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On the set of fixed points of a polynomial automorphism

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We say that a polynomial automorphism $f : \mathbb{K}^n \to \mathbb{K}^n$ is special if the Jacobian of f is equal to 1. We show that every $(n-1)$-dimensional component H of the set $\mathrm{Fix}(f)$ of fixed points of a non-trivial special polynomial automorphism $f : \mathbb{K}^n \to \mathbb{K}^n$ is uniruled. Moreover, we show that if f is non-special and H is an $(n-1)$-dimensional component of the set $\mathrm{Fix}(f)$, then H is smooth, irreducible and $H = \mathrm{Fix}(f)$. Moreover, for $\mathbb{K} = \mathbb{C}$ if f is non-special and $\mathrm{Jac}(f)$ has an infinite order in $\mathbb{C}^*$, then the Euler characteristic of H is equal to 1.

1. Introduction.

Polynomial automorphism of affine space $\mathbb{K}^n$ have always attracted a lot of attention, but the nature of these automorphisms is still not well-known.

Here we are interested in the set of fixed points of such automorphisms. Let us recall that if $f : \mathbb{K}^2 \to \mathbb{K}^2$ is a polynomial automorphism, then the set $\mathrm{Fix}(f)$ of fixed points of f is either finite, or it is a union of smooth, disjoint curves which all are isomorphic to $\mathbb{K}$. This result was proved in [**Jel1**] and later it was partially reproved in [**M-M**]. Moreover, by Kambayashi result, every automorphism of $\mathbb{K}^2$ of finite order is linear in some system of coordinates. We do not know whether Kambayashi result can be extended to higher dimensions. However there is some evidence that the set of fixed points of a polynomial automorphism of finite order should be isomorphic to a linear subspace.

In higher dimensions the situation is more complicated. The set of fixed points can have components of dimension $n-1$ and additionally less dimensional components - an easy example is $f : (x, y, z) \ni \mathbb{K}^4 \to (x + zy, y + zw, z, w) \in \mathbb{K}^4$. Moreover, an $(n-1)$-dimensional component of the set of fixed points of f can be a singular variety- as in the famous Nagata automorphism:

$$N : \mathbb{C}^3 \ni (x, y, z) \mapsto (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z) \in \mathbb{C}^3.$$

Here the set of fixed points is the quadratic cone $\Lambda = \{(x, y, z) : xz + y^2 = 0\}$. We show however that such a strange behavior is possible only for *special automorphisms, i.e., for automorphisms with Jacobian equal to one*. In this paper we focus on $(n-1)$-dimensional components of the set of fixed points of polynomial automorphism of $\mathbb{K}^n$. Our first result is:

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THEOREM 1.1. *Let $\mathbb{K}$ be an arbitrary algebraically closed field. Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a non-special polynomial automorphism. Let H be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of f. Then*

1) *H is smooth and irreducible.*
2) $H = \mathrm{Fix}(f)$.

We can say more if the order of f is infinite and the field $\mathbb{K}$ has characteristic zero. Let $H \subset \mathbb{K}^n$ be a hypersurface and $I(H) = (p)$. We say that H is *super-smooth* if $(\partial p/\partial x_1(x), \ldots, \partial p/\partial x_n(x)) \neq 0$ for every $x \in \mathbb{K}^n$. We have:

THEOREM 1.2. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a non-special polynomial automorphism of infinite order. Let H be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of f. Then*

1) *H is smooth and irreducible.*
2) $H = \mathrm{Fix}(f)$.
3) *If $\lambda = \mathrm{Jac}(f)$ has finite order in $\mathbb{K}^*$, then H is uniruled.*
4) *If $\lambda = \mathrm{Jac}(f)$ has infinite order in $\mathbb{K}^*$, then H is super-smooth. Moreover, if $\mathbb{K} = \mathbb{C}$, then the Euler characteristic of H is equal to* 1.

For special automorphisms of infinite order, the set of fixed points can have many $(n-1)$-dimensional components; the easiest example is a triangular automorphism

$$f(x, y, z) = \left(x + \prod_{i=1}^{r} h_i(y, z), y, z\right).$$

Moreover, as we noticed before, such an $(n-1)$-dimensional component can be singular. Additionally, if H is the union of all $(n-1)$-dimensional components of $\mathrm{Fix}(f)$, then in general $H \neq \mathrm{Fix}(f)$. However, the $(n-1)$-dimensional components of the set of fixed points of a special automorphism have one common property - they are uniruled:

THEOREM 1.3. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a non-trivial special polynomial automorphism. Let H be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of f. Then H is uniruled, i.e., it is covered by rational curves.*

2. Non-special automorphisms.

We first need an elementary lemma from linear algebra.

LEMMA 2.1. *Let $X = \mathbb{K}^n$ and let $F : X \to X$ be a linear isomorphism. Assume that there exists a hyperplane W which is contained in the set of fixed points of F. Then all eigenvalues of F are* 1 *(of multiplicity at least $n-1$) and* $\det(F)$.

Now assume that $\det(F) = \lambda \neq 1$. *If l is a linear form such that* $\ker l = W$, *then the forms proportional to the form l are the only eigenvectors of F^* with eigenvalue λ.*

PROOF. Since W is contained in the set of fixed points of F, we have that 1 is an eigenvalue of F of multiplicity at least $\dim W = n-1$. Hence the remaining eigenvalue has to be equal to $\det(F) = \lambda$.

Now assume that $\lambda \neq 1$ and let $\boldsymbol{g}$ be an eigenvector with eigenvalue λ. Let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}\}$ be a basis of W. It is easy to see that $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}\}$ and $\boldsymbol{g}$ form a basis of X. In this basis F is given by the formula:

$$F : \mathbb{C}^n \in \boldsymbol{v} \mapsto x_1(\boldsymbol{v})\boldsymbol{w}_1 + \cdots + x_{n-1}(\boldsymbol{v})\boldsymbol{w}_{n-1} + \lambda x_n(\boldsymbol{v})\boldsymbol{g} \in \mathbb{C}^n,$$

where $\{x_1, \ldots, x_n\}$ is the dual basis of $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}, \boldsymbol{g}\}$. In particular, the hyperplane W is described by the form x_n, i.e., $l = cx_n$ for some $c \in \mathbb{K}^*$. Moreover,

$$F^*\left(\sum_{i=1}^{n} a_i x_i\right) = a_1 x_1 + \cdots + a_{n-1}x_{n-1} + \lambda a_n x_n.$$

Consequently, only the forms proportional to x_n have eigenvalues equal to λ. □

Moreover, we have:

Lemma 2.2. *Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial automorphism. Let $H \subset Fix(f)$ be a hypersurface. If H is singular, then* $\mathrm{Jac}(f) = 1$.

Proof. Let $a \in \mathrm{Sing}(H)$. Choose a system of coordinates in which $a = (0, \ldots, 0)$. Let h be a reduced equation of H. Since $H \subset \mathrm{Fix}(f)$ we have that $f_i - x_i$ vanishes on H, i.e., $h | f_i - x_i$. Consequently,

$$f_i = x_i + a_i h, \ i = 1, \ldots, n.$$

Since $h = \sum_{|\alpha| \geq 2} h_\alpha x^\alpha$, we have $\mathrm{Jac}(f) = \mathrm{Jac}(\text{identity}) = 1$. □

Lemma 2.3. *Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial automorphism. Let $H \subset \mathbb{K}^n$ be an irreducible hypersurface, such that $H \subset \mathrm{Fix}(f)$. Let h be a reduced equation of H. Then $h \circ f = \lambda h$, where $\lambda = \mathrm{Jac}(f)$.*

Proof. Let $a \in H$ be a smooth point of H. Take $W = T_a H$, $X = T_a\mathbb{K}^n$ and $F = d_a f$. By the assumption, the subspace W is contained in the set of fixed points of the linear isomorphism F. Moreover, W is described by the linear form $l = \sum_{i=1}^n (\partial h/\partial x_i)(a)x_i = 0$. This form can be identified with the vector $\mathrm{grad}_a h = ((\partial h/\partial x_1)(a), \ldots, (\partial h/\partial x_n)(a))$.

Since f is a polynomial automorphism and $H \subset \mathrm{Fix}(f)$, the polynomial h describes the same hypersurface as the polynomial $h \circ f$. Note that these two polynomials are reduced and consequently they are generators of the same ideal $I(H)$. This means that there exists a constant $c \in \mathbb{K}^*$ such that $h \circ f = c \cdot h$. After differentiation, we have $(d_a f)^* \mathrm{grad}_a h = c \cdot \mathrm{grad}_a h$. Hence the vector $\mathrm{grad}_a h$ is an eigenvector of F^*. Now if $\lambda = \mathrm{Jac}(f) = 1$, then by Lemma 2.1, we have that all eigenvalues of F (and hence also of F^*) are equal to 1. Consequently, $c = \lambda = 1$. If $\lambda \neq 1$, then again by Lemma 2.1, we have $c = \lambda$. □

Now we are ready to prove:

Theorem 2.4. *Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a non-special polynomial automorphism. Let H be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of f. Then*

1) *H is smooth and irreducible.*
2) $H = \mathrm{Fix}(f)$.

Proof. Let S be an irreducible component of H with reduced equation $s = 0$. By Lemma 2.3, we have $s \circ f = \lambda s$ and $\lambda \neq 0$. For $t \neq 0$, the hypersurface $S_t := \{x : s(x) = t\}$ is transformed by f onto the hypersurface $S_{\lambda t}$. Since $S_t \cap S_{\lambda t} = \emptyset$ for $t \neq 0$, we have $\mathrm{Fix}(f) = S_0 = S$. In particular, $H = S$ is an irreducible hypersurface, and H is smooth by Lemma 2.2. □

Now we show that non-trivial automorphisms of finite order and with large set of fixed points cannot be special. We start with:

Lemma 2.5. *Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a linear automorphism of finite order $m > 1$ and assume that the set of fixed points of L is a hyperplane W. Then in some coordinates,*

$$L(x_1, x_2, \dots, x_n) = (\epsilon x_1, x_2, \dots, x_n),$$

where $\epsilon^m = 1$ and $\epsilon \neq 1$.

Proof. Take a basis $e_1, \dots, e_n$ in $\mathbb{C}^n$ such that $e_2, \dots, e_n$ span the hyperplane W. Hence $L(e_i) = e_i$ for $i > 1$ and $L(e_1) = \sum_{i=1}^n a_i e_i$. In particular, $L(x_1 e_1 + \dots + x_n e_n) = (a_1 x_1) e_1 + (x_2 + a_2 x_1) e_2 + \dots + (x_n + a_n x_1) e_n$. Since L has finite order, we have $a_i = 0$ for $i > 1$. In particular, $L(x_1, \dots, x_n) = (a_1 x_1, x_2, \dots, x_n)$. However $\det L^m = 1$, i.e., $a_1 = \epsilon$, where $\epsilon^m = 1$ and $\epsilon \neq 1$. □

Now we can state:

Proposition 2.6. *Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial automorphism of finite order $m > 1$. Assume that the set of fixed points of Φ has dimension $n - 1$. Then Φ is not special.*

Proof. Let $H \subset \mathrm{Fix}(\Phi)$ be a hypersurface and let $x \in H$. By the Cartan Theorem (see [**Car**]), the mapping Φ is holomorphically linearizable in some neighborhood of x. Now the proof reduces to Lemma 2.5. □

We conclude this section by:

Theorem 2.7. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $f : \mathbb{K}^n \to \mathbb{K}^n$ be a non-special polynomial automorphism of infinite order. Let H be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of f. Then*

1) *H is smooth and irreducible.*
2) $H = \mathrm{Fix}(f)$.
3) *If $\lambda = \mathrm{Jac}(f)$ has finite order in $\mathbb{K}^*$, then H is uniruled.*

4) *If* $\lambda = \mathrm{Jac}(f)$ *has infinite order in* $\mathbb{K}^*$, *then* H *is super-smooth. Moreover, if* $\mathbb{K} = \mathbb{C}$, *then the Euler characteristic of* H *is equal to* 1.

PROOF. By the Lefschetz principle, we can assume that $\mathbb{K} = \mathbb{C}$. We have 1) and 2) by Theorem 2.4. The point 3) follows from Theorem 3.2 below. Hence it is enough to prove 4). Let $H = \mathrm{Fix}(f)$. As we know the hypersurface H is irreducible. Let $h = 0$ be an irreducible equation for H. By Lemma 2.3 we have $h \circ f = \lambda h$.

If $h : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, then it is proved (see e.g., [**J-K**]) that there is a finite set $B \in \mathbb{C}$ such that h is a locally trivial smooth fibration over the complement of B. The smallest such a set, denoted by $B(h)$, is called the set of *atypical values of* f. Other values of h are called *typical values of* h. It is easy to see that for a typical value a of h the fiber $h = a$ is smooth (see e.g., [**J-K**]).

We show that the polynomial h has no atypical values, except possibly 0. Indeed, since the fiber $h = t$ is transformed by f onto the fiber $h = \lambda t$ and λ has an infinite order we have that for $t \neq 0$ the fiber $h = t$ cannot be atypical. Indeed, there is only a finite number of atypical values of h. In particular, all fibers $h = t$ for $t \neq 0$ are homeomorphic to $H_1 = \{h = 1\}$ and $f : \mathbb{C}^n \setminus H \to \mathbb{C}^*$ is a locally trivial fibration with a fiber H_1. Computing the Euler characteristics, we have $\chi(H_1)\chi(\mathbb{C}^*) + \chi(H) = \chi(\mathbb{C}^n) = 1$, i.e., $\chi(H) = 1$.

Moreover, all fibers $h = t,\ t \in \mathbb{C}$ are smooth. For $t = 0$ it follows from 1); if $t \neq 0$, then $h = t$ is a typical fiber, hence it is smooth. □

3. Special automorphisms.

First we recall the following important fact (see [**Jel3**]):

THEOREM 3.1. *Let* $\mathbb{K}$ *be an algebraically closed field of characteristic zero. Let* X *be a quasi-affine variety. If the group* $\mathrm{Aut}(X)$ *is infinite, then* X *is uniruled.*

Now we can prove our main result:

THEOREM 3.2. *Let* $\mathbb{K}$ *be an algebraically closed field of characteristic zero. Let* $f : \mathbb{K}^n \to \mathbb{K}^n$ *be a non-trivial special polynomial automorphism. Let* H *be a hypersurface that is contained in the set* $\mathrm{Fix}(f)$ *of fixed points of* f. *Then* H *is uniruled, i.e., it is covered by rational curves.*

PROOF. By Proposition 2.6, automorphism f has an infinite order. Let h be an irreducible equation of H. By Lemma 2.3 we have $h \circ f = h$. In particular, f preserves all fibers $h = t$. Let Γ be an irrational affine curve and let $\pi : \Gamma \to \mathbb{K}$ be a finite morphism. Denote by

$$X := \mathbb{K}^n \times_{\mathbb{K}} \Gamma,$$

the fiber product determined by the mappings h and π. On X acts the automorphism $F = (f \times \mathrm{identity})_{|X}$. We have the projection $\Pi : X \to \Gamma$ with $(n-1)$-dimensional fibers $\Pi^{-1}(\gamma) = h^{-1}(\pi(\gamma))$. Because the polynomial h is irreducible, generic fibers of Π are irreducible. Moreover, since the variety X is a hypersurface in $\mathbb{K}^n \times \Gamma$, it has to be an irreducible affine variety of dimension n. It is easy to see that the automorphism F has an infinite order. By virtue of Theorem 3.1, the variety X is uniruled. Since the curve Γ is not uniruled, all fibers of the mapping $\Pi : X \to \Gamma$ are uniruled. The hypersurface H is one of these fibers. $\square$

Example 3.3. Let

$$N : \mathbb{C}^3 \ni (x, y, z) \mapsto (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z) \in \mathbb{C}^3,$$

be the famous Nagata automorphism. The set of fixed points of N is the cone $\Lambda = \{(x, y, z) \in \mathbb{C}^3 : xz + y^2 = 0\}$. Since Λ is a singular variety, we see that the automorphism N is special. Of course Λ is a uniruled (even a rational) surface. $\square$

Example 3.4. We cannot expect that a lower dimensional components of the set $\mathrm{Fix}(f)$ are uniruled. Indeed, let $\Gamma = \{(x, y) \in \mathbb{C}^2 : h(x, y) = 0\}$ be an arbitrary plane curve. Then there is a polynomial automorphism $f : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\mathrm{Fix}(f) \cong \Gamma$. Indeed, take

$$f(x, y, z) = (x, y + z + h(x, y), z + h(x, y)).$$

If we take $g(x, y, z, w) = (f(x, y, z), 2w)$, we obtain a non-special automorphism $g : \mathbb{C}^4 \to \mathbb{C}^4$ with $\mathrm{Fix}(g) \cong \Gamma$.

At the end of this paper we state a conjecture, which is more or less the Masuda-Miyanishi Conjecture (see [**M-M**]):

Conjecture. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a non-special automorphism. Assume that $H \subset \mathrm{Fix}(f)$ is a hypersurface. Then H is isomorphic to $\mathbb{C}^{n-1}$.*

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