©2016 The Mathematical Society of Japan
J. Math. Soc. Japan
Vol. 68, No. 2 (2016) pp. 863–883
doi: 10.2969/jmsj/06820863

Classification of conformal minimal immersions of constant curvature from S^2 to Q_3

By Mingyan Li, Xiaoxiang Jiao and Ling He

(Received Apr. 10, 2014)
(Revised July 21, 2014)

Abstract. In this paper, we study geometry of conformal minimal two-spheres immersed in complex hyperquadric Q_3. We firstly use Bahy-El-Dien and Wood's results to obtain some characterizations of the harmonic sequences generated by conformal minimal immersions from S^2 to $G(2,5;\mathbb{R})$. Then we give a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$, or equivalently, a complex hyperquadric Q_3.

1. Introduction.

The classification of minimal surfaces of constant curvature is an important topic of differential geometry. Bryant [**4**] gave a classification of minimal surfaces with constant curvature in $S^n(1)$. Kenmotsu and Masuda [**12**] classified all minimal surfaces of constant curvature in two-dimensional complex space forms. Bolton et al. [**3**] proved that a linearly full conformal minimal immersion of S^2 in $\mathbb{C}P^n$ with constant curvature belongs to the Veronese sequence, up to a holomorphic isometry of $\mathbb{C}P^n$. Generally, if the ambient space is not a real (or complex) space form, for example, complex Grassmannian $G(k,n;\mathbb{C})$, complex hyperquadric Q_n and quaternionic projective space HP^n and so on, the classification of minimal 2-spheres of constant curvature in them is not easy. It is well known that Hoffman and Osserman [**9**] gave some results about minimal surfaces in $\mathbb{R}^n$ whose Gaussian image in Q_{n-2} has constant curvature, and Chi and Zheng [**7**] classified all holomorphic curves from Riemann spheres into $G(2,4)$ whose curvature is equal to 2 into two families. Recently, J. Wang and the second author ([**10**], [**13**]) determined curvatures and Kähler angles of conformal minimal 2-spheres in Q_2 if their curvature is constant and all the totally real conformal minimal two-spheres of constant curvature in Q_n (only when $n = 2, 3, 4, 5$). Previously, in [**8**], the authors gave a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to HP^2. Here our interest is to study conformal minimal 2-spheres immersed in Q_n with constant curvature.

As is well known, $G(2,n;\mathbb{R})$ may be identified with complex hyperquadric Q_{n-2} in $\mathbb{C}P^{n-1}$ (for detailed descriptions see the Preliminaries below). In 1989 Bahy-El-Dien and Wood [**2**] gave the explicit construction of all harmonic two-spheres in $G(2,n;\mathbb{R})$,

---

2010 *Mathematics Subject Classification.* Primary 53C42, 53C55.
Key Words and Phrases. conformal minimal immersion, Gauss curvature, Kähler angle, complex hyperquadric, classification.
This research was supported by the NSFC (Grant No. 11331002.)

which is considered as totally geodesic submanifolds in complex Grassmann manifolds $G(2,n;\mathbb{C})$. In this paper we study classification of conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ by theory of harmonic maps, and discuss the Kähler angle of conformal minimal immersions of S^2 in Q_n.

Our arrangement is as follows.

In the second section of this paper, firstly we identify Q_{n-2} and $G(2,n;\mathbb{R})$, then we give some fundamental results concerning $G(k,n;\mathbb{C})$ from the view of harmonic sequences, at last we give some brief descriptions of Veronese sequence and the rigidity theorem in $\mathbb{C}P^n$. In the third section, we use Bahy-El-Dien and Wood's results to study some properties of the harmonic sequence generated by a harmonic map from S^2 to $G(2,5;\mathbb{R})$ and obtain some characteristics of the corresponding harmonic map in $G(2,5;\mathbb{R})$. In the last section, we discuss geometric properties of conformal minimal 2-spheres immersed in $G(2,5;\mathbb{R})$ with constant curvature and give a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ (see Theorem 4.9). In addition, we give a formula about Kähler angle of conformal minimal immersions from S^2 to Q_n.

2. Preliminaries.

(A) For $0 \le k \le n$, let $G(k,n;\mathbb{R})$ denote the Grassmannian of all real k-dimensional subspaces of $\mathbb{R}^n$ and

$$\sigma : G(k,n;\mathbb{C}) \to G(k,n;\mathbb{C})$$

denote the complex conjugation of $G(k,n;\mathbb{C})$. It is easy to see that σ is an isometry with the standard Riemannian metric of $G(k,n;\mathbb{C})$. Its fixed point set is $G(k,n;\mathbb{R})$, thus $G(k,n;\mathbb{R})$ lies totally geodesically in $G(k,n;\mathbb{C})$.

Map

$$Q_{n-2} \to G(2,n;\mathbb{R})$$

by

$$q \mapsto \frac{\sqrt{-1}}{2} Z \wedge \overline{Z},$$

where $q \in Q_{n-2}$ and Z is a homogeneous coordinate vector of q. It is clear that the map is well defined. We can easily check that the map is one-to-one and onto, and it is an isometry. Thus we can identify Q_{n-2} and $G(2,n;\mathbb{R})$ (for more details see [**14**]). Here we suppose that the metric on $G(2,n;\mathbb{R})$ is given by Section 2 of [**11**], then the metric is twice as much as the standard metric on Q_{n-2} induced by the inclusion $\tau : Q_{n-2} \to \mathbb{C}P^{n-1}$, where this latter space is given the Fubini-Study metric of constant holomorphic sectional curvature 4.

(B) In this section we simply introduce harmonic maps and harmonic sequences in $G(k,n;\mathbb{C})$ and calculate some corresponding geometric quantities.

Let M be an arbitrary Riemann surface and let $\varphi : M \to G(k,n;\mathbb{C})$ be a map. We shall frequently use one-to-one correspondence between maps $\varphi : M \to G(k,n;\mathbb{C})$ and rank k subbundles $\underline{\varphi}$ of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$ given by setting the fibre $\underline{\varphi}_x = \varphi(x)$ for all $x \in M$. Then $\underline{\varphi}$ is called (a) *harmonic* ((*sub-*) *bundle*) whenever φ is a harmonic map (cf. [**5**]).

Let $(z, \overline{z})$ be a complex coordinate on M. We take the metric $ds_M^2 = dzd\overline{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}}.$$

Let $\varphi : S^2 \to G(k,n;\mathbb{C})$ be a smooth harmonic map. Then from φ two harmonic sequences are derived as follows:

$$\underline{\varphi} = \underline{\varphi}_0 \xrightarrow{\partial'} \underline{\varphi}_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi}_\alpha \xrightarrow{\partial'} \cdots, \tag{2.1}$$

$$\underline{\varphi} = \underline{\varphi}_0 \xrightarrow{\partial''} \underline{\varphi}_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} \underline{\varphi}_{-\alpha} \xrightarrow{\partial''} \cdots, \tag{2.2}$$

where $\underline{\varphi}_\alpha = \partial' \underline{\varphi}_{\alpha-1}$ and $\underline{\varphi}_{-\alpha} = \partial'' \underline{\varphi}_{-\alpha+1}$ are Hermitian orthogonal projections from $S^2 \times \mathbb{C}^n$ onto $\underline{Im}\left(\varphi_{\alpha-1}^{\perp} \partial \varphi_{\alpha-1}\right)$ and $\underline{Im}\left(\varphi_{-\alpha+1}^{\perp} \overline{\partial} \varphi_{-\alpha+1}\right)$ respectively, $\alpha = 1, 2, \ldots$.

As in [**2**] call a harmonic map $\varphi : S^2 \to G(k,n;\mathbb{C})$ (*strongly*) *isotropic* if $\varphi_\alpha \perp \varphi$ $\forall \alpha \in \mathbb{Z},\ \alpha \neq 0$.

For an arbitrary harmonic map $\varphi : S^2 \to G(k,n;\mathbb{C})$, define its *isotropy order* (cf. [**5**]) to be the greatest integer r such that $\varphi_\alpha \perp \varphi$ for all α with $1 \leq \alpha \leq r$; if $\underline{\varphi}$ is isotropic, set $r = \infty$.

Definition 2.1. Let $\varphi : S^2 \to G(k,n;\mathbb{C})$ be a map. φ is *linearly full* if $\underline{\varphi}$ cannot be contained in any proper trivial subbundle $S^2 \times \mathbb{C}^m$ of $S^2 \times \mathbb{C}^n$ $(m < n)$.

In this paper, we always assume that φ is linearly full.

Suppose that $\varphi : S^2 \to G(2,n;\mathbb{C})$ is a linearly full harmonic map and belongs to the following harmonic sequence:

$$\underline{\varphi}_0 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi} = \underline{\varphi}_\alpha \xrightarrow{\partial'} \underline{\varphi}_{\alpha+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi}_{\alpha_0} \xrightarrow{\partial'} 0 \tag{2.3}$$

for $\alpha = 0, \ldots, \alpha_0$. We choose the local unit orthogonal frame $e_1^{(\alpha)}, e_2^{(\alpha)}, \ldots, e_{k_\alpha}^{(\alpha)}$ such that they locally span subbundle $\underline{\varphi}_\alpha$ of $S^2 \times \mathbb{C}^n$, where $k_\alpha = \text{rank } \underline{\varphi}_\alpha$.

Let $W_\alpha = \left(e_1^{(\alpha)}, e_2^{(\alpha)}, \ldots, e_{k_\alpha}^{(\alpha)}\right)$ be $(n \times k_\alpha)$-matrix. Then we have

$$\begin{gathered} \varphi_\alpha = W_\alpha W_\alpha^*, \\ W_\alpha^* W_\alpha = I_{k_\alpha \times k_\alpha}, \quad W_\alpha^* W_{\alpha+1} = 0, \quad W_\alpha^* W_{\alpha-1} = 0. \end{gathered} \tag{2.4}$$

By (2.4), a straightforward computation shows that

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha \Psi_\alpha, \\ \overline{\partial} W_\alpha = -W_{\alpha-1}\Omega_{\alpha-1}^* - W_\alpha \Psi_\alpha^*, \end{cases} \tag{2.5}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix.

Set $L_\alpha = \operatorname{tr}(\Omega_\alpha \Omega_\alpha^*)$. By a straightforward calculation, the metric induced by φ_α is given by

$$ds_\alpha^2 = (L_{\alpha-1} + L_\alpha) dz d\overline{z}. \tag{2.6}$$

The Laplacian $\triangle_\alpha$ and the curvature K_α of ds_α^2 are given by

$$\triangle_\alpha = \frac{4}{L_{\alpha-1} + L_\alpha} \partial\overline{\partial}, \qquad K_\alpha = -\frac{2}{L_{\alpha-1} + L_\alpha} \partial\overline{\partial} \log(L_{\alpha-1} + L_\alpha). \tag{2.7}$$

Set

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_\alpha d\overline{z} \wedge dz. \tag{2.8}$$

In the following, we give a definition of the unramified harmonic map as follows:

DEFINITION 2.2 ([**11**]). If $\det(\Omega_\alpha \Omega_\alpha^*) dz^{k_{\alpha+1}} d\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.3), we say that $\varphi_\alpha : S^2 \to G(k_\alpha, n; \mathbb{C})$ is *unramified*. If $\det(\Omega_\alpha \Omega_\alpha^*) dz^{k_{\alpha+1}} d\overline{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.1) (resp. (2.2)) for each $\alpha = 0, 1, 2, \ldots$, we say that the harmonic sequence (2.1) (resp. (2.2)) is *totally unramified*. If (2.1) and (2.2) are both totally unramified, we say that φ is *totally unramified*.

Now recall ([**5**, Section 3A]) that a harmonic map $\varphi : S^2 \to G(k, n; \mathbb{C})$ in (2.1) (resp. (2.2)) is said to be *∂'-irreducible* (resp. *∂''-irreducible*) if rank $\underline{\varphi}$ = rank $\underline{\varphi}_1$ (resp. rank $\underline{\varphi}$ = rank $\underline{\varphi}_{-1}$) and *∂'-reducible* (resp. *∂''-reducible*) otherwise. In particular, let φ be a harmonic map from S^2 to $G(2, n; \mathbb{R})$, then φ is ∂'-irreducible (resp. ∂'-reducible) if and only if φ is ∂''-irreducible (resp. ∂''-reducible). In this case we simply call that φ is irreducible (resp. reducible). Assume that φ_α in (2.3) is ∂'-irreducible and unramified, then $|\det \Omega_\alpha|^2 dz^{k_\alpha} d\overline{z}^{k_\alpha}$ is a well-defined invariant and has no isolated zeros on S^2, then we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{S^2} \partial\overline{\partial} \log |\det \Omega_\alpha|^2 d\overline{z} \wedge dz = -2k_\alpha. \tag{2.9}$$

(C) In this section, we review the rigidity theorem of conformal minimal immersions with constant curvature from S^2 to $\mathbb{C}P^n$.

Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion, a harmonic sequence is derived as follows

$$0 \xrightarrow{\partial'} \underline{\psi}_0^{(n)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\psi} = \underline{\psi}_p^{(n)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\psi}_n^{(n)} \xrightarrow{\partial'} 0, \tag{2.10}$$

for some $p = 0, 1, \dots, n$.

We define a sequence $f_0^{(n)}, \dots, f_n^{(n)}$ of local sections of $\underline{\psi}_0^{(n)}, \dots, \underline{\psi}_n^{(n)}$ inductively such that $f_0^{(n)}$ is a nowhere zero local section of $\underline{\psi}_0^{(n)}$ (without loss of generality, we assume that $\overline{\partial} f_0^{(n)} \equiv 0$) and $f_{p+1}^{(n)} = \psi_p^{(n)\perp}(\partial f_p^{(n)})$ for $p = 0, \dots, n-1$. Then we have some formulae as follows

$$\partial f_p^{(n)} = f_{p+1}^{(n)} + \frac{\langle \partial f_p^{(n)}, f_p^{(n)} \rangle}{|f_p^{(n)}|^2} f_p^{(n)}, \quad p = 0, \dots, n,$$

$$\overline{\partial} f_p^{(n)} = -\frac{|f_p^{(n)}|^2}{|f_{p-1}^{(n)}|^2} f_{p-1}^{(n)}, \qquad p = 1, \dots, n.$$

Let

$$l_p^{(n)} = |f_{p+1}^{(n)}|^2 / |f_p^{(n)}|^2, \quad p = 0, \dots, n-1, \quad l_{-1}^{(n)} = l_n^{(n)} = 0. \tag{2.11}$$

Then Bolton et al ([**3**]) proved the following unintegrated Plücker formula

$$\partial \overline{\partial} \log l_p^{(n)} = l_{p+1}^{(n)} - 2 l_p^{(n)} + l_{p-1}^{(n)}, \quad p = 0, \dots, n-1.$$

Let $F_p^{(n)} = f_0^{(n)} \wedge \cdots \wedge f_p^{(n)}$ be a local lift of the p-th osculating curve, where $p = 0, \dots, n$. We write $F_p^{(n)} = g(z) \tilde{F}_p^{(n)}$, where $g(z)$ is the greatest common divisor of the $\binom{n+1}{p+1}$ components of $F_p^{(n)}$. Then $\tilde{F}_p^{(n)}$ is a nowhere zero holomorphic curve, and the degree $\delta_p^{(n)}$ of $F_p^{(n)}$ is given by $\delta_p^{(n)} = (1/2\pi\sqrt{-1}) \int_{S^2} \partial\overline{\partial} \log |F_p^{(n)}|^2 d\overline{z} \wedge dz$, which is equal to the degree of the polynomial function $\tilde{F}_p^{(n)}$. By a simple calculation we have

$$\delta_p^{(n)} = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} l_p^{(n)} d\overline{z} \wedge dz, \tag{2.12}$$

which is consistent with (2.8) in the case $k = 1$.

Moreover, if (2.10) is a totally unramified harmonic sequence (i.e. $\psi_p^{(n)}$ is unramified, $p = 0, \dots, n$), then (cf. [**3**])

$$\delta_p^{(n)} = (p+1)(n-p). \tag{2.13}$$

Let

$$0 \longrightarrow \underline{V}_0^{(n)} \xrightarrow{\partial'} \underline{V}_1^{(n)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{V}_n^{(n)} \xrightarrow{\partial'} 0,$$

which is called the Veronese sequence, defined by $V_p^{(n)} = (v_{p,0}, \dots, v_{p,n})^T$, where, for $z \in S^2$,

$$v_{p,r}(z)=\frac{p!}{(1+z\overline{z})^p}\sqrt{\binom{n}{r}}z^{r-p}\sum_k(-1)^k\binom{r}{p-k}\binom{n-r}{k}(z\overline{z})^k,$$

$\max\{0,p-r\}\le k\le\min\{p,n-r\}$, and $|V_p^{(n)}|^2=(n!p!/(n-p)!)(1+z\overline{z})^{n-2p}$. Each map $\underline{V}_p^{(n)}$ has induced metric

$$ds_p^2=\frac{n+2p(n-p)}{(1+z\overline{z})^2}dzd\overline{z}, \tag{2.14}$$

the corresponding constant curvature K_p and constant Kähler angle θ_p are given by

$$K_p=\frac{4}{n+2p(n-p)},\quad \left(\tan\frac{1}{2}\theta_p\right)^2=\frac{p(n-p+1)}{(p+1)(n-p)}. \tag{2.15}$$

By Calabi's rigidity theorem, Bolton et al proved the following rigidity result (cf. [**3**]).

Lemma 2.3 ([**3**]). *Let $\psi:S^2\to\mathbb{C}P^n$ be a linearly full conformal minimal immersion of constant curvature. Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the harmonic sequence determined by ψ is the Veronese sequence.*

3. Characterization of harmonic maps from S^2 to $G(2,5;\mathbb{R})$.

We analyze harmonic maps from S^2 to $G(2,5;\mathbb{R})$ by reducible and irreducible case respectively. It follows from [**2**] that all reducible harmonic maps from S^2 to $G(2,5;\mathbb{R})$ with finite isotropy order have been characterized by harmonic maps from S^2 to $\mathbb{C}P^4$, and for the strongly isotropic ones we will discuss in detail in Subsection 4.1 below.

Now we only consider irreducible harmonic maps $\varphi:S^2\to G(2,5;\mathbb{R})$ of isotropy order r. If φ has finite isotropy order, then $r=1$ by ([**2**, Proposition 2.8 and Lemma 2.15]); if φ is strongly isotropic, then $r=\infty$. But for any irreducible harmonic map from S^2 to $G(2,n;\mathbb{R})$, if it is strongly isotropic, then we have $n\ge 6$. Therefore the isotropy order r of φ must be finite and $r=1$.

Here we state one of Bahy-El-Dien and Wood' results ([**2**, Theorem 4.7]) as follows:

Lemma 3.1 ([**2**]). *Let $\varphi:S^2\to G(2,5;\mathbb{R})$ be an irreducible harmonic map of isotropy order r. We know that $r=1$. Then there is a unique sequence of harmonic maps $\varphi^i:S^2\to G(2,5;\mathbb{C})$, $(i=0,1,2)$ such that*

- (i) *$\underline{\varphi}^0$ is a real mixed pair, in fact $\underline{\varphi}^0=\overline{\underline{f}}_0^{(4)}\oplus\underline{f}_0^{(4)}$, where $f_0^{(4)}\in H_5^1$;*
- (ii) *$\underline{\varphi}=\underline{\varphi}^2$;*
- (iii) *$\underline{\varphi}^1$ is obtained from $\underline{\varphi}^0$ by forward replacement of $\underline{f}_0^{(4)}$;*
- (iv) *$\underline{\varphi}^2$ is obtained from $\underline{\varphi}^1$ by backward replacement of $\underline{V}^\perp\cap\underline{\varphi}^1$, where $\underline{V}$ is a holomorphic line subbundle of $\underline{\varphi}^1$ not equal to the image of the first ∂'-return map of $\underline{\varphi}^1$.*

Firstly we recall ([**2**, Section 4]) that H_n^s denote the set of all holomorphic maps $f_0^{(m)}: S^2 \to \mathbb{C}P^m \subset \mathbb{C}P^{n-1},\ m < n$ satisfying

$$\begin{cases} \langle f_i^{(m)}, \overline{f}_0^{(m)} \rangle = 0 \quad (0 \le i \le 2s+1), \\ \langle f_{2s+2}^{(m)}, \overline{f}_0^{(m)} \rangle \neq 0 \end{cases}$$

for any integers $n \ge 3,\ s \ge 0$, where $0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{2s+1}^{(m)} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0$ is a harmonic sequence in $\mathbb{C}P^m \subset \mathbb{C}P^{n-1}$.

Let $\varphi: S^2 \to G(2,5;\mathbb{R})$ be a linearly full irreducible harmonic map of isotropy order 1. In the following we characterize φ explicitly by Lemma 3.1.

In (i) of Lemma 3.1, φ^0 with isotropy order 3 belongs to the harmonic sequence as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_1^{(4)} \xleftarrow{\partial''} \underline{\varphi}^0 \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{3.1}$$

where $\underline{\varphi}^0 = \underline{\overline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ and

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0 \tag{3.2}$$

is a harmonic sequence in $\mathbb{C}P^4$. Since $f_0^{(4)} \in H_5^1$, then we have

$$\begin{cases} \langle \overline{f}_0^{(4)}, f_i^{(4)} \rangle = 0 \quad \text{for } 0 \le i \le 3, \\ \langle \overline{f}_0^{(4)}, f_4^{(4)} \rangle \neq 0. \end{cases} \tag{3.3}$$

Thus we get

$$\underline{\overline{f}}_0^{(4)} = \underline{f}_4^{(4)}, \quad \underline{\overline{f}}_1^{(4)} = \underline{f}_3^{(4)}, \quad \underline{\overline{f}}_2^{(4)} = \underline{f}_2^{(4)},$$

and

$$l_0^{(4)} = l_3^{(4)}, \quad l_1^{(4)} = l_2^{(4)}.$$

By (iii) of Lemma 3.1, $\underline{\varphi}^1$ is obtained from $\underline{\varphi}^0$ by forward replacement of $\underline{f}_0^{(4)}$, using (3.1) we have

$$\underline{\varphi}^1 = \underline{\overline{f}}_0^{(4)} \oplus \underline{f}_1^{(4)}.$$

The isotropy order of φ^1 is 2, and a harmonic sequence is derived as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_3^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_2^{(4)} \xleftarrow{\partial''} \underline{\varphi}_{-1}^1 \xleftarrow{\partial''} \underline{\varphi}^1 \xrightarrow{\partial'} \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{3.4}$$

where $\underline{\overline{\varphi}}^1_{-1} = \underline{\varphi}^1$.

From (3.4), the image of the first ∂'-return map of $\underline{\varphi}^1$ is $\underline{\overline{f}}_0^{(4)}$. By (iv) of Lemma 3.1, let $V = f_1^{(4)} + x_0 \overline{f}_0^{(4)}$, where x_0 is a smooth function on S^2 expect some isolated points. Moreover, let $X = -|f_0^{(4)}|^2 \overline{x}_0 f_1^{(4)} + |f_1^{(4)}|^2 \overline{f}_0^{(4)}$, it satisfies $\underline{X} = \underline{V}^\perp \cap \underline{\varphi}^1$. Since $\underline{\varphi}^2$ is obtained from $\underline{\varphi}^1$ by backward replacement of $\underline{X}$, then we have $\underline{\varphi}^2 = \underline{V} \oplus \underline{W}$ where $W = \varphi^{1\perp}\overline{\partial} X$. Moreover, $\underline{\varphi}^2$ with isotropy order 1 belongs to the harmonic sequence as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_3^{(4)} \xleftarrow{\partial''} \underline{Y} \oplus \underline{\overline{f}}_2^{(4)} \xleftarrow{\partial''} \underline{\varphi}^2 \xrightarrow{\partial'} \underline{\overline{Y}} \oplus \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{3.5}$$

where $\underline{Y} = \underline{W}^\perp \cap \underline{\overline{\varphi}}^1$. Applying the equation $W = \varphi^{1\perp}\overline{\partial} X$ we obtain

$$W = |f_1^{(4)}|^2 \overline{V}, \tag{3.6}$$

which implies that

$$\underline{W} = \underline{\overline{V}},\ \underline{Y} = \underline{\overline{X}}.$$

Obviously, $\underline{\overline{X}}, \underline{X}, \underline{\overline{V}}$ and $\underline{V}$ are mutually orthogonal. Then we have $\underline{\varphi} = \underline{\overline{V}} \oplus \underline{V}$ and (3.5) becomes

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_3^{(4)} \xleftarrow{\partial''} \underline{\overline{X}} \oplus \underline{\overline{f}}_2^{(4)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{X} \oplus \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0.$$

Since $\underline{V}$ is a holomorphic line subbundle of $\underline{\varphi}^1$, we get

$$\varphi^1(\overline{\partial} V) \in \underline{V}. \tag{3.7}$$

Through a direct computation, condition (3.7) is equivalent to the following equation

$$\partial \overline{x}_0 + \overline{x}_0 \partial \log |f_0^{(4)}|^2 = 0. \tag{3.8}$$

Then we have

PROPOSITION 3.2. *The map $\varphi : S^2 \to G(2,5;\mathbb{R})$ is a linearly full irreducible harmonic map if and only if $\underline{\varphi} = \underline{\overline{V}} \oplus \underline{V}$, where $V = f_1^{(4)} + x_0 \overline{f}_0^{(4)}$, $f_0^{(4)} \in H_5^1$, and the corresponding coefficient x_0 satisfies the equation* (3.8).

PROOF. Through the construction of φ as shown above, the necessity is obvious. Since $f_0^{(4)} \in H_5^1$, using (3.8), this is a straightforward computation $\varphi^\perp \partial\overline{\partial}\varphi\varphi = 0$, which implies that $\underline{\varphi}$ is harmonic. Thus we get the sufficiency. $\square$

4. Conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

In this section, we regard harmonic maps from S^2 to $G(2,5;\mathbb{R})$ as conformal minimal immersions of S^2 in $G(2,5;\mathbb{R})$. Then we consider the harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$ by reducible case and irreducible case. So we divide these two cases into the following two subsections.

4.1. Reducible harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full reducible harmonic map, then by ([**2**, Proposition 2.12]) we know that φ is a real mixed pair with finite isotropy order 1 or 3, or φ is strongly isotropic. In the following we discuss these three cases with φ of constant curvature respectively.

(I) If φ is a linearly full real mixed pair with isotropy order 1, then

$$\underline{\varphi} = \underline{\overline{f}}_0^{(m)} \oplus \underline{f}_0^{(m)}$$

for $2 \le m \le 4$. By using φ, a harmonic sequence is derived as follows

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_1^{(m)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0,$$

where

$$0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0$$

is a harmonic sequence in $\mathbb{C}P^m \subset \mathbb{C}P^4$ satisfies

$$\langle \overline{f}_0^{(m)}, f_0^{(m)} \rangle = 0, \quad \langle \overline{f}_0^{(m)}, f_1^{(m)} \rangle = 0, \quad \langle \overline{f}_0^{(m)}, f_2^{(m)} \rangle \neq 0. \tag{4.1}$$

The induced metric of φ is given by

$$ds^2 = 2l_0^{(m)} dz d\overline{z}, \tag{4.2}$$

where $l_0^{(m)} dz d\overline{z}$ is the induced metric of $f_0^{(m)}$.

Then we prove

Lemma 4.1. *There does not exist linearly full real mixed pair of constant curvature from S^2 to $G(2,5;\mathbb{R})$ with isotropy order* 1.

Proof. Since $\underline{\varphi}$ is of constant curvature, using (4.2) we get that the constant curvature K of φ satisfies $K = 2/m$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(m)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(5)$ and $V_0^{(m)}$ has the standard expression given in part (C) of Section 2 (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^5$). Then from (4.1) we have

$$\begin{cases} \langle UV_0^{(m)}, \overline{UV_0^{(m)}} \rangle = 0, \\ \langle UV_1^{(m)}, \overline{UV_0^{(m)}} \rangle = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathrm{tr} W V_0^{(m)} V_0^{(m)T} = 0, \\ \mathrm{tr} W V_1^{(m)} V_0^{(m)T} = 0, \end{cases} \tag{4.3}$$

where $W = U^T U$, it satisfies $W \in U(5)$ and $W^T = W$.

Define a set

$$G_W \triangleq \{U \in U(5) | U^T U = W\}. \tag{4.4}$$

For a given W, the following can be easily checked

(1) $\forall\ U \in G_W,\ A \in SO(5)$, we have $AU \in G_W$;
(2) $\forall\ U, V \in G_W, \exists\ A \in SO(5),\ s.t.\ U = AV.$

In the following we discuss W in cases $m = 2, 3, 4$ respectively.

(Ia) $m = 4,\ K = 1/2.$

By the standard expression of $V_0^{(4)}$ and $V_1^{(4)}$, we get $V_1^{(4)} V_0^{(4)T}$ is a polynomial matrix in z and $\overline{z}$. But W is a constant matrix. Using the method of indeterminate coefficients by (4.3), assume $W = (a_{ij}),\ 1 \le i, j \le 5$, we get

$$W = \begin{pmatrix} 0 & 0 & a_{13} & -\sqrt{6}a_{23} & a_{15} \\ 0 & (-\sqrt{6}/2)a_{13} & a_{23} & a_{24} & -\sqrt{6}a_{34} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ -\sqrt{6}a_{23} & a_{24} & a_{34} & (-\sqrt{6}/2)a_{35} & 0 \\ a_{15} & -\sqrt{6}a_{34} & a_{35} & 0 & 0 \end{pmatrix},$$

where

$$a_{15} + 3a_{33} + 4a_{24} = 0.$$

Applying the equation $a_{15} + 3a_{33} + 4a_{24} = 0$, using the property of the unitary matrix, this is a straightforward computation

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & -a_{24} \\ 0 & 0 & 0 & a_{24} & 0 \\ 0 & 0 & -a_{24} & 0 & 0 \\ 0 & a_{24} & 0 & 0 & 0 \\ -a_{24} & 0 & 0 & 0 & 0 \end{pmatrix} \in U(5).$$

With a simple test we have

$$\mathrm{tr} W V_2^{(4)} V_0^{(4)T} = 0,$$

i.e. $\langle \overline{f}_0^{(4)}, f_2^{(4)} \rangle = 0$, which contradicts $r = 1$. Thus this case does not exist.

(Ib) $m = 3,\ K = 2/3$.

Similar to (Ia), we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & (-2/\sqrt{3})a_{13} & (-1/3)a_{14} & a_{24} & a_{25} \\ a_{13} & (-1/3)a_{14} & (-2/\sqrt{3})a_{24} & 0 & a_{35} \\ a_{14} & a_{24} & 0 & 0 & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{pmatrix}.$$

Moreover, using the property of the unitary matrix, we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ 0 & (-2/\sqrt{3})a_{13} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & a_{45} & 0 \end{pmatrix},$$

which contradicts $W \in U(5)$, thus this case does not exist.

(Ic) $m = 2,\ K = 1$.

From (4.3), this is a straightforward computation

$$W = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & -a_{13} & 0 & a_{24} & a_{25} \\ a_{13} & 0 & 0 & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{pmatrix}.$$

Moreover, using the property of the unitary matrix, we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ 0 & -a_{13} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{45} & a_{55} \end{pmatrix}, \tag{4.5}$$

where $|a_{13}| = 1$ and $\left(\begin{smallmatrix} a_{44} & a_{45} \\ a_{45} & a_{55} \end{smallmatrix}\right) \in U(2)$. Obviously φ is not linearly full in this condition.

In summary we get the conclusion. □

(II) If φ is a linearly full real mixed pair with isotropy order 3, then we have $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ belongs to the following harmonic sequence:

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_4^{(4)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_1^{(4)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{4.6}$$

where

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0$$

is a harmonic sequence in $\mathbb{C}P^4$ satisfies

$$\overline{\underline{f}}_0^{(4)} = \underline{f}_4^{(4)}, \quad \overline{\underline{f}}_1^{(4)} = \underline{f}_3^{(4)}, \quad \overline{\underline{f}}_2^{(4)} = \underline{f}_2^{(4)}. \tag{4.7}$$

The induced metric of φ is given by $ds^2 = 2l_0^{(4)} dz d\overline{z}$. Since $\underline{\varphi}$ is of constant curvature, then the constant curvature K of φ is $1/2$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(4)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that $f_0^{(4)} = UV_0^{(4)}$, where $U \in U(5)$ and $V_0^{(4)}$ has the standard expression given in part (C) of Section 2.

Then we have

Lemma 4.2. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full real mixed pair with isotropy order 3. If the curvature K of φ is constant, then up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{UV}_0^{(4)}} \oplus \underline{UV}_0^{(4)}$ with $K = 1/2$ for some $U \in G \triangleq \{U \in U(5) | \overline{U} = UW_0\}$, where $W_0 = antidiag\{1,-1,1,-1,1\}$.*

Proof. Equation $\overline{\underline{f}}_0^{(4)} = \underline{f}_4^{(4)}$ is equivalent to

$$\overline{UV_0^{(4)}} = \lambda U V_4^{(4)}, \tag{4.8}$$

where λ is a parameter.

Set $W_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. From part (C) of Section 2, we get

$$V_0^{(4)} = (1,\ 2z,\ \sqrt{6}z^2,\ 2z^3,\ z^4)^T$$

and

$$V_4^{(4)} = \frac{4!}{(1+z\overline{z})^4} (\overline{z}^4,\ -2\overline{z}^3,\ \sqrt{6}\overline{z}^2,\ -2\overline{z},\ 1)^T,$$

which implies $V_4^{(4)} = (4!/(1+z\overline{z})^4) W_0 \overline{V}_0^{(4)}$. Then condition (4.8) becomes

$$\overline{U} = UW_0,$$

Define a set

$$G \triangleq \{U \in U(5) | \overline{U} = UW_0\},$$

then the following can be easily checked

(1) $\forall\ U \in G,\ A \in SO(5)$, we have $AU \in G$;
(2) $\forall\ U, V \in G, \exists\ A \in SO(5),\ s.t.\ U = AV$.

So we get the conclusion. $\square$

REMARK 4.3. $G \neq \emptyset$. Simply choose $U_0 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ \sqrt{-1}/\sqrt{2} & 0 & 0 & 0 & -\sqrt{-1}/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & \sqrt{-1}/\sqrt{2} & 0 & \sqrt{-1}/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, we have $U_0 \in G$ and $\forall\ U \in G$ can be obtained from U_0 by an $SO(5)$-motion. Then up to an isometry of $G(2,5;\mathbb{R})$,

$$\underline{\varphi} = \underline{\overline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$$

with

$$f_0^{(4)} = \big(1 + z^4,\ \sqrt{-1}(1 - z^4),\ 2(z - z^3),\ 2\sqrt{-1}(z + z^3),\ 2\sqrt{3}z^2\big)^T. \tag{4.9}$$

(III) If φ is a linearly full reducible harmonic map with isotropy order ∞. By using φ, a harmonic sequence is derived as follows

$$0 \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_m^{(m)} \overset{\partial''}{\longleftarrow} \cdots \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_{p+1}^{(m)} \overset{\partial''}{\longleftarrow} \underline{\varphi} \overset{\partial'}{\longrightarrow} \underline{f}_{p+1}^{(m)} \overset{\partial'}{\longrightarrow} \cdots \overset{\partial'}{\longrightarrow} \underline{f}_m^{(m)} \overset{\partial'}{\longrightarrow} 0, \tag{4.10}$$

where $m \leq 4$ and $\underline{\overline{f}}_m^{(m)}, \ldots, \underline{\overline{f}}_{p+1}^{(m)}, \underline{\varphi}, \underline{f}_{p+1}^{(m)}, \ldots, \underline{f}_m^{(m)}$ are mutually orthogonal. Since φ is a map from S^2 to $G(2,5;\mathbb{R})$, then $m - p \leq 1$.

Then we have

LEMMA 4.4. *There does not exist linearly full harmonic map of constant curvature from S^2 to $G(2,5;\mathbb{R})$ with isotropy order ∞.*

PROOF. From (4.10) we know that $f_p^{(m)}$ and $\overline{f}_p^{(m)}$ are two local sections of $\underline{\varphi}$.

If $\underline{\overline{f}}_p^{(m)} = \underline{f}_p^{(m)}$, applying the inequality $m - p \leq 1$, we have $p = 1, m = 2$. Then (4.10) becomes

$$0 \overset{\partial''}{\longleftarrow} \underline{f}_0^{(2)} \overset{\partial''}{\longleftarrow} \underline{\varphi} \overset{\partial'}{\longrightarrow} \underline{f}_2^{(2)} \overset{\partial'}{\longrightarrow} 0. \tag{4.11}$$

From (4.11), by a straightforward calculation, we have

$$\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g},$$

where $\underline{g}$ is a constant vector in $\mathbb{C}^5$ and $\overline{\underline{f}}_1^{(2)} = \underline{f}_1^{(2)}$. Obviously φ is included in $G(2,4;\mathbb{R})$, so it is not linearly full.

If $\overline{\underline{f}}_p^{(m)} \neq \underline{f}_p^{(m)}$, this is a straightforward computation

$$\mathrm{tr}\partial\varphi\overline{\partial}\varphi = 2l_p^{(m)},$$

i.e. the induced metric of φ is given by $ds^2 = 2l_p^{(m)} dzd\overline{z}$. Since $\underline{\varphi}$ is of constant curvature, then the constant curvature K of φ satisfies $K = 2/(m-p)(p+1)$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(m)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ so that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(5)$ and $V_0^{(m)}$ has the standard expression given in part (C) of Section 2 (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^5$). Here $m = 3$ or 4.

For $m = 4$, $p = 3$, we can easily check that for any $U \in U(5)$ satisfies $\mathrm{tr}U^TUV_4^{(4)}V_4^{(4)T} = 0$, we also have $\mathrm{tr}U^TUV_3^{(4)}V_3^{(4)T} = 0$. Thus we have $\underline{\varphi} = \overline{\underline{f}}_3^{(4)} \oplus \underline{f}_3^{(4)}$, which implies that φ is irreducible. For $m = 3$, $p = 2$, a straightforward calculation shows that U^TU does not exist.

In summary we get the conclusion. □

Remark 4.5. In the case $\overline{\underline{f}}_p^{(m)} = \underline{f}_p^{(m)}$ in Lemma 4.4, we have $\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g}$, where $\underline{g}$ is a constant vector in $\mathbb{C}^5$ and $\overline{\underline{f}}_1^{(2)} = \underline{f}_1^{(2)}$. Since $\underline{\varphi}$ is of constant curvature, then the curvature of $\underline{f}_1^{(2)}$ is also a constant. By Lemma 2.3, there exists some $U \in U(5)$ so that

$$\underline{f}_1^{(2)} = \underline{UV}_1^{(2)}, \quad \overline{\underline{UV}}_1^{(2)} = \underline{UV}_1^{(2)}.$$

By a straightforward calculation, we have, up to an isometry of $G(2,5;\mathbb{R})$,

$$f_1^{(2)} = \left(\sqrt{-1}(z-\overline{z}),\ z\overline{z}-1,\ z+\overline{z}\right)^T,$$

and the curvature of $\underline{\varphi}$ is 1. Here $\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g}$ is a linearly full harmonic map from S^2 into $G(2,4;\mathbb{R})$. Moreover we can check that φ is totally geodesic.

From Lemma 4.1, 4.2 and 4.4 we have

Proposition 4.6. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full reducible harmonic map with constant curvature K. Then, up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ with $K = 1/2$, where $f_0^{(4)}$ satisfies* (4.9).

4.2. Irreducible harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

In this section, we discuss linearly full irreducible harmonic maps from S^2 to $G(2,5;\mathbb{R})$ with constant curvature in Section 3.

Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full irreducible harmonic map of isotropy order r. From the discussion of Section 3, we know that $r = 1$. By Proposition 3.2, we choose local frame

$$e_1 = \frac{\overline{V}}{|V|},\ e_2 = \frac{V}{|V|},\ e_3 = \frac{X}{|X|},\ e_4 = \frac{f_2^{(4)}}{|f_2^{(4)}|},\ e_5 = \frac{\overline{X}}{|X|},\ e_6 = \frac{\overline{f}_2^{(4)}}{|f_2^{(4)}|},\ e_7 = \frac{f_3^{(4)}}{|f_3^{(4)}|},$$

where $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$ and x_0 is a smooth function on S^2 except some isolate points. Since the isotropy order of φ is 1, the local frame we choose here is not unitary frame.

Set $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_{-1} = (e_5, e_6)$, and $W_2 = (e_7)$, then by (2.5), we obtain

$$\Omega_0 = \begin{pmatrix} -\frac{|f_1^{(4)}|}{|f_0^{(4)}|} & \frac{\langle \partial V, X\rangle}{|X||V|} \\ 0 & \frac{|f_2^{(4)}|}{|V|} \end{pmatrix}, \quad \Omega_{-1} = -\begin{pmatrix} \frac{\langle \partial V, X\rangle}{|X||V|} & \frac{|f_2^{(4)}|}{|V|} \\ -\frac{|f_1^{(4)}|}{|f_0^{(4)}|} & 0 \end{pmatrix}, \quad \Omega_1 = \left(0,\ \frac{|f_3^{(4)}|}{|f_2^{(4)}|}\right). \tag{4.12}$$

From (4.12), applying the equation $L_\alpha = \mathrm{tr}(\Omega_\alpha\Omega_\alpha^*)$, a straightforward computation shows

$$L_0 = L_{-1} = \frac{\langle \partial V, X\rangle\langle X, \partial V\rangle}{|X|^2|V|^2} + \frac{|f_2^{(4)}|^2}{|V|^2} + l_0^{(4)}, \tag{4.13}$$

$$L_1 = l_2^{(4)}, \tag{4.14}$$

$$|\det\Omega_0|^2 dz^2 d\overline{z}^2 = \frac{|f_0^{(4)}|^2}{|V|^2}(l_0^{(4)})^2 l_1^{(4)} dz^2 d\overline{z}^2, \tag{4.15}$$

$$\det\Omega_1\Omega_1^* dzd\overline{z} = l_2^{(4)} dzd\overline{z}. \tag{4.16}$$

Since $\varphi_{-1}, \varphi_0, \varphi_1$ are not mutually orthogonal, we can't use the unintegrated Plücker formula directly. But using (4.13) and (4.14), by a straightforward calculation, we also have

$$\partial\overline{\partial}\log|\det\Omega_0|^2 = L_{-1} - 2L_0 + L_1. \tag{4.17}$$

If φ is totally unramified, then $|\det\Omega_0|^2 dz^2 d\overline{z}^2 \neq 0$ and $\det\Omega_1\Omega_1^* dzd\overline{z} \neq 0$ everywhere on S^2. It follows from (4.15) and (4.16) that $l_p^{(4)} dzd\overline{z} \neq 0$ $(p = 0, 1, 2)$ everywhere on S^2 and $(|f_0^{(4)}|^2/|V|^2)l_0^{(4)}$ is well-defined on S^2. In Section 3 we have $l_0^{(4)} = l_3^{(4)}$ and $l_1^{(4)} = l_2^{(4)}$. So $l_p^{(4)} dzd\overline{z} \neq 0$ $(p = 0, 1, 2, 3)$ everywhere on S^2. Then the harmonic sequence

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0$$

is also totally unramified.

In this case, we prove

Proposition 4.7. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full irreducible totally unramified harmonic map with constant curvature K. Then, up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{UV}_1^{(4)}} \oplus \underline{UV_1^{(4)}}$ with $K = 1/5$ for some $U \in G$.*

Proof. Since the harmonic sequence $\underline{f}_0^{(4)}, \dots, \underline{f}_4^{(4)} : S^2 \to \mathbb{C}P^4$ is totally unramified, it follows from (2.13) that

$$\delta_0^{(4)} = \delta_3^{(4)} = 4, \quad \delta_1^{(4)} = \delta_2^{(4)} = 6. \tag{4.18}$$

From (2.8) and (2.9) we have

$$\delta_1 - 2\delta_0 + \delta_{-1} = -4, \tag{4.19}$$

where $\delta_\alpha = (1/2\pi\sqrt{-1}) \int_{S^2} L_\alpha d\bar{z} \wedge dz$, $\alpha = -1, 0, 1$. It follows from (4.13) and (4.14) that $\delta_0 = \delta_{-1}$ and $\delta_1 = \delta_2^{(4)} = 6$. So that

$$\delta_0 = 10. \tag{4.20}$$

Since φ is of constant curvature K, using (4.20) we know that $K = 1/5$, and we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that the induced metric $ds^2 = 2L_0 dz d\bar{z}$ of φ is given by

$$ds^2 = \frac{20}{(1+z\bar{z})^2} dz d\bar{z},$$

which implies

$$L_0 = \frac{10}{(1+z\bar{z})^2}. \tag{4.21}$$

Consider the local lift of the p-th osculating curve $F_p^{(4)} = f_0^{(4)} \wedge \cdots \wedge f_p^{(4)}$ $(p = 0, \dots, 4)$. We choose a nowhere zero holomorphic $\mathbb{C}^5$-valued function $f_0^{(4)}$, then $F_p^{(4)}$ is a nowhere zero holomorphic curve and is a polynomial function on $\mathbb{C}$ of degree $\delta_p^{(4)}$ satisfying $\partial\bar{\partial} \log |F_p^{(4)}|^2 = l_p^{(4)}$. So using (4.13) (4.14) (4.15) and (4.17), we obtain

$$\partial\bar{\partial} \log \frac{(1+z\bar{z})^{10} |f_0^{(4)}|^2}{|F_0^{(4)}|^6 |V|^2} = 0. \tag{4.22}$$

By (4.15) we know that $(|f_0^{(4)}|^2/|V|^2) l_0^{(4)}$ is a globally defined function without zeros on S^2. Then it follows from (4.18) that $(1+z\bar{z})^{10} |f_0^{(4)}|^2 / |F_0^{(4)}|^6 |V|^2$ is globally defined on $\mathbb{C}$ and has a positive constant limit $1/c$ as $z \to \infty$. Thus from (4.22) we obtain

$$\frac{(1+z\bar{z})^{10} |f_0^{(4)}|^2}{|F_0^{(4)}|^6 |V|^2} = \frac{1}{c}.$$

Moreover we have

$$|V|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^4}. \tag{4.23}$$

Applying the equation $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$, (4.23) becomes

$$|x_0|^2|f_0^{(4)}|^4 + |F_1^{(4)}|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^2}. \tag{4.24}$$

By equation (3.8) we get $\overline{\partial}(x_0|f_0^{(4)}|^2) = 0$. Observing (4.24), we find that $x_0|f_0^{(4)}|^2$ is a holomorphic function on $\mathbb{C}$ at most with the pole $z = \infty$. So it is a polynomial function about z. Without loss of generality, we set

$$x_0|f_0^{(4)}|^2 = h(z), \tag{4.25}$$

then (4.24) becomes

$$|h|^2 + |F_1|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^2}. \tag{4.26}$$

Since both sides of (4.26) are polynomial functions and $\delta_0^{(4)} = 4$, then we have

$$|f_0^{(4)}|^2 = \mu(1+z\overline{z})^4, \tag{4.27}$$

where μ is a real parameter.

If $h \neq 0$, then $1+z\overline{z}$ is a factor of it, which contradicts the fact that h is holomorphic. Thus we have $h = 0$, which implies that $x_0 = 0$. Then we get

$$V = f_1^{(4)}, \quad \underline{\varphi} = \underline{\overline{f}_1^{(4)}} \oplus \underline{f_1^{(4)}}.$$

From (4.27), by Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(4)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ so that $f_0^{(4)} = UV_0^{(4)}$, where $U \in U(5)$ and $V_0^{(4)}$ has the standard expression given in part (C) of Section 2. Thus we have $\underline{\varphi} = \underline{\overline{UV}_1^{(4)}} \oplus \underline{UV_1^{(4)}}$. To determine φ, we just need to determine the matrix U. Since $\underline{\overline{f}_0^{(4)}} = \underline{f_4^{(4)}}$, using the standard expression of $V_0^{(4)}$, we have $\overline{U} = UW_0$. Similar to Lemma 4.2, we get the conclusion. □

REMARK 4.8. We choose the same U_0 as the one shown in Remark 4.3, then

$$\underline{\varphi} = \underline{\overline{f}_1^{(4)}} \oplus \underline{f_1^{(4)}} \in G(2,5;\mathbb{R})$$

with

$$f_1^{(4)} = \big((2(z^3 - \overline{z}),\ -2\sqrt{-1}(z^3 + \overline{z}),\ (1 - 3z\overline{z}) - z^2(3 - z\overline{z}),$$
$$\sqrt{-1}[(1 - 3z\overline{z}) + z^2(3 - z\overline{z})],\ 2\sqrt{3}z(1 - z\overline{z})\big)^T. \tag{4.28}$$

Moreover we can check that φ is totally geodesic.

By Proposition 4.6 and Proposition 4.7, we obtain a classification of conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ as follows:

THEOREM 4.9. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full conformal minimal immersion of constant curvature. Then, up to an isometry of $G(2,5;\mathbb{R})$,*

- (i) *If φ is reducible, $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ with constant curvature $1/2$, where $f_0^{(4)}$ satisfies* (4.9);
- (ii) *If φ is totally unramified irreducible, $\underline{\varphi} = \overline{\underline{f}}_1^{(4)} \oplus \underline{f}_1^{(4)}$ with constant curvature $1/5$, where $f_1^{(4)}$ satisfies* (4.28).

Theorem 4.9 shows that all linearly full totally unramified conformal minimal immersions of two-spheres in Q_3 with constant curvature are presented by the Veronese curves in $\mathbb{C}P^4$. We believe that these maps are homogeneous.

For the isotropy order r of φ, we have

REMARK 4.10. Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full conformal minimal immersion with constant curvature. Suppose that the isotropy order of φ is r. We then have

- (i) If φ is reducible, then $r = 3$;
- (ii) If φ is irreducible, then $r = 1$.

In the following, we discuss the Kähler angle of a curve from S^2 to Q_n. Throughout this section, we agree on the following ranges of indices

$$1 \le \alpha, \beta, \gamma, \ldots \le n, \quad 1 \le A, B, C, \ldots \le n+1.$$

Let $f : S^2 \to Q_n$ be a map, and $\tau : Q_n \to \mathbb{C}P^{n+1}$ denote the inclusion. The algebraic variety is given by

$$(w_0)^2 + (w_1)^2 + \cdots + (w_{n+1})^2 = 0,$$

where $(w_0, w_1, \ldots, w_{n+1})$ are homogeneous coordinate system on $\mathbb{C}P^{n+1}$. If $w^0 \neq 0$, let $z_1 = w_1/w_0, \ldots, z_{n+1} = w_{n+1}/w_0$, then we have

$$1 + z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0, \tag{4.29}$$

where $(z_1, \ldots, z_{n+1})$ are inhomogeneous coordinate system on $\mathbb{C}P^{n+1}$. A natural complex structure J on $\mathbb{C}P^{n+1}$ is defined by $J(\partial/\partial z_A) = \sqrt{-1}(\partial/\partial z_A)$. Suppose $z_{n+1} \neq 0$, then we have complex coordinate system $(\widetilde{z}_1, \ldots, \widetilde{z}_n)$ of Q_n such that $\widetilde{z}_1 = z_1, \ldots, \widetilde{z}_n = z_n$.

Therefore a natural complex structure $\widetilde{J}$ on Q_n is given by $\widetilde{J}(\partial/\partial\widetilde{z}_\alpha) = \sqrt{-1}(\partial/\partial\widetilde{z}_\alpha)$.

By differentiating (4.29) we obtain

$$\frac{\partial z_{n+1}}{\partial\widetilde{z}_\alpha} = -\frac{z_\alpha}{z_{n+1}},$$

which implies that

$$\tau_*\left(\frac{\partial}{\partial\widetilde{z}_\alpha}\right) = \frac{\partial}{\partial z_\alpha} - \frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}}.$$

Then we have

$$(\tau\circ f)_*\left(\frac{\partial}{\partial z}\right) = \sum_\alpha\left(\frac{\partial f^\alpha}{\partial z}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial z}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}} + \frac{\partial\overline{f}^\alpha}{\partial z}\frac{\partial}{\partial\overline{z}_\alpha} - \frac{\partial\overline{f}^\alpha}{\partial z}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial\overline{z}_{n+1}}\right), \quad (4.30)$$

and

$$(\tau\circ f)_*\left(\frac{\partial}{\partial\overline{z}}\right) = \sum_\alpha\left(\frac{\partial f^\alpha}{\partial\overline{z}}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial\overline{z}}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}} + \frac{\partial\overline{f}^\alpha}{\partial\overline{z}}\frac{\partial}{\partial\overline{z}_\alpha} - \frac{\partial\overline{f}^\alpha}{\partial\overline{z}}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial\overline{z}_{n+1}}\right), \quad (4.31)$$

where $f(z) = (f^1(z), \ldots, f^n(z))$.

For a conformal immersion $f : S^2 \to Q_n$, we define the *Kähler angle* of f to be the function $\theta : S^2 \to [0, \pi]$ given in terms of a complex coordinate $z = x + \sqrt{-1}y$ on S^2, where θ is the angle between $\widetilde{J}f_*(\partial/\partial x)$ and $f_*(\partial/\partial y)$. Since τ is a holomorphic isometry, by a simple calculation, θ is also the angle between $J(\tau\circ f)_*(\partial/\partial x)$ and $(\tau\circ f)_*(\partial/\partial y)$. It is clear that θ is globally defined. Thus we have

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{\left|(\tau\circ f)_*\left(\frac{\partial}{\partial y}\right) - J(\tau\circ f)_*\left(\frac{\partial}{\partial x}\right)\right|^2}{\left|(\tau\circ f)_*\left(\frac{\partial}{\partial y}\right) + J(\tau\circ f)_*\left(\frac{\partial}{\partial x}\right)\right|^2}.$$

Let $L = 1 + |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2$, from the metric $ds^2 = \sum_{A,B}((L\delta_{AB} - \overline{z}^A z^B)/L^2)dz^A d\overline{z}^B$ of $\mathbb{C}P^{n+1}$, using (4.30) and (4.31), we directly compute to obtain

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{\sum_\alpha\left|\frac{\partial\overline{f}^\alpha}{\partial z}\frac{\partial}{\partial\overline{z}_\alpha} - \frac{\partial\overline{f}^\alpha}{\partial z}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial\overline{z}_{n+1}}\right|^2}{\sum_\alpha\left|\frac{\partial f^\alpha}{\partial z}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial z}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}}\right|^2}. \quad (4.32)$$

Since

$$\sum_\alpha \left| \frac{\partial f^\alpha}{\partial z} \frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial z} \frac{z_\alpha}{z_{n+1}} \frac{\partial}{\partial z_{n+1}} \right|^2 = -\frac{1}{L^2} \left| \sum_A \partial f_A \overline{f}_A \right|^2 + \frac{1}{L} \Big(\sum_A |\partial f_A|^2 \Big),$$

$$\sum_\alpha \left| \frac{\partial \overline{f}^\alpha}{\partial z} \frac{\partial}{\partial \overline{z}_\alpha} - \frac{\partial \overline{f}^\alpha}{\partial z} \frac{\overline{z}_\alpha}{\overline{z}_{n+1}} \frac{\partial}{\partial \overline{z}_{n+1}} \right|^2 = -\frac{1}{L^2} \left| \sum_A \overline{\partial} f_A \overline{f}_A \right|^2 + \frac{1}{L} \Big(\sum_A |\overline{\partial} f_A|^2 \Big),$$

then (4.32) becomes

$$\Big(\tan \frac{\theta}{2}\Big)^2 = \frac{L\Big(\sum_A |\overline{\partial} f_A|^2\Big) - \Big|\sum_A \overline{\partial} f_A \overline{f}_A\Big|^2}{L\Big(\sum_A |\partial f_A|^2\Big) - \Big|\sum_A \partial f_A \overline{f}_A\Big|^2}. \tag{4.33}$$

Take $\underline{\psi} = \underline{f}_1^{(4)} = \underline{U}_0 \underline{V}_1^{(4)}$ as an example, where U_0 is the one in Remark 4.3 and $f_1^{(4)}$ satisfies (4.28). We can easily checked that $f_1^{(4)}$ is an immersion of S^2 in Q_3. A straightforward calculation shows that

$$L\Big(\sum_A |\partial f_A|^2\Big) - \Big|\sum_A \partial f_A \overline{f}_A\Big|^2 = \frac{3(1+z\overline{z})^6}{2|z^3-\overline{z}|^4},$$

$$L\Big(\sum_A |\overline{\partial} f_A|^2\Big) - \Big|\sum_A \overline{\partial} f_A \overline{f}_A\Big|^2 = \frac{(1+z\overline{z})^6}{|z^3-\overline{z}|^4}.$$

Using (4.33), the Kähler angle θ of $\underline{\psi} = \underline{f}_1^{(4)}$ is given by

$$\tan^2 \frac{\theta}{2} = \frac{2}{3}. \tag{4.34}$$

Remark 4.11. For the example above, we can check that the Kähler angle in (4.34) satisfies (2.15). In fact, the conformal immersion from S^2 into Q_n is also a conformal immersion from S^2 into $\mathbb{C}P^{n+1}$, it is not difficult to check that their Kähler angles are equal.

Acknowledgements. The authors would like to express gratitude for the referee's comments.

References

[1] A. R. Aithal, Harmonic maps from S^2 to $G_2(C^5)$, J. London Math. Soc., **32** (1985), 572–576.
[2] A. Bahy-El-Dien and J. C. Wood, The explicit construction of all harmonic two-spheres in $G_2(R^n)$, J. Reine Angew. Math., **398** (1989), 36–66.
[3] J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward, On conformal minimal immersions of S^2 into CP^n, Math. Ann., **279** (1988), 599–620.
[4] R. L. Bryant, Minimal surfaces of constant curvature in S^n, Trans. Amer. Math. Soc., **290** (1985), 259–271.

[5] F. E. Burstall and J. C. Wood, The construction of harmonic maps into complex Grassmannians, J. Diff. Geom., **23** (1986), 255–297.

[6] S. S. Chern and J. G. Wolfson, Harmonic maps of the two-spheres in a complex Grassmann manifold, II, Ann. Math., **125** (1987), 301–335.

[7] Q. S. Chi and Y. B. Zheng, Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifolds, Trans. Amer. Math. Soc., **313** (1989), 393–406.

[8] L. He and X. X. Jiao, Classification of conformal minimal immersions of constant curvature from S^2 to HP^2, Math. Ann., **359** (2014), 663–694.

[9] D. A. Hoffman and R. Osserman, The geometry of the generalized Gauss map, Mem. Amer. Math. Soc., **28** (236), 1980.

[10] X. X. Jiao and J. Wang, Minimal surfaces in a complex hyperquadric Q_2, Manus. Math., **140** (2013), 597–611.

[11] X. X. Jiao, Pseudo-holomorphic curves of constant curvature in complex Grassmannians, Israel J. Math., **163** (2008), 45–63.

[12] K. Kenmotsu and K. Masuda, On minimal surfaces of constant curvature in two-dimensional complex space form, J. Reine Angew. Math., **523** (2000), 182–191.

[13] J. Wang and X. X. Jiao, Totally real minimal surfaces in the complex hyperquadrics, Diff. Geom. Appl., **31** (2013), 540–555.

[14] J. G. Wolfson, Harmonic maps of the two-sphere into the complex hyperquadric, J. Diff. Geom., **24** (1986), 141–152.

Mingyan Li
School of Mathematics and Statistics
Zhengzhou University
Zhengzhou 450001, China

Xiaoxiang Jiao
(corresponding author)
School of Mathematical Sciences
University of Chinese Academy of Sciences
Beijing 101408, China
E-mail: xxjiao@ucas.ac.cn

Ling He
Center for Applied Mathematics
Tianjin University
Tianjin 300072, China