+\frac{\eta^{-1}}{4}\log(1-x_0)+O(x_0-1), \qquad (95)
\end{aligned}$$

$$\begin{aligned}
&M_3(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_3(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{\delta}{4}\log\frac{\left(p+2\delta^2+2\delta\sqrt{\delta^2+p+\gamma^2}\right)\left(\hat{p}+2\hat{\delta}^2+e^{\pi i}2\hat{\delta}\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2}\right)}{\left(\hat{p}+2\hat{\delta}^2+2\hat{\delta}\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2}\right)\left(p+2\delta^2+e^{\pi i}2\delta\sqrt{\delta^2+p+\gamma^2}\right)}\\
&\qquad-\frac{\eta^{-1}}{4}\log\frac{\hat{p}+2\hat{\delta}^2+e^{\pi i}2\hat{\delta}\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2}}{\hat{p}+2\hat{\delta}^2+2\hat{\delta}\sqrt{\hat{\delta}^2+\hat{p}+\gamma^2}}+O(x_0-1).
\end{aligned}\tag{96}$$

In our computation, we have used reductions:

$$p+2\delta^2+2\delta\sqrt{\delta^2+p+\gamma^2}=4e^{-\pi i}\alpha(\gamma-\alpha),\tag{97}$$

$$p+2\delta^2-2\delta\sqrt{\delta^2+p+\gamma^2}=4e^{-\pi i}\beta(\gamma-\beta),\tag{98}$$

$$p+2\gamma^2-2\gamma\sqrt{\delta^2+p+\gamma^2}=4\alpha\beta,\tag{99}$$

$$p+2\gamma^2+2\gamma\sqrt{\delta^2+p+\gamma^2}=4(\gamma-\alpha)(\gamma-\beta).\tag{100}$$

Hence we have

$$M_1(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_1(\alpha,\beta,\gamma;x_0,\eta)=\frac{\gamma}{4}\log\frac{(\gamma-\alpha)(\alpha+\eta^{-1})}{\alpha(\gamma-\alpha-\eta^{-1})}+O(x_0-1),\tag{101}$$

$$\begin{aligned}
&M_2(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_2(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{\alpha+\beta-\gamma}{4}\log\frac{(\alpha+\beta-\gamma)^4(\alpha+\eta^{-1})(\gamma-\alpha-\eta^{-1})}{\alpha(\gamma-\alpha)(\alpha+\eta^{-1}+\beta-\gamma)^4}\\
&\qquad+\frac{\eta^{-1}}{4}\log\frac{\beta(\gamma-\beta)(\alpha+\eta^{-1})(\gamma-\alpha-\eta^{-1})}{(\alpha+\eta^{-1}+\beta-\gamma)^4}+\frac{\eta^{-1}}{2}\log(x_0-1)+O(x_0-1),
\end{aligned}\tag{102}$$

$$\begin{aligned}
&M_3(\alpha+\eta^{-1},\beta,\gamma;x_0,\eta)-M_3(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{\beta-\alpha}{4}\log\frac{\alpha(\gamma-\alpha)}{(\alpha+\eta^{-1})(\gamma-\alpha-\eta^{-1})}\\
&\qquad-\frac{\eta^{-1}}{4}\log\frac{\beta(\gamma-\beta)}{(\alpha+\eta^{-1})(\gamma-\alpha-\eta^{-1})}+O(x_0-1).
\end{aligned}\tag{103}$$

Then we obtain

$$\begin{aligned}
&I_{-1}(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{1}{2}\big\{\eta^{-1}\log(x_0-1)-\alpha\log\alpha+(\alpha+\eta^{-1})\log(\alpha+\eta^{-1})\\
&\qquad+(\gamma-\alpha)\log(\gamma-\alpha)-(\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})\\
&\qquad+2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)-2(\alpha+\eta^{-1}+\beta-\gamma)\log(\alpha+\eta^{-1}+\beta-\gamma)\big\}\\
&\quad\;+O(x_0-1).
\end{aligned}\tag{104}$$

To obtain the asymptotic behavior for $J_{-1}(\alpha,\beta,\gamma;x_0,\eta)$ as $x_0 \to 1$, we exchange α and $e^{\pi i}$ in (104) for β and $e^{-\pi i}$, respectively:

$$\begin{aligned}
&J_{-1}(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{1}{2}\bigl\{\eta^{-1}\log(x_0-1)+\beta\log\beta+(\beta+\eta^{-1})\log(\beta+\eta^{-1})\\
&\qquad\quad+(\gamma-\beta)\log(\gamma-\beta)-(\gamma-\beta-\eta^{-1})\log(\gamma-\beta-\eta^{-1})\\
&\qquad\quad+2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)-2(\alpha+\beta+\eta^{-1}-\gamma)\log(\alpha+\beta+\eta^{-1}-\gamma)\bigr\}\\
&\qquad+O(x_0-1).
\end{aligned}\tag{105}$$

In a similar manner, we have

$$\begin{aligned}
&K_{-1}(\alpha,\beta,\gamma;x_0,\eta)\\
&\quad=\frac{1}{2}\bigl\{-\eta^{-1}\log(x_0-1)+(\gamma-\alpha)\log(\gamma-\alpha)-(\gamma-\alpha+\eta^{-1})\log(\gamma-\alpha+\eta^{-1})\\
&\qquad\quad+(\gamma-\beta)\log(\gamma-\beta)-(\gamma-\beta+\eta^{-1})\log(\gamma-\beta+\eta^{-1})\\
&\qquad\quad+(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)-2(\alpha+\beta-\gamma-\eta^{-1})\log(\alpha+\beta-\gamma-\eta^{-1})\bigr\}\\
&\qquad+O(x_0-1).
\end{aligned}\tag{106}$$

Similarly, we can compute the asymptotic behavior of I, J, K, I_{-1}, J_{-1} and K_{-1} near $x=0$ (resp. $x=\infty$). Hence we have the following

Proposition 2.5. *The Voros coefficient V_1 satisfies the following system of difference equations as a formal power series solution in η^{-1}:*

$$\begin{aligned}
&V_1(\alpha+\eta^{-1},\beta,\gamma;\eta)-V_1(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{(\alpha+(\eta^{-1}/2))(\gamma-\alpha-(\eta^{-1}/2))}{(\alpha+\beta-\gamma)(\alpha+\beta-\gamma+\eta^{-1})}\\
&\qquad+\frac{\eta}{2}\bigl\{\alpha\log\alpha-(\alpha+\eta^{-1})\log(\alpha+\eta^{-1})-(\gamma-\alpha)\log(\gamma-\alpha)\\
&\qquad\quad+(\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})-2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)\\
&\qquad\quad+2(\alpha+\eta^{-1}+\beta-\gamma)\log(\alpha+\eta^{-1}+\beta-\gamma)\bigr\},
\end{aligned}\tag{107}$$

$$\begin{aligned}
&V_1(\alpha,\beta+\eta^{-1},\gamma;\eta)-V_1(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{(\beta+(\eta^{-1}/2))(\gamma-\beta-(\eta^{-1}/2))}{(\alpha+\beta-\gamma)(\alpha+\beta-\gamma+\eta^{-1})}\\
&\qquad+\frac{\eta}{2}\bigl\{\beta\log\beta-(\beta+\eta^{-1})\log(\beta+\eta^{-1})-(\gamma-\beta)\log(\gamma-\beta)\\
&\qquad\quad+(\gamma-\beta-\eta^{-1})\log(\gamma-\beta-\eta^{-1})-2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)\\
&\qquad\quad+2(\alpha+\beta+\eta^{-1}-\gamma)\log(\alpha+\beta+\eta^{-1}-\gamma)\bigr\},
\end{aligned}\tag{108}$$

$$
\begin{aligned}
&V_1(\alpha,\beta,\gamma+\eta^{-1};\eta)-V_1(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{(\alpha+\beta-\gamma)(\alpha+\beta-\gamma-\eta^{-1})}{(\gamma-\alpha+(\eta^{-1}/2))(\gamma-\beta+(\eta^{-1}/2))}\\
&\qquad+\frac{\eta}{2}\bigl\{-(\gamma-\alpha)\log(\gamma-\alpha)+(\gamma-\alpha+\eta^{-1})\log(\gamma-\alpha+\eta^{-1})-(\gamma-\beta)\log(\gamma-\beta)\\
&\qquad\qquad+(\gamma-\beta+\eta^{-1})\log(\gamma-\beta+\eta^{-1})-2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)\\
&\qquad\qquad+2(\alpha+\beta-\gamma-\eta^{-1})\log(\alpha+\beta-\gamma-\eta^{-1})\bigr\}.
\end{aligned}\tag{109}
$$

Similarly, we obtain

PROPOSITION 2.6. *The Voros coefficient V_0 satisfies the following system of difference equations as a formal power series solution in η^{-1}:*

$$
\begin{aligned}
&V_0(\alpha+\eta^{-1},\beta,\gamma;\eta)-V_0(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{\gamma-\alpha-(\eta^{-1}/2)}{\alpha+(\eta^{-1}/2)}\\
&\qquad-\frac{\eta}{2}\bigl\{\alpha\log\alpha-(\alpha+\eta^{-1})\log(\alpha+\eta^{-1})+(\gamma-\alpha)\log(\gamma-\alpha)\\
&\qquad\qquad-(\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})\bigr\},
\end{aligned}\tag{110}
$$

$$
\begin{aligned}
&V_0(\alpha,\beta+\eta^{-1},\gamma;\eta)-V_0(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{\gamma-\beta-(\eta^{-1}/2)}{\beta+(\eta^{-1}/2)}\\
&\qquad-\frac{\eta}{2}\bigl\{\beta\log\beta-(\beta+\eta^{-1})\log(\beta+\eta^{-1})+(\gamma-\beta)\log(\gamma-\beta)\\
&\qquad\qquad-(\gamma-\beta-\eta^{-1})\log(\gamma-\beta-\eta^{-1})\bigr\},
\end{aligned}\tag{111}
$$

$$
\begin{aligned}
&V_0(\alpha,\beta,\gamma+\eta^{-1};\eta)-V_0(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{\gamma(\gamma+\eta^{-1})}{(\gamma-\alpha+(\eta^{-1}/2))(\gamma-\beta+(\eta^{-1}/2))}\\
&\qquad-\frac{\eta}{2}\bigl\{(\gamma-\alpha)\log(\gamma-\alpha)-(\gamma-\alpha+\eta^{-1})\log(\gamma-\alpha+\eta^{-1})\\
&\qquad\qquad+(\gamma-\beta)\log(\gamma-\beta)-(\gamma-\beta+\eta^{-1})\log(\gamma-\beta+\eta^{-1})\\
&\qquad\qquad-2\gamma\log\gamma+2(\gamma+\eta^{-1})\log(\gamma+\eta^{-1})\bigr\}.
\end{aligned}\tag{112}
$$

PROPOSITION 2.7. *The Voros coefficient V_2 satisfies the following system of difference equations as a formal power series solution in η^{-1}:*

$$
\begin{aligned}
&V_2(\alpha+\eta^{-1},\beta,\gamma;\eta)-V_2(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{(\beta-\alpha)(\beta-\alpha-\eta^{-1})}{(\alpha+(\eta^{-1}/2))(\gamma-\alpha-(\eta^{-1}/2))}\\
&\qquad+\frac{\eta}{2}\big\{-\alpha\log\alpha+(\alpha+\eta^{-1})\log(\alpha+\eta^{-1})+(\gamma-\alpha)\log(\gamma-\alpha)\\
&\qquad\quad-(\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})-2(\beta-\alpha)\log(\beta-\alpha)\\
&\qquad\quad+2(\beta-\eta^{-1}-\alpha)\log(\beta-\eta^{-1}-\alpha)\big\},
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
&V_2(\alpha,\beta+\eta^{-1},\gamma;\eta)-V_2(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{(\beta+(\eta^{-1}/2))(\gamma-\beta-(\eta^{-1}/2))}{(\beta-\alpha)(\beta-\alpha+\eta^{-1})}\\
&\qquad+\frac{\eta}{2}\big\{\beta\log\beta-(\beta+\eta^{-1})\log(\beta+\eta^{-1})-(\gamma-\beta)\log(\gamma-\beta)\\
&\qquad\quad+(\gamma-\beta-\eta^{-1})\log(\gamma-\beta-\eta^{-1})-2(\beta-\alpha)\log(\beta-\alpha)\\
&\qquad\quad+2(\beta+\eta^{-1}-\alpha)\log(\beta+\eta^{-1}-\alpha)\big\},
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
&V_2(\alpha,\beta,\gamma+\eta^{-1};\eta)-V_2(\alpha,\beta,\gamma;\eta)\\
&\quad=\frac{1}{2}\log\frac{\gamma-\alpha+(\eta^{-1}/2)}{\gamma-\beta+(\eta^{-1}/2)}\\
&\qquad+\frac{\eta}{2}\big\{(\gamma-\alpha)\log(\gamma-\alpha)-(\gamma+\eta^{-1}-\alpha)\log(\gamma+\eta^{-1}-\alpha)\\
&\qquad\quad-(\gamma-\beta)\log(\gamma-\beta)+(\gamma+\eta^{-1}-\beta)\log(\gamma+\eta^{-1}-\beta)\big\}.
\end{aligned}
\tag{115}
$$

Proposition 2.8. *The system of difference equations* (110), (111), (112) (*resp.* (107), (108), (109), *resp.* (113), (114), (115)) *satisfies the compatibility conditions and it has a unique formal power series solution* V_0 (*resp.* V_1, *resp.* V_2) *in* η^{-1} *without constant term which is homogeneous in* $(\alpha,\beta,\gamma;\eta^{-1})$.

Proof. Since V_1 (resp. V_0, resp. V_2) satisfies (107), (108), (109) (resp. (110), (111), (112), resp. (113), (114), (115)), the compatibility conditions trivially hold. We consider the following equation

$$V(\alpha+\eta^{-1},\beta,\gamma;\eta)-V(\alpha,\beta,\gamma;\eta)=0. \tag{116}$$

Assume that V has an expansion of the form:

$$V(\alpha,\beta,\gamma;\eta)=\sum_{n=1}^{\infty}v_n(\alpha,\beta,\gamma)\eta^{-n}, \tag{117}$$

where v_n is a homogeneous rational function of order $-n$ ($n=1,2,3,\dots$). Since the first term of the left-hand side of (116) has an expansion

$$V(\alpha+\eta^{-1},\beta,\gamma;\eta)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\partial_\alpha^k v_n(\alpha,\beta,\gamma)}{\eta^{k+n}k!}=\sum_{l=1}^{\infty}\left(\sum_{k+n=l}\frac{\partial_\alpha^k v_n(\alpha,\beta,\gamma)}{\eta^l k!}\right), \tag{118}$$

we obtain

$$V(\alpha+\eta^{-1},\beta,\gamma;\eta)-V(\alpha,\beta,\gamma;\eta)=\sum_{l=1}^{\infty}\sum_{k=1}^{l}\frac{\partial_\alpha^k v_{l-k}(\alpha,\beta,\gamma)}{k!}\frac{1}{\eta^l}=0. \tag{119}$$

Hence v_n is a function which is independent of α. In a similar way, we can prove that v_n does not depend on β and γ. Since v_n is homogeneous of order $-n$ in (α,β,γ), we have $v_n\equiv 0$. This completes the proof of Proposition 2.8. □

We consider V_1. In our computation, we tentatively admit formal power series in η^{-1} starting from the first power:

$$\sum_{n=-1}^{\infty} v_n(\alpha,\beta,\gamma)\eta^{-n} \tag{120}$$

with the coefficients v_{-1} and v_0 containing logarithms of α, β, γ. These extra terms will disappear in the final result. The right-hand side of (107) can be written in the form

$$f(\alpha,\beta,\gamma;\eta)+g(\alpha,\beta,\gamma;\eta) \tag{121}$$

with

$$f(\alpha,\beta,\gamma;\eta)=\frac{1}{2}\log\frac{(\alpha+(\eta^{-1}/2))(\gamma-\alpha-(\eta^{-1}/2))}{(\alpha+\beta-\gamma)(\alpha+\beta-\gamma+\eta^{-1})} \tag{122}$$

and

$$\begin{aligned} g(\alpha,\beta,\gamma;\eta)=\frac{\eta}{2}\big\{&\alpha\log\alpha-(\alpha+\eta^{-1})\log(\alpha+\eta^{-1})-(\gamma-\alpha)\log(\gamma-\alpha)\\ &+(\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})-2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)\\ &+2(\alpha+\eta^{-1}+\beta-\gamma)\log(\alpha+\eta^{-1}+\beta-\gamma)\big\}. \end{aligned} \tag{123}$$

Then (107) is decomposed into the following two equations:

$$V_{11}(\alpha+\eta^{-1},\beta,\gamma;\eta)-V_{11}(\alpha,\beta,\gamma;\eta)=f(\alpha,\beta,\gamma;\eta), \tag{124}$$

$$V_{12}(\alpha+\eta^{-1},\beta,\gamma;\eta)-V_{12}(\alpha,\beta,\gamma;\eta)=g(\alpha,\beta,\gamma;\eta). \tag{125}$$

If we find solutions V_{11} and V_{12}, then $V_1=V_{11}+V_{12}$ satisfies (107). We can easily solve (125):

$$V_{12}(\alpha,\beta,\gamma;\eta)$$
$$= \frac{\eta}{2}(-\alpha\log\alpha + (\gamma-\alpha)\log(\gamma-\alpha) + 2(\alpha+\beta-\gamma)\log(\alpha+\beta-\gamma)). \tag{126}$$

To solve (124), we employ an idea developed by Candelpergher–Coppo–Delabaere [**5**]. We rewrite the left-hand side of (124) as follows:

$$V_{11}(\alpha+\eta^{-1},\beta,\gamma;\eta) - V_{11}(\alpha,\beta,\gamma;\eta) = (e^{\eta^{-1}\partial_\alpha}-1)V_{11}(\alpha,\beta,\gamma;\eta). \tag{127}$$

Here $\partial_\alpha = \partial/\partial\alpha$. The inverse of the difference operator $e^{\eta^{-1}\partial_\alpha}-1$ can be expanded in the form

$$(e^{\eta^{-1}\partial_\alpha}-1)^{-1} = \eta\partial_\alpha^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}\eta^{-n}\partial_\alpha^n, \tag{128}$$

where B_n denotes n-th Bernoulli number and ∂_α^{-1} the indefinite integration operator in α. Using operator (128), we have a solution of (124):

$$V_{11}(\alpha,\beta,\gamma;\eta) = (e^{\eta^{-1}\partial_\alpha}-1)^{-1}f(\alpha,\beta,\gamma;\eta^{-1}) + f_0(\beta,\gamma;\eta^{-1}), \tag{129}$$

where $f_0(\beta,\gamma)$ is an arbitrary formal power series in η^{-1} which does not depend on α. To compute the first term of the right-hand side of (129), we use the following lemma.

Lemma 2.9. *We have the following two formulas*:

$$\partial_\alpha(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\log\left(1+\frac{1}{\alpha\eta}\right) = \frac{1}{\alpha}, \tag{130}$$

$$\partial_\alpha(e^{\eta^{-1}\partial_\alpha/2}-1)^{-1}\log\left(1+\frac{1}{2\eta\alpha}\right) = \frac{1}{\alpha}. \tag{131}$$

Proof. The first formula immediately follows from

$$\partial_\alpha\log\left(1+\frac{1}{\eta\alpha}\right) = \frac{1}{\alpha+\eta^{-1}} - \frac{1}{\alpha} = (e^{\eta^{-1}\partial_\alpha}-1)\frac{1}{\alpha}. \tag{132}$$

Similarly, we have (131). □

We consider the α-derivative of V_{11}:

$$\partial_\alpha V_{11} = -\partial_\alpha(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\frac{1}{2}\biggl\{\log(\alpha+\beta-\gamma) + \log(\alpha+\beta-\gamma+\eta^{-1})$$
$$- \log\left(\alpha+\frac{\eta^{-1}}{2}\right) - \log\left(\gamma-\alpha-\frac{\eta^{-1}}{2}\right)\biggr\}. \tag{133}$$

The right-hand side can be written in the form:

$$\begin{aligned}
&\frac{\partial_\alpha}{2}(e^{\eta^{-1}\partial_\alpha}-1)^{-1}(-2\log(\alpha+\beta-\gamma)+\log\alpha+\log(\gamma-\alpha))\\
&\quad+\frac{\partial_\alpha}{2}(e^{(1/2)\eta^{-1}\partial_\alpha}+1)^{-1}(e^{(1/2)\eta^{-1}\partial_\alpha}-1)^{-1}\left(\log\left(1+\frac{1}{2\alpha\eta}\right)+\log\left(1-\frac{1}{2(\gamma-\alpha)\eta}\right)\right)\\
&\quad-\frac{\partial_\alpha}{2}(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\log\left(1+\frac{1}{\eta(\alpha+\beta-\gamma)}\right).
\end{aligned}\tag{134}$$

The first term turns out to be

$$\begin{aligned}
&-\frac{\eta}{2}\sum_{n=0}^{\infty}\frac{(-1)^nB_n}{n!}\eta^{-n}\partial_\alpha^n\left(2\log(\alpha+\beta-\gamma)-\log\alpha-\log(\gamma-\alpha)\right)\\
&\quad=-\frac{\eta}{2}(2\log(\alpha+\beta-\gamma)-\log\alpha-\log(\gamma-\alpha))+\frac{B_1}{2}\left(-\frac{2}{\alpha+\beta-\gamma}+\frac{1}{\alpha}-\frac{1}{\gamma-\alpha}\right)\\
&\qquad+\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n}\eta^{1-n}\left(-\frac{2}{(\alpha+\beta-\gamma)^n}+\frac{1}{\alpha^n}-\frac{1}{(\gamma-\alpha)^n}\right).
\end{aligned}\tag{135}$$

Using the identity

$$(e^{(1/2)\eta^{-1}\partial_\alpha}+1)^{-1}=(e^{(1/2)\eta^{-1}\partial_\alpha}-1)^{-1}-2(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\tag{136}$$

and Lemma 2.9, we can rewrite the second and third terms of (134) in the form:

$$\begin{aligned}
&-\frac{1}{2}\big((e^{(1/2)\eta^{-1}\partial_\alpha}-1)^{-1}-2(e^{\eta^{-1}\partial_\alpha}-1)^{-1}\big)\left(-\frac{1}{\alpha}+\frac{1}{\gamma-\alpha}\right)-\frac{1}{2(\alpha+\beta-\gamma)}\\
&\quad=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nB_n}{n!}\eta^{1-n}(2^{1-n}-2)\partial_\alpha^{n-1}\left(-\frac{1}{\alpha}-\frac{1}{\gamma-\alpha}\right)+\frac{1}{2(\alpha+\beta-\gamma)}\\
&\quad=-\frac{B_1}{2}\left(-\frac{1}{\alpha}+\frac{1}{\gamma-\alpha}\right)-\frac{1}{2(\alpha+\beta-\gamma)}\\
&\qquad-\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n}\eta^{1-n}(2^{1-n}-2)\left(\frac{1}{\alpha^n}+\frac{1}{(\gamma-\alpha)^n}\right).
\end{aligned}\tag{137}$$

Hence we obtain

$$\begin{aligned}
\partial_\alpha V_{11}(\alpha,\beta,\gamma)=-\frac{1}{2}\bigg\{&\eta(2\log(\alpha+\beta-\gamma)-\log\alpha-\log(\gamma-\alpha))\\
&+\sum_{n=2}^{\infty}\frac{B_n}{n}\eta^{1-n}\left((2^{1-n}-1)\left(\frac{1}{\alpha^n}+\frac{1}{(\gamma-\alpha)^n}\right)-\frac{2}{(\alpha+\beta-\gamma)^n}\right)\bigg\}.
\end{aligned}\tag{138}$$

Thus we have

$$V_1(\alpha,\beta,\gamma;\eta) = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n(n-1)}\eta^{1-n} \times \left(\left(1-\frac{1}{2^{n-1}}\right)\left(\frac{1}{\alpha^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^n}\right)+f_0(\beta,\gamma). \tag{139}$$

Here f_0 is an arbitrary formal power series in η^{-1} whose coefficients depend only on β and γ. Solving (108), we have

$$V_1(\alpha,\beta,\gamma;\eta) = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)} \times \left((1-2^{1-n})\left(\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}+\frac{2}{(\alpha+\beta-\gamma)^n}\right)\right)+f_1(\alpha,\gamma), \tag{140}$$

where $f_1(\alpha,\gamma)$ is an arbitrary formal power series in η^{-1} whose coefficients depend only on α and γ. On the other hand, solving (109), we have

$$V_1(\alpha,\beta,\gamma;\eta) = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)} \times \left((1-2^{1-n})\left(-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^n}\right) + f_2(\alpha,\beta), \tag{141}$$

where $f_2(\alpha,\beta)$ is an arbitrary formal power series in η^{-1} whose coefficients depend only on α and β. Combining (139), (140) and (141) yields (46). In a similar manner, we obtain V_0 and V_2. This completes the proof of Theorem 2.3. □

In the above computation, we have assumed that (α,β,γ) is contained in ω_2. For the other cases, the same discussion works and the explicit forms of V_j are the same as those given in Theorem 2.3 if the branch of $S_{-1}(x)=\sqrt{Q_0(x)}$ is specified as (36), (37), (38). Note that for the choice of the branch in each case, we have to place the branch cut suitably. Some examples of the branch cuts are shown by wavy curves in Figures 2.4–2.8. For later use, we denote by L_1 the branch cut shown by the wavy curve in Figure 2.5.

3. Borel sums of the Voros coefficients.

In this section, we examine the Borel summability of V_j in ω_h and compute the Borel sums V_j^h $(j=0,1,2)$ of the Voros coefficients V_j in ω_h $(h=1,\ldots,4)$. We use the branch of S_{-1} defined by (36), (37) and (38). The following theorem has been announced in [**2**]. Note that the choice of the branch of S_{-1} at the origin in [**2**] is opposite of the current choice.

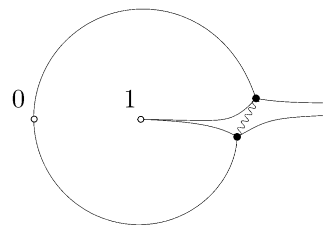

Figure 2.4. $(\alpha, \beta, \gamma) = (0.5, 1 - \epsilon - \delta i, 1)$ in ω_1.

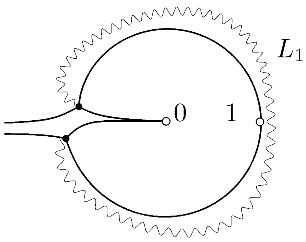

Figure 2.5. $(1 - \epsilon + \delta i, 2, 1)$ in ω_1.

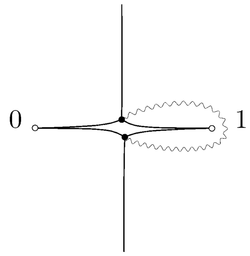

Figure 2.6. $(\epsilon + \delta i, 2, 1)$ in ω_1.

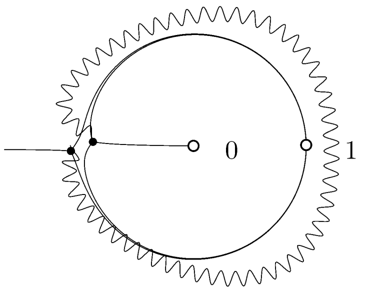

Figure 2.7. $(\alpha, \beta, \gamma) = (1 + \epsilon + \delta i, 1)$ in ω_3.

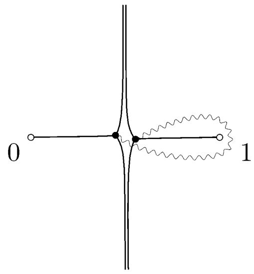

Figure 2.8. $(-\epsilon + \delta i, 2, 1)$ in ω_4.

THEOREM 3.1. *For each $j = 0, 1, 2$ and $h = 1, \ldots, 4$, the Voros coefficient V_j is Borel summable in ω_h. The Borel sums V_j^h of V_j in ω_h have the following forms*:

$$V_0^1 = \frac{1}{2} \log \frac{\Gamma(1/2 + (\beta - \gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)\Gamma(1/2 + (\gamma - \alpha)\eta)(\beta - \gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}, \tag{142}$$

$$V_0^2 = \frac{1}{2} \log \frac{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma-\alpha)\eta}(\gamma - \beta)^{(\gamma-\beta)\eta}2\pi\eta}{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)\Gamma(1/2 + (\gamma - \alpha)\eta)\Gamma(1/2 + (\gamma - \beta)\eta)\gamma^{2\eta\gamma-1}}, \tag{143}$$

$$V_0^3 = \frac{1}{2} \log \frac{\Gamma(1/2 + (\alpha - \gamma)\eta)\Gamma(1/2 + (\beta - \gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)(\alpha - \gamma)^{(\alpha-\gamma)\eta}(\beta - \gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}, \tag{144}$$

$$V_0^4 = \frac{1}{2} \log \frac{\Gamma(1/2 - \alpha\eta)\Gamma(1/2 + (\beta - \gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2 + \beta\eta)\Gamma(1/2 + (\gamma - \alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta - \gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}, \tag{145}$$

$$V_1^1 = \frac{1}{2} \log \frac{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)\Gamma(1/2 + (\beta - \gamma)\eta)(\gamma - \alpha)^{(\gamma-\alpha)\eta}(\alpha + \beta - \gamma)^{2(\alpha+\beta-\gamma)\eta-1}}{\Gamma(1/2 + (\gamma - \alpha)\eta)\Gamma^2((\alpha + \beta - \gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\beta - \gamma)^{(\beta-\gamma)\eta}\eta}, \tag{146}$$

$$V_1^2 = \frac{1}{2} \log \frac{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)(\gamma - \alpha)^{(\gamma-\alpha)\eta}(\gamma - \beta)^{(\gamma-\beta)\eta}(\alpha + \beta - \gamma)^{2(\alpha+\beta-\gamma)\eta-1}2\pi}{\Gamma(1/2 + (\gamma - \alpha)\eta)\Gamma(1/2 + (\gamma - \beta)\eta)\Gamma^2((\alpha + \beta - \gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}, \tag{147}$$

$$V_1^3 = \frac{1}{2} \log \frac{\Gamma(1/2 + \alpha\eta)\Gamma(1/2 + \beta\eta)\Gamma(1/2 + (\alpha - \gamma)\eta)\Gamma(1/2 + (\beta - \gamma)\eta)(\alpha + \beta - \gamma)^{2(\alpha+\beta-\gamma)\eta-1}}{2\pi\Gamma^2((\alpha + \beta - \gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\alpha - \gamma)^{(\alpha-\gamma)\eta}(\beta - \gamma)^{(\beta-\gamma)\eta}\eta}, \tag{148}$$

$$V_1^4 = \frac{1}{2}\log\frac{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\beta-\gamma)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}2\pi}{\Gamma(1/2-\alpha\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}, \tag{149}$$

$$V_2^1 = \frac{1}{2}\log\frac{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\Gamma(1/2+(\beta-\gamma)\eta)\alpha^{\alpha\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma(1/2+\alpha\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}, \tag{150}$$

$$V_2^2 = \frac{1}{2}\log\frac{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\alpha^{\alpha\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}2\pi}{\Gamma(1/2+\alpha\eta)\Gamma(1/2+(\gamma-\beta)\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \tag{151}$$

$$V_2^3 = \frac{1}{2}\log\frac{2\pi\Gamma(1/2+\beta\eta)\Gamma(1/2+(\beta-\gamma)\eta)\alpha^{\alpha\eta}(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma(1/2+\alpha\eta)\Gamma(1/2+(\alpha-\gamma)\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}, \tag{152}$$

$$V_2^4 = \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\Gamma(1/2+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}. \tag{153}$$

Proof. Our proof follows from the proof of Theorem 2.1 in [**14**], but the computation is slightly more complicated than that of [**14**]. We now compute the Borel sums V_0^h of V_0. To find the Borel sums V_0^h, we first take the Borel transform $V_{0,B}(\alpha,\beta,\gamma;y)$ of V_0. By the definition, we have

$$\begin{aligned} V_{0,B}(\alpha,\beta,\gamma;y) &= \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n y^{n-2}}{n!} \\ &\quad\times\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\}. \end{aligned} \tag{154}$$

We use the following functions:

$$\begin{aligned} \tilde{g}(t) &= \sum_{n=2}^{\infty}\frac{B_n}{n!}\left(\frac{y}{t}\right)^n = \frac{y/t}{\exp(y/t)-1} - 1 + \frac{y}{2t} \\ &= \frac{y}{2t}\left(\frac{1}{\exp(y/2t)-1} - \frac{1}{\exp(y/2t)+1}\right) - 1 + \frac{y}{2t}, \end{aligned} \tag{155}$$

$$g_0(t) = \tilde{g}(t)\frac{t}{y^2} = \frac{1}{y}\left(\frac{1}{\exp(y/t)-1} + \frac{1}{2} - \frac{t}{y}\right) \tag{156}$$

and

$$g_1(t) = \frac{1}{\exp(y/2t)-1} + \frac{1}{\exp(y/2t)+1} - \frac{2t}{y}. \tag{157}$$

Then we have

$$V_{0,B}(\alpha,\beta,\gamma;y) = -\frac{1}{4y}\{g_1(\alpha)+g_1(\beta)+g_1(\gamma-\alpha)+g_1(\gamma-\beta)\} + g_0(\gamma). \tag{158}$$

Simplifying computation, we introduce the following auxiliary infinite series:

$$\tilde{V}_0 = V_0 + \mu(\alpha) + \mu(\beta) + \mu(\gamma - \alpha) + \mu(\gamma - \beta), \tag{159}$$

where

$$\mu(t) = -\frac{1}{4} + \frac{t\eta}{2}\log\left(1 + \frac{1}{2t\eta}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n+2}(-2t\eta)^{-(n+1)}. \tag{160}$$

The Borel transform $\mu_B(t; y)$ of $\mu(t)$ is

$$\mu_B(t; y) = -\frac{1}{8t}\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}\left(-\frac{y}{2t}\right)^n = -\frac{t}{4y^2}\left\{\left(-\frac{y}{t} - 2\right)\exp\left(-\frac{y}{2t}\right) + 2\right\}. \tag{161}$$

The Borel transform $\tilde{V}_{0,B}$ of $\tilde{V}_0$ is related to $V_{0,B}$ by

$$\tilde{V}_{0,B} = V_{0,B} + \mu_B(\alpha) + \mu_B(\beta) + \mu_B(\gamma - \alpha) + \mu_B(\gamma - \beta). \tag{162}$$

In view of (162), we define

$$g(t) = \frac{1}{4y}\left\{\left(\frac{1}{\exp(y/2t) - 1} + \frac{1}{\exp(y/2t) + 1}\right) - \left(1 + \frac{2t}{y}\right)\exp\left(-\frac{y}{2t}\right)\right\}. \tag{163}$$

The function $g(t)$ rewrites as follows:

$$g(t) = \frac{1}{2y}\exp\left(-\frac{y}{2t}\right)\left(\frac{1}{\exp(y/t) - 1} + \frac{1}{2} - \frac{t}{y}\right).$$

Then we have

$$\tilde{V}_{0,B} = -g(\alpha) - g(\beta) - g(\gamma - \alpha) - g(\gamma - \beta) + g_0(\gamma). \tag{164}$$

We can compute the Borel transforms $\tilde{V}_{1,B}$ of $\tilde{V}_1$ and $\tilde{V}_{2,B}$ of $\tilde{V}_2$ in a similar manner as the computation of the Borel transform $\tilde{V}_{0,B}$ of $\tilde{V}_0$ and we have the following proposition.

PROPOSITION 3.2. *The Borel transforms $\tilde{V}^1_{j,B}$ of the Voros coefficients $\tilde{V}_j$ have the following forms*:

$$\tilde{V}_{0,B} = -g(\alpha) - g(\beta) - g(\gamma - \alpha) - g(\gamma - \beta) + g_0(\gamma), \tag{165}$$

$$\tilde{V}_{1,B} = g(\alpha) + g(\beta) - g(\gamma - \alpha) - g(\gamma - \beta) + g_0(\alpha + \beta - \gamma), \tag{166}$$

$$\tilde{V}_{2,B} = -g(\alpha) - g(\beta) + g(\gamma - \alpha) - g(\gamma - \beta) - g_0(\beta - \alpha). \tag{167}$$

Next we consider the Borel sums V^1_0 of V_0. We use the following integral representation of the logarithm of the Γ-function [**8**].

$$\int_0^\infty \left(\frac{1}{\exp s - 1} + \frac{1}{2} - \frac{1}{s}\right)\frac{\exp(-\theta s)}{s}\, ds = \log\frac{\Gamma(\theta)}{\sqrt{2\pi}} - \left(\theta - \frac{1}{2}\right)\log\theta + \theta, \tag{168}$$

where $\mathrm{Re}\,\theta$ is positive. Let $L(\theta)$ denote the left-hand side of (168). We can compute the Laplace transform

$$\int_0^\infty g(\alpha)\exp(-y\eta)\, dy \tag{169}$$

of $g(\alpha)$ by using (168) if $\mathrm{Re}\,\alpha$ is positive. If $\mathrm{Re}\,\alpha$ is negative, we make use of the relation

$$g_1(\alpha) = -g_1(-\alpha). \tag{170}$$

The Laplace transforms of $g(\beta)$, $g(\gamma-\alpha)$ and $g(\gamma-\beta)$ are obtained by replacing α by β, $\gamma-\alpha$ and $\gamma-\beta$, respectively. Similarly we can compute the Laplace transform of $g_0(\gamma)$ and we have:

$$\begin{aligned}\tilde{V}_0^1 = &-L\left(\frac{1}{2}+\alpha\eta\right) - L\left(\frac{1}{2}+\beta\eta\right) - L\left(\frac{1}{2}+(\gamma-\alpha)\eta\right)\\ &+ L\left(\frac{1}{2}+(\beta-\gamma)\eta\right) + L(\gamma\eta).\end{aligned} \tag{171}$$

Since μ is a convergent power series of η^{-1}, we have

$$V_0^1 = \tilde{V}_0^1 - \mu(\alpha) - \mu(\beta) - \mu(\gamma-\alpha) + \mu(\beta-\gamma). \tag{172}$$

Noting that the right-hand side of (171) can be written in terms of gamma functions and logarithms, we obtain (142). In a similar way, we can compute the Borel sums V_0^2, V_0^3, V_0^4, V_1^h, V_2^h ($h = 1, \ldots, 4$) and we have (142)–(153). This completes the proof of Theorem 3.1. □

4. Parametric Stokes phenomena.

4.1. The relations between the Borel resummed Voros coefficients in adjacent Stokes regions.

First we consider the relations between V_j^1 and $V_j^h (j = 0, 1, 2; h = 2, 3, 4)$. We take an analytic continuation of V_j^1 to $\omega_h (h = 2, 3, 4)$. Using (73), (76) and (77), we have the following theorem which has been announced in [**2**].

Theorem 4.1. *The Borel sums V_j^1 of Voros coefficients V_j can be analytically continued over ω_h $(h = 2, 3, 4)$. The analytic continuations of the Borel sums V_j^1 to ω_h are related to V_j^h as follows*:

$$V_j^1 = V_j^2 - \frac{1}{2}\log(\exp 2(\gamma-\beta)\eta\pi i + 1) \quad (j = 0, 1, 2), \tag{173}$$

$$V_2^1 = V_2^3 - \frac{1}{2}\log(\exp 2(\alpha-\gamma)\eta\pi i + 1), \tag{174}$$

$$V_j^1 = V_j^3 + \frac{1}{2}\log(\exp 2(\alpha-\gamma)\eta\pi i + 1) \quad (j = 0, 1), \tag{175}$$

$$V_1^1 = V_1^4 - \frac{1}{2}\log(\exp(-2\alpha\eta\pi i) + 1), \tag{176}$$

$$V_j^1 = V_j^4 + \frac{1}{2}\log(\exp(-2\alpha\eta\pi i) + 1) \quad (j = 0, 2). \tag{177}$$

Proof. By the explicit forms of V_j^1 given in Theorem 3.1, analytic continuability to ω_h is clear. We give the proof of (173) ($j = 0$) only. Using $\beta - \gamma = (\gamma - \beta)e^{-\pi i}$, we rewrite the analytic continuation of V_0^1 to ω_2 as follows:

$$\begin{aligned} V_0^1 = \frac{1}{2}\log \frac{\Gamma(1/2+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2+\alpha\eta)\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)(\gamma-\beta)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}} \\ + \frac{(\beta-\gamma)\eta\pi i}{2}, \end{aligned} \tag{178}$$

Combining (143) and (178), we obtain

$$V_0^1 - V_0^2 = \frac{1}{2}\log\frac{\Gamma(1/2+(\gamma-\beta)\eta)\Gamma(1/2+(\beta-\gamma)\eta)}{2\pi} + \frac{(\beta-\gamma)\eta\pi i}{2}.$$

In a similar manner, we have (173), (174), (175), (176) and (177). □

Next we examine the Borel summability of Voros coefficients V_j in ω_{hm} ($h - 1, \ldots, 4$; $m = 0, 1, \ldots, 6$) and relate the Borel sum V_j^h of V_j and the Borel sum of V_j in ω_{hm} ($j = 0, 1, 2$). We define the action of $\tau \in G$ on $V_j^h(\alpha, \beta, \gamma; \eta)$ by

$$\tau_* V_j^h(\alpha,\beta,\gamma;\eta) = V_j^h(\tau(\alpha,\beta,\gamma;\eta)). \tag{179}$$

To unify the notation, we denote V_j^{hm} by $V_j^{h\tau}$ for $\tau = \iota_m \in G$ ($m = 0, 1, \ldots, 6$) and V_j^h by $V_j^{h\tau}$ for $\tau = \mathrm{id} \in G$.

Theorem 4.2. *Let τ be an element of G of the form:*

$$\tau = \iota_0^{\epsilon_0}\iota_1^{\epsilon_1}\iota_2^{\epsilon_2} \quad (\epsilon_j = 0 \text{ or } 1). \tag{180}$$

We define $\mathrm{sgn}(\tau, j)$ *by*

$$\begin{cases} \mathrm{sgn}(\tau,0) = (-1)^{\epsilon_0}, \\ \mathrm{sgn}(\tau,j) = (-1)^{\epsilon_0+\epsilon_j} \quad (j = 1, 2). \end{cases} \tag{181}$$

The Borel resummed Voros coefficients $V_j^{h\tau}$ in $\tau(\omega_h)$ are related to $\tau_* V_j^h$ by

$$V_j^{h\tau} = \operatorname{sgn}(\tau, j)\tau_* V_j^h \tag{182}$$

for $j = 0, 1, 2$; $h = 1, 2, 3, 4$.

Proof. We take $\tau = \iota_1$ and compare $V_j^{4\tau} = V_j^{41}$ with $\iota_{1*} V_j^4$. In a similar manner to the computation of V_j^h $(j = 0, 1, 2;\ h = 1, 2, 3, 4)$, we obtain

$$\begin{aligned} V_0^{41} &= V_0^4 \\ &= \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}, \end{aligned} \tag{183}$$

$$V_1^{41} = \frac{1}{2}\log\frac{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2-\alpha\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}, \tag{184}$$

$$\begin{aligned} V_2^{41} &= V_2^4 \\ &= \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\Gamma(1/2+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}. \end{aligned} \tag{185}$$

By the definition, we have

$$\iota_{1*}V_0^4 = \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}, \tag{186}$$

$$\iota_{1*}V_1^4 = \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}{\Gamma(1/2+\beta\eta)\Gamma(1/2+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \tag{187}$$

$$\iota_{1*}V_2^4 = \frac{1}{2}\log\frac{\Gamma(1/2-\alpha\eta)\Gamma(1/2+\beta\eta)\Gamma(1/2+(\gamma-\alpha)\eta)\Gamma(1/2+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}. \tag{188}$$

Comparing the above equations (186), (187) and (188) with (183)–(185), we obtain

$$V_j^{41} = \iota_{1*}V_j^4 \quad (j = 0, 2), \tag{189}$$

$$V_1^{41} = -\iota_{1*}V_1^4. \tag{190}$$

In a similar way, we can obtain the relations for the other cases. □

4.2. Parametric Stokes phenomena for the WKB solution.

We consider the analytic continuation of the WKB solution ψ_+ with respect to the parameters. Stokes phenomena occur for ψ_+ when the triplet of parameter (α, β, γ) crosses E_1 or E_2 (cf. (15), (16)). We call them parametric Stokes phenomena. When (α, β, γ) lies on E_1 or E_2, the Stokes geometry degenerates:

Theorem 4.3 ([**16**, Theorem 3.1]). *We assume that (α, β, γ) is not contained in E_0.*

(i) *If two distinct turning points are connected by a Stokes curve, then* (α, β, γ) *belongs to* E_1.

(ii) *If a Stokes curve forms a closed curve with a single turning point as the base point, then* (α, β, γ) *belongs to* E_2.

Hereafter, we assume that (α, β, γ) is not contained in E_0. There are two distinct turning points a_0, a_1. We consider the case where (α, β, γ) moves from ω_1 to ω_h ($h = 2, 3, 4$). There are two choices of ways when (α, β, γ) goes from ω_1 to ω_2 (resp. ω_3, resp. ω_4) crossing E_1 avoiding E_0 which correspond to the signature of $\operatorname{Im}(\gamma - \beta)$ (resp. $\operatorname{Im}(\alpha - \gamma)$, resp. $\operatorname{Im} \alpha$). We assume that $\operatorname{Im}(\gamma - \beta) > 0$ (resp. $\operatorname{Im}(\alpha - \gamma) > 0$, resp. $\operatorname{Im} \alpha > 0$).

Let us specify the regions in x-space which are surrounded by Stokes curves.

(1) The case where $(\alpha, \beta, \gamma) \in \omega_1$. Then there are three Stokes curves s_{kj} ($j = 0, 1, 2$) emanating from a_k which flow respectively into b_j for $k = 0, 1$. Let $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_1}$, resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$) denote the open set surrounded by s_{00}, s_{01}, s_{10}, s_{11} (resp. by s_{00}, s_{02}, s_{10}, s_{12}, resp. by s_{01}, s_{02}, s_{11}, s_{12}). (cf. Figures 4.1, 4.4, 4.7.)

(2) The case where $(\alpha, \beta, \gamma) \in \omega_2$. There is a unique Stokes curve s_{01} which flows into b_1. We may assume that s_{01} emanates from a_0 and that a_0 is the analytic continuation of that in the first case. The other Stokes curves emanating from a_0 flow into b_0 which are labeled s^1_{00} and s^2_{00}. From another turning point a_1, three Stokes curves emanate. One of them flows into b_2, which is denoted by s_{12}. Others flow into b_0 and they are labeled s^1_{10} and s^2_{10}. Let $\mathcal{R}^{\mathrm{I}}_{\omega_2}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_2}$, resp. $\mathcal{R}^{\mathrm{IV}}_{\omega_2}$) denote the open set surrounded by s^1_{00}, s^2_{00}, s_{01} (resp. by s^1_{10}, s^2_{10}, s_{12}, resp. by s^1_{00}, s^2_{00}, s^2_{10}, s^1_{10}). (cf. Figure 4.3.)

(3) The case where $(\alpha, \beta, \gamma) \in \omega_3$. There is a unique Stokes curve s_{00} which flows into b_0. We may assume that s_{00} emanates from a_0 and that a_0 is the analytic continuation of that in the first case. The other Stokes curves emanating from a_0 flow into b_1 which are labeled s^1_{01} and s^2_{01}. From another turning point a_1, three Stokes curves emanate. One of them flows into b_2, which is denoted by s_{12}. Others flow into b_1 and they are labeled s^1_{11} and s^2_{11}. Let $\mathcal{R}^{\mathrm{I}}_{\omega_3}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_3}$, resp. $\mathcal{R}^{\mathrm{V}}_{\omega_3}$) denote the open set surrounded by s_{00}, s^1_{01}, s^2_{01} (resp. by s^1_{11}, s^2_{11}, s_{12}, resp. by s^1_{01}, s^2_{01}, s^1_{11}, s^2_{11}). (cf. Figure 4.6.)

(4) The case where $(\alpha, \beta, \gamma) \in \omega_4$. There is a unique Stokes curve s_{00} (resp. s_{11}) which flows into b_0 (resp. b_1). We may assume that s_{00} (resp. s_{11}) emanates from a_0 (resp. a_1) and that a_0 is the analytic continuation of that in the first case. The other Stokes curves emanating from a_0 (resp. a_1) flow into b_2 which are labeled s^1_{02} and s^2_{02} (resp. s^1_{12} and s^2_{12}). Let $\mathcal{R}^{\mathrm{II}}_{\omega_4}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_4}$, resp. $\mathcal{R}^{\mathrm{VI}}_{\omega_4}$) denote the open set surrounded by s_{00}, s^1_{02}, s^2_{02} (resp. by s_{11}, s^1_{12}, s^2_{12}, resp. by s^1_{02}, s^2_{02}, s^1_{12}, s^2_{12}). (cf. Figure 4.9.)

(i) If (α, β, γ) is contained in ω_1 and it is sufficiently close to the boundary between ω_1 and ω_2, we take the same branch cut of $\sqrt{Q_0}$ as Figure 2.4. Then we specify the branch of $S_{-1} = \sqrt{Q_0}$ as (37). In this case, we consider the WKB solution

$$\psi_k = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_{a_k}^{x} S_{\mathrm{odd}}\, dx \right) \tag{191}$$

in a neighborhood of a_k ($k = 0, 1$), where we take the straight line connecting a_k to x as the path of the integration.

Let $D^{\mathrm{I}}_{\omega_1}$ (resp. $D^{\mathrm{II}}_{\omega_1}$) denote the intersection of $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_1}$) and a small neighborhood of a_0 (resp. a_1). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{I}}_{\omega_1}$(resp. $D^{\mathrm{II}}_{\omega_1}$) (cf. [**11**]). We take the Borel sum of the WKB solution ψ_0 in $D^{\mathrm{I}}_{\omega_1}$ (resp. ψ_1 in $D^{\mathrm{II}}_{\omega_1}$). It can be analytically continued with respect to x variable to $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_1}$), which we denote by $\psi^{\mathrm{I}}_{\omega_1}$ (resp. $\psi^{\mathrm{II}}_{\omega_1}$).

Next we assume that (α,β,γ) is contained in ω_2 and it is close to the point (α,β,γ) taken in the preceding case. We take the same branch cut as Section 2.2. Then we specify the branch of $S_{-1}=\sqrt{Q_0}$ as (37). Let us consider the intersection of $\mathcal{R}^{\mathrm{I}}_{\omega_2}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_2}$) and a small neighborhood of a_0 (resp. a_1). It has two connected components. We choose one of them which has a portion of s^1_{00} (resp. s^2_{10}) as a part of its boundary. Let us denote it by $D^{\mathrm{I},1}_{\omega_2}$ (resp. by $D^{\mathrm{II},2}_{\omega_2}$). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{I},1}_{\omega_2}$ (resp. $D^{\mathrm{II},2}_{\omega_2}$). We take the Borel sum of ψ_0 (resp. ψ_1) in $D^{\mathrm{I},1}_{\omega_2}$ (resp. $D^{\mathrm{II},2}_{\omega_2}$). It can be analytically continued in x to $\mathcal{R}^{\mathrm{I}}_{\omega_2}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_2}$), which we denote by $\psi^{\mathrm{I}}_{\omega_2}$ (resp. by $\psi^{\mathrm{II}}_{\omega_2}$).

Let us show examples of Stokes curves of those two cases and a degenerate case between them in Figures 4.1, 4.2 and 4.3 below, where $\epsilon>0$ and $\delta>0$ are sufficiently small.

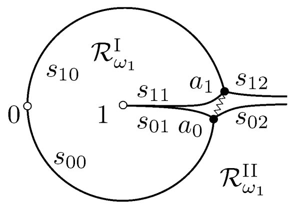

Figure 4.1. $(\alpha,\beta,\gamma)=(0.5, 1+\epsilon-\delta i, 1)$ in ω_1.

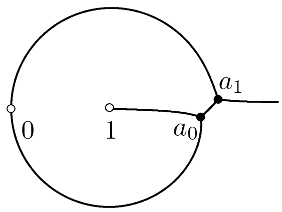

Figure 4.2. $(0.5, 1-\delta i, 1)$.

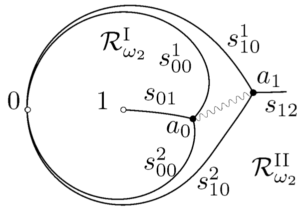

Figure 4.3. $(0.5, 1-\epsilon-\delta i, 1)$ in ω_2.

(ii) We assume that (α,β,γ) is contained in ω_1 and it is sufficiently close to the boundary between ω_1 and ω_3. In this case, the Stokes geometry is shown in Figure 4.4. We take a curve connecting a_0 and a_1 in $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ (denote by L_2 and shown by the wavy segment in Figure 4.4) as a branch cut of $\sqrt{Q_0}$ and specify the branch of $\sqrt{Q_0}$ so that $\sqrt{Q_0}\sim(\beta-\alpha)/(2x)$ holds near $x=\infty$. In this case, the behavior of $\sqrt{Q_0}$ near 0 and 1 are $\sqrt{Q_0}\sim-\gamma/(2x)$ and $\sqrt{Q_0}\sim(\alpha+\beta-\gamma)/(2(x-1))$, respectively. Note that these are different from (37) and (38) because the singular points 0 and 1 are surrounded by the union of L_1 (cf. Section 2.2) and L_2 (cf. Figure 4.4). We denote by $D^{\mathrm{I}}_{\omega_1}$ (resp. by $D^{\mathrm{III}}_{\omega_1}$) the intersection of $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$) and a small neighborhood of a_0 (resp. a_1). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{I}}_{\omega_1}$ (resp. in $D^{\mathrm{III}}_{\omega_1}$). We take the Borel sum of the WKB solution ψ_0 (resp. ψ_1) in $D^{\mathrm{I}}_{\omega_1}$ (resp. in $D^{\mathrm{III}}_{\omega_1}$). It can be analytically continued in x to $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$), which we denote by $\psi^{\mathrm{I}}_{\omega_1}$(resp. $\psi^{\mathrm{III}}_{\omega_1}$).

Next we consider the case where (α,β,γ) is contained in ω_3 and it is close to the point (α,β,γ) taken in the preceding case. We specify a curve in $\mathcal{R}^{\mathrm{IV}}_{\omega_3}$ which connects a_0 and a_1 as a branch cut of $\sqrt{Q_0}$. In this case, we specify the branch of $S_{-1}=\sqrt{Q_0}$ so that $\sqrt{Q_0}\sim(\alpha+\beta-\gamma)/(2(x-1))$ holds near $x=1$. This is different from (38) as the above discussion. Let us consider the intersection of $\mathcal{R}^{\mathrm{I}}_{\omega_3}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_3}$) and a small

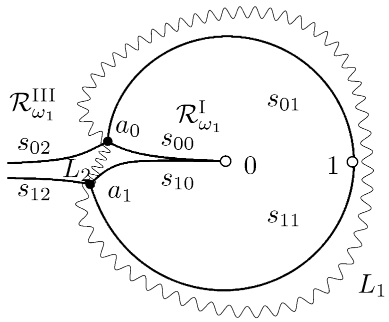

Figure 4.4. $(\alpha, \beta, \gamma) = (1 - \epsilon + \delta i, 2, 1)$ in ω_1.

neighborhood of a_0 (resp. a_1). It has two connected components. We choose one of them which has a portion of s^1_{01} (resp. s^2_{11}) as a part of its boundary. Let us denote it by $D^{\mathrm{I},1}_{\omega_3}$ (resp. by $D^{\mathrm{III},2}_{\omega_3}$). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{I},1}_{\omega_3}$ (resp. in $D^{\mathrm{III},2}_{\omega_3}$). We take the Borel sum of ψ_0 (resp. ψ_1) in $D^{\mathrm{I},1}_{\omega_3}$ (resp. in $D^{\mathrm{III},2}_{\omega_3}$). It can be analytically continued with respect to x variable to $\mathcal{R}^{\mathrm{I}}_{\omega_3}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_3}$), which we denote by $\psi^{\mathrm{I}}_{\omega_3}$ (resp. by $\psi^{\mathrm{III}}_{\omega_3}$).

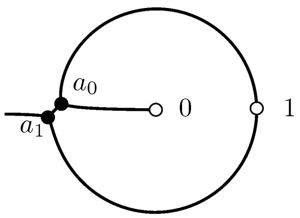

Figure 4.5. $(\alpha, \beta, \gamma) = (1 + \delta i, 2, 1)$.

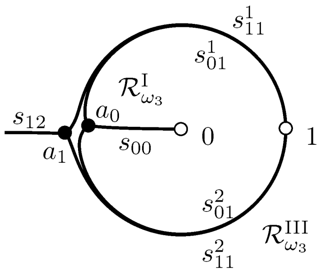

Figure 4.6. $(1 + \epsilon + \delta i, 2, 1)$ in ω_3.

(iii) If (α, β, γ) is contained in ω_1 and it is sufficiently close to the boundary between ω_1 and ω_4, we specify a curve in $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ which connects a_0 and a_1 as a branch cut of $\sqrt{Q_0}$. In this case, we specify the branch of $S_{-1} = \sqrt{Q_0}$ as (36). We denote by $D^{\mathrm{II}}_{\omega_1}$ (resp. by $D^{\mathrm{III}}_{\omega_1}$) the intersection of $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$) and a small neighborhood of a_0 (resp. a_1). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{II}}_{\omega_1}$(resp. in $D^{\mathrm{III}}_{\omega_1}$). We take the Borel sum of ψ_0 (resp. ψ_1) in $D^{\mathrm{II}}_{\omega_1}$ (resp. in $D^{\mathrm{III}}_{\omega_1}$). It can be analytically continued with respect to x variable to $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$), which we denote by $\psi^{\mathrm{II}}_{\omega_1}$ (resp. by $\psi^{\mathrm{III}}_{\omega_1}$).

If (α, β, γ) is contained in ω_4 and it is close to the point (α, β, γ) taken in the preceding case, we specify a curve in $\mathcal{R}^{\mathrm{VI}}_{\omega_4}$ which connects a_0 and a_1 as a branch cut of $\sqrt{Q_0}$. In this case, we specify the branch of $S_{-1} = \sqrt{Q_0}$ as (36). Let us consider the intersection of $\mathcal{R}^{\mathrm{II}}_{\omega_4}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_4}$) and a small neighborhood of a_0 (resp. a_1). It has two connected components. We choose one of them which has a portion of s^1_{02} (resp. s^2_{12}) as a part of its boundary. We denote it by $D^{\mathrm{II},1}_{\omega_4}$ (resp. by $D^{\mathrm{III},2}_{\omega_4}$). The WKB solution ψ_0 (resp. ψ_1) is Borel summable in $D^{\mathrm{II},1}_{\omega_4}$ (resp. in $D^{\mathrm{III},2}_{\omega_4}$). We take the Borel sum of the WKB solution ψ_0 (resp. ψ_1) in $D^{\mathrm{II},1}_{\omega_4}$ (resp. in $D^{\mathrm{III},2}_{\omega_4}$). It can be analytically continued in x to $\mathcal{R}^{\mathrm{II}}_{\omega_4}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_4}$), which is denoted by $\psi^{\mathrm{II}}_{\omega_4}$ (resp. by $\psi^{\mathrm{III}}_{\omega_4}$).

Let us show examples of Stokes curves of those two cases and a degenerate case between them in Figures 4.7, 4.8 and 4.9.

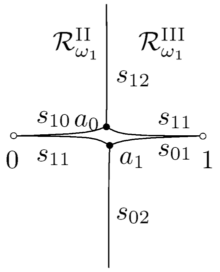

Figure 4.7. $(\alpha, \beta, \gamma) = (\epsilon + \delta i, 2, 1)$ in ω_1.

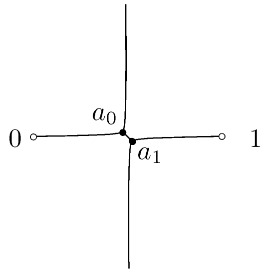

Figure 4.8. $(\delta i, 2, 1)$.

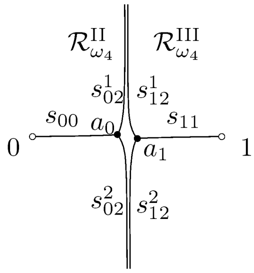

Figure 4.9. $(-\epsilon + \delta i, 2, 1)$ in ω_4.

The Stokes region $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_1}$, resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$) is continuously deformed to $\mathcal{R}^{\mathrm{I}}_{\omega_h}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_h}$, resp. $\mathcal{R}^{\mathrm{III}}_{\omega_h}$) through the variation of (α, β, γ) from ω_1 to ω_h $(h = 2, 3, 4)$. Under the notation given above, we have the following theorem.

Theorem 4.4. (i) *Between the Borel sums* $\psi^{\mathrm{I}}_{\omega_1}$ *and* $\psi^{\mathrm{I}}_{\omega_2}$ *of the WKB solution* ψ_0 *defined by* (191) *the following relation holds*:

$$\psi^{\mathrm{I}}_{\omega_1} = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{1/2} \psi^{\mathrm{I}}_{\omega_2}. \tag{192}$$

Between the Borel sums $\psi^{\mathrm{II}}_{\omega_1}$ *and* $\psi^{\mathrm{II}}_{\omega_2}$ *of the WKB solution* ψ_1 *the following relation holds*:

$$\psi^{\mathrm{II}}_{\omega_1} = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{1/2} \psi^{\mathrm{II}}_{\omega_2}. \tag{193}$$

(ii) *Between the Borel sums* $\psi^{\mathrm{I}}_{\omega_1}$ *and* $\psi^{\mathrm{I}}_{\omega_3}$ *of the WKB solution* ψ_0 *the following relation holds*:

$$\psi^{\mathrm{I}}_{\omega_1} = (1 + \exp(2\pi i(\alpha - \gamma)\eta))^{1/2} \psi^{\mathrm{I}}_{\omega_3}. \tag{194}$$

Between the Borel sums $\psi^{\mathrm{III}}_{\omega_1}$ *and* $\psi^{\mathrm{III}}_{\omega_3}$ *of the WKB solution* ψ_1 *the following relation holds*:

$$\psi^{\mathrm{III}}_{\omega_1} = (1 + \exp(2\pi i(\alpha - \gamma)\eta))^{1/2} \psi^{\mathrm{III}}_{\omega_3}. \tag{195}$$

(iii) *Between the Borel sums* $\psi^{\mathrm{II}}_{\omega_1}$ *and* $\psi^{\mathrm{II}}_{\omega_4}$ *of the WKB solution* ψ_0 *the following relation holds*:

$$\psi^{\mathrm{II}}_{\omega_1} = (1 + \exp(2\pi i\alpha\eta))^{-1/2} \psi^{\mathrm{II}}_{\omega_4}. \tag{196}$$

Between the Borel sums $\psi^{\mathrm{III}}_{\omega_1}$ *and* $\psi^{\mathrm{III}}_{\omega_4}$ *of the WKB solution* ψ_1 *the following relation holds*:

$$\psi_{\omega_1}^{\mathrm{III}} = (1+\exp(2\pi i\alpha\eta))^{-1/2}\psi_{\omega_4}^{\mathrm{III}}. \tag{197}$$

PROOF. We now compare $\psi_{\omega_1}^{\mathrm{I}}$ with $\psi_{\omega_2}^{\mathrm{I}}$. We consider the WKB solution normalized at b_0:

$$\psi^{(0)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\biggl(\int_{b_0}^{x}(S_{\mathrm{odd}}-\eta S_{-1})\,dx + \eta\int_{a_0}^{x} S_{-1}\,dx\biggr). \tag{198}$$

If $(\alpha,\beta,\gamma)\in\omega_1$, $\psi^{(0)}$ is Borel summable in $D_{\omega_1}^{\mathrm{I}}$. We take the Borel sum of $\psi^{(0)}$ in $D_{\omega_1}^{\mathrm{I}}$ and its analytic continuation to $\mathcal{R}_{\omega_1}^{\mathrm{I}}$, which we denote by $\psi_{\omega_1}^{(0),\mathrm{I}}$. Similarly, if $(\alpha,\beta,\gamma)\in\omega_2$, we can take the Borel sum of $\psi^{(0)}$ in $D_{\omega_2}^{\mathrm{I}}$ and its analytic continuation to $\mathcal{R}_{\omega_2}^{\mathrm{I}}$, which we denote by $\psi_{\omega_2}^{(0),\mathrm{I}}$. It follows from a result by Koike and Schäfke [**11**] (see [**6**] also) that

$$\psi_{\omega_1}^{(0),\mathrm{I}} = \psi_{\omega_2}^{(0),\mathrm{I}} \tag{199}$$

holds. Combining (35), (173) and (199), we have

$$\begin{aligned}\psi_{\omega_1}^{\mathrm{I}} &= (\exp(-V_0^1))\psi_{\omega_1}^{(0),\mathrm{I}}\\ &= (1+\exp(2\pi i(\gamma-\beta)\eta))^{1/2}(\exp(-V_0^2))\psi_{\omega_2}^{(0),\mathrm{I}}\\ &= (1+\exp(2\pi i(\gamma-\beta)\eta))^{1/2}\psi_{\omega_2}^{\mathrm{I}}.\end{aligned}$$

In a similar manner, we have the other relations. □

Finally we consider the parametric Stokes phenomena for (191) between ω_{1m} and ω_{hm} ($h=2,3,4;\ m=0,\dots,6$). Note that since the potential Q is invariant under involution ι_m, the Stokes geometry of (1) for $\iota_m(\alpha,\beta,\gamma)\in\omega_{hm}$ is the same as that for $(\alpha,\beta,\gamma)\in\omega_h$. Applying ι_m to the relations (192)–(197), we have the formulas which describe the parametric Stokes phenomena between ω_{1m} and ω_{hm}. For example, we consider the parametric Stokes phenomena for (191) between ω_{11} and ω_{12}. We apply ι_1 to the relations (192) and (193), then we have the following relations:

$$\psi_{\omega_{11}}^{\mathrm{I}} = (1+\exp(2\pi i(-\alpha)\eta))^{-1/2}\psi_{\omega_{21}}^{\mathrm{I}} \tag{200}$$

and

$$\psi_{\omega_{11}}^{\mathrm{II}} = (1+\exp(2\pi i(-\alpha)\eta))^{-1/2}\psi_{\omega_{21}}^{\mathrm{II}}, \tag{201}$$

respectively.

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