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Long time existence of classical solutions for the 3D incompressible rotating Euler equations

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Abstract. We consider the initial value problem of the 3D incompressible rotating Euler equations. We prove the long time existence of classical solutions for initial data in $H^s(\mathbb{R}^3)$ with $s > 5/2$. Also, we give an upper bound of the minimum speed of rotation for the long time existence when initial data belong to $H^{7/2}(\mathbb{R}^3)$.

1. Introduction.

Let us consider the initial value problem of the rotating Euler equations in $\mathbb{R}^3$, describing the motion of perfect incompressible fluids in the rotational framework.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } (0,\infty) \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } (0,\infty) \times \mathbb{R}^3, \\ u(0,x) = \phi(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{E_Ω}$$

Here, $u = (u_1(t,x), u_2(t,x), u_3(t,x))$ and $p = p(t,x)$ denote the velocity field and the pressure of the fluids, respectively, while $\phi = (\phi_1(x), \phi_2(x), \phi_3(x))$ denotes the given initial velocity field satisfying $\operatorname{div} \phi = 0$. The real constant $\Omega \in \mathbb{R}$ represents the speed of rotation of the fluids around the vertical unit vector $e_3 := (0,0,1)$.

In this manuscript, we prove the long time existence of classical solutions to (E_Ω) for initial data in $H^s(\mathbb{R}^3)$ with $s > 5/2$ when the speed of rotation is sufficiently high. More precisely, we shall show that for given initial velocity $\phi \in H^s(\mathbb{R}^3)$ with $s > 5/2$ satisfying $\operatorname{div} \phi = 0$ and for given *finite* time T, there exists a positive number $\Omega_{\phi,T}$ such that the 3D rotating Euler equation admits a unique classical solution on the time interval $[0,T]$ provided $|\Omega| \geq \Omega_{\phi,T}$. Furthermore, we shall give an upper bound of the minimum speed of rotation $\Omega_{\phi,T}$ which ensures the long time existence to (E_Ω) in terms of the norm of initial data and the given time T when initial data belong to $H^{7/2}(\mathbb{R}^3)$. This also gives a lower bound for the maximal existence time of the solution to (E_Ω) in terms of the rotating speed $|\Omega|$.

Before stating our results, we summarize the known results for the existence of classical solutions to the Euler equations for both non-rotating $\Omega = 0$ and rotating $\Omega \in \mathbb{R}$

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cases. Let $\mathbb{P} := (\delta_{jk} + R_j R_k)_{1\le j,k\le 3}$ be the Helmholtz projection onto the divergence-free vector fields, where R_j denotes the Riesz transform in $\mathbb{R}^3$. Applying the projection $\mathbb{P}$ to both sides of the first equation of (E_Ω), we obtain the following abstract evolution equations.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Omega\mathbb{P}(e_3 \times u) + \mathbb{P}(u\cdot\nabla)u = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u(0,x) = \phi(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{E'_Ω}$$

Let us first review the local existence results on the original Euler equations for $\Omega = 0$ in $\mathbb{R}^3$. Kato [**18**] proved that for given integer $m \in \mathbb{Z}$ with $m > 5/2$ and for given divergence-free initial velocity $\phi \in H^m(\mathbb{R}^3)$, there exists a positive time $T = T(m, \|\phi\|_{H^m})$ such that (E'_0) possesses a unique classical solution u in the class $C([0,T]; H^m(\mathbb{R}^3)) \cap C^1([0,T]; H^{m-1}(\mathbb{R}^3))$. Kato–Ponce [**20**] extended this result to the Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ of the fractional order for $s > 3/p + 1$, $1 < p < \infty$. Chae [**8**] and Chen–Miao–Zhang [**12**] gave further extensions to the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^3)$ with $s > 3/p+1$, $1 < p, q < \infty$ and the Besov spaces $B^s_{p,q}(\mathbb{R}^3)$ with $s > 3/p+1$, $1 < p < \infty$, $1 \le q \le \infty$ or $s = 3/p + 1$, $1 < p < \infty$, $q = 1$. The currently-known best result on the local existence was given by Pak–Park [**24**] in the Besov space $B^1_{\infty,1}(\mathbb{R}^3)$.

For the high-speed rotation case $|\Omega| \gg 1$, Dutrifoy [**14**], [**15**] showed the long time existence of classical solutions for the initial data in $H^s(\mathbb{R}^3)$ with $s > 7/2$ or $B^{7/2}_{2,1}(\mathbb{R}^3)$, and proved the asymptotics of solutions to vortex patches or Yudovich solutions as the Rossby number goes to zero for some particular initial data. Similar results are obtained for the quasigeostrophic systems by Dutrifoy [**14**] and Charve [**9**]. Koh–Lee–Takada [**21**] obtained the optimal range of the Strichartz estimate for the linear propagator associated with the Coriolis force, and showed the long time existence results for the initial data in $H^s(\mathbb{R}^3)$ with $s > 7/2$. For the periodic setting in $\mathbb{T}^3$, we refer to Babin–Mahalov–Nicolaenko [**1**], [**2**].

Here we remark that the local existence results for the original Euler equations (E'_0) were obtained in function spaces which are embedded in $C^1(\mathbb{R}^3)$, while the long time existence results for the rotating Euler equations (E'_Ω) have required more regular class of initial data which are embedded in $C^2(\mathbb{R}^3)$. This is due to the loss of derivative in the nonlinear term $(u\cdot\nabla)u$. In the non-rotating case (E'_0), such a loss of derivative can be recovered by the classical energy method and the commutator estimates [**8**], [**18**], [**20**]. In the high-speed rotating case (E'_Ω), the key ingredient for the proof of long time existence results is the Strichartz estimate for the linear propagator $e^{\pm i\Omega t(D_3/|D|)}$ generated by the Coriolis force $\Omega e_3 \times u$ [**9**], [**11**], [**14**], [**15**], [**21**]. However, in the energy methods, we cannot derive such a dispersion effect of the Coriolis force because of its skew-symmetry

$$\int_{\mathbb{R}^3} \Omega e_3 \times u(t,x)\cdot u(t,x)\,dx = 0.$$

Also, the smoothing property of the propagator $e^{\pm i\Omega t(D_3/|D|)}$ is not enough to recover the loss of derivative in the nonlinear term [**21**, Corollary 1.2].

In this paper, we relax the smoothness conditions for initial data, and prove the long time existence of classical solutions to the high-speed rotating Euler equations (E'_Ω) for initial data in $H^s(\mathbb{R}^3)$ with $s > 5/2$, which is the same regularity as the local existence results for the original Euler equations (E'_0).

Our first result reads as follows:

THEOREM 1.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then for every $\phi \in H^s(\mathbb{R}^3)$ satisfying $\operatorname{div} \phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $\Omega_{\phi,T}$ depending on s, T and ϕ such that if $|\Omega| \geq \Omega_{\phi,T}$ then (E'_Ω) possesses a unique classical solution u in the class*

$$u \in C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3)).$$

REMARK 1.2. Theorem 1.1 states that the long time existence of classical solutions to the rotating Euler equations (E'_Ω) can be proved for the initial velocity in $H^s(\mathbb{R}^3)$ with $s > 5/2$. This regularity condition corresponds to the local existence results for the original Euler equations. From the viewpoint of smoothness assumptions for initial data, Theorem 1.1 gives an improvement of the results in [**14**], [**15**], [**21**].

In the proof of Theorem 1.1, we adapt the regularization and the approximation arguments in Kato–Lai [**19**] and Bona–Smith [**5**], where they proved the continuous dependences of solutions on initial data for the original Euler equations and the KdV equations, respectively. Given $\phi \in H^s(\mathbb{R}^3)$ with $s > 5/2$ and $0 < \varepsilon \leq 1$, let ϕ_ε be an regularization of ϕ by an approximate identity. By the previous result [**21**], we can construct a solution u_ε with $u_\varepsilon(0) = \phi_\varepsilon$ on the given time interval $[0,T]$ if $|\Omega| \geq \Omega_{\phi_\varepsilon,T}$. Then, the difference $v_\varepsilon = u_\varepsilon - u$ of u_ε and the local solution u with $u(0) = \phi$ satisfies

$$\partial_t v_\varepsilon + \Omega \mathbb{P}(e_3 \times v_\varepsilon) + \mathbb{P}(v_\varepsilon \cdot \nabla) u_\varepsilon + \mathbb{P}(u \cdot \nabla) v_\varepsilon = 0 \tag{1.1}$$

with $v_\varepsilon(0) = \phi_\varepsilon - \phi$ on some local time interval. We shall show that the H^s-norm of v_ε can be taken arbitrarily small provided that the parameter $\varepsilon > 0$ is small enough depending only on the given data s, T and ϕ. Then, the local solution u has a uniform H^s-bound, and can be continued to the given time interval $[0,T]$. Here, we should remark that similar regularization argument has already been used in Babin–Mahalov–Nicolaenko [**3**] for the 3D rotating Boussinesq equations in the periodic setting. However, the proof in this paper is not the same as in [**3**], and gives an alternative one. We establish an a priori bound for smooth solutions to (E'_Ω), and give detailed analyses for how to recover the loss of derivative in the convection term $(v_\varepsilon \cdot \nabla) u_\varepsilon$ in (1.1) and how to continue the local solution u to the given time interval $[0,T]$.

Besides the smoothness condition on the initial data, we shall consider the relation between the given data (ϕ, T) and the speed of rotaion $\Omega_{\phi,T}$ which ensures the long time existence of solutions to (E'_Ω). In other words, we are interested in a characterization of a lower bound for the maximal existence time T_Ω of the solution to (E'_Ω) in terms of the speed of rotation $|\Omega|$. Dutrifoy [**15**] considered vortex patches or Yudovich solutions and gave lower bounds of maximal existence times by $T_\Omega \gtrsim \log\log\log|\Omega|$ or $T_\Omega \gtrsim \log\log|\Omega|$, respectively. Koh–Lee–Takada [**21**] established a single logarithmic lower bound $T_\Omega \gtrsim \log|\Omega|$ by using a single exponential estimate for the blow-up criterion. Here we remark that in those results [**15**], [**21**] the initial data have to belong to the regular Sobolev spaces $H^s(\mathbb{R}^3)$ with $s > 7/2$. Unfortunately, in Theorem 1.1, it seems to be difficult to characterize a upper bound of the minimum speed of rotation in terms of the norm of initial data and the given time T because of the regularization and the approximation arguments. In this paper, we shall characterize such a relation between the given data (ϕ, T) and the speed of rotation $\Omega_{\phi,T}$ for the initial data ϕ in the Sobolev space $H^{7/2}(\mathbb{R}^3)$.

Our second result reads as follows:

Theorem 1.3. *For every $\phi \in H^{7/2}(\mathbb{R}^3)$ satisfying $\operatorname{div}\phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $\Omega_{\phi,T}$ depending on T and $\|\phi\|_{H^{7/2}}$ such that if $|\Omega| \geq \Omega_{\phi,T}$ then (E'_Ω) possesses a unique classical solution u in the class*

$$u \in C([0,T]; H^{7/2}(\mathbb{R}^3)) \cap C^1([0,T]; H^{5/2}(\mathbb{R}^3)). \tag{1.2}$$

In particular, let $\Omega^*_{\phi,T}$ be the infimum of the set of $|\Omega| \geq 0$ such that (E'_Ω) admits a unique classical solution u in the class (1.2)*. Then, for $2 < q < \infty$ there exist a positive absolute constant $C_* = C_*(7/2)$ and a positive constant C_q depending on q such that*

$$0 < \Omega^*_{\phi,T} \leq C_q T^{q-1}\big\{\|\phi\|_{H^{5/2}} + T(\|\phi\|_{H^{7/2}} + e)^{C_*} e^{C_* T}\big\}^q. \tag{1.3}$$

Remark 1.4. It follows from the characterization (1.3) in Theorem 1.3 that the maximal existence time $T_\Omega \geq 1$ has a lower bound

$$T_\Omega \geq \frac{C'_q}{\log(\|\phi\|_{H^{7/2}} + e)} \log\left(\frac{|\Omega|}{C''_q}\right)$$

for sufficiently high speed of rotation $|\Omega|$ with some positive constants C'_q and C''_q depending on q. This single logarithmic order is due to the use of the logarithmic Sobolev inequality for the blow-up criterion (see Lemma 2.3 and Lemma 6.1).

The key ingredient of the proof of Theorem 1.3 is a refined blow-up criterion in $\mathrm{BMO}(\mathbb{R}^3)$ or $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ of the Beale–Kato–Majda and the Kozono–Taniuchi type [**4**], [**22**], [**23**]. Roughly speaking, in the proof of (1.3), there appears loss of $3/2 + 1 + 1$ derivatives because of the Strichartz estimate

$$\big\|\Delta_j e^{\pm i\Omega t(D_3/|D|)}\big\|_{L^q_t L^\infty_x} \lesssim |\Omega|^{-1/q} 2^{3j/2} \|\Delta_j f\|_{L^2_x}$$

(see Corollary 5.3), the blow-up criterion for the vorticity $\nabla \times u$ and the nonlinear term $(u\cdot\nabla)u$, respectively. Hence, we needed $7/2+$ regularity due to the continuous embedding $B^{0+}_{\infty,2}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. The refined embeddings $\dot{B}^0_{\infty,2}(\mathbb{R}^3) \hookrightarrow \mathrm{BMO}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ allow us to treat the critical case $H^{7/2}(\mathbb{R}^3)$.

This paper is organized as follows. In Section 2, we recall the definitions of function spaces and the bilinear and the commutator estimates in these spaces. In Section 3, we establish an a priori estimate for the solution of (E'_Ω) with smooth initial data, which is one of the key ingredients in the proof of Theorem 1.1. In Section 4, we present the proof of Theorem 1.1. In Section 5, we derive the solution formula for the linear vorticity equations and recall the linear estimates. In Section 6, we state the refined blow-up criterion of Kozono–Taniuchi type for the local solutions to (E'_Ω). In Section 7, we present the proof of Theorem 1.3. In Section 8, we give an alternative derivation of the solution formula to the vorticity equations as an Appendix.

Throughout this paper, we denote by C the constants which may differ from line to line. In particular, $C = C(\cdot, \dots, \cdot)$ will denote the constant which depends only on the quantities appearing in parentheses. For $A, B \geq 0$, $A \lesssim B$ means that there exists some positive constant C such that $A \leq CB$. Also, $A \gtrsim B$ is defined in the same way. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

2. Preliminaries.

We first introduce some function spaces. We denote by $C^\infty_{0,\sigma}(\mathbb{R}^3)$ the set of all C^∞ vector functions $v = (v_1, v_2, v_3)$ with compact support in $\mathbb{R}^3$ satisfying $\operatorname{div} v = 0$. $L^2_\sigma(\mathbb{R}^3)$ denotes the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with respect to the $L^2(\mathbb{R}^3)$-norm $\|\cdot\|_{L^2}$. Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class, and let $\mathscr{S}'(\mathbb{R}^3)$ be the space of tempered distributions. For $f \in \mathscr{S}(\mathbb{R}^3)$, the Fourier transform and the inverse Fourier transform are defined by

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx, \qquad \xi \in \mathbb{R}^3,$$
$$\mathscr{F}^{-1}[f](x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} f(\xi)d\xi, \quad x \in \mathbb{R}^3,$$

respectively. Next, we recall the definition of the Littlewood–Paley decomposition. Let φ_0 be a function in $\mathscr{S}(\mathbb{R}^3)$ satisfying $0 \leq \widehat{\varphi_0}(\xi) \leq 1$ for all $\xi \in \mathbb{R}^3$, $\operatorname{supp} \widehat{\varphi_0} \subset \{\xi \in \mathbb{R}^3 \mid 1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi_j}(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(x) := 2^{3j}\varphi_0(2^j x)$. We set $\widehat{\chi}(\xi) := 1 - \sum_{j\geq 1} \widehat{\varphi_j}(\xi)$. Let $\{\Delta_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley operator defined by $\Delta_j f := \varphi_j * f$ for $f \in \mathscr{S}'(\mathbb{R}^3)$. Then, we recall the definitions of the inhomogeneous and the homogeneous Besov spaces.

Definition 2.1. (i) For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{B^s_{p,q}} := \|\chi * f\|_{L^p} + \left\|\{2^{sj}\|\Delta_j f\|_{L^p}\}_{j=1}^\infty\right\|_{\ell^q} < \infty.$$

(ii) For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \big\|\{2^{sj}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}}\big\|_{\ell^q} < \infty.$$

Let $H^s(\mathbb{R}^3)$ denote the Sobolev space of order $s \in \mathbb{R}$ with the inner product

$$\langle f, g\rangle_{H^s} := \int_{\mathbb{R}^3} (1-\Delta)^{s/2} f(x)\overline{(1-\Delta)^{s/2} g(x)}dx = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

and the norm $\|f\|_{H^s} := \sqrt{\langle f, f\rangle_{H^s}}$. For $s > 0$, it is known that the norm equivalence

$$\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{\dot{H}^s} \tag{2.1}$$

holds, where $\|\cdot\|_{\dot{H}^s}$ denotes the homogeneous Sobolev semi-norm defined by

$$\|f\|_{\dot{H}^s} := \|f\|_{\dot{B}^s_{2,2}} = \Big(\sum_{j\in\mathbb{Z}} 2^{2sj}\|\Delta_j f\|^2_{L^2}\Big)^{1/2}.$$

We end this section by preparing some key inequalities and estimates for the proofs of Theorem 1.1 and Theorem 1.3. We first recall the Gronwall inequality and the logarithmic Sobolev inequality of the Brezis–Gallouet–Wainger type [**6**], [**7**].

Lemma 2.2. *Let $0 < T < \infty, 0 \le A < \infty$, and let f, g and h be non-negative, continuous functions on $[0,T]$ satisfying*

$$f(t) \le A + \int_0^t g(s)ds + \int_0^t h(s)f(s)ds$$

for all $t \in [0,T]$. Then it holds

$$f(t) \le Ae^{\int_0^t h(\tau)d\tau} + \int_0^t e^{\int_s^t h(\tau)d\tau} g(s)ds$$

for all $t \in [0,T]$.

Lemma 2.3 ([**22**], [**23**]). *For $s > 3/2$, there exists a positive constant $C = C(s)$ such that*

$$\|f\|_{L^\infty} \le C\big\{1 + \|f\|_{\dot{B}^0_{\infty,\infty}}(1 + \log^+ \|f\|_{H^s})\big\}$$

holds for all $f \in H^s(\mathbb{R}^3)$, where $\log^+ a := \max\{\log a, 0\}$ for $a > 0$.

Next, we recall several bilinear and commutator estimates in the Sobolev spaces.

Lemma 2.4 ([**8**], [**10**], [**13**]). *For $s > 0$, there exists a positive constant $C = C(s)$*

such that

$$\|fg\|_{\dot{H}^s} \le C(\|f\|_{L^\infty}\|g\|_{\dot{H}^s} + \|g\|_{L^\infty}\|f\|_{\dot{H}^s})$$

holds for all $f, g \in L^\infty(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$.

LEMMA 2.5 ([**12**], [**25**]).

(1) *For* $s > 0$, *there exists a positive constant* $C = C(s)$ *such that*

$$\left(\sum_{j\in\mathbb{Z}} 2^{2sj}\big\|[u\cdot\nabla, \Delta_j]f\big\|_{L^2}^2\right)^{1/2} \le C(\|\nabla u\|_{L^\infty}\|f\|_{\dot{H}^s} + \|\nabla f\|_{L^\infty}\|u\|_{\dot{H}^s})$$

for all $u, f \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$ *with* $\operatorname{div} u = 0$.

(2) *For* $s > -1$, *there exists a positive constant* $C = C(s)$ *such that*

$$\left(\sum_{j\in\mathbb{Z}} 2^{2sj}\big\|[u\cdot\nabla, \Delta_j]f\big\|_{L^2}^2\right)^{1/2} \le C(\|\nabla u\|_{L^\infty}\|f\|_{\dot{H}^s} + \|f\|_{L^\infty}\|u\|_{\dot{H}^{s+1}})$$

for all $f \in L^\infty(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$ *and* $u \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap \dot{H}^{s+1}(\mathbb{R}^3)$ *with* $\operatorname{div} u = 0$.

LEMMA 2.6 ([**20**]). *For* $s \ge 0$, *there exists a positive constant* $C = C(s)$ *such that*

$$\|(1-\Delta)^{s/2}(fg) - f(1-\Delta)^{s/2}g\|_{L^2} \le C(\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|g\|_{L^\infty}\|f\|_{H^s})$$

holds for all $f \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ *and* $g \in L^\infty(\mathbb{R}^3) \cap H^{s-1}(\mathbb{R}^3)$.

Finally, we introduce a mollifier J_ε of the Friedrichs type in $\mathbb{R}^3$, and state some properties of J_ε. For the proof, we refer to [**5**, Lemma 5] (see also [**19**, (7.2)]).

LEMMA 2.7. *Let* $G_t(x)$ *be the Gauss kernel defined by*

$$G_t(x) := \frac{1}{(4\pi t)^{3/2}} \exp\left\{-\frac{|x|^2}{4t}\right\} \quad t > 0, x \in \mathbb{R}^3.$$

For $0 < \varepsilon \le 1$ *and* $f \in \mathscr{S}'(\mathbb{R}^3)$, *let* J_ε *be the mollifier defined by* $J_\varepsilon f := G_{\varepsilon^2} * f$. *Then, the followings hold*:

(1) *For all* $0 < \varepsilon \le 1$ *and all* $f \in H^s(\mathbb{R}^3)$ *with* $s \in \mathbb{R}$, *there hold*

$$\|J_\varepsilon f\|_{H^s} \le \|f\|_{H^s}, \quad \|J_\varepsilon f\|_{H^{s+1}} \le \frac{1}{\varepsilon}\|f\|_{H^s}.$$

(2) *For every* $f \in H^s(\mathbb{R}^3)$ *with* $s \in \mathbb{R}$, *there hold*

$$\lim_{\varepsilon\to 0}\|J_\varepsilon f - f\|_{H^s} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\|J_\varepsilon f - f\|_{H^{s-1}} = 0.$$

3. A priori estimate for the solution with smooth initial data.

In this section, we shall establish an a priori estimate for the solution to (E'_Ω) on the given time interval $[0,T]$ when the initial velocity belongs to $H^{s+1}(\mathbb{R}^3)$ with $s > 5/2$. This is one of the key ingredient for the treatment of the convection term $(v_\varepsilon \cdot \nabla)u_\varepsilon$ in the perturbed system (1.1). More precisely, we shall prove the following.

Theorem 3.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then for every $\phi \in H^{s+1}(\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $\Omega_{\phi,T}$ depending on s, T and $\|\phi\|_{H^{s+1}}$ such that if $|\Omega| \geq \Omega_{\phi,T}$, then ($\mathrm{E}'_\Omega$) possesses a unique classical solution u satisfying*

$$u \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T]; H^s(\mathbb{R}^3)).$$

Furthermore, for every $s \leq r \leq s+1$ there exists positive constant C_r such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^r} \leq C_r \|\phi\|_{H^r}. \tag{3.1}$$

Proof. By [**21**, Theorem 1.4], we see that there exists a positive constant $\Omega^0_{\phi,T}$ depending on $s, T, \|\phi\|_{H^{s+1}}$ such that if $|\Omega| \geq \Omega^0_{\phi,T}$, then ($\mathrm{E}'_\Omega$) possesses a unique classical solution u on the time interval $[0,T]$ in the class

$$u \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T]; H^s(\mathbb{R}^3)).$$

Hence it remains to prove the a priori estimate (3.1). Taking the L^2 inner product of (E'_Ω) with u, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = 0$$

by the skew-symmetry of $e_3 \times u$ and the divergence-free condition. Hence we have

$$\|u(t)\|_{L^2} = \|\phi\|_{L^2} \tag{3.2}$$

for all $0 \leq t \leq T$. Next, applying the Littlewood–Paley projection operator Δ_j to both sides of (E'_Ω), we have

$$\partial_t \Delta_j u + \Omega \mathbb{P}(e_3 \times \Delta_j u) + \mathbb{P}\Delta_j (u \cdot \nabla)u = 0. \tag{3.3}$$

Taking the L^2 inner product of (3.3) with $\Delta_j u$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u(t)\|^2_{L^2} + \langle \Delta_j (u(t) \cdot \nabla)u(t), \Delta_j u(t)\rangle_{L^2} = 0. \tag{3.4}$$

Since it holds

$$\int_{\mathbb{R}^3} (u(t,x)\cdot\nabla)\Delta_j u(t,x)\cdot \Delta_j u(t,x)dx = 0$$

by integration by parts and the divergence-free condition, the Schwartz inequality yields that

$$\begin{aligned}
&\big|\langle \Delta_j (u(t)\cdot\nabla)u(t), \Delta_j u(t)\rangle_{L^2}\big| \\
&\quad = \big|\langle \Delta_j (u(t)\cdot\nabla)u(t) - (u(t)\cdot\nabla)\Delta_j u(t), \Delta_j u(t)\rangle_{L^2}\big| \\
&\quad \le \big\|[u(t)\cdot\nabla, \Delta_j]u(t)\big\|_{L^2}\|\Delta_j u(t)\|_{L^2}. \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.4), we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u(t)\|_{L^2}^2 \le \big\|[u(t)\cdot\nabla, \Delta_j]u(t)\big\|_{L^2}\|\Delta_j u(t)\|_{L^2},$$

which yields that

$$\|\Delta_j u(t)\|_{L^2} \le \|\Delta_j\phi\|_{L^2} + \int_0^t \big\|[u(\tau)\cdot\nabla, \Delta_j]u(\tau)\big\|_{L^2}d\tau. \qquad (3.6)$$

Multiplying both sides of (3.6) by 2^{rj} and then taking the $\ell^2(\mathbb{Z})$-norm, from the Minkowski inequality and Lemma 2.5 (1) we have

$$\begin{aligned}
\|u(t)\|_{\dot{H}^r} &\le \|\phi\|_{\dot{H}^r} + \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{2rj}\big\|[u(\tau)\cdot\nabla, \Delta_j]u(\tau)\big\|_{L^2}^2\right)^{1/2} d\tau \\
&\le \|\phi\|_{\dot{H}^r} + C_r\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot{H}^r}d\tau \qquad (3.7)
\end{aligned}$$

with some positive constant $C_r = C(r)$. Combining (3.2) and (3.7) and using the norm equivalence (2.1), we obtain

$$\|u(t)\|_{H^r} \le C_r\|\phi\|_{H^r} + C_r\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|u(\tau)\|_{H^r}d\tau$$

with some positive constant $C_r = C(r)$. Hence the Gronwall inequality yields that

$$\|u(t)\|_{H^r} \le C_r\|\phi\|_{H^r}\exp\left\{C_r\int_0^t \|\nabla u(\tau)\|_{L^\infty}d\tau\right\} \qquad (3.8)$$

for all $0 \le t \le T$. Now, let us set

$$V(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty}d\tau, \quad 0 \le t \le T.$$

It follows from [**21**, (8.10) in page 743] that for $2 < q < \infty$ there exist positive constants $C_{s,q}$ and C_s such that

$$V(t) \le C_{s,q} t^{1-1/q} |\Omega|^{-1/q} \|\phi\|_{H^{s+1}} \big(1 + \|\phi\|_{H^{s+1}} t e^{C_s V(t)}\big) \tag{3.9}$$

for all $0 \le t \le T$. Let us define

$$\begin{aligned} X_T &:= \{t \in [0,T] \mid V(t) \le 1\}, \\ T_* &:= \sup X_T. \end{aligned}$$

We shall prove that $T_* = T$ provided the speed of rotation is sufficiently high. Assume that $T_* < T$. We take a sequence $\{T_j\}_{j=1}^{\infty} \subset X_T$ such that $T_j \nearrow T_*$ as $j \to \infty$. Since $T_* < T$ and u belongs to $C([0,T]; H^{s+1}(\mathbb{R}^3))$, $V(t)$ is uniformly continuous on $[0, T_*]$ and it holds that

$$V(T_*) = \lim_{j \to \infty} V(T_j) \le 1. \tag{3.10}$$

Take a sufficiently large $\Omega \in \mathbb{R} \setminus \{0\}$ so that $|\Omega| \ge \Omega^0_{\phi,T}$ and

$$|\Omega|^{1/q} \ge 2 C_{s,q} T^{1-1/q} \|\phi\|_{H^{s+1}} \big(1 + \|\phi\|_{H^{s+1}} T e^{C_s}\big). \tag{3.11}$$

Then, since $T_* < T$, by (3.9), (3.10) and (3.11) we have

$$\begin{aligned} V(T_*) &\le C_{s,q} T_*^{1-1/q} |\Omega|^{-1/q} \|\phi\|_{H^{s+1}} \big(1 + \|\phi\|_{H^{s+1}} T_* e^{C_s V(T_*)}\big) \\ &\le C_{s,q} T^{1-1/q} |\Omega|^{-1/q} \|\phi\|_{H^{s+1}} \big(1 + \|\phi\|_{H^{s+1}} T e^{C_s}\big) \\ &\le \frac{1}{2}. \end{aligned}$$

Hence one can take S such that $T_* < S < T$ and $V(S) \le 1$, which contradicts the definition of T_*. Hence we have $T_* = T$ provided the speed of rotation is high enough. Therefore, since $u \in C([0,T]; H^{s+1}(\mathbb{R}^3))$, we obtain

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau = \lim_{t \nearrow T} V(t) \le 1. \tag{3.12}$$

Substituting (3.12) into (3.8), we obtain the desired estimate. □

4. Proof of Theorem 1.1.

We first recall the uniform local existence theorem with respect to $\Omega \in \mathbb{R}$.

Theorem 4.1 ([**21**, Theorem 1.3]). *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then for every $M > 0$ and for every $\phi \in H^s(\mathbb{R}^3)$ satisfying $\operatorname{div} \phi = 0$ and $\|\phi\|_{H^s} \le M$, there exists a positive time $T_0 = T_0(s, M)$ depending only on s and M such that (E'_Ω) possesses a* *unique classical solution u for all $\Omega \in \mathbb{R}$ satisfying*

$$u \in C([0,T_0]; H^s(\mathbb{R}^3)) \cap C^1([0,T_0]; H^{s-1}(\mathbb{R}^3)).$$

In particular, there exist positive constants $C_0 = C_0(s)$ and $C_1 = C_1(s)$ such that

$$T_0 \geq \frac{C_0}{M}, \qquad \sup_{0 \leq t \leq T_0} \|u(t)\|_{H^s} \leq C_1 \|\phi\|_{H^s}. \tag{4.1}$$

Now we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $s > 5/2$, and let $\phi \in H^s(\mathbb{R}^3)$ with $\operatorname{div} \phi = 0$. Also, let $0 < T < \infty$ be the given finite time. We use the regularization of the data by a family $\{J_\varepsilon\}_{0<\varepsilon\leq 1}$ of mollifiers defined in Lemma 2.7. For $0 < \varepsilon \leq 1$, we put $\phi_\varepsilon := J_\varepsilon \phi$. We remark that this small parameter ε is determined later depending only on the given data s, T and ϕ. By Lemma 2.7, we see that $\phi_\varepsilon \in H^{s+1}(\mathbb{R}^3)$ and

$$\|\phi_\varepsilon\|_{H^s} \leq \|\phi\|_{H^s}, \quad \|\phi_\varepsilon\|_{H^{s+1}} \leq \frac{1}{\varepsilon} \|\phi\|_{H^s}.$$

Hence it follows from Theorem 3.1 that for every $0 < \varepsilon \leq 1$, there exists a positive constant $\Omega_{\varepsilon,T}$ depending on s, T and $\|\phi_\varepsilon\|_{H^{s+1}}$ such that if $|\Omega| \geq \Omega_{\varepsilon,T}$ then there exists a unique classical solution u_ε to (E'_Ω) with $u_\varepsilon(0) = \phi_\varepsilon$ in the class

$$u_\varepsilon \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T]; H^s(\mathbb{R}^3)).$$

Furthermore, it follows from (3.1) and Lemma 2.7 (1) that there exists positive constant C_s such that there hold

$$\sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|_{H^s} \leq C_s \|\phi\|_{H^s}, \tag{4.2}$$

$$\sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|_{H^{s+1}} \leq \frac{C_s}{\varepsilon} \|\phi\|_{H^s}. \tag{4.3}$$

Also, since $\phi \in H^s(\mathbb{R}^3)$, it follows from Theorem 4.1 that there exists a positive time $T_0 = T_0(s, \|\phi\|_{H^s})$ such that (E'_Ω) possesses a unique classical solution u in the class

$$u \in C([0,T_0]; H^s(\mathbb{R}^3)) \cap C^1([0,T_0]; H^{s-1}(\mathbb{R}^3))$$

for all $\Omega \in \mathbb{R}$, and there exists a positive constant $C_1 = C_1(s)$ such that

$$\sup_{0 \leq t \leq T_0} \|u(t)\|_{H^s} \leq C_1 \|\phi\|_{H^s}. \tag{4.4}$$

If $T_0 \geq T$, then the proof is completed. We shall consider the case $T_0 < T$ and prove that the local solution u on $[0, T_0]$ can be extended to the solution of (E'_Ω) on the given time interval $[0,T]$ when $|\Omega|$ is sufficiently large.

Let us set $v_\varepsilon := u_\varepsilon - u$ and $|\Omega| \geq \Omega_{\varepsilon,T}$. Then v_ε solves

$$\begin{cases} \partial_t v_\varepsilon + \Omega \mathbb{P}(e_3 \times v_\varepsilon) + \mathbb{P}(v_\varepsilon \cdot \nabla) u_\varepsilon + \mathbb{P}(u \cdot \nabla) v_\varepsilon = 0, \\ \operatorname{div} v_\varepsilon = 0, \\ v_\varepsilon(0) = \phi_\varepsilon - \phi \end{cases} \tag{4.5}$$

on the local time interval $[0,T_0]$. Let us derive the estimates for v_ε. Taking the L^2 inner product of (4.5) with v_ε, we have

$$\frac{1}{2}\frac{d}{dt}\|v_\varepsilon(t)\|_{L^2}^2 + \langle (v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t), v_\varepsilon(t)\rangle_{L^2} = 0. \tag{4.6}$$

The Schwartz inequality gives that

$$\begin{aligned} |\langle (v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t), v_\varepsilon(t)\rangle_{L^2}| &\leq \|(v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t)\|_{L^2} \|v_\varepsilon(t)\|_{L^2} \\ &\leq \|\nabla u_\varepsilon(t)\|_{L^\infty} \|v_\varepsilon(t)\|_{L^2}^2. \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$\frac{1}{2}\frac{d}{dt}\|v_\varepsilon(t)\|_{L^2}^2 \leq \|\nabla u_\varepsilon(t)\|_{L^\infty} \|v_\varepsilon(t)\|_{L^2}^2,$$

which yields with the aid of the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$ that

$$\begin{aligned} \|v_\varepsilon(t)\|_{L^2} &\leq \|v_\varepsilon(0)\|_{L^2} + \int_0^t \|\nabla u_\varepsilon(\tau)\|_{L^\infty} \|v_\varepsilon(\tau)\|_{L^2} d\tau \\ &\leq \|v_\varepsilon(0)\|_{L^2} + C\int_0^t \|u_\varepsilon(\tau)\|_{H^s} \|v_\varepsilon(\tau)\|_{L^2} d\tau \end{aligned} \tag{4.8}$$

for all $0 \leq t \leq T_0$ with some positive constant $C = C(s)$. Next, applying the Littlewood–Paley projection operator Δ_j to both sides of (4.5), we have

$$\partial_t \Delta_j v_\varepsilon + \Omega \mathbb{P}(e_3 \times \Delta_j v_\varepsilon) + \mathbb{P}\Delta_j (v_\varepsilon \cdot \nabla) u_\varepsilon + \mathbb{P}\Delta_j (u \cdot \nabla) v_\varepsilon = 0. \tag{4.9}$$

Taking the L^2 inner product of (4.9) with $\Delta_j v_\varepsilon$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta_j v_\varepsilon(t)\|_{L^2}^2 + \langle \Delta_j (v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t), \Delta_j v_\varepsilon(t)\rangle_{L^2} \\ &\quad + \langle \Delta_j (u(t) \cdot \nabla) v_\varepsilon(t), \Delta_j v_\varepsilon(t)\rangle_{L^2} = 0. \end{aligned} \tag{4.10}$$

For the second term of (4.10), it follows from the Schwartz inequality that

$$|\langle \Delta_j (v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t), \Delta_j v_\varepsilon(t)\rangle_{L^2}| \leq \|\Delta_j (v_\varepsilon(t) \cdot \nabla) u_\varepsilon(t)\|_{L^2} \|\Delta_j v_\varepsilon(t)\|_{L^2}. \tag{4.11}$$

For the third term of (4.10), since it holds

$$\int_{\mathbb{R}^3} (u(t,x)\cdot\nabla)\Delta_j v_\varepsilon(t,x)\cdot\Delta_j v_\varepsilon(t,x)dx = 0$$

by the divergence-free condition, we see that

$$\begin{aligned}
&|\langle \Delta_j (u(t)\cdot\nabla)v_\varepsilon(t), \Delta_j v_\varepsilon(t)\rangle_{L^2}| \\
&\quad = |\langle \Delta_j (u(t)\cdot\nabla)v_\varepsilon(t) - (u(t)\cdot\nabla)\Delta_j v_\varepsilon(t), \Delta_j v_\varepsilon(t)\rangle_{L^2}| \\
&\quad \le \big\|[u(t)\cdot\nabla, \Delta_j]v_\varepsilon(t)\big\|_{L^2}\|\Delta_j v_\varepsilon(t)\|_{L^2}.
\end{aligned}\tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j v_\varepsilon(t)\|_{L^2}^2 \le{}& \|\Delta_j (v_\varepsilon(t)\cdot\nabla)u_\varepsilon(t)\|_{L^2}\|\Delta_j v_\varepsilon(t)\|_{L^2} \\
&+ \big\|[u(t)\cdot\nabla, \Delta_j]v_\varepsilon(t)\big\|_{L^2}\|\Delta_j v_\varepsilon(t)\|_{L^2},
\end{aligned}$$

which yields that

$$\begin{aligned}
\|\Delta_j v_\varepsilon(t)\|_{L^2} \le{}& \|\Delta_j v_\varepsilon(0)\|_{L^2} + \int_0^t \|\Delta_j (v_\varepsilon(\tau)\cdot\nabla)u_\varepsilon(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \big\|[u(\tau)\cdot\nabla, \Delta_j]v_\varepsilon(\tau)\big\|_{L^2}d\tau
\end{aligned}\tag{4.13}$$

for all $0 \le t \le T_0$. Multiplying both sides of (4.13) by $2^{(s-1)j}$ and then taking the $\ell^2(\mathbb{Z})$-norm, we have

$$\begin{aligned}
\|v_\varepsilon(t)\|_{\dot H^{s-1}} \le{}& \|v_\varepsilon(0)\|_{\dot H^{s-1}} + \int_0^t \|(v_\varepsilon(\tau)\cdot\nabla)u_\varepsilon(\tau)\|_{\dot H^{s-1}}d\tau \\
&+ \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{2(s-1)j}\big\|[u(\tau)\cdot\nabla, \Delta_j]v_\varepsilon(\tau)\big\|_{L^2}^2\right)^{1/2} d\tau.
\end{aligned}\tag{4.14}$$

For the second term of the right hand side of (4.14), it follows from Lemma 2.4 and the continuous embeddings $H^{s-1}(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$ that

$$\begin{aligned}
\|(v_\varepsilon\cdot\nabla)u_\varepsilon\|_{\dot H^{s-1}} &\le C(\|v_\varepsilon\|_{L^\infty}\|\nabla u_\varepsilon\|_{\dot H^{s-1}} + \|\nabla u_\varepsilon\|_{L^\infty}\|v_\varepsilon\|_{\dot H^{s-1}}) \\
&\le C\|u_\varepsilon\|_{H^s}\|v_\varepsilon\|_{H^{s-1}}
\end{aligned}\tag{4.15}$$

with some positive constant $C = C(s)$. For the third term of the right hand side of (4.14), it follows from Lemma 2.5 (2) and the continuous embeddings $H^{s-1}(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$ that

$$\left(\sum_{j\in\mathbb{Z}} 2^{2(s-1)j}\big\|[u\cdot\nabla,\Delta_j]v_\varepsilon\big\|_{L^2}^2\right)^{1/2} \le C(\|\nabla u\|_{L^\infty}\|v_\varepsilon\|_{\dot H^{s-1}} + \|v_\varepsilon\|_{L^\infty}\|u\|_{\dot H^s})$$
$$\le C\|u\|_{H^s}\|v_\varepsilon\|_{H^{s-1}} \tag{4.16}$$

with some positive constant $C = C(s)$. Substituting (4.15) and (4.16) into (4.14), we have

$$\|v_\varepsilon(t)\|_{\dot H^{s-1}} \le \|v_\varepsilon(0)\|_{\dot H^{s-1}} + C\int_0^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s})\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau. \tag{4.17}$$

Combining (4.8) and (4.17), by the norm equivalence (2.1) we see that

$$\|v_\varepsilon(t)\|_{H^{s-1}} \le C\|v_\varepsilon(0)\|_{H^{s-1}} + C\int_0^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s})\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau \tag{4.18}$$

for all $0 \le t \le T_0$ with some positive constant $C = C(s)$. Hence by (4.2), (4.4) and (4.18), we have

$$\|v_\varepsilon(t)\|_{H^{s-1}} \le C_s\|v_\varepsilon(0)\|_{H^{s-1}} + C_s\|\phi\|_{H^s}\int_0^t \|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau.$$

Hence the Gronwall inequality yields that

$$\sup_{0\le t\le T_0}\|v_\varepsilon(t)\|_{H^{s-1}} \le C_s\|J_\varepsilon\phi-\phi\|_{H^{s-1}}e^{C_sT\|\phi\|_{H^s}}. \tag{4.19}$$

Next, let us derive the H^s-estimate. Multiplying both sides of (4.13) by 2^{sj} and then taking the $\ell^2(\mathbb{Z})$-norm, we have

$$\|v_\varepsilon(t)\|_{\dot H^s} \le \|v_\varepsilon(0)\|_{\dot H^s} + \int_0^t \|(v_\varepsilon(\tau)\cdot\nabla)u_\varepsilon(\tau)\|_{\dot H^s}d\tau$$
$$+\int_0^t\left(\sum_{j\in\mathbb{Z}} 2^{2sj}\big\|[u(\tau)\cdot\nabla,\Delta_j]v_\varepsilon(\tau)\big\|_{L^2}^2\right)^{1/2}d\tau. \tag{4.20}$$

For the second term of the right hand side of (4.20), it follows from Lemma 2.4 and the continuous embeddings $H^{s-1}(\mathbb{R}^3)\hookrightarrow C(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)\hookrightarrow C^1(\mathbb{R}^3)$ that

$$\|(v_\varepsilon\cdot\nabla)u_\varepsilon\|_{\dot H^s} \le C(\|v_\varepsilon\|_{L^\infty}\|\nabla u_\varepsilon\|_{\dot H^s} + \|\nabla u_\varepsilon\|_{L^\infty}\|v_\varepsilon\|_{\dot H^s})$$
$$\le C(\|u_\varepsilon\|_{H^{s+1}}\|v_\varepsilon\|_{H^{s-1}} + \|u_\varepsilon\|_{H^s}\|v_\varepsilon\|_{H^s}) \tag{4.21}$$

with some positive constant $C = C(s)$. For the third term of the right hand side of (4.20), it follows from Lemma 2.5 (1) and the continuous embeddings $H^{s-1}(\mathbb{R}^3)\hookrightarrow C(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)\hookrightarrow C^1(\mathbb{R}^3)$ that

$$\left(\sum_{j\in\mathbb{Z}} 2^{2sj}\big\|[u\cdot\nabla,\Delta_j]v_\varepsilon\big\|_{L^2}^2\right)^{1/2} \le C(\|\nabla u\|_{L^\infty}\|v_\varepsilon\|_{\dot H^s} + \|\nabla v_\varepsilon\|_{L^\infty}\|u\|_{\dot H^s})$$
$$\le C\|u\|_{H^s}\|v_\varepsilon\|_{H^s} \tag{4.22}$$

with some positive constant $C = C(s)$. Substituting (4.21) and (4.22) into (4.20), we have

$$\|v_\varepsilon(t)\|_{\dot H^s} \le \|v_\varepsilon(0)\|_{\dot H^s} + C\int_0^t \|u_\varepsilon(\tau)\|_{H^{s+1}}\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau$$
$$+ C\int_0^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s})\|v_\varepsilon(\tau)\|_{H^s}d\tau. \tag{4.23}$$

Combining (4.8) and (4.23), by the norm equivalence (2.1) we have

$$\|v_\varepsilon(t)\|_{H^s} \le C\|v_\varepsilon(0)\|_{H^s} + C\int_0^t \|u_\varepsilon(\tau)\|_{H^{s+1}}\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau$$
$$+ C\int_0^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s})\|v_\varepsilon(\tau)\|_{H^s}d\tau \tag{4.24}$$

for all $0 \le t \le T_0$ with some positive constant $C = C(s)$. For the second term of the right hand side of (4.24), since $0 \le t \le T_0 < T$, it follows from (4.3) and (4.19) that

$$\int_0^t \|u_\varepsilon(\tau)\|_{H^{s+1}}\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau \le t\frac{C_s}{\varepsilon}\|\phi\|_{H^s}C_s\|J_\varepsilon\phi - \phi\|_{H^{s-1}}e^{C_sT\|\phi\|_{H^s}}$$
$$\le C_sT\|\phi\|_{H^s}e^{C_sT\|\phi\|_{H^s}}\frac{1}{\varepsilon}\|J_\varepsilon\phi - \phi\|_{H^{s-1}} \tag{4.25}$$

for all $0 \le t \le T_0$. For the third term of the right hand side of (4.24), it follows from (4.2) and (4.4) that

$$\int_0^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u\|_{H^s})\|v_\varepsilon(\tau)\|_{H^s}d\tau \le C_s\|\phi\|_{H^s}\int_0^t \|v_\varepsilon(\tau)\|_{H^s}d\tau \tag{4.26}$$

for all $0 \le t \le T_0$. Substituting (4.25) and (4.26) into (4.24), we have

$$\|v_\varepsilon(t)\|_{H^s} \le C_s\|J_\varepsilon\phi - \phi\|_{H^s} + C_sT\|\phi\|_{H^s}e^{C_sT\|\phi\|_{H^s}}\frac{1}{\varepsilon}\|J_\varepsilon\phi - \phi\|_{H^{s-1}}$$
$$+ C_s\|\phi\|_{H^s}\int_0^t \|v_\varepsilon(\tau)\|_{H^s}d\tau$$

for all $0 \le t \le T_0$. Hence the Gronwall inequality yields that

$$\sup_{0\le t\le T_0} \|v_\varepsilon(t)\|_{H^s}$$
$$\le \left\{ C_s\|J_\varepsilon\phi - \phi\|_{H^s} + C_sT\|\phi\|_{H^s} e^{C_sT\|\phi\|_{H^s}} \frac{1}{\varepsilon}\|J_\varepsilon\phi-\phi\|_{H^{s-1}} \right\} e^{C_sT\|\phi\|_{H^s}}. \tag{4.27}$$

Since

$$\lim_{\varepsilon\to 0}\|J_\varepsilon\phi-\phi\|_{H^s} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\|J_\varepsilon\phi-\phi\|_{H^{s-1}} = 0$$

by Lemma 2.7 (2), it follows from (4.19) and (4.27) that for every $0<\delta\le 1$, there exists a positive constant $\varepsilon_\delta = \varepsilon(\delta, s, T, \phi) \in (0,1]$ such that

$$\sup_{0\le t\le T_0}\|v_\varepsilon(t)\|_{H^s} \le \delta, \quad \frac{1}{\varepsilon}\sup_{0\le t\le T_0}\|v_\varepsilon(t)\|_{H^{s-1}} \le \delta \tag{4.28}$$

for all $0<\varepsilon\le\varepsilon_\delta$.

Next, we shall regard $u(T_0)$ as the initial velocity. Then for every $0<\varepsilon\le\varepsilon_\delta$, we have by (4.2) and (4.28)

$$\|u(T_0)\|_{H^s} \le \delta + \|u_\varepsilon(T_0)\|_{H^s} \le 1 + C_s\|\phi\|_{H^s}. \tag{4.29}$$

Hence by (4.29) and Theorem 4.1 we see that there exists a positive time $T_1 = T_1(s,\|\phi\|_{H^s})$ such that u can be uniquely extended to the solution of (E'_Ω) on the time interval $[T_0,T_1]$ for all $|\Omega|\ge\Omega_{\varepsilon,T}$ (The uniqueness can be proved by the standard argument. See, for example, [**21**, pp. 726–728]). Also, by (4.1) we have

$$T_1 - T_0 \ge \frac{C_0}{1+C_s\|\phi\|_{H^s}}, \tag{4.30}$$
$$\sup_{T_0\le t\le T_1}\|u(t)\|_{H^s} \le C_1(1+C_s\|\phi\|_{H^s}). \tag{4.31}$$

If $T_1\ge T$, the proof is completed. Hence we assume $T_1<T$. Let us derive the estimates for v_ε on $[T_0,T_1]$. Similarly to (4.8), we have

$$\|v_\varepsilon(t)\|_{L^2} \le \|v_\varepsilon(T_0)\|_{L^2} + C\int_{T_0}^t \|u_\varepsilon(\tau)\|_{H^s}\|v_\varepsilon(\tau)\|_{L^2}d\tau \tag{4.32}$$

for all $T_0\le t\le T_1$ with some positive constant $C=C(s)$. Also, similarly to (4.17) and (4.18), it follows from (4.32) that

$$\|v_\varepsilon(t)\|_{H^{s-1}} \le C\|v_\varepsilon(T_0)\|_{H^{s-1}} + C\int_{T_0}^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s})\|v_\varepsilon(\tau)\|_{H^{s-1}}d\tau \tag{4.33}$$

for all $T_0\le t\le T_1$ with some positive constant $C=C(s)$. Hence by (4.2), (4.31) and (4.33), we have

$$\|v_\varepsilon(t)\|_{H^{s-1}} \le C_s \|v_\varepsilon(T_0)\|_{H^{s-1}} + C_s\big(1 + \|\phi\|_{H^s}\big) \int_{T_0}^t \|v_\varepsilon(\tau)\|_{H^{s-1}} d\tau.$$

Hence the Gronwall inequality yields that

$$\sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^{s-1}} \le C_s \|v_\varepsilon(T_0)\|_{H^{s-1}} e^{C_s T (1 + \|\phi\|_{H^s})}. \tag{4.34}$$

For the H^s-estimate, similarly to (4.24), we have

$$\begin{aligned} \|v_\varepsilon(t)\|_{H^s} \le C\|v_\varepsilon(T_0)\|_{H^s} &+ C\int_{T_0}^t \|u_\varepsilon(\tau)\|_{H^{s+1}} \|v_\varepsilon(\tau)\|_{H^{s-1}} d\tau \\ &+ C\int_{T_0}^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u(\tau)\|_{H^s}) \|v_\varepsilon(\tau)\|_{H^s} d\tau \end{aligned} \tag{4.35}$$

for all $T_0 \le t \le T_1$ with some positive constant $C = C(s)$. For the second term of the right hand side of (4.35), since $T_0 \le t \le T_1 < T$, it follows from (4.3) and (4.34) that

$$\begin{aligned} \int_{T_0}^t \|u_\varepsilon(\tau)\|_{H^{s+1}} \|v_\varepsilon(\tau)\|_{H^{s-1}} d\tau &\le (t - T_0) \frac{C_s}{\varepsilon} \|\phi\|_{H^s} C_s \|v_\varepsilon(T_0)\|_{H^{s-1}} e^{C_s T(1+\|\phi\|_{H^s})} \\ &\le C_s T \|\phi\|_{H^s} e^{C_s T(1+\|\phi\|_{H^s})} \frac{1}{\varepsilon} \|v_\varepsilon(T_0)\|_{H^{s-1}} \end{aligned} \tag{4.36}$$

for all $T_0 \le t \le T_1$. For the third term of the right hand side of (4.35), it follows from (4.2) and (4.31) that

$$\int_{T_0}^t (\|u_\varepsilon(\tau)\|_{H^s} + \|u\|_{H^s}) \|v_\varepsilon(\tau)\|_{H^s} d\tau \le C_s(1 + \|\phi\|_{H^s}) \int_{T_0}^t \|v_\varepsilon(\tau)\|_{H^s} d\tau. \tag{4.37}$$

Substituting (4.36) and (4.37) into (4.35), we have

$$\begin{aligned} \|v_\varepsilon(t)\|_{H^s} \le C_s \|v_\varepsilon(T_0)\|_{H^s} &+ C_s T \|\phi\|_{H^s} e^{C_s T(1+\|\phi\|_{H^s})} \frac{1}{\varepsilon} \|v_\varepsilon(T_0)\|_{H^{s-1}} \\ &+ C_s(1 + \|\phi\|_{H^s}) \int_{T_0}^t \|v_\varepsilon(\tau)\|_{H^s} d\tau \end{aligned}$$

for all $T_0 \le t \le T_1$. Hence the Gronwall inequality yields that

$$\begin{aligned} &\sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^s} \\ &\le \left\{ C_s \|v_\varepsilon(T_0)\|_{H^s} + C_s T \|\phi\|_{H^s} e^{C_s T(1+\|\phi\|_{H^s})} \frac{1}{\varepsilon} \|v_\varepsilon(T_0)\|_{H^{s-1}} \right\} e^{C_s T(1+\|\phi\|_{H^s})} \end{aligned} \tag{4.38}$$

Substituting (4.28) into (4.34) and (4.38), we obtain

$$\frac{1}{\varepsilon} \sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^{s-1}} \le C_s \delta e^{C_s T(1+\|\phi\|_{H^s})} \tag{4.39}$$

and

$$\sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^s} \le C_s \delta \big\{1 + T\|\phi\|_{H^s} e^{C_s T(1+\|\phi\|_{H^s})}\big\} e^{C_s T(1+\|\phi\|_{H^s})} \tag{4.40}$$

for all $0 < \varepsilon \le \varepsilon_\delta$. Since $0 < \delta \le 1$ is arbitrary, it follows from (4.39) and (4.40) that for given $0 < \lambda \le 1$, one can take $\delta_\lambda = \delta(\lambda, s, T, \|\phi\|_{H^s}) \in (0,1]$ such that for every $0 < \delta \le \delta_\lambda$ there exists a positive number $\varepsilon_{\lambda,\delta} = \varepsilon(\lambda, \delta, s, T, \phi) \in (0,1]$ such that

$$\sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^s} \le \lambda, \quad \frac{1}{\varepsilon} \sup_{T_0 \le t \le T_1} \|v_\varepsilon(t)\|_{H^{s-1}} \le \lambda \tag{4.41}$$

holds for all $0 < \varepsilon \le \varepsilon_{\lambda,\delta}$. In particular, it follows from by (4.2) and (4.41) that

$$\|u(T_1)\|_{H^s} \le \lambda + \|u_\varepsilon(T_1)\|_{H^s} \le 1 + C_s \|\phi\|_{H^s}. \tag{4.42}$$

Note that the above bound (4.42) is exactly same as (4.29). Hence by Theorem 4.1 and (4.1), u can be uniquely extended to the solution of (E'_Ω) on the time interval $[T_1, T_1 + (T_1 - T_0)]$ (defined in (4.30)) for all $|\Omega| \ge \Omega_{\varepsilon,T}$ with $0 < \varepsilon \le \varepsilon_{\lambda,\delta}$ and satisfies

$$\sup_{T_1 \le t \le 2T_1 - T_0} \|u(t)\|_{H^s} \le C_1(1 + C_s \|\phi\|_{H^s}). \tag{4.43}$$

Also note that the bound (4.43) is exactly same as (4.31). Since T is arbitrary *finite* time, we repeat a finite number of the same procedures in the above, and continue the solution u to the time interval $[0, T]$. This completes the proof of Theorem 1.1. $\square$

5. The linear rotating vorticity equations.

In this section, we consider the linear problem for the vorticity equations in the rotational framework. The linear equations for (E'_Ω) are described as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Omega \mathbb{P}(e_3 \times u) = 0, \quad \operatorname{div} u = 0, \\ u(0,x) = \phi(x). \end{cases} \tag{5.1}$$

Taking curl to (5.1) and using the divergence-free condition, we have

$$\frac{\partial \omega}{\partial t} - \Omega \frac{\partial u}{\partial x_3} = 0, \quad \omega(0,x) = \psi(x), \tag{5.2}$$

where $\omega := \operatorname{curl} u = \nabla \times u$ and $\psi := \operatorname{curl} \phi$. By the Biot–Savart law, the gradient of the velocity ∇u has the representation in terms of the vorticity ω such that

$$\frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}(-\Delta)^{-1}\operatorname{curl}\omega = R_j(R\times\omega), \quad j=1,2,3, \tag{5.3}$$

where $R=(R_1,R_2,R_3)$ and R_j denotes the Riesz transform in $\mathbb{R}^3$. Then the linear vorticity equations (5.2) can be rewritten as

$$\frac{\partial\omega}{\partial t} - \Omega\frac{\partial}{\partial x_3}(-\Delta)^{-1}\operatorname{curl}\omega = 0, \quad \omega(0,x)=\psi(x). \tag{5.4}$$

Taking the Fourier transform to (5.4) yields

$$\frac{\partial}{\partial t}\widehat{\omega}(t,\xi) - \Omega\frac{\xi_3}{|\xi|}\begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}\widehat{\omega}(t,\xi)=0, \quad \widehat{\omega}(0,\xi)=\widehat{\psi}(\xi). \tag{5.5}$$

Let us define

$$R(\xi) := \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}, \quad S(\xi) := \frac{\xi_3}{|\xi|}R(\xi) \tag{5.6}$$

for $\xi \in \mathbb{R}^3 \setminus \{0\}$. Then the solution to (5.5) is written as

$$\widehat{\omega}(t,\xi) = e^{\Omega t S(\xi)}\widehat{\psi}(\xi),$$

where $e^{\Omega t S(\xi)}$ is defined by the convergent series

$$e^{\Omega t S(\xi)} := \sum_{j=0}^{\infty}\frac{1}{j!}(\Omega t)^j S(\xi)^j \quad \text{on } \{\xi\}^{\perp}.$$

Let I be the 3×3 identity matrix. Note that since it holds

$$S(\xi)^2 v(\xi) = -\frac{\xi_3^2}{|\xi|^2}Iv(\xi)$$

for $v(\xi)\in\mathbb{R}^3$ with $\xi\cdot v(\xi)=0$, the solution of (5.5) is explicitly given by

$$\begin{aligned}\widehat{\omega}(t,\xi) &= \cos\left(\Omega t\frac{\xi_3}{|\xi|}\right)I\widehat{\psi}(\xi) + \sin\left(\Omega t\frac{\xi_3}{|\xi|}\right)R(\xi)\widehat{\psi}(\xi)\\ &= \frac{1}{2}e^{i\Omega t(\xi_3/|\xi|)}\{I - iR(\xi)\}\widehat{\psi}(\xi) + \frac{1}{2}e^{-i\Omega t(\xi_3/|\xi|)}\{I + iR(\xi)\}\widehat{\psi}(\xi). \qquad (5.7)\end{aligned}$$

We remark that the explicit formula (5.7) has already been derived in [**1**], [**16**], [**17**] for the original equations of velocity fields. By (5.7), we have the following proposition.

Proposition 5.1. *For every $\Omega \in \mathbb{R}$ and every $\psi \in L^2_\sigma(\mathbb{R}^3)$, there exists a unique solution ω to* (5.4) *which is given explicitly by*

$$\omega(t,x) = T(\Omega t)\psi(x) := \frac{1}{2}e^{i\Omega t(D_3/|D|)}(I+\mathcal{R})\psi(x) + \frac{1}{2}e^{-i\Omega t(D_3/|D|)}(I-\mathcal{R})\psi(x) \quad (5.8)$$

for $t \geq 0$ and $x \in \mathbb{R}^3$, where

$$e^{\pm it(D_3/|D|)}f(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it(\xi_3/|\xi|)}\widehat{f}(\xi)d\xi, \quad \mathcal{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}.$$

We end this section by recalling the linear estimates for $T(t)$ given in (5.8). Since the phase $\xi_3/|\xi|$ is homogeneous function of degree 0, by the Littlewood–Paley decomposition and scaling, the matter is reduced to the frequency localized case. Now let us consider the operator

$$\mathscr{G}_\pm(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it(\xi_3/|\xi|)}\widehat{\Phi}(\xi)\widehat{f}(\xi)d\xi, \quad (t,x) \in \mathbb{R}^{1+3},$$

where $\Phi \in \mathscr{S}(\mathbb{R}^3)$ satisfies $\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^3 \mid 2^{-2} \leq |\xi| \leq 2^2\}$ and $\widehat{\Phi} = 1$ on $\{\xi \in \mathbb{R}^3 \mid 2^{-1} \leq |\xi| \leq 2\}$. The sharp Strichartz estimates for $\mathscr{G}_\pm(t)$ were obtained in [**21**]:

Theorem 5.2 ([**21**, Theorem 1.1]). *Let $2 \leq q, r \leq \infty$ with $(q,r) \neq (2,\infty)$. Then the space-time estimate*

$$\|\mathscr{G}_\pm(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}$$

holds if and only if

$$\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

Corollary 5.3. *Let $2 \leq q, r \leq \infty$ satisfy $(q,r) \neq (2,\infty)$ and $1/q + 1/r \leq 1/2$. Then, there exists a positive constant $C = C(q,r)$ such that*

$$\left\|\Delta_j e^{\pm i\Omega t(D_3/|D|)}f\right\|_{L^q_t L^r_x} \leq C|\Omega|^{-1/q}(2^j)^{3/2-3/r}\|\Delta_j f\|_{L^2}$$

holds for all $\Omega \in \mathbb{R} \setminus \{0\}$, $j \in \mathbb{Z}$ *and* $f \in L^2(\mathbb{R}^3)$.

PROOF. Since $\widehat{\Phi}(\xi) = 1$ on the support of $\widehat{\varphi_0}$, we have

$$\begin{aligned}\mathscr{G}_\pm(t)\Delta_0 f(x) &= \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it(\xi_3/|\xi|)}\widehat{\Phi}(\xi)\widehat{\varphi_0}(\xi)\widehat{f}(\xi)d\xi \\ &= (2\pi)^3 e^{\pm it(D_3/|D|)}\Delta_0 f(x).\end{aligned}$$

Hence by Theorem 5.2 there exists a positive constant $C = C(q,r)$ such that

$$\left\|e^{\pm it(D_3/|D|)}\Delta_0 f\right\|_{L^q_t L^r_x} \le C\|\Delta_0 f\|_{L^2}. \tag{5.9}$$

Since the phase $\xi_3/|\xi|$ is homogeneous of degree 0 and $\widehat{\varphi_j}(\xi) = \widehat{\varphi_0}(\xi/2^j)$, by the change of variable $\xi \mapsto 2^j\xi$, we have

$$\begin{aligned}e^{\pm it(D_3/|D|)}\Delta_j f(x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i2^j x\cdot\xi \pm it(\xi_3/|\xi|)}\widehat{\varphi_0}(\xi)\mathscr{F}\left[f\left(\frac{\cdot}{2^j}\right)\right](\xi)d\xi \\ &= e^{\pm it(D_3/|D|)}\Delta_0\left[f\left(\frac{\cdot}{2^j}\right)\right](2^j x)\end{aligned}$$

for $j \in \mathbb{Z}$. Also it holds $\Delta_j f(x) = \Delta_0\big[f\big(\frac{\cdot}{2^j}\big)\big](2^j x)$. Hence by (5.9) and scaling, we have

$$\begin{aligned}\left\|\Delta_j e^{\pm it(D_3/|D|)} f\right\|_{L^q_t L^r_x} &\le C(2^j)^{-3/r}\left\|\Delta_0\left[f\left(\frac{\cdot}{2^j}\right)\right]\right\|_{L^2} \\ &= C(2^j)^{3/2-3/r}\|\Delta_j f\|_{L^2}.\end{aligned}$$

The scaling in time $t \mapsto \Omega t$ gives the desired estimate. □

6. A priori estimate and the blow-up criterion.

In this section, we shall establish the blow-up criterion of the Beale–Kato–Majda and the Kozono–Taniuchi type [**4**], [**22**], [**23**] for the local solution to (E'_Ω).

LEMMA 6.1. *Let* $s > 5/2, \Omega \in \mathbb{R}$ *and* $\phi \in H^s(\mathbb{R}^3)$ *with* $\operatorname{div}\phi = 0$. *Let* u *be the solution to* (E'_Ω) *in the class* $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$ *with some* $T > 0$. *Then, there exists a positive constant* $C = C(s)$ *depending only on* s *such that*

$$\|u(t)\|_{H^s} + e \le (\|\phi\|_{H^s} + e)^{\alpha(t)}\exp\{Ct\alpha(t)\} \tag{6.1}$$

holds for all $0 \le t < T$, *where*

$$\alpha(t) := \exp\left\{C\int_0^t \|\operatorname{curl} u(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau\right\}.$$

By Theorem 4.1, for the given initial velocity $\phi \in H^s(\mathbb{R}^3)$ with $s > 5/2$, the time interval $[0, T)$ of the existence of the solution u to (E'_Ω) in $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$ depends only on s and $\|\phi\|_{H^s}$. Hence by the standard argument of continuation of local solutions, Lemma 6.1 yields the following blow-up criterion.

Lemma 6.2. *Let $s > 5/2, \Omega \in \mathbb{R}$ and $\phi \in H^s(\mathbb{R}^3)$ with $\operatorname{div} \phi = 0$. Let u be the solution to (E'_Ω) in the class $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$. Assume that T is maximal, that is, u cannot be continued to the solution in the class $C([0,T'); H^s(\mathbb{R}^3)) \cap C^1([0,T'); H^{s-1}(\mathbb{R}^3))$ for any $T' > T$. Then, it holds*

$$\int_0^T \|\operatorname{curl} u(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty.$$

Proof of Lemma 6.1. The proof is based on the energy method with the commutator estimates and the Gronwall inequality with the logarithmic Sobolev inequality as in [**4**], [**22**], [**23**]. Taking the H^s inner product of (E'_Ω) with u, by the skew-symmetry of the operator $e_3\times$ we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s}^2 + \langle (u(t)\cdot\nabla)u(t), u(t)\rangle_{H^s} = 0. \tag{6.2}$$

Here, it follows from the divergence-free condition, the Schwartz inequality and Lemma 2.6 that

$$\begin{aligned} |\langle (u\cdot\nabla)u, u\rangle_{H^s}| &= |\langle (1-\Delta)^{s/2}(u\cdot\nabla)u - (u\cdot\nabla)(1-\Delta)^{s/2}u, (1-\Delta)^{s/2}u\rangle_{L^2}| \\ &\le \|(1-\Delta)^{s/2}(u\cdot\nabla)u - (u\cdot\nabla)(1-\Delta)^{s/2}u\|_{L^2}\|u\|_{H^s} \\ &\le C(\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{s-1}} + \|\nabla u\|_{L^\infty}\|u\|_{H^s})\|u\|_{H^s} \\ &\le C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2 \end{aligned} \tag{6.3}$$

with some constant $C = C(s) > 0$. Substituting (6.3) into (6.2), we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s}^2 \le C\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^s}^2,$$

which yields

$$\frac{d}{dt}\|u(t)\|_{H^s} \le C\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^s}.$$

Hence it follows from the Gronwall inequality that

$$\|u(t)\|_{H^s} \le \|\phi\|_{H^s}\exp\left\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right\} \tag{6.4}$$

for all $0 \le t < T$ with some constant $C = C(s) > 0$.

Let $\omega := \operatorname{curl} u$ be the vorticity of u. Since $s > 5/2$, it follows from Lemma 2.3, the Biot–Savart law (5.3) and the boundedness of the Riesz transforms in $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\big\{1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}(1 + \log^+ \|\nabla u\|_{H^{s-1}})\big\} \\ &\le C\big\{1 + \|\omega\|_{\dot{B}^0_{\infty,\infty}}(1 + \log^+ \|u\|_{H^s})\big\} \end{aligned} \tag{6.5}$$

with $C = C(s) > 0$. Substituting (6.5) into (6.4), we have

$$\|u(t)\|_{H^s} \le \|\phi\|_{H^s} \exp\left[C\int_0^t \big\{1 + \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}(1 + \log^+ \|u(\tau)\|_{H^s})\big\}d\tau\right] \tag{6.6}$$

for all $0 \le t < T$. Moreover, since $\log^+ \|u(\tau)\|_{H^s} \le \log(\|u(\tau)\|_{H^s} + e)$, (6.6) implies that

$$\|u(t)\|_{H^s} + e \le (\|\phi\|_{H^s} + e) \exp\left[C\int_0^t \big\{1 + \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}} \log(\|u(\tau)\|_{H^s} + e)\big\}d\tau\right]. \tag{6.7}$$

Let us define $z(t) := \log(\|u(t)\|_{H^s} + e)$. Then it follows from (6.7) that

$$z(t) \le z(0) + Ct + C\int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}} z(\tau)d\tau$$

for all $0 \le t < T$. Hence by Lemma 2.2, we obtain

$$\begin{aligned} z(t) &\le z(0)\exp\left\{C\int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau\right\} + C\int_0^t \exp\left\{C\int_s^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau\right\}ds \\ &\le (z(0) + Ct)\exp\left\{C\int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau\right\}, \end{aligned}$$

which implies that

$$\log(\|u(t)\|_{H^s} + e) \le \log\big\{(\|\phi\|_{H^s} + e)^{\alpha(t)} e^{Ct\alpha(t)}\big\},$$

where

$$\alpha(t) := \exp\left\{C\int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau\right\}.$$

This completes the proof of Lemma 6.1. □

7. Proof of Theorem 1.3.

PROOF OF THEOREM 1.3. We shall prove that the local solution u to (E'_Ω) constructed in Theorem 4.1 with $s = 7/2$ can be extended to any time interval $[0, T]$ provided the speed of rotation is sufficiently high. To this end, we adapt the argument in [**9**], [**15**], [**21**].

Let $\phi \in H^{7/2}(\mathbb{R}^3)$ with $\operatorname{div} \phi = 0$, and let u be the solution to (E'_Ω) in the class $u \in C([0, T_\Omega); H^{7/2}(\mathbb{R}^3)) \cap C^1([0, T_\Omega); H^{5/2}(\mathbb{R}^3))$, where $0 < T_\Omega < \infty$ denotes the maximal time of existence. Taking curl to (E'_Ω) and using the Biot–Savart law (5.3), we have the vorticity equation

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \Omega \dfrac{\partial}{\partial x_3} (-\Delta)^{-1} \operatorname{curl} \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0, \\ \omega(0, x) = \psi(x), \end{cases} \tag{7.1}$$

where $\omega := \operatorname{curl} u = \nabla \times u$ and $\psi := \operatorname{curl} \phi$. By the Plancherel theorem and the Lebesgue dominated convergence theorem, we see that $\Omega(\partial/\partial x_3)(-\Delta)^{-1} \operatorname{curl}$ is the infinitesimal generator of the C_0 semigroup $T(\Omega t)$ (defined in Proposition 5.1) on $L^2_\sigma(\mathbb{R}^3)$ with the domain of generator $L^2_\sigma(\mathbb{R}^3)$. Therefore by the Duhamel principle, the solution ω to (7.1) can be represented as

$$\begin{aligned} \omega(t) = T(\Omega t)\psi &- \int_0^t T(\Omega(t-\tau))(u(\tau) \cdot \nabla)\omega(\tau) d\tau \\ &+ \int_0^t T(\Omega(t-\tau))(\omega(\tau) \cdot \nabla) u(\tau) d\tau \end{aligned} \tag{7.2}$$

for $0 < t < T_\Omega$. We shall derive the $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$-estimates for the vorticity ω. Let $2 < q < \infty$. Then, by the Minkowski inequality and Corollary 5.3, we have

$$\begin{aligned} \|T(\Omega t)\psi\|_{L^q_t(0,\infty; \dot{B}^0_{\infty,\infty})} &\le \left\| \left(\sum_{j \in \mathbb{Z}} \|\Delta_j T(\Omega t)\psi\|_{L^\infty}^2 \right)^{1/2} \right\|_{L^q_t(0,\infty)} \\ &\le \left(\sum_{j \in \mathbb{Z}} \|\Delta_j T(\Omega t)\psi\|^2_{L^q_t(0,\infty; L^\infty)} \right)^{1/2} \\ &\le C|\Omega|^{-1/q} \left\{ \sum_{j \in \mathbb{Z}} \left(2^{3j/2} \|\Delta_j \psi\|_{L^2} \right)^2 \right\}^{1/2} \\ &= C|\Omega|^{-1/q} \|\psi\|_{\dot{H}^{3/2}} \end{aligned} \tag{7.3}$$

with some constant $C = C(q) > 0$. Next, let us consider the Duhamel terms in (7.2). It follows from the Minkowski inequality and Corollary 5.3 that

$$\begin{aligned} &\left\| \Delta_j \int_0^t T(\Omega(t-\tau))(u(\tau) \cdot \nabla)\omega(\tau) d\tau \right\|_{L^q_t(0,T; L^\infty)} \\ &\quad \le \int_0^T \|\Delta_j T(\Omega(t-\tau))(u(\tau) \cdot \nabla)\omega(\tau)\|_{L^q_t(\tau,T; L^\infty)} d\tau \end{aligned}$$

$$\leq C|\Omega|^{-1/q}\int_0^T 2^{3j/2}\|\Delta_j(u(\tau)\cdot\nabla)\omega(\tau)\|_{L^2}d\tau \tag{7.4}$$

for all $j\in\mathbb{Z}$ and $0<T<T_\Omega$ with some constant $C=C(q)>0$. Hence, by the Minkowski inequality and (7.4), we have

$$\begin{aligned}
&\left\|\int_0^t T(\Omega(t-\tau))(u(\tau)\cdot\nabla)\omega(\tau)d\tau\right\|_{L^q_t(0,T;\dot{B}^0_{\infty,\infty})}\\
&\leq \left\|\left(\sum_{j\in\mathbb{Z}}\left\|\Delta_j\int_0^t T(\Omega(t-\tau))(u(\tau)\cdot\nabla)\omega(\tau)d\tau\right\|_{L^\infty}^2\right)^{1/2}\right\|_{L^q_t(0,T)}\\
&\leq \left(\sum_{j\in\mathbb{Z}}\left\|\Delta_j\int_0^t T(\Omega(t-\tau))(u(\tau)\cdot\nabla)\omega(\tau)d\tau\right\|_{L^q_t(0,T;L^\infty)}^2\right)^{1/2}\\
&\leq C|\Omega|^{-1/q}\left\{\sum_{j\in\mathbb{Z}}\left(\int_0^T 2^{3j/2}\|\Delta_j(u(\tau)\cdot\nabla)\omega(\tau)\|_{L^2}d\tau\right)^2\right\}^{1/2}\\
&\leq C|\Omega|^{-1/q}\int_0^T\left\{\sum_{j\in\mathbb{Z}}\left(2^{3j/2}\|\Delta_j(u(\tau)\cdot\nabla)\omega(\tau)\|_{L^2}\right)^2\right\}^{1/2}d\tau\\
&= C|\Omega|^{-1/q}\int_0^T\|(u(\tau)\cdot\nabla)\omega(\tau)\|_{\dot{H}^{3/2}}d\tau.
\end{aligned}\tag{7.5}$$

Similarly to (7.4) and (7.5), we also have

$$\begin{aligned}
&\left\|\int_0^t T(\Omega(t-\tau))(\omega(\tau)\cdot\nabla)u(\tau)d\tau\right\|_{L^q_t(0,T;\dot{B}^0_{\infty,\infty})}\\
&\qquad\leq C|\Omega|^{-1/q}\int_0^T\|(\omega(\tau)\cdot\nabla)u(\tau)\|_{\dot{H}^{3/2}}d\tau.
\end{aligned}\tag{7.6}$$

Therefore, by (7.2), (7.3), (7.5) and (7.6), we see that for every $2<q<\infty$ there exists a positive constant $C=C(q)$ such that

$$\begin{aligned}
&\|\omega\|_{L^q(0,T;\dot{B}^0_{\infty,\infty})}\\
&\leq C|\Omega|^{-1/q}\left\{\|\psi\|_{\dot{H}^{3/2}}+\int_0^T\left(\|(u(\tau)\cdot\nabla)\omega(\tau)\|_{\dot{H}^{3/2}}+\|(\omega(\tau)\cdot\nabla)u(\tau)\|_{\dot{H}^{3/2}}\right)d\tau\right\}
\end{aligned}\tag{7.7}$$

for all $0<T<T_\Omega$.

Now, let us define

$$V(t):=\int_0^t\|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}d\tau,\quad 0\leq t<T_\Omega.$$

Since it holds

$$\begin{aligned}\|(u\cdot\nabla)\omega\|_{\dot H^{3/2}}+\|(\omega\cdot\nabla)u\|_{\dot H^{3/2}}&\le C\big(\|\omega\otimes u\|_{\dot H^{5/2}}+\|u\otimes\omega\|_{\dot H^{5/2}}\big)\\&\le C\big(\|\omega\otimes u\|_{H^{5/2}}+\|u\otimes\omega\|_{H^{5/2}}\big)\\&\le C\|\omega\|_{H^{5/2}}\|u\|_{H^{5/2}}\end{aligned}\tag{7.8}$$

by the divergence-free conditions, it follows from the Hölder inequality, (7.7), (7.8) and (6.1) that

$$\begin{aligned}V(t)&\le t^{1-1/q}\|\omega\|_{L^q(0,t;\dot B^0_{\infty,\infty})}\\&\le Ct^{1-1/q}|\Omega|^{-1/q}\left(\|\psi\|_{\dot H^{3/2}}+\int_0^t\|\omega(\tau)\|_{H^{5/2}}\|u(\tau)\|_{H^{5/2}}d\tau\right)\\&\le Ct^{1-1/q}|\Omega|^{-1/q}\left(\|\phi\|_{\dot H^{5/2}}+\int_0^t\|u(\tau)\|_{H^{7/2}}\|u(\tau)\|_{H^{5/2}}d\tau\right)\\&\le Ct^{1-1/q}|\Omega|^{-1/q}\left(\|\phi\|_{H^{5/2}}+\int_0^t\|u(\tau)\|^2_{H^{7/2}}d\tau\right)\\&\le Ct^{1-1/q}|\Omega|^{-1/q}\left\{\|\phi\|_{H^{5/2}}+\int_0^t\big(\|\phi\|_{H^{7/2}}+e\big)^{2\alpha(\tau)}\exp\{C\tau\alpha(\tau)\}d\tau\right\}.\end{aligned}$$

Hence, we see that there exist an absolute constant $C_*=C_*(7/2)>0$ and a constant $C_q>0$ depending on q such that

$$V(t)\le C_qt^{1-1/q}|\Omega|^{-1/q}\big\{\|\phi\|_{H^{5/2}}+t\big(\|\phi\|_{H^{7/2}}+e\big)^{2\alpha(t)}\exp\{C_*t\alpha(t)\}\big\}\tag{7.9}$$

for all $0\le t<T_\Omega$, where

$$\alpha(t):=\exp\{C_*V(t)\}.$$

Now, for given time $0<T<\infty$, we define

$$\begin{aligned}X_{T,\Omega}&:=\{t\in[0,T]\cap[0,T_\Omega)\mid V(t)\le 1\},\\T_\Omega^*&:=\sup X_{T,\Omega}.\end{aligned}$$

We shall prove that $T_\Omega^*=\min\{T,T_\Omega\}$ when $|\Omega|$ is sufficiently large by contradiction argument. Assume that $T_\Omega^*<\min\{T,T_\Omega\}$. Then, we can take $\widetilde T$ satisfying $T_\Omega^*<\widetilde T<\min\{T,T_\Omega\}$. Since u belongs to $C([0,\widetilde T];H^{7/2}(\mathbb{R}^3))$, we see that $V(t)$ is uniformly continuous on $[0,\widetilde T]$, and then it holds

$$V(T_\Omega^*)\le 1.\tag{7.10}$$

Since $T_\Omega^*<\min\{T,T_\Omega\}\le T$, it follows from (7.9) and (7.10) that

$$\begin{aligned}
V(T_\Omega^*) &\le C_q (T_\Omega^*)^{1-1/q} |\Omega|^{-1/q} \big[\|\phi\|_{H^{5/2}} + T_\Omega^* \big(\|\phi\|_{H^{7/2}} + e \big)^{2\alpha(T_\Omega^*)} \exp\{C_* T_\Omega^* \alpha(T_\Omega^*)\} \big] \\
&\le C_q T^{1-1/q} |\Omega|^{-1/q} \big[\|\phi\|_{H^{5/2}} + T \big(\|\phi\|_{H^{7/2}} + e \big)^{2\exp\{C_*\}} \exp\{C_* e^{C_* T}\} \big] \\
&\le |\Omega|^{-1/q} C_q T^{1-1/q} \big\{ \|\phi\|_{H^{5/2}} + T \big(\|\phi\|_{H^{7/2}} + e \big)^{C_*'} e^{C_*' T} \big\}. \qquad (7.11)
\end{aligned}$$

Hence taking a sufficiently large $\Omega \in \mathbb{R} \setminus \{0\}$ so that

$$|\Omega|^{1/q} \ge 2 C_q T^{1-1/q} \big\{ \|\phi\|_{H^{5/2}} + T \big(\|\phi\|_{H^{7/2}} + e \big)^{C_*'} e^{C_*' T} \big\}, \qquad (7.12)$$

by (7.11) we have

$$V(T_\Omega^*) \le \frac{1}{2}.$$

Then, one can take a time S such that $T_\Omega^* < S < \widetilde{T}$ and $V(S) \le 1$, which contradicts the definition of T_Ω^*. Therefore, we have $T_\Omega^* = \min\{T, T_\Omega\}$ provided the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies (7.12).

Let $\Omega \in \mathbb{R} \setminus \{0\}$ satisfy (7.12), and assume that $T_\Omega < T$. Then it follows from the above argument that $T_\Omega = T_\Omega^* = \sup X_{T,\Omega}$. Therefore we have

$$V(t) = \int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}} d\tau \le 1 < \infty$$

for all $0 \le t < T_\Omega$. However, by Lemma 6.2, this contradicts the maximality of T_Ω. Hence we obtain $T_\Omega \ge T$ if the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ is high enough as in (7.12). This completes the proof of Theorem 1.3. □

8. Appendix.

In this section, we shall give an alternative derivation of the solution formula (5.7) to the linear vorticity equations (5.4). Let $S(\xi)$ be the skew-symmetric matrix defined in (5.6). The direct calculation gives that

$$\det(\lambda I - S(\xi)) = \lambda \left(\lambda^2 + \frac{\xi_3^2}{|\xi|^2} \right).$$

Hence, the matrix $S(\xi)$ possesses the eigenvalues $\pm i(\xi_3/|\xi|)$ and 0. In order to derive the corresponding eigenvectors, we shall use the Craya-Herring decomposition. Let $\{e_1(\xi), e_2(\xi), e_3(\xi)\}$ be an orthonormal system in $\mathbb{R}^3$ defined by

$$e_1(\xi) := \begin{pmatrix} -\frac{\xi_2}{|\xi_h|} \\ \frac{\xi_1}{|\xi_h|} \\ 0 \end{pmatrix}, \quad e_2(\xi) := \begin{pmatrix} \frac{\xi_1\xi_3}{|\xi||\xi_h|} \\ \frac{\xi_2\xi_3}{|\xi||\xi_h|} \\ -\frac{\xi_1^2+\xi_2^2}{|\xi||\xi_h|} \end{pmatrix}, \quad e_3(\xi) := \frac{\xi}{|\xi|} = \begin{pmatrix} \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} \\ \frac{\xi_3}{|\xi|} \end{pmatrix},$$

where $\xi_h := (\xi_1, \xi_2)$. Then, it is easy to see that

$$S(\xi)e_1(\xi) = \frac{\xi_3}{|\xi|}e_2(\xi), \quad S(\xi)e_2(\xi) = -\frac{\xi_3}{|\xi|}e_1(\xi), \quad S(\xi)e_3(\xi) = 0. \tag{8.1}$$

Therefore, putting

$$\begin{aligned} v_1(\xi) &:= \frac{1}{\sqrt{2}}(e_2(\xi) + ie_1(\xi)) = \frac{1}{\sqrt{2}|\xi||\xi_h|}\begin{pmatrix} \xi_1\xi_3 - i\xi_2|\xi| \\ \xi_2\xi_3 + i\xi_1|\xi| \\ -(\xi_1^2+\xi_2^2) \end{pmatrix}, \\ v_2(\xi) &:= \frac{1}{\sqrt{2}}(e_2(\xi) - ie_1(\xi)) = \frac{1}{\sqrt{2}|\xi||\xi_h|}\begin{pmatrix} \xi_1\xi_3 + i\xi_2|\xi| \\ \xi_2\xi_3 - i\xi_1|\xi| \\ -(\xi_1^2+\xi_2^2) \end{pmatrix}, \\ v_3(\xi) &:= e_3(\xi), \end{aligned}$$

we have by (8.1)

$$S(\xi)v_1(\xi) = i\frac{\xi_3}{|\xi|}v_1(\xi), \quad S(\xi)v_2(\xi) = -i\frac{\xi_3}{|\xi|}v_2(\xi), \quad S(\xi)v_3(\xi) = 0.$$

Also, by the orthonormality of $\{e_1(\xi), e_2(\xi), e_3(\xi)\}$ we see that $\{v_1(\xi), v_2(\xi), v_3(\xi)\}$ is an orthonormal basis in $\mathbb{C}^3$. Hence we see that $U(\xi) := (v_1(\xi), v_2(\xi), v_3(\xi))$ is a unitary matrix, and it holds that

$$U(\xi)^*S(\xi)U(\xi) = \begin{pmatrix} i\frac{\xi_3}{|\xi|} & 0 & 0 \\ 0 & -i\frac{\xi_3}{|\xi|} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which yields that

$$e^{\Omega tS(\xi)} = U(\xi)\begin{pmatrix} e^{i\Omega t(\xi_3/|\xi|)} & 0 & 0 \\ 0 & e^{-i\Omega t(\xi_3/|\xi|)} & 0 \\ 0 & 0 & 1 \end{pmatrix}U(\xi)^*. \tag{8.2}$$

Then, by direct calculations, we have

$$\begin{aligned} &U(\xi)\begin{pmatrix} e^{i\Omega t(\xi_3/|\xi|)} & 0 & 0 \\ 0 & e^{-i\Omega t(\xi_3/|\xi|)} & 0 \\ 0 & 0 & 1 \end{pmatrix} U(\xi)^* \\ &\quad = \frac{1}{2}e^{i\Omega t(\xi_3/|\xi|)}\{I - F(\xi) - iR(\xi)\} + \frac{1}{2}e^{i\Omega t(\xi_3/|\xi|)}\{I - F(\xi) + iR(\xi)\} + F(\xi), \end{aligned} \tag{8.3}$$

where I is the identity matrix, $R(\xi)$ is the skew-symmetric matrix defined in (5.6) and $F(\xi)$ is defined by

$$F(\xi) := \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix}.$$

Note that $F(\xi)u(\xi) = 0$ for all $u(\xi) \in \mathbb{R}^3$ satisfying $\xi \cdot u(\xi) = 0$. Hence by (8.2) and (8.3), we obtain

$$e^{\Omega t S(\xi)}\widehat{\psi}(\xi) = \frac{1}{2}e^{i\Omega t(\xi_3/|\xi|)}\{I - iR(\xi)\}\widehat{\psi}(\xi) + \frac{1}{2}e^{-i\Omega t(\xi_3/|\xi|)}\{I + iR(\xi)\}\widehat{\psi}(\xi)$$

for $\psi \in L^2_\sigma(\mathbb{R}^3)$. This completes an another derivation of the solution formula (5.7).

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