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Some remarks on cubature formulas with linear operators

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Abstract. In this paper we consider a novel type of cubature formulas called operator-type cubature formulas. The notion originally goes back to a famous work by G. D. Birkhoff in 1906 on Hermite interpolation problem. A well-known theorem by Sobolev in 1962 on invariant cubature formulas is generalized to operator-type cubature, which provides a systematic treatment of Lebedev's works in the 1970s and some related results by Shamsiev in 2006. We give a lower bound for the number of points needed, and discuss analytic conditions for equality, together with tight illustrations for Laplacian-type cubature.

1. Introduction.

In this paper, we define operator-type cubature formulas as a generalization of Laplacian-type cubature formulas and classical polynomial-type cubature formulas, and study the following three topics:

- A Stroud-type inequality for operator-type cubature. Especially a lower bound for Laplacian-type cubature is given.
- A series of Laplacian-type cubature attaining the bound.
- Generalizing Sobolev's Theorem on invariant polynomial-type cubature to operator-type cubature.

A *cubature formula* is an approximation of the definite integral of a multivariate function, expressed as a weighted average of the function values at finitely many specified points within the domain of integration. The term quadrature is often used to refer to one-dimensional cubature formulas. A t-point Gaussian quadrature is a quadrature formula of degree $2t-1$, meaning, a formula that is exact for all polynomials of degree at most $2t-1$. A Euclidean t-design, an important research object in combinatorics and statistics, can be regarded as a cubature formula of degree t for an integral with a rotational symmetry property; for example, see [**4**].

A t-point Gaussian quadrature is "tight" among all quadrature formulas of the same degree, that is, the number of points in any quadrature of degree $2t-1$ is bounded from below by t. This bound is often called *Stroud bound* in analysis, or *Fisher-type bound*

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in combinatorics and statistics. It is well known (cf. [**6**]) that the points of a Gaussian quadrature are uniquely expressed by zeros of an orthogonal polynomial. Similarly, the theory of cubature has been developed in analysis and related areas, in parallel with the study of common zeros of multivariate orthogonal polynomials; for the details, we refer the reader to the comprehensive textbooks by Dunkl and Xu [**6**] and Sobolev and Vaskevich [**15**]. Specifically, a cubature with equality in the Stroud bound naturally carries rich algebraic and geometric structures [**2**], [**5**], [**10**], [**19**].

One of the most important problems in numerical analysis, combinatorics and other related areas is, to find a cubature formula with the smallest possible number of points for a rotationally symmetric integral. Many publications have been devoted to this subject [**1**], [**3**], [**5**], [**19**], [**20**]. However, only a few such examples have been reported in high dimension.

In this paper we consider a novel type of cubature formula which we call an *operator-type cubature formula*. Such a cubature often has fewer points, compared with the usual cubature of the same degree; see Remark 3.4 of this paper.

Historically, Turán [**17**, Problem XXXIII] has initiated the question of the existence of a special class of operator-type quadrature formulas, motivated by a classical work of G. D. Birkhoff on Hermite interpolation. Varma [**18**] gave a positive answer to Turán's problem. Moreover, in higher dimension, Pizzetti's formula for polyharmonic functions, a natural generalization of Gauss mean-value property for harmonic functions, is a particular case of operator-type cubature formulas.

The existence of operator-type cubature formulas was extensively studied by Lebedev [**7**], [**8**] who considered the Laplacian operator and found many interesting examples of such formulas for the uniform measure on the unit sphere S^2. Following the work of Lebedev, Shamsiev [**13**] took a linear combination of powers of the Laplacian operator and determined the maximum degree of a "Laplacian-type cubature" on the unit disk B^2. The results by Lebedev and Shamsiev are based on Sobolev's Theorem for the usual cubature formula [**14**].

This paper is organized as follows. In Section 2 we introduce some basic notions and related facts that are often used throughout this paper. In Section 3 a Stroud-type inequality is proved for operator-type cubature formulas, which in particular provides the first theoretic lower bound for Laplacian-type cubature formulas. In Section 4 Sobolev's Theorem is generalized to operator-type cubature formulas, to present a systematic treatment of the works of Lebedev and Shamsiev. In Section 5 a new family of Laplacian-type cubature is given in two dimension, together with a geometric characterization.

2. Definitions.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We denote the set of polynomials on $\mathbb{R}^n$ of degree at most t by $\mathcal{P}_t(\mathbb{R}^n)$. Let Ω be a subset of $\mathbb{R}^n$ and μ be a finite, strictly positive measure on Ω defined on the Borel σ-algebra of Ω. We assume that for any $f \in \mathcal{P}_t(\mathbb{R}^n)$, the restricted function $f|_\Omega$ is L^1-integrable on (Ω, μ).

Definition 2.1. Let t be a non-negative integer and $\mathcal{A}$ be a subspace of $\mathcal{P}_t(\mathbb{R}^n)$.

A pair of a finite subset X of $\mathbb{R}^n$ and a set $\mathcal{T} = \{T_{\boldsymbol{x}} \in \mathrm{End}_{\mathbb{R}}(\mathcal{P}_t(\mathbb{R}^n)) \mid \boldsymbol{x} \in X\}$ of linear operators on $\mathcal{P}_t(\mathbb{R}^n)$ is called an *operator-type cubature formula for $\mathcal{A}$* if

$$\int_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X} (T_{\boldsymbol{x}} f)(\boldsymbol{x})$$

for every $f \in \mathcal{A}$; in particular, $(X, \mathcal{T})$ is said to have *degree t* if $\mathcal{A} = \mathcal{P}_t(\mathbb{R}^n)$.

If X is contained in Ω and $T_{\boldsymbol{x}}$ is a positive scalar multiplication on $\mathcal{P}_t(\mathbb{R}^n)$ for any $\boldsymbol{x} \in X$, then $(X, \mathcal{T})$ is the usual cubature formula.

Historically, Turán [**17**, Problem XXXIII] has raised the question of the existence of a *quadrature formula of type* $(0, 2)$ for the Lebesgue measure on the interval $[-1, 1]$, meaning,

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{N} \left(a_i f(x_i) + b_i \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \Big|_{x=x_i} \right) \quad \text{for every } f \in \mathcal{P}_{2N}(\mathbb{R}). \tag{2.1}$$

There are many papers that describe an affirmative answer to Turán's problem; for example, see [**18**]. A quadrature formula of type $(0, 2)$ is a special case of Birkhoff-type quadrature formula whose motivation lies in the study of Hermite interpolation. Note that, in (2.1), we can never replace the second derivatives $d^2 f/dx^2$ with the first derivatives, that is, there is no quadrature formula of the form

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{N} \left(a_i f(x_i) + b_i \frac{\mathrm{d}}{\mathrm{d}x} f(x) \Big|_{x=x_i} \right) \quad \text{for every } f \in \mathcal{P}_{2N}(\mathbb{R}).$$

This observation will clarify the importance of which operators we choose, even in the one-dimensional case.

In higher dimension, Lebedev [**7**], [**8**] took the Laplacian operator for $T_{\boldsymbol{x}}$ and constructed many examples of such formulas for the uniform measure on the sphere S^2. The points of Lebedev's formulas are invariant under the Weyl group of type B. Following the work of Lebedev, Shamsiev [**13**] took a linear combination of powers of the Laplacian operator at the origin and determined the maximum degree of a "Laplacian-type cubature formula" on the unit disk B^2, with points invariant under a dihedral group (see Example 4.4 in Section 4).

The following terminology will often appear in this paper:

Definition 2.2. Let us denote the Laplacian on $\mathbb{R}^n$ by $\Delta = \sum_{i=1}^{n} (\partial^2/\partial x_i^2)$. We say that an operator-type cubature $(X, \mathcal{T})$ on (Ω, μ) for $\mathcal{A} \subset \mathcal{P}_t(\mathbb{R}^n)$ is of *Laplacian-type of order* $2s$ if the following holds:

- $\mathbf{0} \in X$ and $T_{\mathbf{0}} = T := \sum_{k=0}^{s} \lambda_k \Delta^k$ for a positive integer s and some real numbers λ_i, and
- $T_{\boldsymbol{x}}$ is a positive scalar multiplication on $\mathcal{P}_t(\mathbb{R}^n)$ for each $\boldsymbol{x} \in X \setminus \{\mathbf{0}\}$.

Let us recall Pizzetti's formula as a typical example of Laplacian-type cubature formula on the unit ball $B^n := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| < 1\}$ equipped with the Lebesgue measure:

Theorem 2.3 (Pizzetti's formula).

$$\int_{B^n} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \pi^{n/2} \sum_{k=0}^{s} \frac{1}{k! 2^{2k} \Gamma(n/2+k+1)} (\Delta^k f) \Big|_{\boldsymbol{x}=\mathbf{0}}$$

for each polynomial f with $\Delta^{s+1} f = 0$, where Γ means the gamma function.

Pizzetti's formula gives a Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2s$ and degree at least $2s$, with $X = \{\mathbf{0}\}$ and

$$\mathcal{T} = \left\{ T_{\mathbf{0}} = \pi^{n/2} \sum_{k=0}^{s} \frac{1}{k! 2^{2k} \Gamma(n/2+k+1)} \Delta^k \right\}.$$

It is easily checked that a Laplacian-type cubature of order $2s$ and degree $2s$ with $X = \{\mathbf{0}\}$ is uniquely determined.

In Section 3.2 (see Corollary 3.8), for fixed positive integers e and s, we give a lower bound for the cardinality of X in a Laplacian-type cubature $(X, \mathcal{T})$ of order $2s$ and degree $2e$, when $n = 2$. In Section 5 we study "tight" Laplacian-type cubature formula for $e - s = 2$.

3. Stroud-type inequality for operator-type cubature.

In this section we generalize the Stroud bound for classical polynomial-type cubature to operator-type cubature:

Theorem 3.1 (The Stroud bound (cf. [**16**])). *Let (X, λ) be a cubature formula of degree $2e$ on (Ω, μ). Then*

$$|X| \geq \dim_{\mathbb{R}} \mathcal{P}_e(\Omega),$$

where $\mathcal{P}_e(\Omega)$ denotes the restriction of $\mathcal{P}_e(\mathbb{R}^n)$ to a subset Ω of $\mathbb{R}^n$.

Note that $\dim_{\mathbb{R}} \mathcal{P}_e(\Omega) \leq \dim_{\mathbb{R}} \mathcal{P}_e(\mathbb{R}^n)$ in general.

By Theorem 3.1, for each pair of positive integers e and s, we obtain a lower bound for the cardinality of X in a Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2s$ and degree $2e$ in $\mathbb{R}^2$ (see Corollary 3.8).

3.1. Stroud-type inequality.

In this subsection we present a Stroud-type inequality for an operator-type cubature of even degree. We use the notation Ω, μ as in Definition 2.1.

In order to describe our theorem, let us fix a terminology as follows: Let V be an N-dimensional real vector space equipped with a (possibly degenerate) symmetric bilinear form $(\cdot,\cdot)$. Sylvester's law of inertia states that there exists a unique triple (p,q,r) of non-negative integers for which

(i) $p+q+r=N$,

(ii) there exists an ordered basis $v_1,\dots,v_N$ such that

$$(v_i,v_j)=\begin{cases}\delta_{ij} & \text{if } i\le p,\\ -\delta_{ij} & \text{if } p+1\le i\le p+q,\\ 0 & \text{otherwise,}\end{cases}$$

where $\delta_{ij}=1$ or 0 according on whether $i=j$ or not. The triple (p,q,r) is called the *signature* of the symmetric form $(\cdot,\cdot)$.

Let us take a non-negative integer e, a point $\boldsymbol{x}\in\mathbb{R}^n$ and an $\mathbb{R}$-linear operator T on $\mathcal{P}_{2e}(\mathbb{R}^n)$. We define a (possibly degenerate) symmetric form $(\cdot,\cdot)_{\boldsymbol{x},T}$ on $\mathcal{P}_e(\mathbb{R}^n)$ by

$$(f,f')_{\boldsymbol{x},T}:=(T(f\cdot f'))(\boldsymbol{x}) \tag{3.1}$$

where $f\cdot f'$ is the product of polynomials f and f' on $\mathbb{R}^n$. We denote the signature of $(\cdot,\cdot)_{\boldsymbol{x},T}$ on $\mathcal{P}_e(\mathbb{R}^n)$ by $(p(e,\boldsymbol{x},T),q(e,\boldsymbol{x},T),r(e,\boldsymbol{x},T))$. Note that

$$\dim_{\mathbb{R}}\mathcal{P}_e(\mathbb{R}^n)=p(e,\boldsymbol{x},T)+q(e,\boldsymbol{x},T)+r(e,\boldsymbol{x},T).$$

Clearly, $(p(e,\boldsymbol{x},0),q(e,\boldsymbol{x},0))=(0,0)$.

The following inequality for operator-type cubature generalizes the Stroud bound given in Theorem 3.1.

Theorem 3.2 (Stroud-type inequality for operator-type cubature). *Let e be a non-negative integer and $(X,\mathcal{T})$ be an operator-type cubature formula of degree $2e$. Then the following inequality holds*:

$$\sum_{\boldsymbol{x}\in X}p(e,\boldsymbol{x},T_{\boldsymbol{x}})\ge\dim_{\mathbb{R}}\mathcal{P}_e(\Omega). \tag{3.2}$$

As a corollary to Theorem 3.2, we give a lower bound of Laplacian-type cubature formulas on $\mathbb{R}^2$ (see Corollary 3.8).

Note that if $T_{\boldsymbol{x}}$ is a positive scalar multiplication for any $\boldsymbol{x}\in X$, then the right hand side of (3.2) is the cardinality of X by the example below. This implies Theorem 3.1.

Example 3.3. Let us fix a non-negative integer e, $\boldsymbol{x}\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}\setminus\{0\}$. If T is the scalar multiplication λ on $\mathcal{P}_e(\mathbb{R}^n)$, then

$$(f,f')_{\boldsymbol{x},\lambda}=\lambda f(\boldsymbol{x})\cdot f'(\boldsymbol{x})$$

for $f,f'\in\mathcal{P}_e(\mathbb{R}^n)$. Then we have

$$(p(e, \boldsymbol{x}, \lambda), q(e, \boldsymbol{x}, \lambda)) = \begin{cases} (1, 0) & \text{if } \lambda > 0, \\ (0, 1) & \text{if } \lambda < 0. \end{cases}$$

This can be easily seen by taking an orthogonal decomposition

$$\mathcal{P}_e(\mathbb{R}^n) = \mathbb{R}\{\mathbf{1}\} \oplus \{f \in \mathcal{P}_e(\mathbb{R}^n) \mid f(\boldsymbol{x}) = 0\}$$

with respect to the form $(\cdot, \cdot)_{\boldsymbol{x},\lambda}$, where $\mathbf{1} \in \mathcal{P}_e(\mathbb{R}^n)$ is the constant polynomial taking value 1 everywhere.

Remark 3.4. Our theorem is designed only for even-degree operator-type cubature. Then, what a lower bound should be for the odd degree cases? A good answer for this was given by Möller [**10**] for classical polynomial-type cubature formulas. In Appendix A, we give some examples of operator-type cubature of odd degree for Gaussian integrals, all of which have smaller number of points than classical polynomial-type cubature of the same degree. These illustrations will lead to a question of what a Möller-type bound should be for operator-type cubature, which will be left for future works. Just for reader's informations, cubature for Gaussian integrals are of particular interest in algebraic combinatorics [**1**] and probability theory [**9**].

Now, we prove Theorem 3.2.

Lemma 3.5. *Let V, W be finite-dimensional real* [*resp. complex*] *vector spaces equipped with* (*possibly degenerate*) *symmetric* [*resp. Hermitian*] *bilinear forms $(\cdot, \cdot)_V$ on V and $(\cdot, \cdot)_W$ on W, respectively. We denote the signature of $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ by (p_V, q_V, r_V) and (p_W, q_W, r_W), respectively. If there exists an $\mathbb{R}$-linear* [*resp. $\mathbb{C}$-linear*] *map $L : V \to W$ such that $(v, v')_V = (Lv, Lv')_W$ for every $v, v' \in V$, then $p_V \leq p_W$ and $q_V \leq q_W$. Furthermore, if L is surjective, then $p_V = p_W$ and $q_V = q_W$.*

The above lemma is easy and so proof is omitted.

Lemma 3.6. *Let V be a finite-dimensional real* [*resp. complex*] *vector space, and for each $i = 1, \ldots, m$, let V_i be a copy of V with symmetric* [*resp. Hermitian*] *forms $(\cdot, \cdot)_i$ of signatures (p_i, q_i, r_i). We define a symmetric* [*resp. Hermitian*] *form $(\cdot, \cdot)_V$ by*

$$(v, w)_V := \sum_{i=1}^{m} (v, w)_i,$$

and denote the signature of $(\cdot, \cdot)_V$ by (p_V, q_V, r_V). Then $p_V \leq \sum_{i=1}^m p_i$ and $q_V \leq \sum_{i=1}^m q_i$.

Proof. Let us take a direct sum $\bigoplus_{i=1}^m V_i$ of them. Then $\bigoplus_{i=1}^m V_i$ has a form

$$\left(\sum_{i=1}^{m} v_i, \sum_{i=1}^{m} w_i \right) := \sum_{i=1}^{m} (v_i, w_i)_i$$

where $v_i, w_i \in V_i$. One can easily show that the signature of the form on $\bigoplus_{i=1}^m V_i$ is

$(\sum_{i=1}^m p_i, \sum_{i=1}^m q_i, \sum_{i=1}^m r_i)$. By applying Lemma 3.5 to the diagonal embedding $V \to \bigoplus_{i=1}^m V_i$, we have $p_V \leq \sum_{i=1}^m p_i$ and $q_V \leq \sum_{i=1}^m q_i$. $\square$

LEMMA 3.7. *For a non-negative integer e, consider the restriction map*

$$\pi : \mathcal{P}_e(\mathbb{R}^n) \to \mathcal{P}_e(\Omega), \quad f \mapsto f|_\Omega$$

and define a symmetric form $(\cdot,\cdot)_\pi$ on $\mathcal{P}_e(\mathbb{R}^n)$ by

$$(f, f')_\mu := \int_\Omega \pi(f \cdot f')\, \mathrm{d}\mu. \tag{3.3}$$

Then the signature of $(\cdot,\cdot)_\mu$ is $(\dim_{\mathbb{R}} \mathcal{P}_e(\Omega), 0, \dim_{\mathbb{R}} \operatorname{Ker}\pi)$.

PROOF. Since any polynomial is continuous on Ω and μ is a positive Borel measure on Ω, we see that $(f,f)_\mu \geq 0$ for any $f \in \mathcal{P}_e(\mathbb{R}^n)$, and $(f,f)_\mu = 0$ implies that $f|_\Omega = 0$ in $\mathcal{P}_e(\Omega)$. Hence $\operatorname{Ker}\pi$ is the radical of $(\cdot,\cdot)_\mu$. Let us fix any complement V of $\operatorname{Ker}\pi$ in $\mathcal{P}_e(\mathbb{R}^n)$. Then $\pi|_V : V \to \mathcal{P}_e(\Omega)$ is bijective and $(\cdot,\cdot)_\mu$ is positive definite on V. This completes the proof of Lemma 3.7. $\square$

PROOF OF THEOREM 3.2. Since $(X, \mathcal{T})$ is a cubature formula for $\mathcal{P}_{2e}(\mathbb{R}^n)$,

$$(f, f')_\mu = \sum_{\boldsymbol{x} \in X} (f, f')_{\boldsymbol{x}, T_{\boldsymbol{x}}}$$

for every $f, f' \in \mathcal{P}_e(\mathbb{R}^n)$ (see (3.1) and (3.3) for the notation). Thus, the theorem follows by Lemma 3.6 and Lemma 3.7. $\square$

3.2. A lower bound for Laplacian-type cubature in two dimension.

In this subsection, we present a lower bound for the cardinality of even-degree Laplacian-type cubature formulas on $\mathbb{R}^2$:

COROLLARY 3.8 (Corollary of Theorem 3.2). *Let us consider 2-dimensional Euclidean space $\mathbb{R}^2$ and take a measure space (Ω, μ) as in Definition 2.1. Then for any Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2s$ and degree $2e$ on (Ω, μ) (see Definition 2.2 for the definition of Laplacian-type cubature), the following inequality holds*:

$$|X \setminus \{\mathbf{0}\}| \geq \dim_{\mathbb{R}} \mathcal{P}_e(\Omega) - \frac{1}{2}(s+1)(s+2).$$

In particular, if Ω has an interior point, then

$$|X \setminus \{\mathbf{0}\}| \geq \frac{1}{2}(e-s)(e+s+3). \tag{3.4}$$

DEFINITION 3.9. A Laplacian-type cubature $(X, \mathcal{T})$ with equality in (3.4) is said to be *tight*.

Remark 3.10. (i) As far as the authors know, the bound in Corollary 3.8 is the first theoretic lower bound for Laplacian-type cubature of even degree. (ii) In (3.4), if $e = s$ then $|X \setminus \{\mathbf{0}\}| = 0$. Recall that Pizzetti's formula (see Theorem 2.3) gives a Laplacian-type cubature formula $(X, \mathcal{T})$ with $e = s$ and $|X \setminus \{\mathbf{0}\}| = 0$, which is a unique tight Laplacian-type cubature with $X = \{\mathbf{0}\}$. In Section 5, we study tight Laplacian-type cubature formulas in two dimension.

Now, we show that

$$p(e, \mathbf{0}, T) \leq \frac{1}{2}(s+1)(s+2) \tag{3.5}$$

for $T := \sum_{k=0}^{s} \lambda_k \Delta^k$ (see Section 3.1 for the notation); if this is the case, then by combining this with Theorem 3.2, we obtain Corollary 3.8.

To establish (3.5), we need some preparations. For each t, we denote the complexification of $\mathcal{P}_t(\mathbb{R}^2)$ by $\mathcal{P}_t^{\mathbb{C}}(\mathbb{R}^2)$, meaning, $\mathcal{P}_t^{\mathbb{C}}(\mathbb{R}^2)$ is the set of polynomials over $\mathbb{C}$ of degree at most t in variables x and y. By $T^{\mathbb{C}}$ we also denote the complexification, which is a $\mathbb{C}$-linear operator on $\mathcal{P}_{2e}^{\mathbb{C}}(\mathbb{R}^2)$, of the operator T on $\mathcal{P}_{2e}(\mathbb{R}^2)$. Then

$$(f, f')_{\mathbf{0},T^{\mathbb{C}}} := (T^{\mathbb{C}}(f \cdot \overline{f'}))(\mathbf{0}), \quad f, f' \in \mathcal{P}_e^{\mathbb{C}}(\mathbb{R}^2)$$

defines a Hermitian form on $\mathcal{P}_e^{\mathbb{C}}(\mathbb{R}^2)$, where $\overline{f'}$ is the complex conjugation of the polynomial f' over $\mathbb{C}$. By $(p(e, \mathbf{0}, T^{\mathbb{C}}), q(e, \mathbf{0}, T^{\mathbb{C}}), r(e, \mathbf{0}, T^{\mathbb{C}}))$ we denote the signature of the Hermitian form $(\cdot,\cdot)_{\mathbf{0},T^{\mathbb{C}}}$ on $\mathcal{P}_e^{\mathbb{C}}(\mathbb{R}^2)$. Clearly,

$$(p(e, \mathbf{0}, T^{\mathbb{C}}), q(e, \mathbf{0}, T^{\mathbb{C}}), r(e, \mathbf{0}, T^{\mathbb{C}})) = (p(e, \mathbf{0}, T), q(e, \mathbf{0}, T), r(e, \mathbf{0}, T)).$$

Let us put

$$\partial := \frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}, \quad \bar{\partial} := \frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}.$$

Then $\Delta = \partial\bar{\partial}$ and

$$\overline{\partial f} = \bar{\partial}\bar{f}, \quad \overline{\bar{\partial} f} = \partial\bar{f}$$

for any polynomial f over $\mathbb{C}$. By using the Leibniz rule, we have

$$\begin{aligned}(T^{\mathbb{C}}(f \cdot \overline{f'}))(\mathbf{0}) &= \sum_{k=0}^{s} \lambda_k \sum_{0 \leq i,j \leq k} \binom{k}{i}\binom{k}{j} (\partial^i \bar{\partial}^j f)\big|_{(x,y)=(0,0)} \cdot (\partial^{k-i}\bar{\partial}^{k-j}\overline{f'})\big|_{(x,y)=(0,0)} \\ &= \sum_{k=0}^{s} \lambda_k \sum_{0 \leq i,j \leq k} \binom{k}{i}\binom{k}{j} (\partial^i \bar{\partial}^j f)\big|_{(x,y)=(0,0)} \cdot \overline{(\partial^{k-j}\bar{\partial}^{k-i} f')}\big|_{(x,y)=(0,0)}\end{aligned}$$

for $f, f' \in \mathcal{P}_e^{\mathbb{C}}(\mathbb{R}^2)$.

For each integer $0 \leq k \leq s$, let $I_k := \{0, 1, \dots, k\}$. Let $\mathbb{C}^{I_s \times I_s}$ be the direct product of copies of $\mathbb{C}$ indexed by $I_s \times I_s$, which is an $(s+1)^2$-dimensional $\mathbb{C}$-vector space. We define a Hermitian form $(\cdot, \cdot)_T$ on $\mathbb{C}^{I_s \times I_s}$ by

$$(v, v')_T := \sum_{k=0}^{s} \lambda_k \sum_{(i,j) \in I_k \times I_k} \binom{k}{i} \binom{k}{j} v_{i,j} \overline{v'_{k-j,k-i}} \tag{3.6}$$

for $v := (v_{i,j})_{i,j \in I_s}$, $v' := (v'_{i,j})_{i,j \in I_s} \in \mathbb{C}^{I_s \times I_s}$. Then the $\mathbb{C}$-linear map

$$L : \mathcal{P}_e^{\mathbb{C}}(\mathbb{R}^n) \to \mathbb{C}^{I_s \times I_s}, \quad f \mapsto ((\partial^i \bar{\partial}^j f)|_{(x,y)=(0,0)})_{i,j \in I_s} \tag{3.7}$$

preserves the Hermitian forms.

LEMMA 3.11. *Let V be a finite-dimensional $\mathbb{C}$-vector space equipped with a (possibly degenerate) Hermitian form $(\cdot, \cdot)$ of signature (p, q, r). Let U be a totally isotropic subspace of V. Then*

$$\dim_{\mathbb{C}} U_{\mathbb{C}} \leq \min\{p, q\} + r.$$

In particular, $p \leq \dim_{\mathbb{C}} V - \dim_{\mathbb{C}} U$.

PROOF. Let U be a maximal totally isotropic subspace of V. Then the radical V_0 of $(\cdot, \cdot)$ is included in U, and U/V_0 is also a totally isotropic subspace of V/V_0 with respect to the non-degenerate form $(\cdot, \cdot)_{V/V_0}$ on V/V_0 induced by the form on V. For simplicity, let $V' := V/V_0$ and $U' := U/V_0$. It suffices to show that

$$\dim_{\mathbb{C}} U' \leq \min\{p, q\}.$$

Since $(\cdot, \cdot)_{V'}$ is a non-degenerate form with signature (p, q), we obtain a decomposition $V' = V'_p \oplus V'_q$ with $\dim_{\mathbb{C}} V'_p = p$ and $\dim_{\mathbb{C}} V'_q = q$, such that $(\cdot, \cdot)_{V'}$ is positive definite on V'_p and negative definite on V'_q. By $\pi_p : V' \to V'_p$ and $\pi_q : V' \to V'_q$ we denote the projection with respect to the decomposition. Let us take any $u \in U'$. Then $(\cdot, \cdot)_{V'} = 0$. Let us consider the case where $\pi_p(u) = \mathbf{0}$. Then $u \in V'_q$ and hence $u = \mathbf{0}$ since the form $(\cdot, \cdot)_{V'}$ on V'_q is negative definite. Therefore, $\pi_p|_{U'}$ is injective and $\dim_{\mathbb{C}} U' \leq p$. The same arguments show that $\pi_q|_{U'}$ is injective and $\dim_{\mathbb{C}} U' \leq q$. □

Now we denote the signature of $(\cdot, \cdot)_T$ on $\mathbb{C}^{I_s \times I_s}$ by (p_T, q_T, r_T). Since the map L defined by (3.7) preserves the Hermitian forms, we have $p(e, 0, T) \leq p_T$. Let us take

$$U := \{v \in \mathbb{C}^{I_s \times I_s} \mid v_{i,j} = 0 \text{ for } 0 \leq i + j \leq s\}.$$

Then, $\dim_{\mathbb{C}} U = s(s+1)/2$ and

$$\dim_{\mathbb{C}} \mathbb{C}^{I_s \times I_s} - \dim_{\mathbb{C}} U = \frac{1}{2}(s+1)(s+2).$$

Clearly, U is a totally isotropic subspace of $\mathbb{C}^{I_s \times I_s}$ with respect to $(\cdot,\cdot)_T$. Therefore, by Lemma 3.11,

$$p(e,\mathbf{0},T) \le p_T \le \frac{1}{2}(s+1)(s+2),$$

which completes the proof of (3.5).

Example 3.12. Let us consider the case where $\boldsymbol{x} = \mathbf{0} \in \mathbb{R}^2$, $2s \le e$, and $T = \Delta^s$. We shall prove that

$$(p(e,\mathbf{0},T), q(e,\mathbf{0},T)) = \left(\frac{1}{2}(s+1)(s+2), \frac{1}{2}s(s+1)\right).$$

First, note that the map L is surjective since $2s \le e$. Thus Lemma 3.5 implies that $(p(e,\mathbf{0},T), q(e,\mathbf{0},T)) = (p_T, q_T)$. Here we denote the standard basis of $\mathbb{C}^{I_s \times I_s}$ by $\{\boldsymbol{e}_{i,j} \mid i,j \in I_s\}$. Then the Hermitian form $(\cdot,\cdot)_T$ defined by (3.6) is positive definite on

$$V_p := \mathrm{Span}_{\mathbb{C}}(\{\boldsymbol{e}_{i,j} + \boldsymbol{e}_{s-j,s-i} \mid i+j < s\} \cup \{\boldsymbol{e}_{l,s-l} \mid l \in I_s\})$$

and negative definite on

$$V_q := \mathrm{Span}_{\mathbb{C}}\{\boldsymbol{e}_{i,j} - \boldsymbol{e}_{s-j,s-i} \mid i+j < s\}.$$

Since $\mathbb{C}^{I_s \times I_s} = V_p \oplus V_q$, we have

$$(p(e,\mathbf{0},T), q(e,\mathbf{0},T)) = (p_T, q_T) = \left(\frac{1}{2}(s+1)(s+2), \frac{1}{2}s(s+1)\right).$$

Remark 3.13. In the proof of the bound (3.4), the choice of the totally isotropic subspace U is key. Then, what about higher-dimensional cases? In this case, the authors gave a choice of U in Appendix B, but we suspect that the resulting bound would not be best and it might be possible to pick up a "better" totally isotropic subspace U. It seems that to choose a good totally isotropic subspace U is hard, which is thus beyond the scope of this paper and left for future work. We note that a family of two-dimensional "tight" Laplacian-type cubature is given, which implies our bound is good in two dimension; see Section 5 for the details.

4. Sobolev's theorem for operator-type cubature.

In this section we generalize a famous theorem due to Sobolev [**14**] on invariant cubature formulas to operator-type invariant cubature, in order to construct tight Laplacian-type cubature formulas on $\mathbb{R}^2$ (see Theorem 5.3).

Let G be a finite subgroup of the orthogonal group $O(\mathbb{R}^n)$, and let $f \in \mathcal{P}_t(\mathbb{R}^n)$. We consider the action of $g \in G$ on f as follows:

$$f^g(\boldsymbol{x}) = f(\boldsymbol{x}^{g^{-1}}), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

A polynomial f is said to be *G-invariant* if $f^g = f$ for every $g \in G$. By $\mathcal{P}_t(\mathbb{R}^n)^G$ we denote the set of G-invariant polynomials in $\mathcal{P}_t(\mathbb{R}^n)$; a similar notation will be used for other polynomial spaces later.

For a measure space (Ω, μ) in $\mathbb{R}^n$ as in Definition 2.1, we say that (Ω, μ) is *G-invariant* if Ω and μ are invariant under G, respectively. The similar notation is used for a pair (X, λ) of a finite subset X and a positive weight function λ on X.

The following is known as Sobolev's theorem [**14**].

THEOREM 4.1 (Sobolev's theorem). *Let us take a G-invariant measure space (Ω, μ) in $\mathbb{R}^n$ and a G-invariant pair (X, λ) as above. Then the following conditions on (X, λ) are equivalent*:

1. *(X, λ) is a cubature formula of degree t.*
2. *(X, λ) is a cubature formula for $\mathcal{P}_t(\mathbb{R}^n)^G$.*

Let us generalize Theorem 4.1 for operator-type cubature formulas.

For a linear operator T on $\mathcal{P}_t(\mathbb{R}^n)$ and $g \in G$, we define a linear operator T^g on $\mathcal{P}_t(\mathbb{R}^n)$ by

$$T^g f = (T(f^{g^{-1}}))^g, \quad f \in \mathcal{P}_t(\mathbb{R}^n).$$

Note that the Laplacian $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is $O(\mathbb{R}^n)$-invariant, i.e., $\Delta^g = \Delta$ for any $g \in O(\mathbb{R}^n)$.

We say that a pair $(X, \mathcal{T})$ of a finite subset X of $\mathbb{R}^n$ and a set of linear operator $\mathcal{T} = \{T_{\boldsymbol{x}} \mid \boldsymbol{x} \in X\}$ on $\mathcal{P}_t(\mathbb{R}^n)$ indexed by X is *G-invariant* if X is a union of G-orbits and $T_{\boldsymbol{x}}^g = T_{\boldsymbol{x}^g}$ for any $\boldsymbol{x} \in X$ and $g \in G$.

Sobolev's theorem can be generalized for operator-type cubature.

THEOREM 4.2. *Let us take a G-invariant measure space (Ω, μ) in $\mathbb{R}^n$ and a G-invariant pair $(X, \mathcal{T})$ as above. Then the following conditions on $(X, \mathcal{T})$ are equivalent*:

1. *$(X, \mathcal{T})$ is an operator-type cubature formula of degree t.*
2. *$(X, \mathcal{T})$ is an operator-type cubature formula for $\mathcal{P}_t(\mathbb{R}^n)^G$.*

Since the Laplacian operator Δ on $\mathbb{R}^n$ is $O(n)$-invariant, we get the following corollary for Laplacian-type cubature formulas:

COROLLARY 4.3. *Let us take a G-invariant measure space (Ω, μ) in $\mathbb{R}^n$, a G-invariant finite subset X' in Ω with G-invariant positive weights $\{\lambda_{\boldsymbol{x}} \mid \boldsymbol{x} \in X'\}$ on X' and an operator $T_{\mathbf{0}} := T = \sum_{k=0}^{s} \lambda_k \Delta^k$ for real coefficients $\lambda_1, \ldots, \lambda_s$. Let $X := X' \cup \{\mathbf{0}\}$ and $\mathcal{T} := \{\lambda_{\boldsymbol{x}} \mid \boldsymbol{x} \in X'\} \cup \{T_{\mathbf{0}} = T\}$. Then the following conditions on $(X, \mathcal{T})$ are equivalent*:

1. *$(X, \mathcal{T})$ is a Laplacian-type cubature formula of degree t.*
2. *$(X, \mathcal{T})$ is a Laplacian-type cubature formula for $\mathcal{P}_t(\mathbb{R}^n)^G$ (see Definition 2.2 for the notation).*

We recall Shamsiev's results in [**13**] as an example of Corollary 4.3:

Example 4.4. Let l and s be non-negative integers no more than k. Shamsiev found real coefficients $\lambda_1, \dots, \lambda_{2s}$, positive weights $W_1, \dots, W_l$, $\tilde{W}_1, \dots, \tilde{W}_{l+k-s}$ and positive radii $r_1, \dots, r_l$, $\tilde{r}_1, \dots, \tilde{r}_{l+k-s}$ such that the following equation holds for every $f \in \mathcal{P}_{4k+4l+1}(\mathbb{R}^2)$:

$$\begin{aligned}\int_{B^2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} &= \sum_{k=0}^{2s} \lambda_k \Delta^k f(x,y) \Big|_{(x,y)=(0,0)} \\ &+ \sum_{i=1}^{l} W_i \sum_{m=0}^{4k+1} f\left(r_i \cos \frac{2m\pi}{4k+2}, r_i \sin \frac{2m\pi}{4k+2} \right) \\ &+ \sum_{i=1}^{l+k-s} \tilde{W}_i \sum_{m=0}^{4k+1} f\left(\tilde{r}_i \cos \frac{(2m+1)\pi}{4k+2}, \tilde{r}_i \sin \frac{(2m+1)\pi}{4k+2} \right).\end{aligned}$$

Shamsiev found the above Laplacian-type cubature formula by implicitly using Corollary 4.3 with respect to the dihedral group $G = D_{4k+2}$. He also found many other examples of Laplacian-type cubature formulas in [**13**].

We shall prove Theorem 4.2.

Proof of Theorem 4.2. The implication "(1) to (2)" is trivial and so we may assume (2). For $f \in \mathcal{P}_t(\mathbb{R}^n)$, define

$$\phi = \frac{1}{|G|} \sum_{g \in G} f^{g^{-1}}.$$

Clearly, ϕ is G-invariant and $\int_\Omega f\, \mathrm{d}\mu = \int_\Omega \phi\, \mathrm{d}\mu$. By the assumption, we have

$$\begin{aligned}\int_\Omega f\, \mathrm{d}\mu &= \int_\Omega \phi\, \mathrm{d}\mu \\ &= \sum_{\boldsymbol{x} \in X} (T_{\boldsymbol{x}} \phi)(\boldsymbol{x}) \\ &= \frac{1}{|G|} \sum_{\boldsymbol{x} \in X} \sum_{g \in G} (T_{\boldsymbol{x}} f^{g^{-1}})(\boldsymbol{x}) \\ &= \frac{1}{|G|} \sum_{\boldsymbol{x} \in X} \sum_{g \in G} (T_{\boldsymbol{x}}^g f)(\boldsymbol{x}^g) \\ &= \sum_{\boldsymbol{x} \in X} (T_{\boldsymbol{x}} f)(\boldsymbol{x}),\end{aligned}$$

which completes the proof. $\square$

5. Some results for tight Laplacian-type cubature formulas in two-dimension.

In this section, we study tight Laplacian-type cubature on a certain domain in $\mathbb{R}^2$ (see Definition 3.9 for the notation).

5.1. A characterization of tight Laplacian-type cubature.

Let a, b be are real numbers with $0 \leq a < b$, and Ω be the annulus domain in $\mathbb{R}^2$ defined by

$$\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 \mid a \leq \|\boldsymbol{x}\| < b\}.$$

Let $W(\|\boldsymbol{x}\|)$ be a rotationally invariant probability density function on Ω (cf. [**11**, p. 325]).

Let e, s be integers with $e > s \geq 0$. Let X' be a finite subset of $\Omega \setminus \{\boldsymbol{0}\}$, where Ω may not possibly contain the origin, with a positive weight function λ on X. Fix a linear operator $T := \sum_{k=0}^{s} \lambda_k \Delta^k$. We denote

$$X := X' \cup \{\boldsymbol{0}\} \quad \text{and} \quad \mathcal{T} := \{T_{\boldsymbol{0}} := T\} \cup \{\lambda_{\boldsymbol{x}} \mid \boldsymbol{x} \in X'\}.$$

For each non-negative integer l, we write $\mathrm{Hom}_l(\mathbb{R}^2)$ for the set of homogeneous polynomials on $\mathbb{R}^2$ of homogeneous degree l and put

$$\mathrm{Harm}_l(\mathbb{R}^2) := \{f \in \mathrm{Hom}_l(\mathbb{R}^2) \mid \Delta f = 0\}.$$

Sometimes we use the symbol $\|\boldsymbol{x}\|^2$ to mean the polynomial $x^2 + y^2$ in $\mathrm{Hom}_2(\mathbb{R}^2)$ (when a point $\boldsymbol{x} \in \mathbb{R}^2$ is fixed, $\|\boldsymbol{x}\|^2$ means the square of the norm).

Then we get the following easy (albeit important) result.

PROPOSITION 5.1. *The following conditions on $(X, \mathcal{T})$ are equivalent*:

(i) $(X, \mathcal{T})$ *is a Laplacian-type cubature formula of order* $2s$ *and degree* $2e$ *for* $\int_\Omega \cdot W(\|\boldsymbol{x}\|)\, \mathrm{d}\boldsymbol{x}$.

(ii) *For each* k,

$$\lambda_k = \frac{1}{(k!)^2 2^{2k}} \left(2\pi \int_a^b r^{2k+1} W(r)\, \mathrm{d}r - \sum_{\boldsymbol{x} \in X \setminus \{\boldsymbol{0}\}} \|\boldsymbol{x}\|^{2k} \lambda_{\boldsymbol{x}} \right)$$

and (X', λ) *is a cubature formula for the space*

$$\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2) \oplus \bigoplus_{k=0}^{s} \bigoplus_{m=1}^{2s+1-2k} \|\boldsymbol{x}\|^{2k}\, \mathrm{Harm}_m(\mathbb{R}^2). \tag{5.1}$$

We are interested in a Laplacian-type cubature formula $(X, \mathcal{T})$ of degree $2e$ with equality in the bound of Corollary 3.8, or equivalently, a cubature (X', λ) for the space (5.1) with

$$|X'| = \frac{1}{2}(e-s)(e+s+3) \tag{5.2}$$

points.

Now, we establish a characterization of the Laplacian-type cubature of Theorem 5.3 by investigating the structure of a cubature formula for

$$\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2)$$

with $(e-s)(e+s+3)/2$ points. A useful tool for this purpose is the reproducing kernel of $\bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2)$. Let us consider a scalar product on $\mathcal{P}_e(\mathbb{R}^2)$, defined by

$$(f,g) = \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})W(\|\boldsymbol{x}\|)\,\mathrm{d}\boldsymbol{x}.$$

For each $\boldsymbol{x} \in \mathbb{R}^2$, the point evaluation of f is continuous. Hence by Riesz's representation theorem, there exists a function $f_{\boldsymbol{x}} \in \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2)$ such that $f(\boldsymbol{x}) = (f, f_{\boldsymbol{x}})$ for every $f \in \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2)$. The function $K : \Omega \times \Omega \to \mathbb{R}$ defined by

$$K(\boldsymbol{x},\boldsymbol{y}) = (f_{\boldsymbol{x}}, f_{\boldsymbol{y}}) \quad \text{for } \boldsymbol{x},\boldsymbol{y} \in \Omega$$

is called the *reproducing kernel* of the space $\bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2)$. Clearly we have $K(\boldsymbol{x},\boldsymbol{y}) = f_{\boldsymbol{x}}(\boldsymbol{y}) = f_{\boldsymbol{y}}(\boldsymbol{x})$.

The following proposition gives a characterization of cubature formulas (X',λ) for $\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2)$ with $|X'| = (e-s)(e+s+3)/2$:

PROPOSITION 5.2. *Let X' be a finite subset of $\Omega \setminus \{\mathbf{0}\}$, where Ω may not possibly contain the origin. Let λ be a positive weight function on X'. Suppose that (X',λ) is a cubature formula for $\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2)$. Then*

$$\frac{1}{2}(e-s)(e+s+3) \le |X'|$$

and the equality holds in this bound if and only if

$$\sqrt{\lambda_{\boldsymbol{x}}\lambda_{\boldsymbol{x}'}}\,K(\boldsymbol{x},\boldsymbol{x}') = \delta_{\boldsymbol{x}\boldsymbol{x}'} \quad \textit{for } \boldsymbol{x},\boldsymbol{x}' \in X'.$$

The above proposition can also be proved in more general settings [**12**].

PROOF. First, note that

$$\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2) = \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2) \cdot \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2).$$

So (X', λ) is a cubature formula for $\bigoplus_{l=2s+2}^{2e} \mathrm{Hom}_l(\mathbb{R}^2)$ if and only if the restriction map

$$\rho : \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2) \to \ell^2(X', \lambda)$$

is an isometry, where $\ell^2(X', \lambda)$ denotes the Hilbert space of $\mathbb{R}$-valued functions on X', with scalar product defined by

$$\langle f, g \rangle = \sum_{\boldsymbol{x} \in X'} \lambda_{\boldsymbol{x}} f(\boldsymbol{x}) g(\boldsymbol{x}).$$

We thus obtain

$$\begin{aligned}
\frac{1}{2}(e-s)(e+s+3) &= \sum_{l=s+1}^{e} (l+1) \\
&= \dim_{\mathbb{R}} \bigoplus_{l=s+1}^{e} \mathrm{Hom}_l(\mathbb{R}^2) \\
&\le \dim_{\mathbb{R}} \ell^2(X', \lambda) = |X'|.
\end{aligned}$$

The equality implies that ρ is an isomorphism of metric linear spaces, and so is the adjoint map ρ^*. It is easily seen that the point-wise Dirac measures $\delta_{\boldsymbol{x}}$ for all $\boldsymbol{x} \in X'$ form a basis of $\ell^2(X', \lambda)$ and $\rho^*(\delta_{\boldsymbol{x}}) = \lambda_{\boldsymbol{x}} f_{\boldsymbol{x}}$. In summary, for every $x, y \in X'$,

$$\begin{aligned}
\sqrt{\lambda_{\boldsymbol{x}} \lambda_{\boldsymbol{y}}} \delta_{\boldsymbol{x}\boldsymbol{y}} &= \langle \delta_{\boldsymbol{x}}, \delta_{\boldsymbol{y}} \rangle \\
&= (\rho^*(\delta_{\boldsymbol{x}}), \rho^*(\delta_{\boldsymbol{y}})) \\
&= \lambda_{\boldsymbol{x}} \lambda_{\boldsymbol{y}} (f_{\boldsymbol{x}}, f_{\boldsymbol{y}}) \\
&= \lambda_{\boldsymbol{x}} \lambda_{\boldsymbol{y}} K(\boldsymbol{x}, \boldsymbol{y}),
\end{aligned}$$

which completes the proof. □

How to use the above result will soon be clear in Subsection 5.2.

5.2. A family of tight Laplacian-type cubature formulas.

In this subsection, we fix an integer $e \ge 2$ and provide a family of tight Laplacian-type cubature $(X, \mathcal{T})$ with $e - s = 2$, together with a geometric characterization. The readers may be first interested in what would happen for $e - s = 1$; the answer will be given in the next subsection. Remember that Pizzetti's formula (Theorem 2.3) gives a unique tight Laplacian-type cubature with $e - s = 0$ and $X = \{\mathbf{0}\}$.

Recall that Corollary 3.8 gives a lower bound

$$|X \setminus \{\mathbf{0}\}| \ge 2e + 1, \tag{5.3}$$

and a Laplacian-type cubature formula $(X,\mathcal{T})$ of order $2e-4$ and degree $2e$ is said to be tight if the equation above holds.

Theorem 5.3. *Let X' be the set of vertices of a regular $(2e+1)$-gon inscribed in the circle of radius $\sqrt{\iota_e/\iota_{e-1}}$ where*

$$\iota_e := \int_{\boldsymbol{x}\in\Omega} \|\boldsymbol{x}\|^{2e} W(\|\boldsymbol{x}\|)\,\mathrm{d}\boldsymbol{x},$$

and $X = X' \cup \{\boldsymbol{0}\}$. For each $0 \le k \le e-2$, let

$$w_k = \frac{1}{(k!)^2 2^{2k}}\left(\iota_k - \frac{\iota_{e-1}^{e-k}}{\iota_e^{e-k-1}}\right)$$

and $\mathcal{T} = \{T_{\boldsymbol{x}} \mid \boldsymbol{x} \in X\}$ be such that

$$T_{\boldsymbol{x}} = \begin{cases} \displaystyle\sum_{k=0}^{e-2} w_k \Delta^k & \text{if } \boldsymbol{x} = \boldsymbol{0}, \\ \displaystyle\frac{1}{2e+1}\cdot\frac{\iota_{e-1}^{e}}{\iota_e^{e-1}} & \text{if } \boldsymbol{x} \in X'. \end{cases}$$

Then $(X,\mathcal{T})$ is a tight Laplacian-type cubature formula of order $2e-4$ and degree $2e$.

Proof. We only need to show that $(X,\mathcal{T})$ is a Laplacian-type cubature formula since $|X\setminus\{\boldsymbol{0}\}| = |X'| = 2e+1$. Let G be the cyclic group of $2e+1$ rotations about the origin through angles that are multiples of $2\pi/(2e+1)$. Since $(X,\mathcal{T})$ is G-invariant, it suffices by Theorem 4.2 to show that

$$\sum_{\boldsymbol{x}\in X} (T_{\boldsymbol{x}} f)(\boldsymbol{x}) = \int_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}) W(\|\boldsymbol{x}\|)\,\mathrm{d}\boldsymbol{x}$$

for every $f \in \mathcal{P}_{2e}(\mathbb{R}^2)^G$. Note that

$$\mathcal{P}_{2e}(\mathbb{R}^2)^G = \mathrm{Span}_{\mathbb{R}}\{\|\boldsymbol{x}\|^{2m} \mid 0 \le m \le e\},$$

where we use the symbol $\|\boldsymbol{x}\|^2$ as the polynomial x^2+y^2 in $\mathrm{Hom}_2(\mathbb{R}^2)$. Moreover, by elementary calculations, we have

$$(\Delta^k \|\boldsymbol{x}\|^{2m})|_{\boldsymbol{x}=\boldsymbol{0}} = (k!)^2 2^{2k} \delta_{k,m}.$$

Therefore, it suffices to prove that for every $0 \le m \le e$,

$$\sum_{\boldsymbol{x}\in X'} \frac{1}{2e+1}\cdot\frac{\iota_{e-1}^{e}}{\iota_e^{e-1}}\|\boldsymbol{x}\|^{2m} + (m!)^2 2^{2m} w_m = \iota_m,$$

where $w_m = 0$ for $m = e-1, e$. The theorem thus follows by the definition of X' and $\mathcal{T}$. $\square$

Hereafter we focus on an open unit ball $B^2 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}\| < 1\}$. In this case,

$$\iota_e := \int_{B^2} (x^2+y^2)^e \,\mathrm{d}x\mathrm{d}y = \frac{\pi}{e+1}.$$

So, our example of tight Laplacian-type cubature formula in Theorem 5.3 has points consisting the origin and vertices of a regular $(2e+1)$-gon on the circle of radius $\sqrt{\iota_e/\iota_{e-1}} = \sqrt{e/(e+1)}$.

Now, a natural question asks whether there exists other examples of tight Laplacian-type cubature formulas $(X, \mathcal{T})$ such that $X \setminus \{\boldsymbol{0}\}$ is contained in the circle of radius $\sqrt{e/(e+1)}$. Below we give the following negative answer:

THEOREM 5.4. *Let $(X, \mathcal{T})$ be a tight Laplacian-type cubature formula of order $2e-4$ and degree $2e$ on B^2 (see Definition 3.9 for the notation of tightness). Suppose that $\|\boldsymbol{x}\| = \sqrt{e/(e+1)}$ for any $\boldsymbol{x} \in X \setminus \{\boldsymbol{0}\}$. Then the point set $X \setminus \{\boldsymbol{0}\}$ consists of the vertices of a regular $(2e+1)$-gon and the positive weights $T_{\boldsymbol{x}}$ are a constant on $X \setminus \{\boldsymbol{0}\}$.*

PROOF. First, for each $\boldsymbol{x} = (x, y) \in B^2 \setminus \{(0,0)\}$, let

$$z_{\boldsymbol{x}} = r_{\boldsymbol{x}} \exp\left(\sqrt{-1}\theta_{\boldsymbol{x}}\right) := x + \sqrt{-1}y \in \mathbb{C},$$

where $r_{\boldsymbol{x}} := |z_{\boldsymbol{x}}| = \|\boldsymbol{x}\|$ and $\theta_{\boldsymbol{x}} := \mathrm{Arg}\; z_{\boldsymbol{x}}$. Note that

$$\{z_{\boldsymbol{x}}^i \overline{z_{\boldsymbol{x}}}^{\,e-i}, z_{\boldsymbol{x}}^j \overline{z_{\boldsymbol{x}}}^{\,e-1-j} \mid i = 0, \dots, e,\; j = 0, \dots, e-1\}$$

is an orthogonal basis of $\bigoplus_{l=e-1}^{e} \mathrm{Hom}_l^{\mathbb{C}}(\mathbb{R}^2)$, where $\mathrm{Hom}_l^{\mathbb{C}}(\mathbb{R}^2) := \mathrm{Hom}_l(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$. Therefore the reproducing kernel is given by

$$\begin{aligned} K(\boldsymbol{x}, \boldsymbol{x}') &= \frac{1}{\pi} \sum_{l=e-1}^{e} (l+1) \sum_{i=0}^{l} z_{\boldsymbol{x}}^i \overline{z_{\boldsymbol{x}}}^{\,l-i} z_{\boldsymbol{x}'}^{l-i} \overline{z_{\boldsymbol{x}'}}^{\,i} \\ &= \frac{1}{\pi} \sum_{l=e-1}^{e} (l+1) r_{\boldsymbol{x}}^l r_{\boldsymbol{x}'}^l \sum_{i=0}^{l} \exp\left(\sqrt{-1}(2i-l)(\theta_{\boldsymbol{x}} - \theta_{\boldsymbol{x}'})\right) \\ &= \frac{1}{\pi} \sum_{l=e-1}^{e} (l+1) r_{\boldsymbol{x}}^l r_{\boldsymbol{x}'}^l \sum_{i=0}^{l} \cos(2i-l)(\theta_{\boldsymbol{x}} - \theta_{\boldsymbol{x}'}). \end{aligned} \tag{5.4}$$

Since the point set X' is contained in the circle of radius $\sqrt{e/(e+1)}$, the kernel (5.4) can be simplified as follows: For any distinct $\boldsymbol{x}, \boldsymbol{x}'$ with $\|\boldsymbol{x}\| = \|\boldsymbol{x}'\| = \sqrt{e/(e+1)}$ and $\cos\alpha = \langle \boldsymbol{x}, \boldsymbol{x}' \rangle / (\|\boldsymbol{x}\| \|\boldsymbol{x}'\|)$,

$$\begin{aligned}\pi K(\boldsymbol{x},\boldsymbol{x}') &= e r_{\boldsymbol{x}}^{2e-2}\sum_{i=0}^{e-1} T_{2i-e+1}(\cos\alpha) + (e+1) r_{\boldsymbol{x}}^{2e}\sum_{i=0}^{e} T_{2i-e}(\cos\alpha)\\ &= \frac{e^e}{(e+1)^{e-1}}\left\{\sum_{i=0}^{e-1} T_{2i-e+1}(\cos\alpha) + \sum_{i=0}^{e} T_{2i-e}(\cos\alpha)\right\}\\ &= \frac{e^e}{(e+1)^{e-1}}(U_{e-1}(\cos\alpha)+U_e(\cos\alpha)),\end{aligned}$$

where T_i, U_i denote the first and second Chebyshev polynomial of degree i, respectively. We note that

$$T_{2e+1}(\cos\alpha) - 1 = (\cos\alpha - 1)(U_e(\cos\alpha) + U_{e-1}(\cos\alpha))^2,$$

by suitably "perturbing" $\boldsymbol{x}$ and $\boldsymbol{x}'$, and by using the fact that $U_n(\cos\alpha) = \sin(n+1)\alpha/\sin\alpha$. This implies, by Proposition 5.2, that X' is the set of vertices of a regular $(2e+1)$-gon inscribed in the circle of radius $\sqrt{e/(e+1)}$. Finally by using Proposition 5.2 again, we have $\lambda_{\boldsymbol{x}} = \lambda_{\boldsymbol{x}'}$ for every $\boldsymbol{x}, \boldsymbol{x}' \in X'$, which proves Theorem 5.4. □

5.3. Some results on tight Laplacian-type cubature with $e - s \neq 2$.

In the previous subsection we study tight Laplacian-type cubature for $e-s=2$, but the readers may be first interested in what would happen for $e-s=1$. In this subsection we give an answer for this and provide some related remarks.

Before mentioning the main result in this subsection (Theorem 5.6), we give the definition of Euclidean design. For $i = 1, \ldots, p$, let S_i be the sphere with radius r_i. Without loss of generality, we may assume $r_1 > \cdots > r_p \geq 0$. Let $S = \bigcup_{i=1}^p S_i$, and $X_i = X \cap S_i$. By σ_i we denote the normalized surface measure on S_i; in particular when $r_p = 0$, we let $\int_{S_p} f(\boldsymbol{x})\, \mathrm{d}\sigma_p(\boldsymbol{x}) = f(\boldsymbol{0})$. The pair (X, λ) is called a *Euclidean t-design supported by p concentric spheres S* if

$$\sum_{\boldsymbol{x}\in X}\lambda_{\boldsymbol{x}} f(\boldsymbol{x}) = \sum_{i=1}^{p}\left(\sum_{\boldsymbol{x}\in X_i}\lambda_{\boldsymbol{x}}\right)\int_{\boldsymbol{x}\in S_i} f(\boldsymbol{x})\,\mathrm{d}\sigma_i(\boldsymbol{x})$$

for every polynomial f in $\mathcal{P}_t(S)$.

Lemma 5.5 ([**11**]). *Let X be a finite subset, which may possibly contain $\boldsymbol{0}$, with a weight function w. Then the following* (1) *and* (2) *are equivalent*:

(i) *(X, λ) is a Euclidean t-design.*
(ii) *$\sum_{\boldsymbol{x}\in X}\lambda_{\boldsymbol{x}} f(\boldsymbol{x}) = 0$ for any polynomial $f \in \|\boldsymbol{x}\|^{2j}\mathrm{Harm}_l(\mathbb{R}^n)$ with $1 \leq l \leq t$, $0 \leq j \leq \lfloor (t-l)/2 \rfloor$.*

Theorem 5.6. *There exists no tight Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2e-2$ and degree $2e$ on B^2.*

Proof. Assume there exists a tight Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2e-2$ and degree $2e$ on B^2. Then, it follows from (5.2) that

$$|X'| = e + 1,$$

and

$$\int_{B^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{k=0}^{e-1} \lambda_k(\Delta^k f)\Big|_{\boldsymbol{x}=\boldsymbol{0}} + \sum_{\boldsymbol{x}\in X'} \lambda_{\boldsymbol{x}} f(\boldsymbol{x}), \tag{5.5}$$

for any polynomial $f \in \mathcal{P}_{2e}(\mathbb{R}^2)$.

For each $l \geq 1$, $f \in \mathrm{Harm}_l(\mathbb{R}^2)$, and $m \geq 1$, we have

$$\Delta(\|\boldsymbol{x}\|^{2m} f) = 2m(m+1+2l)\|\boldsymbol{x}\|^{2(m-1)} f. \tag{5.6}$$

Since $f(\boldsymbol{0}) = 0$ and $\Delta f \equiv 0$ for any $f \in \mathrm{Harm}_l(\mathbb{R}^2)$, it holds that for $0 \leq k \leq e-1$,

$$(\Delta^k f)|_{\boldsymbol{x}=\boldsymbol{0}} = 0 \tag{5.7}$$

for every polynomial $f \in A := \bigoplus_{m=0}^{e-1} \bigoplus_{l=1}^{2e-2m} \|\boldsymbol{x}\|^{2m}\mathrm{Harm}_l(\mathbb{R}^2) \subset \mathcal{P}_{2e}(\mathbb{R}^2)$. Thus, by (5.5) and (5.7), we have

$$\sum_{\boldsymbol{x}\in X'} \lambda_{\boldsymbol{x}} f(\boldsymbol{x}) = \int_{B^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

for every polynomial $f \in A$, which implies by Lemma 5.5 that X' is a Euclidean $2e$-design. However, a lower bound for number of points of a Euclidean $2e$-design is given by

$$|X'| \geq 2e + 1$$

(cf. [**11**]), which is a contradiction. □

In general, let us assume that (X, w) is a tight Laplacian-type cubature formula of (order $2s$ and) degree $2e$ on p concentric spheres. Then, by the similar arguments, we can prove that $e \leq p$, and therefore there exists no tight Laplacian-type cubature formula on 2 concentric spheres for $e - s = 3$ with $e \geq 3$. Then, what about other cases of p, e and s? Though we investigated many triples of small p, e and s, we could not get systematic results on the existence/non-existence of tight formulas. To study such general cases are left for future work.

Finally, in Corollary 3.8, we explicitly calculated the Stroud bound only for Laplacian-type cubature formulas. However, similar calculations will also work for other types of operator-type cubature formulas. For example, if one considers an operator-type cubature formula for $\mathcal{P}_{2e}(B^2)$ given by

$$\int_{B^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{x}\in X} (\Delta^2 f)(\boldsymbol{x}), \tag{5.8}$$

then by the same argument as in Example 3.12 we get

$$|X| \geq \frac{(e+1)(e+2)}{6}.$$

We shall call (5.8) a *cubature formula of type* $(0,2)$, following the one-dimensional terminology by Turán [**17**].

The authors do not know whether there exists a high-dimensional Turán-type cubature formula with equality in the above lower bound.

Remark 5.7. As mentioned before, there exists no tight Laplacian-type cubature formula $(X, \mathcal{T})$ with $e - s = 1$. But, after many trial and errors, we have found several examples of a Laplacian-type cubature formula which seem to nearly attain the Stroud-type bound, some of which are given as follows:

- For the case $(e,s) = (2,1)$, the following holds for every polynomial at most 4:

$$\begin{aligned}\int_{B^2} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \Lambda_0 f(0,0) + \Lambda_1 \Delta f(x,y)|_{(x,y)=(0,0)} \\ &\quad + \sum_{l=0}^{2} \Lambda_2 f\left(\sqrt{R_1}\cos\frac{2l\pi}{3}, \sqrt{R_1}\sin\frac{2l\pi}{3}\right) \\ &\quad + \sum_{l=0}^{2} \Lambda_3 f\left(\sqrt{R_2}\cos\frac{(2l+1)\pi}{3}, \sqrt{R_2}\sin\frac{(2l+1)\pi}{3}\right),\end{aligned}$$

  where $0 < \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3 < 1$, $0 < R_1, R_2$, and

$$\Lambda_0 = \frac{(R_1R_2)^{1/2} - R_1 - R_2 + 3(R_1R_2)^{3/2}}{3(R_1R_2)^{3/2}}, \quad \Lambda_1 = \frac{3(R_1R_2)^{1/2} - 2}{24(R_1R_2)^{1/2}},$$

$$\Lambda_2 = \frac{1}{9R_1^{3/2}(\sqrt{R_1}+\sqrt{R_2})}, \qquad \Lambda_3 = \frac{1}{9R_2^{3/2}(\sqrt{R_1}+\sqrt{R_2})}.$$

  The constructed formula has 7 points, which is larger than the Stroud-type bound by 4 points, and which has the structure of tight Euclidean 4-designs without the origin.
- For the case $(e,s) = (3,2)$, the following holds for every polynomial at most 6:

$$\begin{aligned}\int_{B^2} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \Lambda_0 f(0,0) + \Lambda_1 \Delta f(x,y)|_{(x,y)=(0,0)} + \Lambda_2 \Delta f(x,y)|_{(x,y)=(0,0)} \\ &\quad + \sum_{l=0}^{2} \Lambda_3 f\left(\sqrt{R_1}\cos\frac{2l\pi}{5}, \sqrt{R_1}\sin\frac{2l\pi}{5}\right) \\ &\quad + \sum_{l=0}^{2} \Lambda_4 f\left(\sqrt{R_2}\cos\frac{(2l+1)\pi}{5}, \sqrt{R_2}\sin\frac{(2l+1)\pi}{5}\right),\end{aligned}$$

where $0 < \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 < 1$, $0 < R_1, R_2$, and

$$\Lambda_0 = \frac{-R_1^2 + R_1^{3/2}\sqrt{R_2} - R_1R_2 + \sqrt{R_1}R_2^{3/2} - R_2^2 + 4(R_1R_2)^{5/2}}{4(R_1R_2)^{5/2}},$$

$$\Lambda_1 = \frac{-R_1 + \sqrt{R_1R_2} - R_2 + 2(R_1R_2)^{3/2}}{16(R_1R_2)^{3/2}}, \quad \Lambda_2 = \frac{-3 + 4\sqrt{R_1R_2}}{768\sqrt{R_1R_2}},$$

$$\Lambda_3 = \frac{1}{20R_1^{5/2}(\sqrt{R_1} + \sqrt{R_2})}, \quad \Lambda_4 = \frac{1}{20R_2^{5/2}(\sqrt{R_1} + \sqrt{R_2})}.$$

The constructed formula has 11 points, which is larger than the Stroud-type bound by 5 points, and which has the structure of tight Euclidean 6-designs without the origin.

A. Some new examples of Laplacian-type cubature.

We give some new examples of Laplacian-type cubature of degrees 7, 9, 11 for Gaussian integral

$$\mathcal{I}[f] := \int_{\mathbb{R}^2} f(x,y)\pi^{-1}e^{-(x^2+y^2)}\, \mathrm{d}x\mathrm{d}y.$$

- The following holds for every polynomial $f(x,y)$ of degree at most 7:

$$\mathcal{I}[f] = \Lambda_0 f(0,0) + \Lambda_1 \Delta f(x,y)|_{(x,y)=(0,0)} + \sum_{l=0}^{5} \Lambda_2 f\left(\sqrt{R_1}\cos\frac{2l\pi}{6}, \sqrt{R_1}\sin\frac{2l\pi}{6}\right)$$
$$+ \sum_{l=0}^{5} \Lambda_3 f\left(\sqrt{R_2}\cos\frac{(2l+1)\pi}{6}, \sqrt{R_2}\sin\frac{(2l+1)\pi}{6}\right),$$

where $3(2-\sqrt{2}) < R_1 < 3(2+\sqrt{2})$, $R_2 = 3R_1/(-3+2R_1)$, and $\Lambda_0 = (-54+36R_1+R_1^2)/9R_1^2$, $\Lambda_1 = (-18+12R_1-R_1^2)/12R_1^2$, $\Lambda_2 = 1/2R_1^3$, $\Lambda_3 = (-3+2R_1)^3/54R_1^3$.
- The following holds for every polynomial $f(x,y)$ of degree at most 9:

$$\mathcal{I}[f] = \Lambda_0 f(0,0) + \Lambda_1 \Delta f(x,y)|_{(x,y)=(0,0)} + \sum_{l=0}^{7} \Lambda_2 f\left(\sqrt{R_1}\cos\frac{2l\pi}{8}, \sqrt{R_1}\sin\frac{2l\pi}{8}\right)$$
$$+ \sum_{l=0}^{7} \Lambda_3 f\left(\sqrt{R_2}\cos\frac{(2l+1)\pi}{8}, \sqrt{R_2}\sin\frac{(2l+1)\pi}{8}\right),$$

where $R_1 = 2(3+\sqrt{3})$, $R_2 = 2(3-\sqrt{3})$, and $\Lambda_0 = 17/24$, $\Lambda_1 = 13/16$, $\Lambda_2 = (7-4\sqrt{3})/384$, $\Lambda_3 = (7+4\sqrt{3})/384$.
- The following holds for every polynomial $f(x,y)$ of degree at most 11:

$$\mathcal{I}[f] = \Lambda_0 f(0,0) + \Lambda_1 \Delta f(x,y)|_{(x,y)=(0,0)} + \sum_{l=0}^{7} \Lambda_2 f\left(\sqrt{R_1}\cos\frac{2l\pi}{8}, \sqrt{R_1}\sin\frac{2l\pi}{8}\right)$$
$$+ \sum_{l=0}^{7} \Lambda_3 f\left(\sqrt{R_2}\cos\frac{(2l+1)\pi}{8}, \sqrt{R_2}\sin\frac{(2l+1)\pi}{8}\right)$$
$$+ \sum_{l=0}^{7} \Lambda_4 f\left(\sqrt{R_3}\cos\frac{2l\pi}{8}, \sqrt{R_3}\sin\frac{2l\pi}{8}\right)$$

where $R_1 = 5(10+\sqrt{34})/11$, $R_2 = 5$, $R_3 = 5(10-\sqrt{34})/11$, and $\Lambda_0 = 3334/5625$, $\Lambda_1 = 29/750$, $\Lambda_2 = (74222 - 12557\sqrt{34})/3060000$, $\Lambda_3 = 3/1250$, $\Lambda_4 = (74222 + 12557\sqrt{34})/3060000$.

B. Lower bounds for Laplacian-type cubature in higher dimensional space.

Theorem B.1. *Let us consider the n-dimensional Euclidean space $\mathbb{R}^n$ and take a measure space (Ω, μ) as in Definition* 2.1. *Then, for any Laplacian-type cubature formula $(X, \mathcal{T})$ of order $2s$ and degree $2e$ on (Ω, μ), the following inequality holds*:

$$|X \setminus \{\mathbf{0}\}| \geq \dim_{\mathbb{R}} \mathcal{P}_e(\Omega) - \sum_{m=0}^{\lfloor s/2 \rfloor} \frac{2e-4m+n-1}{e-2m+n-1}\binom{e-2m+n-1}{n-1}.$$

In particular, if Ω has an interior point, then

$$|X \setminus \{\mathbf{0}\}| \geq \sum_{m=\lceil (s+1)/2 \rceil}^{\lfloor e/2 \rfloor} \frac{2e-4m+n-1}{e-2m+n-1}\binom{e-2m+n-1}{n-1}.$$

By Theorem 3.2, we only need to show

$$p(e, \mathbf{0}, T) \geq \sum_{m=0}^{\lfloor s/2 \rfloor} \frac{2e-4m+n-1}{e-2m+n-1}\binom{e-2m+n-1}{n-1}$$

for $T := \sum_{k=0}^{s} \lambda_k \Delta^k$ (see Section 3.1 for the notation).

Recall that

$$\mathcal{P}_e(\mathbb{R}^n) = \bigoplus_{l=0}^{e} \mathrm{Hom}_l(\mathbb{R}^n)$$
$$= \bigoplus_{l=0}^{e} \bigoplus_{m=0}^{\lfloor l/2 \rfloor} \|\boldsymbol{x}\|^{2m}\, \mathrm{Harm}_{l-2m}(\mathbb{R}^n)$$

$$= \bigoplus_{m=0}^{\lfloor e/2 \rfloor} \bigoplus_{j=0}^{e-2m} \|\boldsymbol{x}\|^{2m} \operatorname{Harm}_j(\mathbb{R}^n).$$

Let us take

$$U := \bigoplus_{m=\lceil (s+1)/2 \rceil}^{\lfloor e/2 \rfloor} \bigoplus_{j=0}^{e-2m} \|\boldsymbol{x}\|^{2m} \operatorname{Harm}_j(\mathbb{R}^n)$$

as a subspace of $\mathcal{P}_e(\mathbb{R}^n)$.

We note that, by recalling (5.6), if $m > k$,

$$(\Delta^k(\|\boldsymbol{x}\|^{2m} f))|_{\boldsymbol{x}=\mathbf{0}} = 0$$

for any $f \in \operatorname{Harm}_l(\mathbb{R}^n)$. So, we can easily check that U is a totally isotropic subspace of $\mathcal{P}_e(\mathbb{R}^n)$ for the bilinear form $(\cdot,\cdot)_{\mathbf{0},T}$. Now, by the argument similar to that of Lemma 3.11, we have

$$\dim U \le \min\{p(e,\mathbf{0},T), q(e,\mathbf{0},T)\} + r(e,\mathbf{0},T)$$

It thus follows that

$$\begin{aligned} p(e,\mathbf{0},T) &\le \dim \mathcal{P}_e(\mathbb{R}^n) - \dim U \\ &= \dim \bigoplus_{m=0}^{\lfloor s/2 \rfloor} \bigoplus_{j=0}^{e-2m} \|\boldsymbol{x}\|^{2m} \operatorname{Harm}_j(\mathbb{R}^n) \\ &= \sum_{m=0}^{\lfloor s/2 \rfloor} \sum_{j=0}^{e-2m} \left(\binom{j+n-1}{n-1} - \binom{j+n-3}{n-1} \right) \\ &= \sum_{m=0}^{\lfloor s/2 \rfloor} \left(\binom{e-2m+n-1}{n-1} + \binom{e-2m+n-2}{n-1} \right) \\ &= \sum_{m=0}^{\lfloor s/2 \rfloor} \frac{2e-4m+n-1}{e-2m+n-1} \binom{e-2m+n-1}{n-1} \end{aligned}$$

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References

[1] Ei. Bannai and Etsu. Bannai, Tight Gaussian 4-designs, J. Algebraic Combin., **22** (2005), 39–63.

[2] Ei. Bannai and Etsu. Bannai, Euclidean designs and coherent configurations, In: Combinatorics and graphs, Contemp. Math., **531**, AMS, Providence, RI, 2010, pp. 59–93.

[3] Ei. Bannai and Etsu. Bannai, Tight 9-designs on two concentric spheres, J. Math. Soc. Japan, **63** (2011), 1359–1376.

[4] Ei. Bannai, Etsu. Bannai, M. Hirao and M. Sawa, Cubature formulas in numerical analysis and Euclidean tight designs, European J. Combin., **31** (2010), 419–422.

[5] P. Delsarte, J. M. Goethals and J. J. Seidel, Spherical codes and designs, Geom. Dedicate, **6** (1977), 363–388.

[6] C. F. Dunkl and Y. Xu, Orthogonal Polynomials of Several Variables, Encyclopedia of Mathematics and its Applications, **81**, Cambridge University Press, 2001.

[7] V. I. Lebedev, Quadratures on the sphere of the highest algebraic accuracy range, In: Theory of Cubature Formulas and Application of Functional Analysis to Certain Mathematical Physics Problems (Novosibirsk, 1973), pp. 31–35, (in Russian).

[8] V. I. Lebedev, Spherical quadrature formulas exact to orders 25–29, Sibirskii Math. Zh., **18** (1975), pp. 132–142, (in Russian).

[9] T. Lyons and N. Victoir, Cubature on Wiener space, Proc. R. Soc. A, **460** (2004), 69–198.

[10] H. M. Mőller, Lower bounds for the number of nodes in cubature formulae, In: Numerische Integration, Oberwolfach, 1978 (ed. G. Hằmmerlin), Birkhằuser, 1979, 221–230.

[11] A. Neumaier and J. J. Seidel, Discrete measures for spherical designs, eutactic stars and lattices, Nederl. Akad. Wetensch, Indag. Math., **50** (1988), 321–334.

[12] M. Sawa, The theory of cubature formula, Sugaku, **68** (2016), 24–52.

[13] E. A. Shamsiev, Cubature formulas for a disk that are invariant with respect to groups of transformations of regular polyhedra into themselves, (in Russian), Zh. Vychisl. Mat. Mat. Fiz., **46** (2006), 1211–1218; translation in Comput. Math. Math. Phys., **46** (2006), 1147–1154.

[14] S. L. Sobolev, Cubature formulas on the sphere which are invariant under transformations of finite rotation groups, (in Russian), Dokl. Akad. Nauk SSSR, **146** (1962), 310–313.

[15] S. L. Sobolev and V. L. Vaskevich, The Theory of Cubature Formulas, Kluwer Academic Publishers, 1997.

[16] H. Stroud, Quadrature methods for functions of more than one variable, Ann. New York Acad. Sci., **86** (1960), 776–791.

[17] P. Turán, On some open problems of approximation theory, P. Turán memorial volume. Translated from the Hungarian by P. Szusz, J. Approx. Theory, **29** (1980), 23–85.

[18] A. K. Varma, On some open problems of P. Turán concerning Birkhoff interpolation, Trans. Amer. Math. Soc., **274** (1982), 797–808.

[19] P. Verlinden and R. Cools, On cubature formulae of degree $4k+1$ attaining Mőller's lower bound for integrals with circular symmetry, Numer. Math., **61** (1992), 395–407.

[20] Y. Xu, Minimal cubature formulae for a family of radial weight functions, Adv. Comput. Math., **8** (1998), 367–380.

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