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Special Lagrangian submanifolds invariant under the isotropy action of symmetric spaces of rank two

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Abstract. We study special Lagrangian submanifolds of the cotangent bundle T^*S^n of the sphere in the tangent space of Riemannian symmetric space of rank two. We show that the special Lagrangian submanifolds correspond to the solution of a differential equation on $\mathbb{R}^2$ under the assumption that the submanifold is of cohomogeneity one. Our result is the generalization of the former work of Sakai and the first author [**5**]. We study the qualitative properties of the solution for the special Lagrangian submanifolds and give some examples.

1. Introduction.

The notion of *calibration*, which was defined by Harvey and Lawson, plays an important role in the study of minimal submanifolds. As an example, special Lagrangian calibration was given in [**4**]. Let $(z_1, \ldots, z_n)$ be the canonical coordinate of $\mathbb{C}^n$ and θ be a real constant. We put

$$\alpha_\theta = \mathrm{Re}\big(e^{\sqrt{-1}\theta} dz_1 \wedge \cdots \wedge dz_n\big).$$

The differential n-form α_θ on $\mathbb{C}^n$ is called the *special Lagrangian n-form of phase θ*. Since the n-form α_θ is invariant under the action of $SU(n)$, it is extended to the calibration on a Riemannian manifold M if the holonomy group of M is a subgroup of $SU(n)$.

A Riemannian manifold M is called a *Calabi–Yau manifold* if the holonomy group is a subgroup of $SU(n)$. The calibration on Calabi–Yau manifold, which is obtained as the parallel extension of α_θ defined on the tangent space at some point, is called the *special Lagrangian calibration of phase θ* and is denoted by Ω_θ. A submanifold N of a Calabi–Yau manifolds is called a *special Lagrangian submanifold of phase θ* if and only if

$$\Omega_\theta(T_x N) = 1 \qquad (x \in N). \tag{1.1}$$

The study of special Lagrangian submanifolds of Calabi–Yau manifolds attracted many mathematicians interest.

In 1993, Stenzel [**11**] gave a complete Ricci-flat Kähler metric, called the *Stenzel metric*, on the cotangent bundle of a compact Riemannian symmetric space of rank one.

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We review some results concerning the special Lagrangian submanifolds of the cotangent bundle with the Stenzel metric of a Riemannian symmetric space of rank one.

Karigiannis and Min-Oo [**9**] proved that the conormal bundle of a submanifold M of the sphere S^n is a special Lagrangian submanifold in T^*S^n if and only if M is an austere submanifold. Anciaux [**1**] constructed special Lagrangian submanifolds of some phase θ in T^*S^n invariant under the natural action of $SO(n)$. Ionel and Min-Oo [**7**] obtained special Lagrangian submanifolds (with respect to some phase) in T^*S^3 invariant under the action of $SO(2)\times SO(2)$ and $SO(3)$. Note that, in [**1**], [**7**], the value θ are not arbitrary. With the aid of moment map, they reduced the partial differential equation (1.1) to an ordinary differential equation.

Sakai and the first author [**5**] generalized the results by Ionel and Min-Oo to the study of special Lagrangian submanifolds in T^*S^n which are invariant under the natural action of $SO(p)\times SO(q)$ $(p+q=n+1)$.

We briefly review the method in [**5**]. Let S^{p+q-1} be the unit sphere of $\mathbb{R}^{p+q}$ centered at the origin and T^*S^{p+q-1} be its cotangent bundle. The action of $SO(p)\times SO(q)$ on S^{p+q-1} is naturally extended to the action on T^*S^{p+q-1}. We can take a cone $\mathcal{V}$ in a subspace $\mathbb{C}^2\subset\mathbb{C}^{p+q}$ as the orbit space of the action. Let N be the special Lagrangian submanifold of phase θ of T^*S^{p+q-1}. Since the codimension of the regular orbit of the action of $SO(p)\times SO(q)$ on T^*S^{p+q-1} is $p+q-1$, we can assume that the intersection $\mathcal{V}\cap N$ is generically a regular curve. The condition that N is a special Lagrangian submanifold of phase θ is equivalent to that N is an isotropic submanifold and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)(T_xN)=0$ holds at any point $x\in N$. From the first condition, there exists a two dimensional real subspace which contains $\mathcal{V}\cap N$, and the second condition induces an ordinary differential equation. The action used in [**5**] is the isotropy action of a reducible Riemannian symmetric space of rank two.

The aim of this paper is to generalize the results in [**5**] to an arbitrary isotropy action of Riemannian symmetric spaces of rank two. Our main result in this paper is Theorem 5.2, in which we give ordinary differential equations by using a method similar to [**5**]. We also investigate the behaviour of solutions to the ordinary differential equations around a singular point (Theorem 6.5) and their asymptotic lines (Theorem 7.1).

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2. Preliminaries.

2.1. Calabi–Yau manifolds and special Lagrangian submanifolds.

We shall review some definitions and basic notions of Calabi–Yau manifolds and special Lagrangian submanifolds. See [**8**] for details.

There are several different definitions of Calabi–Yau manifolds. In this paper, we use the following definition.

Definition 2.1. Let $n\geq 2$. An *almost Calabi–Yau* manifold is a quadruple (M,J,ω,Ω) such that (M,J,ω) is a Kähler manifold of complex dimension n with a complex structure J and a Kähler form ω, and Ω is a non-vanishing holomorphic $(n,0)$-form on M. In addition, if ω and Ω satisfy

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega}, \tag{2.1}$$

then we call (M, J, ω, Ω) a *Calabi–Yau* manifold.

If ω and Ω satisfy (2.1), then the Kähler metric g of (M, J, ω) is Ricci-flat. The condition that the Kähler metric g of (M, J, ω) is Ricci-flat is equivalent to the condition that the holonomy group $\mathrm{Hol}(g)$ is a subgroup of $SU(n)$.

A closed p-form φ on an oriented Riemannian manifold (M, g) is called a *calibration* if

$$\max_{V \subset T_xM,\ \dim V = p} \varphi(V) = 1$$

holds for each $x \in M$. An oriented p-dimensional submanifold N of M is said to be *calibrated* by the calibration φ if $\varphi|_{T_xN} = \mathrm{vol}|_{T_xN}$ holds for all $x \in N$.

Remark 2.2. The constant factor in (2.1) is chosen so that the comass of $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ is equal to one for any $\theta \in \mathbb{R}$.

Definition 2.3. Let (M, J, ω, Ω) be a Calabi–Yau manifold and L be a real n-dimensional submanifold of M. Then, for $\theta \in \mathbb{R}$, L is called a *special Lagrangian submanifold* of phase θ if it is calibrated by the calibration $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$.

Harvey and Lawson gave the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.4 (Harvey-Lawson [**4**]). *Let (M, J, ω, Ω) be a Calabi–Yau manifold and L be a real n-dimensional submanifold of M. Then L is a special Lagrangian submanifold of phase θ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)|_L \equiv 0$.*

2.2. Stenzel metric on the cotangent bundle of the sphere.

In [**11**], Stenzel constructed complete Ricci-flat Kähler metrics on the cotangent bundles of rank one symmetric spaces of compact type. We shall recall the Stenzel metric on the cotangent bundle of the sphere.

We identify the cotangent bundle T^*S^n and the tangent bundle TS^n of the unit sphere in $\mathbb{R}^{n+1}$;

$$T^*S^n = \big\{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x| = 1, (x, \xi) = 0\big\}.$$

We denote by Q^n the complex quadric;

$$Q^n = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

The mapping $\Phi : T^*S^n \longrightarrow Q^n$ defined by

$$\Phi(x,\xi) = x\cosh(|\xi|) + \sqrt{-1}\frac{\xi}{|\xi|}\sinh(|\xi|)$$

is an $SO(n+1)$-equivalent diffeomorphism (Szöke [**12**]).

We introduce the complex structure on T^*S^n by pulling back the complex structure on the complex quadric Q^n. With respect to the complex structure, Stenzel [**11**] gave a complete Ricci-flat Kähler metric on $Q^n \cong T^*S^n$. Let U be the solution of the ordinary differential equation

$$\frac{d}{dt}(U'(t))^n = cn(\sinh t)^{n-1} \qquad (c>0). \tag{2.2}$$

If we denote by ω_{Stz} the Kähler potential of the Käler metric given by Stenzel, then we have

$$\omega_{Stz} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=1}^{n+1}\frac{\partial^2}{\partial z_i\partial\bar{z}_j}u(r^2)dz_i\wedge d\bar{z}_j$$

where $r^2 = |z|^2 = \sum_{i=1}^{n+1} z_i\bar{z}_i$ and u is the function with $U(t) = u(\cosh t)$.

We denote by Ω_0 the standard holomorphic $(n+1,0)$-form $dz_0\wedge dz_1\wedge\cdots\wedge dz_{n+1}$ on $\mathbb{C}^{n+1}$ and define the holomorphic $(n,0)$-form Ω_{Stz} on Q^n by

$$\Omega_{Stz}(v_1,\dots,v_n) = \Omega_0(z,v_1,\dots,v_n), \quad v_1,\dots,v_n\in T_zQ^n, z\in Q^n. \tag{2.3}$$

Then we can show that there exists a constant $\lambda\in\mathbb{R}$ such that

$$\frac{\omega_{Stz}^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n\lambda^2\Omega_{Stz}\wedge\bar{\Omega}_{Stz}$$

holds. Namely the complex $(n,0)$-form $\Omega = \Omega_{Stz}$ satisfies (2.1) for $\omega = \omega_{Stz}$. Hence $(T^*S^n\cong Q^n, J, \omega_{Stz}, \lambda\Omega_{Stz})$ is a Calabi–Yau manifold.

The Lie group $SO(n+1)$ acts transitively on S^n and the isotropy subgroup at a point of S^n acts transitively on the unit tangent sphere. The action of $SO(n+1)$ on T^*S^n is of cohomogeneity one. It is easily verified that the action of $SO(n+1)$ on T^*S^n preserves J, ω_{Stz} and Ω_{Stz}.

3. Moment map and Lagrangian submanifolds.

Let (M,ω) be a symplectic manifold. For any smooth function H on M, the vector field X_H, called the *Hamiltonian vector field* with Hamiltonian function H, is defined by

$$dH = \iota(X_H)\omega.$$

Lemma 3.1. *The one-parameter subgroup generated by Hamiltonian vector field*

X_H preserves H and ω.

Let G be a Lie group acting on M. We denote by $\mathfrak{g}$ the Lie algebra of G and by exp the exponential map $\exp : \mathfrak{g} \to G$. Each element $X \in \mathfrak{g}$ induces a vector field;

$$X^*|_x = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot x \qquad (x \in M).$$

If ω is a G-invariant form on M, the 1-form $\iota(X^*)(\omega)$ is a closed form for each $X \in \mathfrak{g}$, for $(\mathcal{L}(X^*))(\omega) = d(\iota(X^*)(\omega)) + \iota(X^*)(d\omega) = d(\iota(X^*)(\omega)) = 0$.

An action of G on the symplectic manifold (M, ω) is said to be a *Hamiltonian action* if $\iota(X^*)(\omega)$ is an exact form for each $X \in \mathfrak{g}$. If the action of G on (M, ω) is a Hamiltonian action, there exists some function μ_X such that X^* is Hamiltonian vector field of μ_X for each $X \in \mathfrak{g}$. A *moment map*, for the Hamiltonian action of G on the symplectic manifold (M, ω), is a G-equivariant map

$$\mu : M \to \mathfrak{g}^*$$

such that the vector field X^* is the Hamiltonian vector field of the function $\mu_X : x \mapsto (\mu(x), X)$ for each $X \in \mathfrak{g}$. We denote by $(,)$ the pairing of $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$. For a moment map we have

$$d\mu_X(Y) = (d\mu(Y), X) = \omega(X^*, Y) \quad (\text{i.e., } d(\mu, X) = \iota(X^*)(\omega))$$

for every $x \in M$, $Y \in T_xM$ and $X \in \mathfrak{g}$.

We denote by Ad the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The Lie group G acts on $\mathfrak{g}^*$ by the coadjoint action;

$$(\mathrm{Ad}^*(g)(\alpha), Y) = (\alpha, \mathrm{Ad}(g^{-1})(Y)) \qquad (g \in G,\ \alpha \in \mathfrak{g}^*,\ Y \in \mathfrak{g}).$$

Since the moment map μ is a G-equivariant map, we have

$$(\mu(g \cdot x), X) = (\mathrm{Ad}^*(g)\mu(x), X) = (\mu(x), \mathrm{Ad}(g^{-1})(X)) \qquad (g \in G).$$

The center of $\mathfrak{g}^*$ is, by definition, $Z(\mathfrak{g}^*) = \{X \in \mathfrak{g}^* \ :\ \mathrm{Ad}^*(g)X = X \text{ for all } g \in G\}$.

The following propositions, obtained by Sakai and the first author in [**5**], plays an essential role also in our discussion.

Proposition 3.2 (Hashimoto-Sakai [**5**]). *Let G be a compact Lie group acting on a symplectic manifold (M, ω). Assume that the action of G on (M, ω) is a Hamiltonian action.*

(1) *The image $\mu(L)$ of each G-orbit $L = G \cdot x$ is a contained in $Z(\mathfrak{g}^*)$.*
(2) *If the orbit $L = G \cdot x$ is a connected isotropic submanifold of M, i.e., $\omega|_L \equiv 0$ holds, then the moment map μ is constant on L.*

Proof. (1) For every X, $Y \in \mathfrak{g}$, we have

$$0 = \frac{d}{dt}\bigg|_{t=0} (\mu(\exp(tX) \cdot x), Y) = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}^*(\exp(tX))\mu(x), Y) = (\mu(x), [X, Y]).$$

Since G is connected, we have $\mu(G \cdot x) \in Z(\mathfrak{g}^*)$.

(2) For $Y \in T_x L$ at $x \in L$, we have

$$Y \cdot \mu_Y = d\mu_X(Y) = \omega(X^*, Y) = 0.$$

Since L is connected, this implies that μ_X is constant on L for all $X \in \mathfrak{g}$, hence μ is also constant on L. □

Proposition 3.3 (Hashimoto–Sakai [**5**]). *Let L be a connected submanifold of M invariant under the action of G. Suppose the action of G on L is of cohomogeneity one. Then L is an isotropic submanifold, if and only if $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

4. Stenzel metric and moment map.

Let G be a Lie subgroup of $SO(n+1)$ and $\mathfrak{g}$ be its Lie algebra. The action of G on Q^n is a Hamiltonian action. The moment map $\mu : Q^n \to \mathfrak{g}^*$ of the action of G on Q^n is given by

$$(\mu(z))(X) = \mu_X(z) = u'(r^2)(Jz, Xz) \quad (z \in Q^n, X \in \mathfrak{g}). \tag{4.1}$$

For each $(x, \xi) \in T^*S^n$, we put

$$z(x, \xi) = x \cosh(|\xi|) + \sqrt{-1}\frac{\xi}{|\xi|}\sinh(|\xi|).$$

After Szöke [**12**], we identify T^*S^n with Q^n by the diffeomorphism

$$\Phi : T^*S^n \to Q^n; (x, \xi) \mapsto z(x, \xi)$$

where we put $\Phi(x, 0) = \lim_{\xi \to 0} z(x, \xi) = x$. For $z = z(x, \xi)$, we have

$$Jz = -\frac{\sinh|\xi|}{|\xi|}\,\xi + \sqrt{-1}\,(\cosh|\xi|)\,x, \tag{4.2}$$

$$Xz = (\cosh|\xi|)\,Xx + \sqrt{-1}\frac{\sinh|\xi|}{|\xi|}\,X\xi \tag{4.3}$$

and the right hand side of (4.1) is evaluated as follows;

$$(\mu(z))(X) = \mu_X(z) = u'(r^2)(Jz, Xz) = -\frac{\sinh(2|\xi|)}{|\xi|}u'(r^2)(Xx, \xi), \tag{4.4}$$

since we have $(Xx, x) = (X\xi, \xi) = 0$. It is easy to see that the inverse image $\mu^{-1}(c)$ of $c \in \mathfrak{g}^*$ is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$.

5. Special Lagrangian submanifolds of the cotangent bundle of the sphere.

Let G/K be a Riemannian symmetric space of rank two and $\mathfrak{m}$ be the tangent space of G/K. The Lie group G acts naturally on the cotangent bundle T^*S_1 of the unit sphere S_1 in $\mathfrak{m}$ centered at the origin. Let L be a submanifold of T^*S_1 of cohomogeneity one. In this section, we study the conditions that L is a special Lagrangian submanifold of phase θ, namely the conditions in Proposition 3.2 and 3.3.

5.1. Root system and the Weyl group.

Let $\mathfrak{u}$ be a compact simple Lie algebra and σ be an involutive automorphism of $\mathfrak{u}$. We put

$$\mathfrak{g} = \{X \in \mathfrak{u} : \sigma(X) = X\}, \quad \mathfrak{m} = \{X \in \mathfrak{u} : \sigma(X) = -X\}.$$

The Lie subgroup G of $U = \mathrm{Int}(\mathfrak{u})$ generated by $\mathfrak{g}$ acts on $\mathfrak{m}$ by the restriction to $\mathfrak{g}$ of the adjoint action of U on $\mathfrak{u}$. We denote by κ the Killing form of $\mathfrak{u}$ and define the G-invariant inner product on $\mathfrak{m}$ by $(u, v) = -\kappa(u, v)$.

Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{m}$. If we denote by $\mathfrak{b}$ the centralizer $\{X \in \mathfrak{g} : [X, \mathfrak{a}] = \{0\}\}$ of $\mathfrak{a}$ in $\mathfrak{g}$, $\mathfrak{a} + \mathfrak{b}$ is a maximal abelian subalgebra of $\mathfrak{u}$. For a non-zero element λ of $\mathfrak{a}$ we put

$$\begin{aligned}
\mathfrak{g}_\lambda &= \{X \in \mathfrak{g} : [H, [H, X]] = -(\lambda, H)^2 X \quad ({}^\forall H \in \mathfrak{a})\}, \\
\mathfrak{m}_\lambda &= \{X \in \mathfrak{m} : [H, [H, X]] = -(\lambda, H)^2 X \quad ({}^\forall H \in \mathfrak{a})\}.
\end{aligned}$$

By its definition, we have $\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, $\mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}$. The element λ is called a *root* of $(\mathfrak{u}, \mathfrak{g})$ with respect to $\mathfrak{a}$ if and only if $\mathfrak{g}_\lambda \neq \{0\}$ holds. We denote by $\Sigma(\mathfrak{u}, \mathfrak{g})$ the set of roots of $(\mathfrak{u}, \mathfrak{g})$ with respect to $\mathfrak{a}$. An element $H \in \mathfrak{a}$ is called a *regular* element if it satisfies $(\lambda, H) \neq 0$ for all $\lambda \in \Sigma(\mathfrak{u}, \mathfrak{g})$.

Take a regular element $H_0 \in \mathfrak{a}$ and fix it once for all. We denote by $\Sigma^+(\mathfrak{u}, \mathfrak{g})$ the set of all root λ of $(\mathfrak{u}, \mathfrak{g})$ with respect to $\mathfrak{a}$ which satisfies $(\lambda, H_0) > 0$. We have orthogonal direct sum decomposition;

$$\mathfrak{g} = \mathfrak{b} \oplus \left(\bigoplus_{\lambda \in \Sigma^+(\mathfrak{u}, \mathfrak{g})} \mathfrak{g}_\lambda \right), \quad \mathfrak{m} = \mathfrak{a} \oplus \left(\bigoplus_{\lambda \in \Sigma^+(\mathfrak{u}, \mathfrak{g})} \mathfrak{m}_\lambda \right). \tag{5.1}$$

If the dimension of the maximal abelian subspace of $\mathfrak{m}$ is equal to two, then the G-orbit through a unit regular element of $\mathfrak{a}$ is a hypersurface of the unit sphere of $\mathfrak{m}$ centered at the origin.

If H is an element of $\mathfrak{a}$, we have $\mathrm{ad}_H(\mathfrak{m}_\lambda) \subset \mathfrak{g}_\lambda$, $\mathrm{ad}_H(\mathfrak{g}_\lambda) \subset \mathfrak{m}_\lambda$ for any $\lambda \in \Sigma(\mathfrak{u}, \mathfrak{g})$. Let $H_0 \in \mathfrak{a}$ be a regular element. Since $(\mathrm{ad}_{H_0})^2 = -(\kappa(\lambda, H_0))^2$ id. holds on $\mathfrak{g}_\lambda$ and $\mathfrak{m}_\lambda$, ad_{H_0} is an isomorphism of $\mathfrak{g}_\lambda$ to $\mathfrak{m}_\lambda$.

We put $\Sigma^+(\mathfrak{u},\mathfrak{g}) = \{\lambda_1,\dots,\lambda_k\}$ and denote by m_j the multiplicity of λ_j. For each λ_j, we can take an orthonormal basis $X_i^{(j)}$, $Y_i^{(j)}$ $(1 \le i \le m_j)$ of $\mathfrak{g}_{\lambda_j}$ and $\mathfrak{m}_{\lambda_j}$ such that

$$[H, X_i^{(j)}] = -\lambda(H)\, Y_i^{(j)}, \quad [H, Y_i^{(j)}] = \lambda(H)\, X_i^{(j)}, \quad ({}^\forall H \in \mathfrak{a})$$

([**10**, p. 62, Lemma 1.5]). We put $N_G(\mathfrak{a}) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{a}) \subset \mathfrak{a}\}$ and $Z_G(\mathfrak{a}) = \{g \in G : \mathrm{Ad}(g)|_{\mathfrak{a}} = \mathrm{id.}\}$. The quotient group $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is called the *Weyl group* of $(\mathfrak{u},\mathfrak{g})$ and we denote it by $W(\mathfrak{u},\mathfrak{g})$. The restriction to $\mathfrak{m}$ of $\mathrm{Ad}_U(G)$ induces an injective isomorphism $\mathrm{Ad} : W(\mathfrak{u},\mathfrak{g}) \to O(\mathfrak{a})$. For an element $\lambda \in \mathfrak{a}$ we denote by S_λ the reflection with respect to the hyperplane orthogonal to λ. The subgroup $\mathrm{Ad}(W(\mathfrak{u},\mathfrak{g}))$ of $O(\mathfrak{a})$ coincides with the subgroup generated by all the reflections corresponding to $\lambda \in \Sigma^+(\mathfrak{u},\mathfrak{g})$ (Takeuchi [**13**]).

5.2. Group action.

Let $\mathcal{C}$ be one of the connected component of $\mathfrak{a} \setminus \{H : \kappa(\lambda, H) = 0\ ({}^\exists\lambda \in \Sigma^+(\mathfrak{u},\mathfrak{g}))\}$. We denote by S_1 the unit sphere of $\mathfrak{m}$ centered at the origin and put $S_1 \cap \mathcal{C} = \mathcal{O}_0$. Each orbit of the action of G on $\mathfrak{m}$ intersects with the closure $\overline{\mathcal{O}_0}$ of $\mathcal{O}_0$. The G-orbit through the boundary point of $\overline{\mathcal{O}_0}$ is a singular orbit.

Proposition 5.1. *Assume that the root system $\Sigma(\mathfrak{u},\mathfrak{g})$ is irreducible and of rank two.*

(1) *Let μ be the moment map of the action of G on T^*S_1. If x is an element of $\overline{\mathcal{O}_0}$, then $(x,\xi) \in T^*S_1$ satisfies $\mu(\Phi(x,\xi)) \in Z(\mathfrak{g}^*)$ if and only if*

$$\xi \in (\mathfrak{a} \cap x^\perp) \oplus \Bigg(\bigoplus_{(\lambda,x)=0} \mathfrak{m}_\lambda \Bigg).$$

(2) *Let x be a regular element. The element $\mu(\Phi(x,\xi))$ is contained in $Z(\mathfrak{g}^*)$ if and only if $\xi \in \mathfrak{a} \cap x^\perp$. If ξ is an element of $\mathfrak{a}$ with $\mu(\Phi(x,\xi)) = 0$, then the G-orbit through $\Phi(x,\xi)$ does not contain $\Phi(x,-\xi)$.*

Proof. (1) If $\mu(z)$ is an element of $Z(\mathfrak{g}^*)$, we have

$$(\mu(z), Y) = (\mathrm{Ad}^*(\exp(tX))\mu(z), Y) = (\mu(z), \mathrm{Ad}(\exp(-tX))(Y))$$

for any element X, Y of $\mathfrak{g}$. Differentiating both sides of the above equation with respect to t at $t = 0$, we obtain

$$(\mu(z), [X,Y]) = \mu(z)([X,Y]) = 0.$$

Since G is connected, $\mu(z) \in Z(\mathfrak{g}^*)$ holds if and only if $\mu(z)([\mathfrak{g},\mathfrak{g}]) = \{0\}$.

Each element $z \in Q$ is of the form $z = \cosh(|\xi|)\, x + \sqrt{-1}\,(\sinh(|\xi|)\,/\,|\xi|)\,\xi$ where x is an element of $\overline{\mathcal{O}_0}$ and ξ is an element of $\mathfrak{m}$ with $(x,\xi) = 0$. Thus $\mu(z) \in Z(\mathfrak{g}^*)$ is equivalent to $([x,[\mathfrak{g},\mathfrak{g}]],\xi) = \{0\}$.

The center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is a subspace of $\mathfrak{b}$. Since $\mathfrak{g}$ is a compact Lie algebra, we have

$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{z}(\mathfrak{g})^{\perp}$. Thus, from (4.2), (4.3) and (5.1), we have

$$\mu(z)(X) = -\frac{\sinh(2|\xi|)}{|\xi|}\, u'(r^2)\,(Xx, \xi) = \frac{\sinh(2|\xi|)}{|\xi|}\, u'(r^2)\,([x, X], \xi).$$

Since $\mathfrak{b}$ is the centralizer of $\mathfrak{a}$, $[x, \mathfrak{b}] = \{0\}$ holds. The restriction $\mathrm{ad}_x : \mathfrak{g}_\lambda \to \mathfrak{m}_\lambda$ of ad_x to $\mathfrak{g}_\lambda$ is a zero map if $(\lambda, x) = 0$, and is an isomorphism if $(\lambda, x) \neq 0$. Thus we have

$$[x, [\mathfrak{g}, \mathfrak{g}]] = \bigoplus_{(\lambda, x) \neq 0} \mathfrak{m}_\lambda.$$

Since (5.1) is an orthogonal direct sum decomposition, we obtain the conclusion.

(2) The number of elements $\xi \in \mathcal{O}_0$ satisfying $\mu(z(x, \xi)) \in Z(\mathfrak{g}^*)$, is at most two; ξ and $-\xi$. If there exists an element $g \in G$ which satisfy $g \cdot (x, \xi) = (x, -\xi)$, then g is an element of $N_G(\mathfrak{a})$. And the element $s = [\mathrm{Ad}(g)]$ of the Weyl group $W(\mathfrak{u}, \mathfrak{g})$ is the reflection with respect to the line perpendicular to the regular element x. We get a contradiction, for there does not exist any such element in any Weyl group of irreducible root system of rank two. □

5.3. Differential equation.

Let $\Sigma(\mathfrak{u}, \mathfrak{g})$ be an irreducible root system of rank two. Let λ_1, λ_2 be the simple roots of $\Sigma(\mathfrak{u}, \mathfrak{g})$ with $|\lambda_1| \geq |\lambda_2|$. The multiplicity p, q and r of the root λ_1, λ_2 and $2\lambda_2$ are given in Table 1.

Table 1. multiplicity.

Type	$(\mathfrak{u}, \mathfrak{g})$		(p, q, r)
A_2	$(\mathfrak{su}(3), \mathfrak{so}(3))$		$(1, 1, 0)$
	$(\mathfrak{su}(3) + \mathfrak{su}(3), \mathfrak{su}(3))$		$(2, 2, 0)$
	$(\mathfrak{su}(6), \mathfrak{sp}(6))$		$(4, 4, 0)$
	$(\mathfrak{e}_6, \mathfrak{f}_4)$		$(8, 8, 0)$
B_2	$(\mathfrak{so}(5) + \mathfrak{so}(5), \mathfrak{so}(5))$		$(2, 2, 0)$
	$(\mathfrak{so}(2 + m), \mathfrak{so}(2) + \mathfrak{so}(m))$	$(m \geq 3)$	$(1, m - 2, 0)$
BC_2	$(\mathfrak{su}(2 + m), \mathfrak{su}(2) + \mathfrak{su}(m) + \mathbb{R})$	$(m \geq 3)$	$(2, 2(m - 2), 1)$
	$(\mathfrak{sp}(2 + m), \mathfrak{sp}(2) + \mathfrak{sp}(m))$	$(m \geq 2)$	$(4, 4(m - 2), 3)$
	$(\mathfrak{so}(10), \mathfrak{u}(5))$		$(4, 4, 1)$
	$(\mathfrak{e}_6, \mathfrak{so}(10) + \mathbb{R})$		$(6, 8, 1)$
G_2	$(\mathfrak{g}_2, \mathfrak{so}(4))$		$(1, 1, 0)$
	$(\mathfrak{g}_2 + \mathfrak{g}_2, \mathfrak{g}_2)$		$(2, 2, 0)$

If we define the subset Σ of the cotangent bundle T^*S_1 of the unit sphere S_1 by

$$\Sigma = \{(x, \xi) \in \mathcal{O}_0 \times \mathfrak{a} : \xi \neq 0,\ x \perp \xi\} \subset T^*S_1,$$

then $G \cdot \Sigma$ is an open submanifold of T^*S_1. For any smooth curve $C : c(s) =$

$(x(s), \xi(s))$ $(s \in I)$ in Σ, the submanifold $\bigcup_{s\in I} G \cdot c(s)$ is an isotropic submanifold of T^*S_1 by Proposition 5.1.

If we take an orthonormal basis e_1, e_2 of $\mathfrak{a}$, then each element (x, ξ) of Σ is expressed as follows:

$$(x, \xi) = (\cos\varphi\, e_1 + \sin\varphi\, e_2,\ \rho(-\sin\varphi\, e_1 + \cos\varphi\, e_2))\ (\rho \neq 0).$$

Let $\tau(t)$ be a regular curve in $\mathbb{C}$ and $\varphi(t)$ [resp. $\rho(t)$] be the real [resp. imaginary] part of $\tau(t)$;

$$\tau(t) = \varphi(t) + \sqrt{-1}\,\rho(t).$$

Define a pair of curves in $\mathfrak{a}$ by

$$x(t) = \cos\varphi(t)\, e_1 + \sin\varphi(t)\, e_2, \quad v(t) = -\sin\varphi(t)\, e_1 + \cos\varphi(t)\, e_2$$

which are mutually orthogonal to each other. If we denote by $c_\tau(t)$ the regular curve $c_\tau(t) = (x(t), \rho(t)\, v(t))$ in $\Sigma \subset T^*S_1$, then we have

$$\Phi(c_\tau(t)) = \cos(\tau(t))\, x(t) + \sin(\tau(t))\, v(t).$$

We put $n = \dim \mathfrak{m} - 1$ and define a mapping $\widehat{\Phi}$ of $\mathbb{C} \setminus \{0\}$ to Q^n by

$$\widehat{\Phi}\big(\varphi + \sqrt{-1}\,\rho\big) = \cosh(\rho)(\cos\varphi\, e_1 + \sin\varphi\, e_2) + \sqrt{-1}\ \sinh(\rho)(-\sin\varphi\, e_1 + \cos\varphi\, e_2).$$

We have

$$\widehat{\Phi}(\tau(t)) = \cos(\tau(t))\, e_1 + \sin(\tau(t))\, e_2, \tag{5.2}$$

$$\begin{aligned} d\widehat{\Phi}(\tau'(t)) = &\big(-\tau'(t)\sin(\tau(t)) + \sqrt{-1}\varphi'(t)\cos(\tau(t))\big)e_1 \\ &+ \big(\tau'(t)\cos(\tau(t)) - \sqrt{-1}\varphi'(t)\sin(\tau(t))\big)e_2. \end{aligned} \tag{5.3}$$

If we put $e_3 = Y_1^{(1)}, \dots, e_n = Y_{m_k}^{(k)}$, then $e_1, \dots, e_n$ is an orthonormal basis of $\mathfrak{m}$.

Define the complex valued function z_i on $\mathfrak{m}^{\mathbb{C}}$ by $z_i\big(\sum_{j=1}^n c_j\, e_j\big) = c_i$. If we extend the G-invariant inner product $(\,,)$ on $\mathfrak{m}$ to the Hermitian inner product on $\mathfrak{m}^{\mathbb{C}}$ and denote it also by $(\,,)$, then we have

$$z_i(X) = (e_i, X) \quad (X \in \mathfrak{m}^{\mathbb{C}}).$$

By (5.2) and (5.3) we have

$$dz_1 \wedge dz_2(\Phi(c_\tau), d\Phi(c_\tau')) = \tau'.$$

The tangent space of the G-orbit through $x \in S_1$ coincides with

$$[\mathfrak{g}, x] = \sum_{j=1}^{k} \sum_{i=1}^{m_k} \left[Y_i^{(\lambda_j)},\ x\right].$$

If, furthermore, x is a regular element, from

$$\left[Y_i^{(\lambda_j)},\ x\right] = \lambda_j(x)\, X_i^{(\lambda_j)},$$

it coincides with $\mathfrak{a}^{\perp} \cap \mathfrak{m}$. Thus if $x(t)$ is a regular element we have

$$dz_3 \wedge \cdots \wedge dz_n(e_3, \dots, e_n) = \prod_{\lambda \in \Sigma^+(\mathfrak{u},\mathfrak{g})} (\lambda, \Phi(c(t)))$$

and, by (2.3), we have

$$\Omega_{Stz}(d\Phi(c'), e_3, \dots, e_n) = \tau' \cdot \Bigg(\prod_{\lambda \in \Sigma^+(\mathfrak{u},\mathfrak{g})} \lambda(\Phi(c(t))) \Bigg). \tag{5.4}$$

From (5.4) and

$$(e_1, \Phi(c(t))) = \cos(\tau(t)), \quad (e_2, \Phi(c(t))) = \sin(\tau(t)) \tag{5.5}$$

we can obtain the explicit expressions of Ω_{Stz}.

Let (p, q, r) be those in Table 1.

Case 1: Assume that $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type A_2.

Take an orthonormal basis e_1, e_2 of $\mathfrak{a}$, so that the elements of $\Sigma^+(\mathfrak{u}, \mathfrak{g})$ are as follows

$$\alpha_1 = \frac{1}{2}\big(\sqrt{6}e_1 - \sqrt{2}e_2\big), \quad \alpha_2 = \sqrt{2}e_2, \quad \alpha_1 + \alpha_2 = \frac{1}{2}\big(\sqrt{6}e_1 + \sqrt{2}e_2\big). \tag{5.6}$$

Since the multiplicities of α_1, α_2, $\alpha_1 + \alpha_2$ are p $(= q)$, we have

$$\Omega_{Stz}(d\Phi(c'), e_3, \dots, e_n) = K\, e^{\sqrt{-1}\theta} \tau' \left(\sin\tau\, (3\cos^2\tau - \sin^2\tau) \right)^p$$

for some positive constant K from (5.4) and (5.5).

Case 2: Assume that $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type B_2 or BC_2.

Take an orthonormal basis e_1, e_2 of $\mathfrak{a}$, so that the elements of $\Sigma^+(\mathfrak{u}, \mathfrak{g})$ are as follows

$$\begin{cases} \alpha_1 = e_1 - e_2, \quad \alpha_1 + 2\alpha_2 = e_1 + e_2 & (\text{mult.} = p), \\ \alpha_1 + \alpha_2 = e_1, \quad \alpha_2 = e_2 & (\text{mult.} = q), \\ 2(\alpha_1 + \alpha_2) = 2e_1, \quad 2\alpha_2 = 2e_2 & (\text{mult.} = r). \end{cases} \tag{5.7}$$

There exists a positive constant K such that

$$\Omega_{Stz}(d\Phi(c'), e_3, \dots, e_n) = K\, e^{\sqrt{-1}\theta}\tau'\, \sin^{q+r}(2\tau)\, \cos^p(2\tau)$$

holds.

Case 3: Assume that $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type G_2.

Take an orthonormal basis e_1, e_2 of $\mathfrak{a}$, so that the elements of $\Sigma^+(\mathfrak{u}, \mathfrak{g})$ are as follows

$$\begin{cases} 2\alpha_1 + 3\alpha_2 = \sqrt{6}e_1, \quad \alpha_1 = \dfrac{1}{2}\big(\sqrt{6}e_1 - 3\sqrt{2}e_2\big), \\ \quad \alpha_1 + 3\alpha_2 = \dfrac{1}{2}\big(\sqrt{6}e_1 + 3\sqrt{2}e_2\big) \qquad\qquad (\text{mult.} = p), \\ \alpha_2 = \sqrt{2}e_2, \quad \alpha_1 + 2\alpha_2 = \dfrac{1}{2}\big(\sqrt{6}e_1 + \sqrt{2}e_2\big), \\ \quad \alpha_1 + \alpha_2 = \dfrac{1}{2}\big(\sqrt{6}e_1 - \sqrt{2}e_2\big) \qquad\qquad (\text{mult.} = q). \end{cases} \tag{5.8}$$

There exists a positive constant K such that

$$\Omega_{Stz}(d\Phi(c'), e_3, \dots, e_n) = K\, e^{\sqrt{-1}\theta}\tau'\, \big(\sin(2\tau)\,(3\cos^2\tau - \sin^2\tau)\,(\cos^2\tau - 3\sin^2\tau)\big)^p.$$

Then we obtain the following.

Theorem 5.2. *Let $\mathrm{Ad} : G \to SO(\mathfrak{m})$ be an isotropy representation of an irreducible Riemannian symmetric space of rank two, and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{m}$. Let e_1, e_2 be the orthonormal basis of $\mathfrak{a}$ satisfying* (5.6), (5.7) *or* (5.8) *according to the type of root system. Let $\tau(s)$ be a regular curve in $\mathbb{C}$.*

Let $F(\tau)$ be the holomorphic function given in Table 2, *where p, q and r be the integers given in Table* 1. *The image of the mapping*

$$\mathbb{R} \times G \to Q; (t, g) \mapsto \mathrm{Ad}(g)(\cos\tau(t)\, e_1 + \sin\tau(t)\, e_2)$$

is a special Lagrangian submanifold of phase θ of T^*S_1 if and only if

$$\mathrm{Im}(F(\tau)\,\tau') = 0 \tag{5.9}$$

holds.

Table 2. $F(\tau)$.

Type	$F(\tau)$
A_2	$e^{\sqrt{-1}\theta}\ \big(\sin\tau\,(3 - 4\sin^2\tau)\big)^p$
B_2	$e^{\sqrt{-1}\theta}\ \sin^q(2\tau)\ \cos^p(2\tau)$
BC_2	$e^{\sqrt{-1}\theta}\ \sin^{q+r}(2\tau)\ \cos^p(2\tau)$
G_2	$e^{\sqrt{-1}\theta}\ \big(\sin(2\tau)\,(3\cos^2\tau - \sin^2\tau)\,(\cos^2\tau - 3\sin^2\tau)\big)^p$

5.4. The action of the Weyl group.

Let $\tau(t)$, $\varphi(t)$, $\rho(t)$ etc. be those defined in the previous subsection. We denote by S_{α_1} [resp. S_{α_2}] the reflection with respect to the line perpendicular to α_1 [resp. S_{α_2}].

Assume that the restricted root system $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type A_2. Since the images of $x(t)$ and $v(t)$ by S_{α_1} are

$$S_{\alpha_1}(x(t)) = \cos\left(\frac{2\pi}{3} - \varphi\right) e_1 + \sin\left(\frac{2\pi}{3} - \varphi\right) e_2,$$
$$S_{\alpha_1}(v(t)) = -\sin\left(\frac{2\pi}{3} - \varphi\right) e_1 + \cos\left(\frac{2\pi}{3} - \varphi\right) e_2$$

thus we have

$$S_{\alpha_1}(\Phi(c_\tau(t))) = \cosh(\rho(t))\, x\left(\frac{2\pi}{3} - \varphi\right) + \sqrt{-1}\,\sinh(\rho(t)) v\left(\frac{2\pi}{3} - \varphi\right).$$

Since there exists an element $g \in N_G(\mathfrak{a}) \subset G$ such that S_α coincides with the restriction of $\mathrm{Ad}(g)$ on $\mathfrak{a}$ the orbit through the point $\widehat{\Phi}(\varphi + \sqrt{-1}\rho)$ coincides with the orbit through the point $\widehat{\Phi}((2\pi/3 - \varphi) + \sqrt{-1}\rho)$.

Similarly we have

$$S_{\alpha_2}(\Phi(c_\tau)(t)) = \cosh(-\rho)\, x(-\varphi) + \sqrt{-1}\,\sinh(-\rho) v(-\varphi)$$

and the orbit through the point $\widehat{\Phi}(\varphi + \sqrt{-1}\rho)$ coincides with the orbit through the point $\widehat{\Phi}(-(\varphi + \sqrt{-1}\rho))$.

Proposition 5.3. *If we define $S_1(\tau)$ and $S_2(\tau)$ as follows for $\tau = \varphi + \sqrt{-1}\rho \in \mathbb{C}$ $(\varphi, \rho \in \mathbb{R})$, then G-orbits through $\widehat{\Phi}(\tau)$, $\widehat{\Phi}(S_1(\tau))$ and $\widehat{\Phi}(S_2(\tau))$ coincides with each other.*

(1) *If the root system $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type A_2, then*

$$S_1(\tau) = \left(\frac{2\pi}{3} - \varphi\right) + \sqrt{-1}\rho, \quad S_2(\tau) = -(\varphi + \sqrt{-1}\,\rho).$$

(2) *If the root system $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type B_2 or BC_2, then*

$$S_1(\tau) = \left(\frac{\pi}{2} - \varphi\right) + \sqrt{-1}\,\rho, \quad S_2(\tau) = -(\varphi + \sqrt{-1}\,\rho).$$

(3) *If the root system $\Sigma(\mathfrak{u}, \mathfrak{g})$ is of type G_2, then*

$$S_1(\tau) = \left(\frac{\pi}{3} - \varphi\right) + \sqrt{-1}, \rho, \quad S_2(\tau) = -(\varphi + \sqrt{-1}\,\rho).$$

6. Behaviour of solutions.

We study the behaviour of solutions to the differential equation (5.9). In this section, the behaviour around the singular point, and in the following section the asymptotic lines will be investigated.

Let θ be a real number. Let $F(\tau)$ be the complex valued function given in Table 2 of Theorem 5.2. If we denote by $u(\varphi, \rho)$ [resp. $v(\varphi, \rho)$] the real [resp. imaginary] part of $F(\tau)$. Since $F(\tau)$ is a holomorphic function, (5.9) is expressed as an exact differential equation on $\mathbb{R}^2 = \mathbb{C}$

$$v(\varphi, \rho)\, d\varphi + u(\varphi, \rho)\, d\rho = 0. \tag{6.1}$$

The condition that (φ, ρ) is the singular point of the differential equation (6.1) is equivalent to that $\tau = \varphi + \sqrt{-1}\,\rho$ is the zero of $F(\tau)$.

Let $\widetilde{F}(\tau)$ be a primitive function of $F(\tau)$. If we denote the real part and the imaginary part of $\widetilde{F}(\tau)$ by $U(\varphi, \rho)$ and $V(\varphi, \rho)$, then we have

$$u = U_\varphi = V_\rho, \quad v = -U_\rho = V_\varphi.$$

The solution of (6.1) is obtained as the implicit function define by $U(\varphi, \rho) = 0$.

We consider the autonomous system of differential equation

$$\frac{d\varphi}{dt} = u(\varphi, \rho), \ \frac{d\rho}{dt} = -v(\varphi, \rho) \tag{6.2}$$

corresponding to (6.1).

The following is known about the behaviour of the solution around the singularity of autonomous differential equation.

Lemma 6.1 (Yamaguchi [**14**]). *Let $(\varphi(t), \rho(t))$ be the solution of* (6.2). *If $\lim_{t\to\infty} \varphi(t) = \varphi_0$, $\lim_{t\to\infty} \rho(t) = \rho_0$ hold, then we have*

$$u(\varphi_0, \rho_0) = v(\varphi_0, \rho_0) = 0.$$

On the other hand, if there exists a real number t_0, such that

$$\lim_{t\to t_0} \varphi(t) = \varphi_0, \quad \lim_{t\to t_0} \rho(t) = \rho_0,$$

where (φ_0, ρ_0) is the singular point of the differential equation (6.2), *namely $u(\varphi_0, \rho_0) = v(\varphi_0, \rho_0) = 0$ holds, then t_0 is equal to ∞ or $-\infty$.*

Proposition 6.2. *The autonomous system of differential equations* (6.2) *does not have any periodic solution.*

Proof. Let

$$C : \tau = \tau(t) = \varphi(t) + \sqrt{-1}\,\rho(t) \quad (a \le t \le b)$$

be a simply closed curve, where φ and ρ are real valued functions. If we assume that $\tau = \tau(t)$ is a solution of (6.2), we have

$$\begin{aligned}\int_C e^{\sqrt{-1}\theta} F\, d\tau &= e^{\sqrt{-1}\theta}\Big(\int_C (u\, d\varphi - v\, d\rho) + \sqrt{-1}\int_C (u\, d\rho + v\, d\varphi)\Big)\\ &= e^{\sqrt{-1}\theta}\int_C (u^2 + v^2)\, dt = 0\end{aligned}$$

and the holomorphic function $F(\tau)$ is identically zero on C. Thus $F(\tau)$ is identically zero on the interior region bounded by C, which is a contradiction. □

Let $\tau_0 = \varphi_0 + \sqrt{-1}\,\rho_0$ ($\varphi_0, \rho_0 \in \mathbb{R}$) be a zero of $\zeta = F(\tau)$. The point (φ_0, ρ_0) is a singular point of the differential equation (6.1). We consider the behaviour of solution of (6.1) around the singular point (φ_0, ρ_0) using the following theorem.

THEOREM 6.3 (Hartman [**3**, p. 220]). *Consider the autonomous system*

$$\frac{dx}{dt} = P(x, y) + p(x, y), \quad \frac{dy}{dt} = Q(x, y) + q(x, y), \tag{6.3}$$

where P, Q are homogeneous polynomials of degree m and

$$p^2(x, y) + q^2(x, y) = o(r^{2m}) \quad \textit{as} \quad r^2 = x^2 + y^2 \to 0, \tag{6.4}$$

$$(P + p)^2 + (Q + q)^2 \geq 0 \quad \textit{according as} \quad x^2 + y^2 \geq 0. \tag{6.5}$$

In terms of polar coordinate $x = r\cos\alpha$ and $y = r\sin\alpha$, we put

$$S(\alpha) = r^{-m}(Q\cos\alpha - P\sin\alpha).$$

Assume that (6.4), (6.5) *and $S(\alpha) \not\equiv 0$ hold. If $(x(t), y(t))$ is a solution of* (6.3) *for large $t > 0$ [or $-t > 0$] satisfying*

$$0 < x^2(t) + y^2(t) \to 0 \quad \textit{as} \quad t \to \infty \quad [\textit{or } t \to -\infty],$$

then a continuous determination of $\alpha(t) = \tan^{-1} y(t)/x(t)$ satisfies either $\alpha_0 = \lim_{t\to\infty} \alpha(t)$ [or $\alpha_0 = \lim_{t\to-\infty} \alpha(t)$] exists (and is finite) and

$$S(\alpha_0) = 0$$

or

$$|\alpha(t)| \to \infty \quad \textit{as} \quad t \to \infty \quad [\textit{or } t \to -\infty]. \tag{6.6}$$

The above theorem was proved originally by Bendixon [**2**]. From the proof of the above theorem we can conclude the following:

Remark 6.4 (Yamaguchi [**14**]). If the function $S(\alpha_0)$ takes both positive and negative values, then (6.6) does not occur.

Theorem 6.5. *Let $F(\tau)$ be one of the holomorphic function in Table 2. Let $\tau_0 = \varphi_0 + \sqrt{-1}\,\rho_0$ $(\varphi_0, \rho_0 \in \mathbb{R})$ be a zero of $F(\tau)$ and m (> 0) be the order of τ_0.*

(1) *There exists exactly $2(m+1)$ solutions of (6.1) such that*

$$\lim_{t\to\infty} \tau(t) = \tau_0 \quad [\mathit{or} \lim_{t\to-\infty} \tau(t) = \tau_0] \tag{6.7}$$

holds.

(2) *Let $(\varphi(t), \rho(t))$ be a solution of (6.1) defined for $t \le L$ or $t \le -L$, where L is a sufficiently large number, satisfying*

$$0 < (\varphi(t) - \varphi_0)^2 + (\rho(t) - \rho_0)^2 \to 0 \quad \mathit{as} \quad t \to \infty \quad [\mathit{or}\ t \to -\infty].$$

If we define a continuous function $\theta(t)$ by $\tan\theta(t) = (\rho(t) - \rho_0)/(\varphi(t) - \varphi_0)$, then it converges as $t \to \infty$ [or $t \to -\infty$]. The limit $\alpha_0 = \lim\ \alpha(t)$ satisfies

$$\arg(c_m) + (m+1)\theta_0 = 0 \ (\mathrm{mod}\ (\pi\,\mathbb{Z})),$$

where c_m is the coefficient of $(\tau - \tau_0)^m$ of the Taylor expansion of $F(\tau)$ around τ_0.

Proof. We denote by c_j the coefficient of the Taylor expansion of $F(\tau)$ around τ_0 and put

$$\widetilde{F}(\tau) = \sum_{j=m}^{\infty} \frac{c_j}{j+1}\,(\tau - \tau_0)^{j+1}. \tag{6.8}$$

The solution of the equation (5.9) is given by $\widetilde{F}(\tau) = s$ $(s \in \mathbb{R})$. We denote by $\widetilde{F}_1$ the principal term $c_m(\tau - \tau_0)^{m+1}/(m+1)$ of (6.8) and by $\widetilde{F}_2$ the remainder part $\widetilde{F} - \widetilde{F}_1$.

We denote by $B_r(\tau_0)$ the open disk of radius r around τ_0. Take a positive number R so small that the right hand side of the (6.8) is convergent and $|\widetilde{F}_2(\tau)/\widetilde{F}_1(\tau)| < 1/2$ holds for all $|\tau - \tau_0| < R$. We put R' the radius of the circle $\widetilde{F}_1(\tau_0 + R\,e^{\sqrt{-1}\,t})$ $(t \in \mathbb{R})$. For a real number r with $0 < 2r < R' = |c_m|R^{m+1}/(m+1)$, take r' so that $|c_m|(r')^{m+1}/(m+1) = 2r$. If τ runs through the circle $\tau = \tau_0 + r'\,e^{\sqrt{-1}\,t}$ $(t \in \mathbb{R})$, we have $|\widetilde{F}_1(\tau)| = 2r$ and $|\widetilde{F}_1(\tau) - r| \ge r$, and from $|\widetilde{F}_2(\tau)/\widetilde{F}_1(\tau)| < 1/2$ we have $|\widetilde{F}_2| < r$. Thus we have

$$|\widetilde{F}_1(\tau) - r| > |\widetilde{F}_2(\tau)|, \quad |\tau - \tau_0| = r'.$$

By the theorem of Rouché there exists $m+1$ points in $B_{r'}(\tau_0)$ satisfying $\widetilde{F}(\tau) = r$.

Let τ_1 be one of the solution $\widetilde{F}(\tau_1) = r$ with $|\tau_1 - \tau_0| < r'$. The coefficient $F(\tau_1)$ of the first order term of the taylor expansion of $F(\tau)$ is not zero, since τ_0 is the only zero of $F(\tau)$ in $B_{2r'}(\tau_0)$. Thus τ_1 is a zero of order 1 which imply that there exists exactly $m+1$ points in $|\tau - \tau_0| < r'$ satisfying $\widetilde{F}(\tau) = r$. Similarly we can show that there exists exactly $m+1$ points in $|\tau - \tau_0| < r'$ satisfying $\widetilde{F}(\tau) = -r$.

Let r be a real number with $0 < 2r < R'$ and τ ($|\tau - \tau_0| < r'$) be a solution of the equation $\widetilde{F}(\tau) = r$. Let $\tau = G_r(\zeta)$ be the inverse function of $\zeta = \widetilde{F}(\tau)$ defined around $\zeta = r$. Let $G_t(\zeta)$ be the holomorphic function defined around t which is the analytic continuation of $G_r(\zeta)$ along the curve $\zeta = t$ ($0 < t \leq r$). If we put $\tau(t) = G_t(t)$, τ_t is a solution of (6.1).

Take a real constant r_1 with $0 < 2r_1 < R'$. There exists mutually different complex numbers $\tau^{(j)}$ ($1 \leq j \leq m+1$) with $|\tau^{(j)} - \tau_0| < (r_1)'$ and $\widetilde{F}(\tau^{(j)}) = r_1$. We denote by $\tau^{(j)}(t)$ ($0 < t \leq r_1$) the curve obtained by the discussion above (analytic continuation of the inverse function of $\widetilde{F}$).

We denote by $n(r)$ the number of j's with the property $|\tau^{(j)}(r) - \tau_0| < r'$. It is obvious that $n(r) \leq m+1$ and $n(r_1) = m+1$. We put r_2 the infimum of the set of r's such that $n(t) = m+1$ holds for all r with $r < t \leq r_1$. If $r_2 = 0$ then we have $\lim_{t\to 0} \tau_t^{(j)} = 0$ ($1 \leq j \leq m+1$).

If we assume that $0 < r_2$ holds, then we have $n(r_2) < m+1$. There exists at least one τ which is the solution of the equation $\widetilde{F}(\tau^{(0)}) = r_2$ and $\tau \notin \{\tau^{(j)} \ : \ 1 \leq j \leq m+1\}$. There exists a solution $\tau = \tau^0(t)$ of (5.9) which satisfy $\widetilde{F}(\tau^{(0)}(t)) = t$. But this imply $n(t) > m+1$ in some interval $r_2 < t < r_2 + \varepsilon$, which is a contradiction.

If we express the curve $\tau = \tau^{(j)}(t)$ as a solution of (5.9), by Lemma 6.1 we have (6.7).

Let τ_t be a regular curve with $\widetilde{F}(\tau_t) = t$ or $\widetilde{F}(\tau_t) = -t$. We put r_t and $\alpha(t)$ the absolute value and the argument of $\tau_t - \tau_0$ respectively. We shall show the existence of $\lim_{t\to\infty} \alpha(t)$.

If we put $x = \varphi - \varphi_0$, $y = \rho - \rho_0$, $\tau = \varphi + \sqrt{-1}\,\rho$, $\tau_0 = \varphi_0 + \sqrt{-1}\,\rho_0$ and

$$\begin{aligned} P(x,y) &= \ \ \mathrm{Re}(c_m(\tau - \tau_0)^m), \quad & p(x,y) &= \ \ \mathrm{Re}(F(\tau) - c_m(\tau-\tau_0)^m), \\ Q(x,y) &= -\mathrm{Im}(c_m(\tau - \tau_0)^m), \quad & q(x,y) &= -\mathrm{Im}(F(\tau) - c_m(\tau-\tau_0)^m). \end{aligned}$$

The autonomous differential equation (6.2) is of the form (6.3). It is obvious that P and Q are homogeneous polynomials of degree m and (6.4) and (6.5) hold. From

$$Q\,x - P\,y = r(Q\,\cos\alpha - P\,\sin\alpha) = -\mathrm{Im}(c_m(\tau-\tau_0)^{m+1})$$

we conclude that $S(\alpha)$ takes both positive and negative values. Thus by Theorem 6.3 and Remark 6.4, the limit

$$\lim_{t\to\infty} \alpha(t) \quad [\text{or } \lim_{t\to-\infty} \alpha(t)]$$

exists, which we denote α_0.

If we substitute the Taylor expansion of $\widetilde{F}(\tau)$ to (6.8), divide the both sides by $(r(t))^{m+1}$ and take the limits as $t \to +0$, we obtain $\mathrm{Im}\big(c_m\, e^{\sqrt{-1}\,(m+1)\alpha_0}\big) = 0$ namely

$$\arg(c_m) + (m+1)\alpha_0 \in \pi\mathbb{Z}$$

for $\lim_{t \to +0} r(t) = 0$. □

7. Asymptotic lines.

Let $\varphi(t)$ be a real part and $\rho(t)$ be a imaginary part for the solution $\tau = \tau(s)$ of (5.9). We assume that $\tau = \varphi_0$ is the asymptotic line of $\tau = \tau(t)$, that is

$$\lim_{t \to \infty} \rho(t) = \pm\infty, \qquad \lim_{t \to \infty} \varphi(t) = \varphi_0$$

hold. If we put the additional condition

$$\lim_{t \to \infty} \frac{d\varphi}{dt} \Big/ \frac{d\rho}{dt} = 0,$$

we have

$$\lim_{t \to \infty} \frac{\sin \tau(t)}{\cosh \rho(t)} = \pm\sqrt{-1} e^{-\sqrt{-1}\,\varphi_0}, \quad \lim_{t \to \infty} \frac{\cos \tau(t)}{\cosh \rho(t)} = \pm e^{-\sqrt{-1}\,\varphi_0}.$$

If the restricted root system is of type A_2, divide the both sides of (5.9) by $(\cosh \rho(t))^{3p}\, d\rho/dt$, we have

$$\lim_{t \to \pm\infty} \frac{dt}{d\rho} \frac{1}{(\cosh \rho(t))^{3p}} \mathrm{Im}\,(F(\tau(t))\,\tau') = \mathrm{Im}\big((\sqrt{-1})^p\, e^{\sqrt{-1}(\theta \mp p\varphi_0)}\big) = 0$$

as $t \to \infty$.

Similarly, if the restricted root system is of type B_2 or BC_2, we have

$$\mathrm{Im}\big((\sqrt{-1})^{q+r} 2^{p+q+r} e^{\sqrt{-1}(\theta - 2(p+q+r)\varphi_0)}\big) = 0,$$

and if the restricted root system is of type G_2, we have

$$\mathrm{Im}\big((\sqrt{-1})^p 2^{5p} e^{\sqrt{-1}(\theta - 6p\varphi_0)}\big) = 0.$$

We obtain the following results.

Theorem 7.1. *Let $\varphi(t)$ be a real part and $\rho(t)$ be an imaginary part of the solution $\tau(t)$ of* (5.9) *defined on an open set (a, ∞). If there is φ_0 satisfying*

$$\lim_{t \to \infty} \rho(t) = \infty, \quad \lim_{t \to \infty} \varphi(t) = \varphi_0, \quad \lim_{t \to \infty} \frac{d\varphi}{dt} \Big/ \frac{d\rho}{dt} = 0,$$

then, φ_0 *is one in Table* 3.

Table 3.

Type	φ_0	
A_2	$\varphi_0 = \pm\dfrac{\theta - k\pi}{3}$	$(p = 1)$
	$\varphi_0 = \pm\dfrac{2\theta - (2k+1)\pi}{6p}$	$(p = 2,\ 4,\ 8)$
B_2 or BC_2	$\varphi_0 = \pm\dfrac{\theta - k\pi}{2(p+q+r)}$	$(q + r : \text{odd})$
	$\varphi_0 = \pm\dfrac{2\theta - (2k+1)\pi}{4(p+q+r)}$	$(q + r : \text{even})$
G_2	$\varphi_0 = \pm\dfrac{\theta - k\pi}{6}$	$(p = 1)$
	$\varphi_0 = \pm\dfrac{2\theta - (2k+1)\pi}{24}$	$(p = 2)$

8. Examples.

Let $\mathrm{Ad} : G \to SO(\mathfrak{m})$ be the isotropy representation of an irreducible Riemannian symmetric space of rank two and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{m}$. Let e_1, e_2 be the orthonormal basis of $\mathfrak{a}$ satisfying (5.6), (5.7) or (5.8) according to the type of root system.

Example 8.1 ($(\mathfrak{u}, \mathfrak{g}) = (\mathfrak{su}(3), \mathfrak{so}(3))$). In this case the equation (5.9) is

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta}\tau' \sin(\tau)(3 - 4\sin^2\tau)\big) = \frac{1}{3}\mathrm{Im}\big(e^{\sqrt{-1}\theta}(\cos(3\,\tau))'\big) = 0. \tag{8.1}$$

Most general solution is given by the relation

$$\sin(\theta - 3\varphi)\, e^{3\rho} + \sin(\theta + 3\varphi)\, e^{-3\rho} = 2C \tag{8.2}$$

where C is an arbitrary constant. After dividing both sides of (8.2) by $e^{3\rho}$, we take the limit as $\rho \to \infty$. Then we have $\lim_{\rho\to\infty} \sin(\theta - 3\varphi) = 0$. Similarly we have $\lim_{\rho\to-\infty} \sin(\theta + 3\varphi) = 0$. Thus we have the following conclusion;

let $(\varphi(t), \rho(t))$ be a solution curve of (8.2) with $\lim_{t\to\infty} |\rho(t)| = \infty$ [or $\lim_{t\to-\infty} |\rho(t)| = \infty$].

- If $\lim_{t\to\infty} \rho(t) = \infty$ [or $\lim_{t\to-\infty} \rho(t) = \infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta - k\pi)/3$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].
- If $\lim_{t\to\infty} \rho(t) = -\infty$ [or $\lim_{t\to-\infty} \rho(t) = -\infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta + k\pi)/3$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].

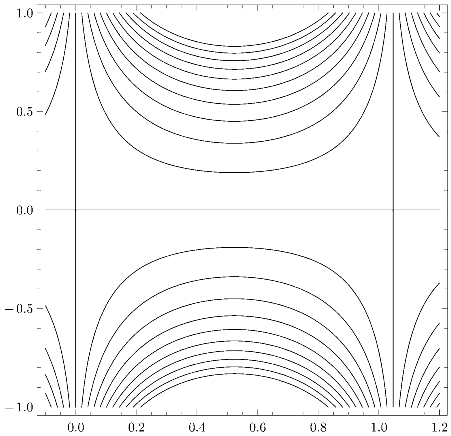

Figure 1. $\theta = 0$.

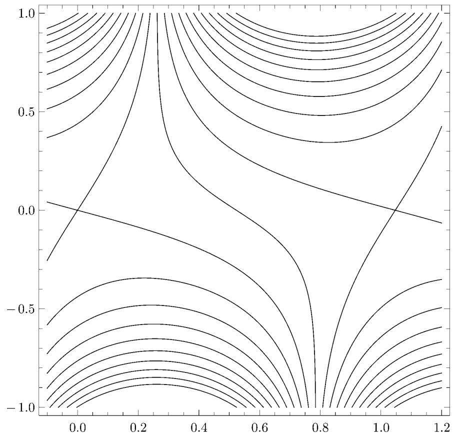

Figure 2. $\theta = \pi/4$.

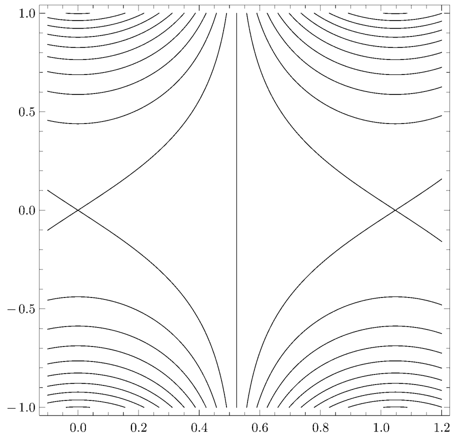

Figure 3. $\theta = \pi/2$.

From (8.2), we easily obtain the explicit expression of the solution for (8.1);

$$\rho = \frac{1}{3} \log \left(\frac{C \pm \sqrt{C^2 - \sin(\theta - 3\varphi)\sin(\theta + 3\varphi)}}{\sin(\theta - 3\varphi)} \right).$$

In Figures 1, 2 and 3, the solution curves of equation (8.2) are given, for $\theta = 0, \pi/4$ and $\pi/2$ respectively.

For each singular point τ_0 of (8.1), there exists exactly 4 solutions of $\tau(t)$ satisfying (6.7) from Theorem 6.5. Actually Figures 1, 2 and 3 illustrates the phenomena.

Example 8.2 $((\mathfrak{u}, \mathfrak{g}) = (\mathfrak{so}(2+3), \mathfrak{so}(2) + \mathfrak{so}(3)))$. In this case the equation (5.9) is

$$\operatorname{Im}\bigl(e^{\sqrt{-1}\theta}\tau' \sin(2\tau)\cos(2\tau)\bigr) = \frac{1}{4}\operatorname{Im}\bigl(e^{\sqrt{-1}\theta}(\sin^2(2\tau))'\bigr) = 0. \tag{8.3}$$

Most general solution is given by the relation

$$\sin(\theta - 4\varphi)\, e^{4\rho} - 2\,\sin\theta + \sin(\theta + 4\varphi)\, e^{-4\rho} = C \tag{8.4}$$

where C is an arbitrary constant. By similar argument to that in Example 8.1, we have the following conclusion;

let $(\varphi(t), \rho(t))$ be a solution curve of (5.9) with $\lim_{t\to\infty} |\rho(t)| = \infty$ [or $\lim_{t\to-\infty} |\rho(t)| = \infty$].

- If $\lim_{t\to\infty} \rho(t) = \infty$ [or $\lim_{t\to-\infty} \rho(t) = \infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta - k\pi)/4$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].
- If $\lim_{t\to\infty} \rho(t) = -\infty$ [or $\lim_{t\to-\infty} \rho(t) = -\infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta + k\pi)/4$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].

From (8.4), we easily obtain the explicit expression of the solution for (8.1);

$$\rho = \frac{1}{4}\log\left(\frac{C + \sin\theta \pm \sqrt{(C + \sin\theta)^2 - \sin(\theta - 4\varphi)\sin(\theta + 4\varphi)}}{\sin(\theta - 4\varphi)}\right).$$

In Figures 4, 5 and 6, the solution curves of equation (8.4) are given, for $\theta = 0, \pi/4$ and $\pi/2$ respectively.

For each singular point τ_0 of (8.1), there exists exactly 4 solutions of $\tau(t)$ satisfying (6.7) from Theorem 6.5. Actually Figures 4, 5 and 6 illustrates the phenomena.

EXAMPLE 8.3 ($(\mathfrak{u}, \mathfrak{g}) = (\mathfrak{g}_2, \mathfrak{so}(4))$). In this case, the equation (5.9) reduces to

$$\frac{1}{6}\mathrm{Im}\big(e^{\sqrt{-1}\theta}(\cos(6\,\tau))'\big) = 0. \tag{8.5}$$

Most general solution is given by the relation

$$\sin(\theta - 6\varphi)\, e^{6\rho} + \sin(\theta + 6\varphi)\, e^{-6\rho} = 2C \tag{8.6}$$

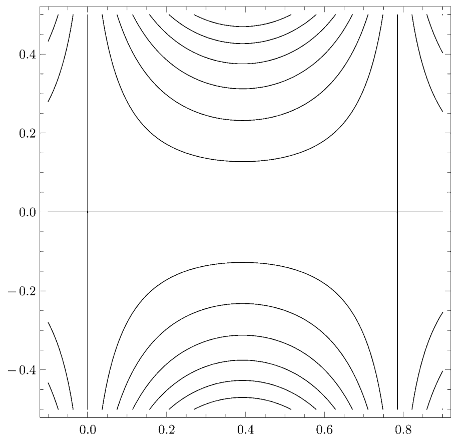

Figure 4. $\theta = 0$.

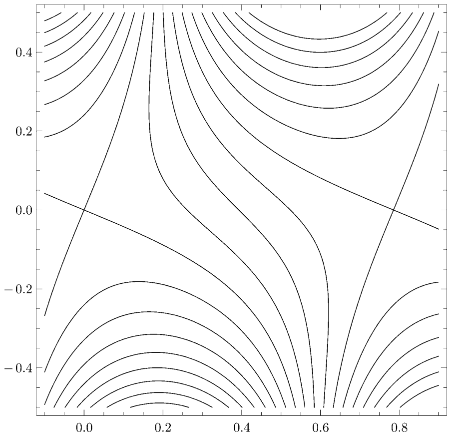

Figure 5. $\theta = \pi/4$.

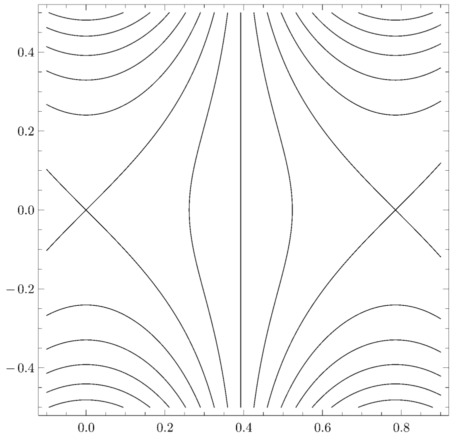

Figure 6. $\theta = \pi/2$.

where C is an arbitrary constant. By similar argument to that in Example 8.1, we have the following conclusion;

let $(\varphi(t), \rho(t))$ be a solution curve of (5.9) with $\lim_{t\to\infty} |\rho(t)| = \infty$ [or $\lim_{t\to-\infty} |\rho(t)| = \infty$].

- If $\lim_{t\to\infty} \rho(t) = \infty$ [or $\lim_{t\to-\infty} \rho(t) = \infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta - k\pi)/6$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].
- If $\lim_{t\to\infty} \rho(t) = -\infty$ [or $\lim_{t\to-\infty} \rho(t) = -\infty$], then the solution curve $(\varphi(t), \rho(t))$ is asymptotic to the line $\varphi = (\theta + k\pi)/6$ $(k \in \mathbb{Z})$ for $t \to \infty$ [or $t \to -\infty$].

From (8.6), we easily obtain the explicit expression of the solution for (8.5);

$$\rho = \frac{1}{6} \log \left(\frac{C \pm \sqrt{C^2 - \sin(\theta - 6\varphi)\sin(\theta + 6\varphi)}}{\sin(\theta - 6\varphi)} \right).$$

In Figures 7, 8 and 9, the solution curves of equation (8.2) are given, for $\theta = 0, \pi/4$ and $\pi/2$ respectively.

For each singular point τ_0 of (8.1), there exists exactly 4 solutions of $\tau(t)$ satisfying (6.7) from Theorem 6.5. Actually Figures 7, 8 and 9 illustrates the phenomena.

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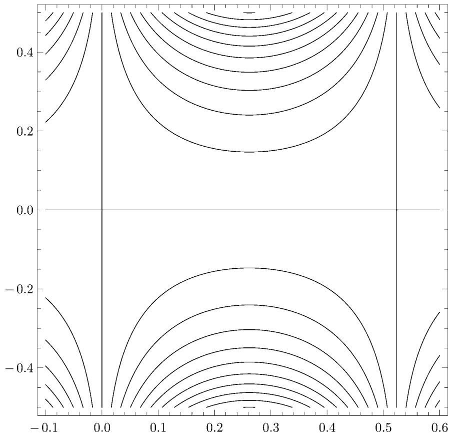

Figure 7. $\theta = 0$.

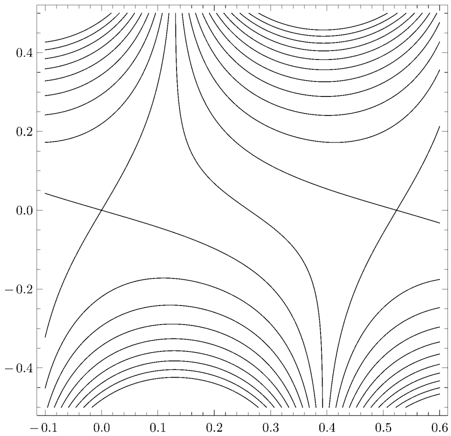

Figure 8. $\theta = \pi/4$.

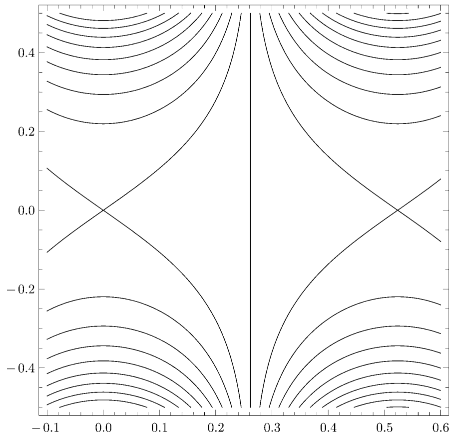

Figure 9. $\theta = \pi/2$.

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