J. Math. Soc. Japan
Vol. 68, No. 2 (2016) pp. 535–577
doi: 10.2969/jmsj/06820535

Short time kernel asymptotics for Young SDE by means of Watanabe distribution theory

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(Received Dec. 1, 2013)
(Revised May 23, 2014)

Abstract. In this paper we study short time asymptotics of a density function of the solution of a stochastic differential equation driven by fractional Brownian motion with Hurst parameter H $(1/2 < H < 1)$ when the coefficient vector fields satisfy an ellipticity condition at the starting point. We prove both on-diagonal and off-diagonal asymptotics under mild additional assumptions. Our main tool is Malliavin calculus, in particular, Watanabe's theory of generalized Wiener functionals.

1. Introduction.

Let $(w_t)_{t\geq 0}$ be the standard d-dimensional Brownian motion and let V_i $(0 \leq i \leq d)$ be smooth vector fields on $\mathbf{R}^n$ with sufficient regularity. Consider the following stochastic differential equation (SDE) of Stratonovich-type;

$$dy_t = \sum_{i=1}^{d} V_i(y_t) \circ dw_t^i + V_0(y_t)dt \quad \text{with} \quad y_0 = a \in \mathbf{R}^n.$$

If the set of vector fields satisfies a hypoellipticity condition, the solution $y_t = y_t(a)$ has a smooth density $p_t(a, a')$ with respect to Lebesgue measure on $\mathbf{R}^n$. From an analytic point of view, $p_t(a, a')$ is a fundamental solution of the parabolic equation $\partial u/\partial t = Lu$, where $L = V_0 + (1/2)\sum_{i=1}^{d} V_i^2$, and is also called a heat kernel of L.

In many fields of mathematics such as probability, analysis, mathematical physics, and differential geometry, short time asymptotic of $p_t(a, a')$ is a very important problem and has been studied extensively. Although analytic methods are also well-known, we only discuss a probabilistic approach via Feynman–Kac formula in this paper. Malliavin calculus is a very powerful theory and was used in many papers on this problem.

Among them, S. Watanabe's result seems to be one of the best. (See [**21**] or Sections 5.8–5.10, [**9**].) His theory of distributional Malliavin calculus is not only very powerful, but also user-friendly. Many heuristic operations are made rigorous in this theory and consequently the theory gives us a good view. Moreover, this theory is quite self-contained in the sense that all the argument, from an explicit expression of the heat kernel to the final asymptotic result, is constructed without much help from other theories.

2010 *Mathematics Subject Classification.* Primary 60H07; Secondary 60F99, 60H10, 60G15.

Key Words and Phrases. fractional Brownian motion, Young integral, stochastic differential equation, Malliavin calculus, Watanabe distribution, short time asymptotics.

The theory goes as follows. First, he constructed a theory of generalized Wiener functionals (i.e., Watanabe distributions) in Malliavin calculus. Then, he gave a representation of the heat kernel by using the pullback of Dirac's delta function; $p_t(a, a') = \mathbb{E}[\delta_{a'}(y_t(a))]$, where the right hand side is the generalized expectation with respect to Wiener measure. Finally, by establishing an asymptotic expansion theory in the spaces of generalized Wiener functionals, he obtained a short time expansion of $p_t(a, a')$ under very mild assumptions. In this method, an asymptotic expansion is actually obtained before taking the generalized expectation.

In this paper we consider the following problem. Let $(w_t^H)_{t\geq 0}$ be d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$. Instead of the above SDE, we consider

$$dy_t = \sum_{i=1}^{d} V_i(y_t) dw_t^{H,i} + V_0(y_t)dt \quad \text{with} \quad y_0 = a \in \mathbf{R}^n.$$

This is an ordinary differential equation (ODE) in the sense of Young integral (see Lyons [**13**]). In fact, this is actually an ODE with a random driving path, but we call this SDE for simplicity. Some researchers have studied the solution of the above SDE with Malliavin calculus. See [**17**], [**8**], [**18**], [**1**], [**6**] and references therein. Under the ellipticity or the hypoellipticity condition, the solution $y_t = y_t(a)$ has a smooth density $p_t(a, a')$. See [**8**], [**18**], [**1**].

In this paper, by using Malliavin calculus and, in particular, Watanabe distribution theory, we will prove a short asymptotic expansion of this density in the elliptic case under mild assumptions. This kind of asymptotics was already studied in [**1**], [**2**], but without Malliavin calculus. In [**1**], they showed on-diagonal short time asymptotics when $V_0 \equiv 0$. In [**2**], by using Laplace's method, they showed off-diagonal short time asymptotics when $V_0 \equiv 0$ and the vector fields V_i's satisfy a rather special condition. Our results is a generalization of these preceding ones. Notice that we do not assume the drift term V_0 is zero. One may think this is just a minor generalization, but this makes the asymptotic expansion much more complicated.

The organization of this paper is as follows: In Section 2, we give settings, assumptions, and precise statements of two main theorems. In Section 3, we recall basic properties of a Young ODE and its Jacobian process for later use. In Section 4, we review Watanabe's theory of generalized Wiener functionals in Malliavin calculus. In Section 5, we discuss the solution of Young ODE driven by fBm with Hurst parameter $H \in (1/2, 1)$ from the viewpoint of Malliavin calculus. We also prove uniform non-degeneracy of Malliavin covariance matrix of the solution under the ellipticity condition. In Section 6, we prove one of our main theorems, namely, on-diagonal asymptotics of the kernel. In section 7, we show the shifted solution of the Young SDE admits an asymptotic expansion in the sense of Watanabe distribution theory. In Section 8, we prove the other of our main theorems, namely, off-diagonal asymptotics of the kernel. In Section 9, we prove that, under the ellipticity assumption at the stating point, our main result (the off-diagonal asymptotics) holds when the end point is close enough to the starting point. We also make sure that Baudoin and Ouyang's result in [**2**] is basically included in ours.

2. Setting and main results.

2.1. Setting.

In this subsection, we introduce a stochastic process that will play a main role in this paper. From now on, dropping the superscript "H", we denote by $(w_t)_{t\geq 0} = (w_t^1, \ldots, w_t^d)_{t\geq 0}$ the d-dimensional fractional Brownian motion (fBm) with Hurst parameter H $(1/2 < H < 1)$. It is a unique d-dimensional mean-zero Gaussian process with covariance

$$\mathbb{E}[w_s^i w_t^j] = \frac{\delta_{ij}}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \quad (s, t \geq 0).$$

Note that, for any $c > 0$, $(w_{ct})_{t\geq 0}$ and $(c^H w_t)_{t\geq 0}$ have the same law. This property is called self-similarity or scale invariance.

Let $V_i : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞, that is, V_i is a bounded smooth function with bounded derivatives of all order $(0 \leq i \leq d)$. We consider the following stochastic ODE in the sense of Young;

$$dy_t = \sum_{i=1}^{d} V_i(y_t)dw_t^i + V_0(y_t)dt \quad \text{with} \quad y_0 = a \in \mathbf{R}^n. \tag{2.1}$$

We will sometimes write $y_t = y_t(a) = y_t(a, w)$ etc. to make explicit the dependence on a and w.

2.2. Assumptions.

In this subsection we introduce assumptions of the main theorems. First, we assume the ellipticity of the coefficient of (2.1) at the starting point $a \in \mathbf{R}^n$.

(A1): The set of vectors $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbf{R}^n$.

It is known that, under Assumption (A1), the law of the solution y_t has a density $p_t(a, a')$ with respect to the Lebesgue measure on $\mathbf{R}^n$ for any $t > 0$ (see [**1**], [**18**]). Hence, for any measurable set $U \subset \mathbf{R}^n$, $\mathbb{P}(y_t \in U) = \int_U p_t(a, a')da'$.

Let $\mathcal{H} = \mathcal{H}^H$ be the Cameron–Martin space of fBm (w_t). For $\gamma \in \mathcal{H}$, we denote by $\phi_t^0 = \phi_t^0(\gamma)$ be the solution of the following Young ODE;

$$d\phi_t^0 = \sum_{i=1}^{d} V_i(\phi_t^0)d\gamma_t^i \quad \text{with} \quad \phi_0^0 = a \in \mathbf{R}^n. \tag{2.2}$$

Set, for $a \neq a'$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

If we assume (A1) for all a, this set $K_a^{a'}$ is not empty. If $K_a^{a'}$ is not empty, it is a Hilbert submanifold of $\mathcal{H}$. From the Schilder-type large deviation theory, it is easy to see that $\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$. Now we introduce the following assumption;

(A2): $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathcal{H}$-norm exists uniquely.

In the sequel, $\bar{\gamma}$ denotes the minimizer in Assumption (A2). We also assume that $\| \cdot \|_{\mathcal{H}}^2/2$ is not so degenerate at $\bar{\gamma}$ in the following sense.

(A3): At $\bar{\gamma}$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathcal{H}}^2/2$ is strictly positive in the form sense. More precisely, if $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$, then $(d/du)^2|_{u=0}\|f(u)\|_{\mathcal{H}}^2/2 > 0$.

Later we will give a more analytical condition (A3)', which is equivalent to (A3) under (A2). In [**21**], Watanabe used (A3)'. We will also use (A3)' in the proof. In order to state (A3)', however, we have to introduce a lot of notations. So, we presented (A3) here for ease of presentation.

2.3. Index sets.

In this subsection we introduce several index sets for the exponent of the small parameter $\varepsilon > 0$, which will be used in the asymptotic expansion. Unlike in the preceding papers, index sets in this paper are not the set of natural numbers and are rather complicated. Set

$$\Lambda_1 = \left\{ n_1 + \frac{n_2}{H} \;\middle|\; n_1, n_2 \in \mathbf{N} \right\},$$

where $\mathbf{N} = \{0, 1, 2, \ldots\}$. We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ all the elements of Λ_1 in increasing order. Several smallest elements are explicitly given as follows;

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{H}, \quad \kappa_3 = 2, \quad \kappa_4 = 1 + \frac{1}{H}, \quad \kappa_5 = 3 \wedge \frac{2}{H}, \ldots$$

As usual, using the scale invariance (i.e., self-similarity) of fBm, we will consider the scaled version of (2.1). (See the scaled Young ODE (6.1) below). From its explicit form, one can easily see why Λ_1 appears.

We also set

$$\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\} = \left\{ 0,\ \frac{1}{H} - 1,\ 1,\ \frac{1}{H},\ \left(3 \wedge \frac{2}{H}\right) - 1, \ldots \right\}$$

and

$$\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1, 1/H\}\} = \left\{ 0,\ \frac{1}{H} - 1,\ \left(3 \wedge \frac{2}{H}\right) - 2, \ldots \right\}.$$

Next we set

$$\Lambda_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2\}.$$

In the sequel, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ stands for all the elements of Λ_3 in increasing order. Similarly,

$$\Lambda_3' = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2'\}.$$

In the sequel, $\{0 = \rho_0 < \rho_1 < \rho_2 < \cdots\}$ stands for all the elements of Λ_3' in increasing order. Finally,

$$\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}.$$

We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ all the elements of Λ_4 in increasing order.

2.4. Statement of the main results.

In this subsection we state two main results of ours, which are basically analogous to the corresponding ones in Watanabe [**21**]. However, there are some differences. First, the exponents of t are not (a constant multiple of) natural numbers. Second, cancellation of "odd terms" as in p. 20 and p. 34, [**21**] does not happen in general in our case. (If the drift term in Young ODE (2.1) is zero, then this kind of cancellation takes place as in [**1**], [**2**]).

The following is a short time asymptotic expansion of the diagonal of the kernel function. This is much easier than the off-diagonal case.

Theorem 2.1. *Assume* (A1). *Then, the diagonal of the kernel* $p(t, a, a)$ *admits the following asymptotics as* $t \searrow 0$;

$$p(t, a, a) \sim \frac{1}{t^{nH}}\big(c_0 + c_{\nu_1} t^{\nu_1 H} + c_{\nu_2} t^{\nu_2 H} + \cdots\big)$$

for certain real constants $c_0, c_{\nu_1}, c_{\nu_2}, \ldots$. *Here,* $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ *are all the elements of* Λ_3 *in increasing order.*

We also have off-diagonal short time asymptotics of the kernel function.

Theorem 2.2. *Assume* $a \neq a'$ *and* (A1)–(A3). *Then, we have the following asymptotic expansion as* $t \searrow 0$;

$$p(t, a, a') \sim \exp\bigg(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2t^{2H}} + \frac{\beta}{t^{2H-1}}\bigg)\frac{1}{t^{nH}}\big\{\alpha_{\lambda_0} + \alpha_{\lambda_1} t^{\lambda_1 H} + \alpha_{\lambda_2} t^{\lambda_2 H} + \cdots\big\}$$

for certain real constants $\beta, \alpha_{\lambda_j}$ $(j = 0, 1, 2, \ldots)$. *Here,* $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ *are all the elements of* Λ_4 *in increasing order.*

Remark 2.3. (i) Consider the following simplest case; $n = d = 1$ and $y_t = a + w_t + bt$ with $b \in \mathbf{R}$. Then, for each $t > 0$, this induces a Gaussian measure with mean $a + bt$ and variance t^{2H}. Hence, the kernel is given by

$$p(t, a, a') = \frac{1}{\sqrt{2\pi} t^H}\exp\bigg(-\frac{(a + bt - a')^2}{2t^{2H}}\bigg)$$

$$= \frac{1}{\sqrt{2\pi}t^H} e^{-(a-a')^2/(2t^{2H})} e^{-b(a-a')/t^{2H-1}} e^{-b^2 t^{2-2H}/2}$$
$$= e^{-(a-a')^2/(2t^{2H}) - b(a-a')/t^{2H-1}}$$
$$\times \frac{1}{\sqrt{2\pi}t^H} \left(1 - \frac{b^2}{2} t^{2(H^{-1}-1)H} + \frac{b^4}{2^2 2!} t^{4(H^{-1}-1)H} - \cdots \right).$$

This example may illustrates that the asymptotics in Theorem 2.2 are not so strange.

(ii) Some of the constants in Theorems 2.1 and 2.2 can be obtained explicitly. For example, in Theorems 2.1, $c_0 = [(2\pi)^{n/2} \det(\sigma(a)\sigma(a)^*)]^{-1}$ and

$$c_{\nu_1} = c_{(1/H)-1} = \sum_{j=1}^{n} \partial_j \delta_0 \big(V_1(a) w_1^1 + \cdots + V_d(a) w_1^d \big) \cdot V_0(a)^j = 0.$$

Here, $\sigma(a)\sigma(a)^*$ is the covariance matrix of the n-dimensional Gaussian random variable $\sum_{j=1}^d V_j(a) w_1^j$. In Theorems 2.2, $\beta = \langle \bar{\nu}, \phi_1^{1/H} \rangle$. The notations in this remark will be given later.

2.5. Outline of proof of off-diagonal asymptotics.

In this subsection we outline the proof of Theorem 2.2 in a heuristic way so that the reader would not get lost in technical details. The argument in this subsection is not rigorous. For $\varepsilon \in (0,1]$ and $\bar{\gamma}$ as in (A2), consider the following SDE;

$$d\tilde{y}_t^\varepsilon = \sum_{i=1}^d V_i(\tilde{y}_t^\varepsilon)(\varepsilon dw_t^i + d\bar{\gamma}_t) + V_0(\tilde{y}_t^\varepsilon)\varepsilon^{1/H} dt \quad \text{with} \quad \tilde{y}_0^\varepsilon = a$$

(We denote by y^ε the solution of the above ODE with $\bar{\gamma} = 0$.)

From the scaling property of fBm and a routine argument in Watanabe's theory,

$$p(\varepsilon^{1/H}, a, a') = \mathbb{E}\big[\delta_{a'}(y_{\varepsilon^{1/H}})\big] = \mathbb{E}\big[\delta_{a'}(y_1^\varepsilon)\big] = \mathbb{E}\big[\delta_{a'}(y_1^\varepsilon)\chi_\eta(\varepsilon, w)\big] + (\text{a small term}).$$

Here, $\chi_\eta(\varepsilon, w)$ is a $\mathbf{D}_\infty$-functional which looks like the indicator of a small ball of a certain radius $\eta > 0$ centered at $\bar{\gamma}$. By Schilder-type large deviations, the second term above is negligible. By Cameron–Martin theorem, the fisrt term is equal to

$$\exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2} \right) \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\langle \bar{\gamma}, w \rangle \right) \delta_{a'}(\tilde{y}_1^\varepsilon) \chi_\eta\left(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}\right) \right].$$

Here, $\chi_\eta(\varepsilon, w + \bar{\gamma}/\varepsilon)$ does not contribute to the asymptotic expansion since it is of the form $1 + O(\varepsilon^N)$ for any large $N \in \mathbf{N}$. So, it is sufficient to consider the two factors; $\delta_{a'}(\tilde{y}_1^\varepsilon)$ and $\exp(-\langle \bar{\gamma}, w \rangle / \varepsilon)$.

We will prove in Section 7 that $\tilde{y}_1^\varepsilon$ admits the following expansion for certain ϕ^{κ_j}'s both in $\mathbf{D}_\infty(\mathbf{R}^n)$-sense and the deterministic sense.

$$\tilde{y}_1^\varepsilon \sim \phi_1^0 + \varepsilon^{\kappa_1}\phi_1^{\kappa_1} + \varepsilon^{\kappa_2}\phi_1^{\kappa_2} + \cdots \quad \text{as } \varepsilon \searrow 0, \quad \Big(\kappa_i \in \Lambda_1 = \mathbf{N} + \frac{1}{H}\mathbf{N}\Big).$$

From the SDE for $\tilde{y}^\varepsilon$, one can easily see that the index set for this Taylor expansion of Itô map should be Λ_1. Set $R^{2,\varepsilon} = \tilde{y}^\varepsilon - (\phi^0 + \varepsilon\phi^1 + \varepsilon^{1/H}\phi^{1/H})$. In fact, ϕ^0, $\phi^{1/H}$ do not depend on w. Then, we see from $\phi_1^0 = a'$ that

$$\delta_{a'}(\tilde{y}_1^\varepsilon) = \delta_0\Big(\varepsilon \cdot \frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\Big) = \varepsilon^{-n}\delta_0\big(\phi_1^1 + \varepsilon^{(1/H)-1}\phi_1^{1/H} + \varepsilon^{-1}R_1^{2,\varepsilon}\big).$$

Since $(\tilde{y}_1^\varepsilon - a')/\varepsilon = \phi_1^1 + \varepsilon^{(1/H)-1}\phi_1^{1/H} + \varepsilon^{-1}R_1^{2,\varepsilon}$ is uniformly non-degenerate in ε in the sense of Malliavin under (A1) and indexed by Λ_2, its composition with the Dirac measure δ_0 is well-defined and admits a Taylor-like expansion with the index set Λ_3.

Next we consider the other factor. We will show that there exists $\bar{\nu} \in \mathbf{R}^n$ such that $\langle \bar{\gamma}, w\rangle = \langle \bar{\nu}, \phi_1^1\rangle$, where the right hand side is the inner product of $\mathbf{R}^n$. Under the condition that $\phi^1 + \varepsilon^{(1/H)-1}\phi^{1/H} + \varepsilon^{-1}R^{2,\varepsilon} = 0$, we have

$$\exp\Big(-\frac{1}{\varepsilon}\langle \bar{\gamma}, w\rangle\Big) = \exp\Big(\frac{\langle \bar{\nu}, \phi_1^{1/H}\rangle}{\varepsilon^{2-1/H}}\Big) \cdot \exp\Big(\frac{\langle \bar{\nu}, R^{2,\varepsilon}\rangle}{\varepsilon^2}\Big).$$

It is obvious that the index set for $R^{2,\varepsilon}/\varepsilon^2$ is Λ_2', which implies that the index set for $\exp(\langle \bar{\nu}, R^{2,\varepsilon}/\varepsilon^2\rangle)$ is Λ_3'. From this heuristic explanation, we see that $p(\varepsilon^{1/H}, a, a')$ admits an asymptotic expansion and why $\Lambda_4 = \Lambda_3 + \Lambda_3'$ appears as the index set of the asymptotics. By setting $\varepsilon = t^H$, we have the desired short time expansion.

When we try to make the above argument rigorous, the most difficult part is to prove integrability of various Wiener functionals of exponential-type. This is highly non-trivial and we will prove a few lemmas for that purpose in Subsection 8.2. Assumption (A3) is actually a sufficient condition for those lemmas to hold.

3. Basic properties of Young ODE and L^q-integrability of Jacobian process.

In this section we recall the basic properties of a Young ODE and its Jacobian process (i.e., derivative process). There is no new result in this section. These facts are scattered across many literatures and it is not so easy to find a suitable one. (In this sense, Lejay [**11**] may be useful.) Here, we summarize some results, in particular, L^q-integrability of the Jacobian process driven by fBm with Hurst parameter $H > 1/2$ for later use. (Zähle [**22**] generalized Young integral and ODE by using fractional calculus, but we do not use it in this paper.)

We always assume that $1/2 < \alpha \le 1$ and the time interval is $[0,1]$. Let $C^{\alpha-hld}([0,1];\mathbf{R}^d)$ be the spaces of $\mathbf{R}^d$-valued α-Hölder continuous paths. The Banach norms are defined by

$$\|x\|_{\alpha-hld} = |x_0| + \sup_{0\le s<t\le 1} \frac{|x_t - x_s|}{(t-s)^\alpha},$$

The closed subspaces of paths that starts at the origin is denoted by $C_0^{\alpha-hld}([0,1];\mathbf{R}^d)$.

Let $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n,d)$ and $b : \mathbf{R}^n \to \mathbf{R}^n$ be sufficiently regular. Consider the following ODE in the Young sense;

$$dy_t = \sigma(y_t)dx_t + b(y_t)dt \quad \text{with} \quad y_0 = a. \tag{3.1}$$

Here, $x \in C_0^{\alpha-hld}([0,1];\mathbf{R}^d)$ and $a \in \mathbf{R}^n$ is the initial value. Let $V_i : \mathbf{R}^n \to \mathbf{R}^n$ be the ith column vector of σ $(1 \le i \le d)$ and set $V_0 = b$. Then, ODE (3.1) can be rewritten equivalently as follows;

$$dy_t = \sum_{i=1}^{d} V_i(y_t)dx_t^i + V_0(y_t)dt \quad \text{with} \quad y_0 = a. \tag{3.2}$$

Some researchers prefer this style. In this paper we will use both (3.1) and (3.2).

Assume σ and b are C_b^2, that is, $\max_{0\le i\le 2}(\|\nabla^i\sigma\|_\infty + \|\nabla^i b\|_\infty) < \infty$, where $\|\cdot\|_\infty$ stands for the sup-norm. Then the above ODE has a unique solution for any given x and a in α-Hölder setting. Moreover, the map

$$C_0^{\alpha-hld}([0,1];\mathbf{R}^d) \times \mathbf{R}^n \ni (x,a) \mapsto y \in C^{\alpha-hld}([0,1];\mathbf{R}^n) \tag{3.3}$$

is locally Lipschitz continuous (i.e., Lipschitz continuous on any bounded set). We will sometimes write $y = I(x,\lambda)$, where $\lambda_t = t$. (In this paper a is fixed.)

Now we discuss the Jacobian process (i.e., the derivative process) J of the ODE (3.1), or equivalently (3.2). J_t is a (formal) derivative of the solution flow $a \mapsto y_t = y_t(a)$ of the Young ODE (3.1).

For $v \in \mathbf{R}^n$, we denote the directional derivative along v by $\nabla_v\sigma(y) = \nabla\sigma(y)\langle v, \cdot\rangle$, etc. So, $\nabla\sigma$ takes its values in $L^{(2)}(\mathbf{R}^n,\mathbf{R}^d;\mathbf{R}^n) = (\mathbf{R}^n)^*\otimes(\mathbf{R}^d)^*\otimes\mathbf{R}^n$, which is equipped with the usual Hilbert–Schmidt norm. Notations such as $\nabla^i V_j$, $\nabla^2\sigma = \nabla\nabla\sigma$, $\nabla^2 b$, etc. should be understood in a similar way.

The Jacobian process J takes its values in $\mathrm{Mat}(n,n) = L(\mathbf{R}^n,\mathbf{R}^n)$ and satisfies

$$dJ_t = \nabla\sigma(y_t)\langle J_t, dx_t\rangle + \nabla b(y_t)\langle J_t\rangle dt \quad \text{with} \quad J_0 = \mathrm{Id}_n. \tag{3.4}$$

More precisely, by setting $M_t = \int_0^t \{\nabla\sigma(y_s)\langle\,\cdot\,, dx_s\rangle + \nabla b(y_s)\langle\,\cdot\,\rangle ds\}$, we may rewrite this equation as follows;

$$dJ_t = dM_t \cdot J_t \quad \text{with} \quad J_0 = \mathrm{Id}_n. \tag{3.5}$$

The dot on the right hand denotes the matrix multiplication. When we need to specify the driving path, we will write $J(x,\lambda)$, where $\lambda_t = t$. The equivalent equation for J that corresponds to (3.2) is as follows;

$$dJ_t = \sum_{i=1}^{d} \nabla V_i(y_t)\langle J_t\rangle dx_t^i + \nabla V_0(y_t)\langle J_t\rangle dt \quad \text{with} \quad J_0 = \mathrm{Id}_n. \tag{3.6}$$

Assume for safety that σ and b are C_b^3. It is known that the system of Young ODEs (3.1) and (3.4) has a unique solution (y, J) for given $x \in C_0^{\alpha-hld}([0,1];\mathbf{R}^d)$ and $a \in \mathbf{R}^n$ in α-Hölder setting and local Lipschitz continuity of $(x,a) \mapsto (y,J)$ also holds in this case.

Now let us consider the moment estimate for Hölder norms of J and J^{-1}, when the driving path x is the d-dimensional fBm $w = (w_t)_{0\le t\le 1}$ with Hurst parameter $H \in (1/2, 1)$. Take any $\alpha \in (1/2, H)$. Then, almost surely, $\|w\|_{\alpha-hld} < \infty$. (By the way, $\|w\|_{1/H-var} = \infty$, a.s. See [**7**], [**19**]. Hence, $\|w\|_{H-hld} = \infty$, a.s.)

The differential equations are given as follows;

$$dy_t = \sigma(y_t)dw_t + b(y_t)dt \text{ with } y_0 = a \quad \text{and} \quad dJ_t = dM_t \cdot J_t \text{ with } J_0 = \mathrm{Id}_n, \tag{3.7}$$

where $M_t = \int_0^t \{\nabla\sigma(y_s)\langle\,\cdot\,, dw_s\rangle + \nabla b(y_s)\langle\,\cdot\,\rangle ds\}$. For simplicity we call them SDEs, though they are just deterministic Young ODEs driven by a random input w (and λ).

Proposition 3.1. *Let $1/2 < \alpha < H$ and assume that the coefficients σ and b are C_b^3. Let J be as in* (3.7) *above. Then, $\|J\|_{\alpha-hld}$ and $\|J^{-1}\|_{\alpha-hld}$ have moments of all order, i.e., $\|J\|_{\alpha-hld}, \|J^{-1}\|_{\alpha-hld} \in \bigcap_{1\le q<\infty} L^q$.*

Proof. This is already known. Here, we give a sketch of proof only.

Since (3.4) is linear, the solution can be written explicitly as follows.

$$J_t = \left(\mathrm{Id}_n + \sum_{k=1}^{\infty} M_{s,t}^{[k]}\right) J_s \quad (0 \le s \le t \le 1), \tag{3.8}$$

where

$$M_{s,t}^{[k]} = \int_{s\le t_1\le\cdots\le t_k\le t} dM_{t_k}\cdots dM_{t_2}dM_{t_1}. \tag{3.9}$$

We can apply the same argument as in the proof of Lyons' extension theorem (p. 35, [**14**]) to obtain

$$\|J\|_{\alpha-hld} \le 1 + c'\big(1 + \|w\|_{\alpha-hld}^{1/\alpha}\big)\exp\big(c\|w\|_{\alpha-hld}^{1/\alpha}\big). \tag{3.10}$$

Here, positive constants c, c' depend only on α, σ, b. Since $1/\alpha < 2$, we can apply Fernique's square exponential integrability theorem for Gaussian measures.

J^{-1} has a series expansion similar to (3.8)–(3.9) and can be dealt with in the same way.

It is also possible to prove Proposition 3.1 by using Hu and Nualart's result on integrability of $\sup_{0\le t\le 1}|J_t|$ in [**8**] plus a cutoff argument. $\square$

Remark 3.2. This kind integrability problem for Jacobian process becomes very difficult when $H < 1/2$. Cass, Litterer, and Lyons [**5**] recently proved it in rough path setting for Gaussian rough path including fractional Brownian rough path with $1/4 < H \le 1/2$.

4. Preliminaries from Watanabe's asymptotic theory of generalized Wiener functionals.

We recall Watanabe's theory of generalized Wiener functionals in Malliavin calculus. Most of the contents and the notations in this section are borrowed from [**21**] or Sections 5.8–5.10, Ikeda and Watanabe [**9**] with trivial modifications. Shigekawa [**20**] and Nualart [**16**] are also good textbooks of Malliavin calculus and we will sometimes refer to them. There is no new result in this section.

Let $(W, \mathcal{H}, \mu)$ be an abstract Wiener space. (The results in [**21**] or Sections 5.8–5.10, [**9**] also holds on any abstract Wiener space.) The following are of particular importance in this paper:

(a) Basics of Sobolev spaces $\mathbf{D}_{q,r}(\mathcal{K})$ of $\mathcal{K}$-valued (generalized) Wiener functionals, where $q \in (1,\infty)$, $r \in \mathbf{R}$, and $\mathcal{K}$ is a real separable Hilbert space. As usual, we will use the spaces $\mathbf{D}_\infty(\mathcal{K})$, $\tilde{\mathbf{D}}_\infty(\mathcal{K})$ of test functions and the spaces $\mathbf{D}_{-\infty}(\mathcal{K})$, $\tilde{\mathbf{D}}_{-\infty}(\mathcal{K})$ of generalized Wiener functionals (i.e., Watanabe distributions) as in [**9**].
(b) Meyer's equivalence of Sobolev norms. (Theorem 8.4, [**9**]. A stronger version can be found in Theorem 4.6, [**20**])
(c) Pullback $T \circ F$ of tempered Schwartz distribution $T \in \mathcal{S}'(\mathbf{R}^n)$ on $\mathbf{R}^n$ by a non-degenerate Wiener functional $F \in \mathbf{D}_\infty(\mathbf{R}^n)$. (see Sections 5.9, [**9**].)
(d) A generalized version of integration by parts formula in the sense of Malliavin calculus for Watanabe distribution. (p. 7, [**21**] or p. 377, [**9**])

Now we consider a family of Wiener functionals indexed by a small parameter $\varepsilon \in (0,1]$. When the index set of asymptotics is $\mathbf{N}$, it is explained in Sections 5.9, [**9**]. This is just a slight generalization of it.

Consider a family of $\mathcal{K}$-valued Wiener functionals $\{F(\varepsilon, w)\}_{0<\varepsilon\le 1}$ and assume $F(\varepsilon,\cdot\,) \in \mathbf{D}_\infty(\mathcal{K})$ for each ε. We say $F(\varepsilon,\cdot\,) = O(\varepsilon^\kappa)$ in $\mathbf{D}_{q,k}(\mathcal{K})$, $\kappa \in \mathbf{R}$, as $\varepsilon \searrow 0$, if $\|F(\varepsilon,\cdot\,)\|_{q,k} = O(\varepsilon^\kappa)$. We say $F(\varepsilon,\cdot\,) = O(\varepsilon^\kappa)$ in $\mathbf{D}_\infty(\mathcal{K})$ as $\varepsilon \searrow 0$, if $F(\varepsilon,\cdot\,) = O(\varepsilon^\kappa)$ in $\mathbf{D}_{p,k}(\mathcal{K})$ for any $1<q<\infty$ and $k \in \mathbf{N}$.

Let $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots \nearrow \infty$ and $f_0, f_{\kappa_1}, f_{\kappa_2}, \ldots \in \mathbf{D}_\infty(\mathcal{K})$. We write

$$F(\varepsilon,\cdot\,) \sim f_0 + \varepsilon^{\kappa_1} f_{\kappa_1} + \varepsilon^{\kappa_2} f_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty(\mathcal{K}) \text{ as } \varepsilon \searrow 0,$$

if, for any $m \in \mathbf{N}$, it holds that

$$F(\varepsilon,\cdot\,) - (f_0 + \varepsilon^{\kappa_1} f_{\kappa_1} + \cdots + \varepsilon^{\kappa_m} f_{\kappa_m}) = O(\varepsilon^{\kappa_{m+1}}) \quad \text{in } \mathbf{D}_\infty(\mathcal{K}) \text{ as } \varepsilon \searrow 0.$$

In a similar way, we can define asymptotic expansions in $\mathbf{D}_{-\infty}(\mathcal{K})$, $\tilde{\mathbf{D}}_\infty(\mathcal{K})$, $\tilde{\mathbf{D}}_{-\infty}(\mathcal{K})$ for a general index set, too, but we omit them.

We recall basic facts for such asymptotic expansions in the Sobolev spaces. Let $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots \nearrow \infty$ be as above. In Proposition 4.1 below, $0 = \nu_0 < \nu_1 < \nu_2 < \cdots \nearrow \infty$ are all the elements of $\{\kappa_i + \kappa_j \mid i,j \in \mathbf{N}\}$ in increasing order. The fundamental case $\kappa_j = j$ is treated in Proposition 9.3, Section 5.9, [**9**]. The following is a straight forward modification of it.

Proposition 4.1. (i) *Suppose that* $F(\varepsilon,\cdot\,) \in \mathbf{D}_\infty(\mathcal{K})$ *admits an expansion such* *as*

$$F(\varepsilon,\,\cdot\,) \sim f_0 + \varepsilon^{\kappa_1} f_{\kappa_1} + \varepsilon^{\kappa_2} f_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty(\mathcal{K}) \text{ as } \varepsilon \searrow 0,$$

with $f_{\kappa_j} \in \mathbf{D}_\infty(\mathcal{K})$ *for all* $j \in \mathbf{N}$. *Suppose also that* $G(\varepsilon,\,\cdot\,) \in \mathbf{D}_\infty$ (*or* $\tilde{\mathbf{D}}_\infty$) *admits an expansion such as*

$$G(\varepsilon,\,\cdot\,) \sim g_0 + \varepsilon^{\kappa_1} g_{\kappa_1} + \varepsilon^{\kappa_2} g_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty \quad (\text{or resp. } \tilde{\mathbf{D}}_\infty) \text{ as } \varepsilon \searrow 0,$$

with $g_{\kappa_j} \in \mathbf{D}_\infty$ (*or resp.* $\tilde{\mathbf{D}}_\infty$) *for* $j \in \mathbf{N}$. *Then,* $H(\varepsilon, w) = F(\varepsilon, w)G(\varepsilon, w)$ *satisfies that*

$$H(\varepsilon,\,\cdot\,) \sim h_0 + \varepsilon^{\nu_1} h_{\nu_1} + \varepsilon^{\nu_2} h_{\nu_2} + \cdots \quad \text{in } \mathbf{D}_\infty(\mathcal{K}) \quad (\text{or resp. } \tilde{\mathbf{D}}_\infty(\mathcal{K})) \text{ as } \varepsilon \searrow 0,$$

where $h_{\nu_n} \in \mathbf{D}_\infty(\mathcal{K})$ (*or resp.* $\tilde{\mathbf{D}}_\infty(\mathcal{K})$) *are given by the following formal multiplication;*

$$h_{\nu_n} = \sum_{(i,j);\kappa_i+\kappa_j=\nu_n} g_{\kappa_i} f_{\kappa_j}.$$

(ii) *Suppose that* $G(\varepsilon,\,\cdot\,) \in \mathbf{D}_\infty$ (*or* $\tilde{\mathbf{D}}_\infty$) *admits an expansion such as*

$$G(\varepsilon,\,\cdot\,) \sim g_0 + \varepsilon^{\kappa_1} g_{\kappa_1} + \varepsilon^{\kappa_2} g_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty \quad (\text{or resp. } \tilde{\mathbf{D}}_\infty) \text{ as } \varepsilon \searrow 0,$$

with $g_{\kappa_j} \in \mathbf{D}_\infty$ (*or resp.* $\tilde{\mathbf{D}}_\infty$) *for all* $j \in \mathbf{N}$. *Suppose also that* $\Phi(\varepsilon,\,\cdot\,) \in \tilde{\mathbf{D}}_{-\infty}(\mathcal{K})$ *admits an expansion such as*

$$\Phi(\varepsilon,\,\cdot\,) \sim \phi_0 + \varepsilon^{\kappa_1} \phi_{\kappa_1} + \varepsilon^{\kappa_2} \phi_{\kappa_2} + \cdots \quad \text{in } \tilde{\mathbf{D}}_{-\infty}(\mathcal{K}) \text{ as } \varepsilon \searrow 0,$$

with $\phi_{\kappa_j} \in \tilde{\mathbf{D}}_{-\infty}(\mathcal{K})$ *for all* $j \in \mathbf{N}$). *Then,* $\Psi(\varepsilon, w) = G(\varepsilon, w)\Phi(\varepsilon, w)$ *satisfies that*

$$\Psi(\varepsilon,\,\cdot\,) \sim \psi_0 + \varepsilon^{\nu_1} \psi_{\nu_1} + \varepsilon^{\nu_2} \psi_{\nu_2} + \cdots \\ \text{in } \tilde{\mathbf{D}}_{-\infty}(\mathcal{K}) \quad (\text{or resp. } \mathbf{D}_{-\infty}(\mathcal{K})) \text{ as } \varepsilon \searrow 0, \tag{4.1}$$

where $\psi_{\nu_n} \in \tilde{\mathbf{D}}_{-\infty}(\mathcal{K})$ (*or resp.* $\mathbf{D}_{-\infty}(\mathcal{K})$) *are given by the following formal multiplication*;

$$\psi_{\nu_n} = \sum_{(i,j);\kappa_i+\kappa_j=\kappa_n} g_{\kappa_i} \phi_{\kappa_j}. \tag{4.2}$$

(iii) *Suppose that* $G(\varepsilon,\,\cdot\,) \in \mathbf{D}_\infty$ *admits an expansion such as*

$$G(\varepsilon,\,\cdot\,) \sim g_0 + \varepsilon^{\kappa_1} g_{\kappa_1} + \varepsilon^{\kappa_2} g_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty \text{ as } \varepsilon \searrow 0,$$

with $g_{\kappa_j} \in \mathbf{D}_\infty$ for all $j \in \mathbf{N}$. Suppose also that $\Phi(\varepsilon, \cdot\,) \in \mathbf{D}_{-\infty}(\mathcal{K})$ admits an expansion such as

$$\Phi(\varepsilon, \cdot\,) \sim \phi_0 + \varepsilon^{\kappa_1}\phi_{\kappa_1} + \varepsilon^{\kappa_2}\phi_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_{-\infty}(\mathcal{K}) \text{ as } \varepsilon \searrow 0,$$

with $\phi_{\kappa_j} \in \mathbf{D}_{-\infty}(\mathcal{K})$ for all $j \in \mathbf{N}$. Then, (4.1) *and* (4.2) *hold in $\mathbf{D}_{-\infty}(\mathcal{K})$.*

Remark 4.2. In (i) of the above Proposition, the index sets $\{\kappa_j\}_{j=0,1,2,\dots}$ for the asymptotic expansions for $F(\varepsilon, \cdot\,)$ and $G(\varepsilon, \cdot\,)$ are the same. However, these index sets for F and G may differ, because the union of the two index sets can be regarded as a new index set. Similar remarks hold for (ii) and (iii), too.

Next we consider asymptotic expansions for the pullback. Let $F(\varepsilon, \cdot\,) \in \mathbf{D}_\infty(\mathbf{R}^n)$ for $0 < \varepsilon \le 1$. We say F is uniformly non-degenerate in the sense of Malliavin if

$$\sup_{0<\varepsilon\le 1} \left\| \det\bigl(\langle DF^i(\varepsilon, \cdot\,), DF^j(\varepsilon, \cdot\,)\rangle_{\mathcal{H}}\bigr)^{-1}_{1\le i,j\le n} \right\|_q < \infty \quad \text{for all } 1 < q < \infty.$$

Here, D stands for the $\mathcal{H}$-derivative.

The following is a straight forward modification of Theorem 9.4, [**9**]. In this theorem, $0 = \nu_0 < \nu_1 < \nu_2 < \cdots \nearrow \infty$ are all the elements of

$$\{\kappa_{j_1} + \cdots + \kappa_{j_n} \mid n = 1, 2, \dots, \text{ and } j_1, \dots, j_n \in \mathbf{N}\}$$

in increasing order.

Theorem 4.3. *Let $F(\varepsilon, \cdot\,) \in \mathbf{D}_\infty(\mathbf{R}^n)$ $(0 < \varepsilon \le 1)$ satisfy the following;*

$$F(\varepsilon, \cdot\,) \sim f_0 + \varepsilon^{\kappa_1} f_{\kappa_1} + \varepsilon^{\kappa_2} f_{\kappa_2} + \cdots \quad \text{in } \mathbf{D}_\infty(\mathbf{R}^n) \text{ as } \varepsilon \searrow 0,$$

with $f_{\kappa_j} \in \mathbf{D}_\infty(\mathbf{R}^n)$ for all $j \in \mathbf{N}$. We also assume that F is uniformly non-degenerate in the sense of Malliavin. Then, for any $T \in \mathcal{S}'(\mathbf{R}^n)$, $\Phi(\varepsilon, w) := T \circ F(\varepsilon, w)$ has the following asymptotic expansion;

$$\Phi(\varepsilon, \cdot\,) \sim \phi_0 + \varepsilon^{\kappa_1}\phi_{\kappa_1} + \varepsilon^{\kappa_2}\phi_{\kappa_2} + \cdots \quad \text{in } \tilde{\mathbf{D}}_{-\infty} \text{ as } \varepsilon \searrow 0,$$

where $\phi_{\kappa_j} \in \tilde{\mathbf{D}}_{-\infty}$ is determined by a formal Taylor expansion as follows;

$$\Phi(\varepsilon, \cdot\,) = \sum_\alpha \frac{1}{\alpha!}(\partial^\alpha T)(f_0)[\varepsilon^{\kappa_1} f_{\kappa_1} + \varepsilon^{\kappa_2} f_{\kappa_2} + \cdots]^\alpha = \phi_0 + \varepsilon^{\nu_1}\phi_{\nu_1} + \varepsilon^{\nu_2}\phi_{\nu_2} + \cdots,$$

where the (formal) summation is over all multi-indexes $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$. (We set $\partial^\alpha = \prod_j (\partial/\partial x^j)^{\alpha_j}$ and $b^\alpha = \prod_j b_j^{\alpha_j}$ for $b = (b_1, \dots, b_n) \in \mathbf{R}^n$ as usual.) For instance, $\phi_0 = T(f_0)$ and $\phi_{\kappa_1} = \sum_{j=1}^n f^j_{\kappa_1} \cdot (\partial T/\partial x^j)(f_0)$ and so on.

Unlike in the usual stochastic analysis, almost every Wiener functional in this paper is continuous with respect to the topology of an abstract Wiener space, because we work in the framework of Young integration. Therefore, the following proposition will be very useful. For Banach spaces $\mathcal{X}_1, \ldots, \mathcal{X}_m, \mathcal{Y}$, $L^{(m)}(\mathcal{X}_1, \ldots, \mathcal{X}_m; \mathcal{Y})$ denotes the space of bounded m-multilinear maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ to $\mathcal{Y}$.

PROPOSITION 4.4. *Let $(W, \mathcal{H}, \mu)$ be an abstract Wiener space. Then, we have the following bounded inclusions;*

$$L^{(m)}(\underbrace{W, \ldots, W}_{m}; \mathbf{R}) \hookrightarrow L^{(m)}(\underbrace{W, \ldots, W}_{m-1}, \mathcal{H}; \mathbf{R}) \hookrightarrow (\mathcal{H}^*)^{\otimes m}.$$

Here, the tensor product on the right hand side is Hilbert–Schmidt as usual.

PROOF. The left bounded inclusion is obvious. The right one is in p. 103, Kuo [**10**]. □

5. Some results on Malliavin calculus for the solution of Young ODE driven by fBm with $H > 1/2$.

In this section we discuss the solution of Young ODE driven by fBm with Hurst parameter $H \in (1/2, 1)$. We give moment estimates for the derivatives of the solution and prove uniform non-degeneracy of Malliavin covariance matrix of the solution.

Take $\alpha \in (1/2, H)$. We denote by $\mu = \mu^H$ the law of d-dimensional fBm starting at 0. This Gaussian measure is supported in $C_0^{\alpha-hld}([0,1]; \mathbf{R}^d)$. Cameron–Martin space is denoted by $\mathcal{H} = \mathcal{H}^H$. We set $\mathcal{W}$ to be the closure of $\mathcal{H}$ in $C_0^{\alpha-hld}([0,1]; \mathbf{R}^d)$. Then, $(\mathcal{W}, \mathcal{H}, \mu)$ becomes an abstract Wiener space. (Note that the separable Hilbert space $\mathcal{H}$ is not dense in $C_0^{\alpha-hld}([0,1]; \mathbf{R}^d)$, which is not separable.) We denote by $(w_t)_{0 \le t \le 1} = (w_t^H)_{0 \le t \le 1}$ the canonical realization of fBm.

From now on, we assume that $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ and $b : \mathbf{R}^n \to \mathbf{R}^n$ are C_b^∞. We recall Young SDE (2.1) driven by fBm (w_t) in the following form;

$$dy_t = \sigma(y_t) dw_t + b(y_t) dt \quad \text{with} \quad y_0 = a. \tag{5.1}$$

Then $y(w) = I(w, \lambda)$, where $\lambda_t = t$ and I is the Itô map corresponding to the coefficients $[\sigma; b] = [V_1, \ldots, V_d; V_0]$. I is everywhere-defined and continuous from $C_0^{\alpha-hld}([0,1]; \mathbf{R}^{d+1})$ to $C^{\alpha-hld}([0,1]; \mathbf{R}^d)$, as we have explained in Section 3.

Moreover, I is smooth in Fréchet sense (See Li and Lyons [**12**]) and, in particular, $y = I(\,\cdot\,, \lambda)$ is infinitely differentiable in $\mathcal{H}$-direction (see Nualart and Saussereau [**18**]). These are deterministic results. In the sense of Malliavin calculus, it is shown in Hu and Nualart [**8**] that $y_T : \mathcal{W} \to \mathbf{R}^n$ is $\mathbf{D}_\infty$ for any $T \in [0,1]$.

We can obtain an explicit form of the directional derivative $\xi_t^h := D_h y_t$ ($h \in \mathcal{H}$) by differentiating (5.1);

$$d\xi_t^h - \nabla\sigma(y_t)\langle \xi_t^h, dw_t \rangle - \nabla b(y_t)\langle \xi_t^h \rangle dt = \sigma(y_t) dh_t \quad \text{with} \quad \xi_0^h = 0, \tag{5.2}$$

or equivalently,

$$\xi^h_T = J(w,\lambda)_T \int_0^T J(w,\lambda)_t^{-1}\sigma(y_t)dh_t. \tag{5.3}$$

Note that all the integrations above are in the Young sense. An ODE for $J = J(w,\lambda)$ is given in (3.4). Let $h,k \in \mathcal{H}$. By differentiating the above ODE, we see that $\xi^{k,h}_t := D_k D_h y_t$ satisfies the following ODE;

$$\begin{aligned} &d\xi^{k,h}_t - \nabla\sigma(y_t)\langle \xi^{k,h}_t, dw_t\rangle - \nabla b(y_t)\langle\xi^{k,h}_t\rangle dt \\ &\quad = \nabla^2\sigma(y_t)\langle\xi^k_t,\xi^h_t,dw_t\rangle + \nabla\sigma(y_t)\langle\xi^h_t,dk_t\rangle + \nabla\sigma(y_t)\langle\xi^k_t,dh_t\rangle + \nabla^2 b(y_t)\langle\xi^k_t,\xi^h_t\rangle dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } \xi^{k,h}_0 = 0. \end{aligned} \tag{5.4}$$

Equivalently, we have

$$\begin{aligned} \xi^{k,h}_T = J(w,\lambda)_T \int_0^T J(w,\lambda)_t^{-1}\big\{&\nabla^2\sigma(y_t)\langle\xi^k_t,\xi^h_t,dw_t\rangle + \nabla\sigma(y_t)\langle\xi^h_t,dk_t\rangle \\ &+ \nabla\sigma(y_t)\langle\xi^k_t,dh_t\rangle + \nabla^2 b(y_t)\langle\xi^k_t,\xi^h_t\rangle dt\big\}. \end{aligned} \tag{5.5}$$

We can also obtain higher order directional derivatives in a similar way, but we omit them.

In a proof for the main theorem, we need to consider $\tilde{y}^\varepsilon(w) = I(\varepsilon w + \gamma, \varepsilon^{1/H}\lambda)$, where $\gamma \in \mathcal{H}$ is a fixed element and $\varepsilon \in (0,1]$. This process satisfies the following Young SDE;

$$d\tilde{y}^\varepsilon_t = \sigma(\tilde{y}^\varepsilon_t)\varepsilon dw_t + \sigma(\tilde{y}^\varepsilon_t)d\gamma_t + b(\tilde{y}^\varepsilon_t)\varepsilon^{1/H}dt \quad \text{with} \quad \tilde{y}^\varepsilon_0 = a. \tag{5.6}$$

When $\gamma = 0$, we write y^ε for $\tilde{y}^\varepsilon$. In that case, self-similarity of (w_t) implies that the two processes $(y_{\varepsilon^{1/H}t})_{0\le t\le 1}$ and $(y^\varepsilon_t)_{0\le t\le 1}$ have the same law.

In the next proposition we give estimates for the derivatives $D^k\tilde{y}^\varepsilon_T$. As we stated above, it is known that y_T (and hence $\tilde{y}^\varepsilon_T$) is $\mathbf{D}_\infty$. In that sense, this proposition is not new. But, the estimate in powers of ε in (5.7) may be new. Also, the proof is slightly different from the preceding papers, because Proposition 4.4 is used.

Proposition 5.1. *Take any $\gamma \in \mathcal{H}$ and fix it. Then, for any $q \in (1,\infty)$ and $k = 0,1,2,\ldots$, there exists a positive constant $C_{q,k}$ such that*

$$\mathbb{E}\big[\|D^k\tilde{y}^\varepsilon_T\|^q_{(\mathcal{H}^*)^{\otimes k}}\big]^{1/q} \le C_{q,k}\varepsilon^k \quad \textit{for all } \varepsilon \in (0,1] \textit{ and } T \in [0,1]. \tag{5.7}$$

Proof. In this proof, an unimportant positive constant C may change from line to line. First, consider the case $k = 0$. Since $\omega(s,t) = (\|w\|^p_{\alpha-hld} + \|\gamma\|^p_{\alpha-hld} + 1)(t-s)$ satisfies

$$|(\varepsilon w_t + \gamma_t) - (\varepsilon w_s + \gamma_s)| + \big|\varepsilon^{1/H}t - \varepsilon^{1/H}s\big| \le \omega(s,t)^{1/p}, \quad (0 \le s \le t \le 1,\ p = 1/\alpha),$$

we can use a well-known estimate for the solutions of Young ODEs to obtain that

$$|\tilde{y}^{\varepsilon}_T| \le \|\tilde{y}^{\varepsilon}\|_{\alpha-hld} \le |a| + C(1 + \|w\|^p_{\alpha-hld} + \|\gamma\|^p_{\alpha-hld}) \tag{5.8}$$

for some constant $C = C_K$. Fernique's theorem immediately implies (5.7) for $k = 0$.

Next let us consider the case $k = 1$. By slightly modifying (5.2)–(5.3), we can easily see that $\tilde{\xi}^{\varepsilon;h}_t := D_h \tilde{y}^{\varepsilon}_t$ satisfies the following (5.9)–(5.10);

$$d\tilde{\xi}^{\varepsilon;h}_t - \nabla\sigma(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;h}_t, d(\varepsilon w_t + \gamma_t)\rangle - \nabla b(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;h}_t\rangle \varepsilon^{1/H} dt = \sigma(\tilde{y}^{\varepsilon}_t)\varepsilon dh_t$$
$$\text{with} \quad \tilde{\xi}^{\varepsilon;h}_0 = 0, \tag{5.9}$$

or equivalently,

$$\tilde{\xi}^{\varepsilon;h}_T = \tilde{J}_T \int_0^T \tilde{J}^{-1}_t \sigma(\tilde{y}^{\varepsilon}_t)\varepsilon dh_t, \tag{5.10}$$

where $\tilde{J} = J(\varepsilon w + \gamma, \varepsilon^{1/H}\lambda)$. From this, we can easily see that

$$\begin{aligned} |\tilde{\xi}^{\varepsilon;h}_T| \le \|\tilde{\xi}^{\varepsilon;h}\|_{\alpha-hld} &\le C\varepsilon \|\tilde{J}\|_{\infty} \|\tilde{J}^{-1}_{\cdot}\sigma(\tilde{y}^{\varepsilon}_{\cdot})\|_{\alpha-hld} \|h\|_{\alpha-hld} \\ &\le C\varepsilon \|\tilde{J}\|_{\infty} \|\tilde{J}^{-1}\|_{\alpha-hld}(1 + \|\tilde{y}^{\varepsilon}\|_{\alpha-hld})\|h\|_{\mathcal{H}} \end{aligned} \tag{5.11}$$

and, hence,

$$\|D\tilde{y}^{\varepsilon}_T\|_{\mathcal{H}^*} \le C\varepsilon \|\tilde{J}\|_{\infty} \|\tilde{J}^{-1}\|_{\alpha-hld}(1 + \|\tilde{y}^{\varepsilon}\|_{\alpha-hld}).$$

By a slight modification of Proposition 3.1, L^q-norm of $\|\tilde{J}^{\pm 1}\|_{\alpha-hld}$ is finite and bounded in ε for any fixed $q \in (1, \infty)$. (Just replace w and λ in Proposition 3.1 by $\varepsilon w + \gamma$ and $\varepsilon^{1/H}\lambda$, respectively.) Hence, using Hölder's inequality, we obtain (5.7) for $k = 1$.

We prove the case $k = 2$. Set $D_k D_h \tilde{y}^{\varepsilon}_T = \tilde{\xi}^{\varepsilon;k,h}_T$ for simplicity. Then, in the same way as in (5.4)–(5.5), we have

$$\begin{aligned} \tilde{\xi}^{\varepsilon;k,h}_T = \tilde{J}_T \int_0^T \tilde{J}^{-1}_t \big\{ &\nabla^2\sigma(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;k}_t, \tilde{\xi}^{\varepsilon;h}_t, d(\varepsilon w_t + \gamma_t)\rangle + \nabla\sigma(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;h}_t, \varepsilon dk_t\rangle \\ &+ \nabla\sigma(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;k}_t, \varepsilon dh_t\rangle + \nabla^2 b(\tilde{y}^{\varepsilon}_t)\langle \tilde{\xi}^{\varepsilon;k}_t, \tilde{\xi}^{\varepsilon;h}_t\rangle dt \big\}. \end{aligned} \tag{5.12}$$

From this, we have

$$\begin{aligned} &\|\tilde{\xi}^{\varepsilon;k,h}\|_{\alpha-hld} \\ &\quad \le C\|\tilde{J}\|_{\infty}\|\tilde{J}^{-1}\|_{\alpha-hld}\big\{ \|\nabla^2\sigma(\tilde{y}^{\varepsilon}_{\cdot})\|_{\alpha-hld}\|\tilde{\xi}^{\varepsilon;k}_T\|_{\alpha-hld}\|\tilde{\xi}^{\varepsilon;h}_T\|_{\alpha-hld}\|\varepsilon w + \gamma\|_{\alpha-hld} \\ &\qquad + \|\nabla\sigma(\tilde{y}^{\varepsilon}_{\cdot})\|_{\alpha-hld}(\|\tilde{\xi}^{\varepsilon;k}\|_{\alpha-hld}\|\varepsilon h\|_{\alpha-hld} + \|\tilde{\xi}^{\varepsilon;h}\|_{\alpha-hld}\|\varepsilon k\|_{\alpha-hld}) \\ &\qquad + \|\nabla^2\sigma(\tilde{y}^{\varepsilon}_{\cdot})\|_{\alpha-hld}\|\tilde{\xi}^{\varepsilon;k}\|_{\alpha-hld}\|\tilde{\xi}^{\varepsilon;h}\|_{\alpha-hld}\big\} \end{aligned}$$

$$\leq C\varepsilon^2 \|\tilde{J}\|_\infty^3 \|\tilde{J}^{-1}\|_{\alpha-hld}^3 (1 + \|w\|_{\alpha-hld}^p + \|\gamma\|_{\alpha-hld}^p)^4 \|h\|_{\alpha-hld} \|k\|_{\alpha-hld}. \tag{5.13}$$

Here, we used (5.8) and (5.11) From Proposition 4.4, we see

$$\|\tilde{\xi}_T^{\varepsilon;k,h}\|_{\mathcal{H}^*\otimes\mathcal{H}^*} \leq C\varepsilon^2 \|\tilde{J}\|_\infty^3 \|\tilde{J}^{-1}\|_{\alpha-hld}^3 (1 + \|w\|_{\alpha-hld}^p + \|\gamma\|_{\alpha-hld}^p)^4. \tag{5.14}$$

Using the moment estimate for $\|\tilde{J}^{\pm 1}\|_{\alpha-hld}$ again, we show (5.7) for $k = 2$.

Finally, we briefly explain the higher order cases ($k \geq 3$). We can show it in a similar way by induction. (We assume α-Hölder norm of $D_{h_1,\ldots,h_m}^m \tilde{y}^\varepsilon$ is dominated by $\prod_{j=1}^m \|h_j\|_{\alpha-hld} \times O(\varepsilon^m)$ in any L^q-sense for $m \leq k-1$ (as in (5.13) for $m = 2$) and then we will prove that α-Hölder norm of $D_{h_1,\ldots,h_k}^k \tilde{y}^\varepsilon$ also does.)

For simplicity, set $\tilde{\eta}_t^\varepsilon = D_{h_1,\ldots,h_k}^k \tilde{y}_t^\varepsilon$. It satisfies the following simple linear ODE similar to (5.9);

$$d\tilde{\eta}_t^\varepsilon - \nabla\sigma(\tilde{y}_t^\varepsilon)\langle \tilde{\eta}_t^\varepsilon, d(\varepsilon w_t + \gamma_t)\rangle - \nabla b(\tilde{y}_t^\varepsilon)\langle \tilde{\eta}_t^\varepsilon\rangle \varepsilon^{1/H} dt = dG_t^\varepsilon \quad \text{with} \quad \tilde{\eta}_0^\varepsilon = 0.$$

Here, G^ε is of the form

$$G_t^\varepsilon = G^\varepsilon\big(\tilde{y}^\varepsilon, D_{h_{j_1}} \tilde{y}^\varepsilon, \ldots, D_{h_{j_1},\ldots,h_{j_{k-1}}}^{k-1} \tilde{y}^\varepsilon, w, \gamma, h_1, \ldots, h_k\big)_t$$

and is of order k in ε. Note that there is no derivative of order k on the right hand side. As in (5.12), we have $\tilde{\eta}_T^\varepsilon = \tilde{J}_T \int_0^T \tilde{J}_t^{-1} dG_t^\varepsilon$. Using this we can estimate α-Hölder norm of $\tilde{\eta}^\varepsilon$ for k in the same way as in (5.13). $\square$

Remark 5.2. We already have (i) Fréchet smoothness of $\tilde{y}_T^\varepsilon$ in the deterministic sense and (ii) L^q-estimates for derivatives as in this proposition. From these, we can easily verify that $\tilde{y}_T^\varepsilon \in \mathbf{D}_\infty$ as follows. (For simplicity of notations, we only consider the case $\gamma = 0, \varepsilon = 1$.) By using Taylor expansion, we have

$$\frac{y_T(w + rh) - y_T(w)}{r} - D_h y_T(w) = r \int_0^1 d\theta (1-\theta) D^2 y_T(w + r\theta h)\langle h, h\rangle$$

for all $w \in \mathcal{W}$, $h \in \mathcal{H}^H$, and $r \in \mathbf{R}$. Note that the derivative D on the both sides of the above equation is in the deterministic sense. By Proposition 5.1 and Cameron–Martin formula, the right hand side is $O(r)$ as $r \to 0$ in L^q-norm for any $q \in (1, \infty)$. This implies that $y_T \in \mathbf{D}_{q,1}$ for any $q \in (1, \infty)$ and the derivative Dy_T in (deterministic) Fréchet sense is also the derivative in the sense of Malliavin calculus. (See Proposition 4.21, [**20**] for instance.) The higher order derivatives can be dealt with in the same way.

Now we show that, under the ellipticity condition (A1) for σ (i.e., for $V_1, \ldots, V_d$), the Malliavin covariance matrix for

$$\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon} \tag{5.15}$$

is uniformly non-degenerate in the sense of Malliavin as $\varepsilon \searrow 0$. Here, we set $a' = \phi_1^0$ for the solution of the following ODE; $d\phi_t^0 = \sigma(\phi_t^0)d\gamma_t$ with $\phi_0^0 = a$.

(A1): The set of vectors $\{V_1(a), \dots, V_d(a)\}$ linearly spans $\mathbf{R}^n$.

Nualart and Saussereau [**18**] showed non-degeneracy of Malliavin covariance matrix of y_T under (A1). Baudoin and Hairer [**1**], proved non-degeneracy under Hörmander's hypoellipticity condition for vector fields $\{V_1, \dots, V_d; V_0\}$.

In the next proposition, we will prove uniform non-degeneracy of (5.15) under (A1) by slightly modifying Baudoin–Hairer's argument. (The special case $\gamma = 0$ has already appeared in Baudoin and Ouyang [**3**].)

Proposition 5.3. *Let $\tilde{y}^\varepsilon = (\tilde{y}^{\varepsilon,1}, \dots, \tilde{y}^{\varepsilon,n})$ be the solution of* (5.6) *and assume* (A1). *Then, $(\tilde{y}_1^\varepsilon - a')/\varepsilon$ is uniformly non-degenerate in the sense of Malliavin as $\varepsilon \searrow 0$.*

Proof. Let $y = (y_t)$ be the solution of (5.1). In p. 388–389, [**1**], an explicit form of the Malliavin covariance matrix for y_1 is given. By replacing the vector fields $[\sigma; b] = [V_1, \dots, V_d; V_0]$ with $[\varepsilon\sigma; \varepsilon^{1/H} b] = [\varepsilon V_1, \dots, \varepsilon V_d; \varepsilon^{1/H} V_0]$, we can easily see that

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\big(\langle Dy_1^{\varepsilon;i}, Dy_1^{\varepsilon;j}\rangle_{\mathcal{H}}\big)_{1\le i,j\le n} &= H(2H-1)J(\varepsilon w, \varepsilon^{1/H}\lambda)_1 \\
&\quad \times \int_0^1\int_0^1 J(\varepsilon w, \varepsilon^{1/H}\lambda)_u^{-1}\sigma(y_u^\varepsilon)\sigma(y_v^\varepsilon)^* \\
&\quad \times J(\varepsilon w, \varepsilon^{1/H}\lambda)_v^{-1,*}|u-v|^{2H-2}dudv J(\varepsilon w, \varepsilon^{1/H}\lambda)_1^*.
\end{aligned}
$$

Here, $\lambda_t = t$ and A^* denotes the transposed matrix of A. By shifting $w \mapsto w + (\gamma/\varepsilon)$, we have

$$
\frac{1}{\varepsilon^2}\big(\langle D\tilde{y}_1^{\varepsilon;i}, D\tilde{y}_1^{\varepsilon;j}\rangle_{\mathcal{H}}\big)_{1\le i,j\le n} = H(2H-1)\tilde{J}_1 C \tilde{J}_1^*, \tag{5.16}
$$

where $\tilde{J}_t = J(\varepsilon w + \gamma, \varepsilon^{1/H}\lambda)_t$ as before and we set

$$
C = \int_0^1\int_0^1 \tilde{J}_s^{-1}\sigma(\tilde{y}_s^\varepsilon)\sigma(\tilde{y}_t^\varepsilon)^*\tilde{J}_t^{-1,*}|s-t|^{2H-2}dsdt.
$$

Since $\sup_{0<\varepsilon\le 1}\|\tilde{J}_1^{\pm 1}\|_q < \infty$ for any $q \in (1,\infty)$, it is sufficient to prove

$$
\sup_{0<\varepsilon\le 1}\||\det C|^{-1}\|_q < \infty \quad \text{for any } 1 < q < \infty. \tag{5.17}
$$

We will follow the argument in pp. 340–387, [**1**]. In order to show (5.17) above, it is sufficient to prove that, for any $1 < q < \infty$, there exists $\rho_0(q)$, which is independent of ε and satisfies that,

$$
\sup_{\boldsymbol{a}\in\mathbf{R}^n, \|\boldsymbol{a}\|=1} \mu\big(\langle \boldsymbol{a}, C\boldsymbol{a}\rangle \le \rho\big) \le \rho^q \quad \text{for any } \rho \in (0, \rho_0(q)) \text{ and } \varepsilon \in (0,1]. \tag{5.18}
$$

(For a proof, see Lemma 2.3.1, Nualart [**16**]). As in [**1**],

$$\langle \boldsymbol{a}, C\boldsymbol{a}\rangle = \sum_{j=1}^{d}\int_0^1\int_0^1 \langle \boldsymbol{a}, \tilde{J}_s^{-1}V_j(\tilde{y}_s^\varepsilon)\rangle\langle \boldsymbol{a}, \tilde{J}_t^{-1}V_j(\tilde{y}_t^\varepsilon)\rangle |s-t|^{2H-2}dsdt$$

$$= \sum_{j=1}^{d}\big\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{\hat{\mathcal{H}}}^2. \tag{5.19}$$

By a Norris-type lemma (Corollary 4.5, [**1**]), there exists $0<\beta<1/2$ such that for any $r<H-(1/2)$ and $0<\rho\le 1$, the following inequalities hold;

$$\begin{aligned}
&\mu\big(\langle \boldsymbol{a}, C\boldsymbol{a}\rangle \le \rho\big)\\
&\quad\le \min_{1\le j\le d}\mu\big(\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{\hat{\mathcal{H}}}\le \rho^{1/2}\big)\\
&\quad\le \min_{1\le j\le d}\Big[\mu\big(\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{L^\infty}<\rho^{\beta/2}\big)+\mu\big(\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{r-hld}>\rho^{-\beta/2}\big)\Big]\\
&\quad\le \min_{1\le j\le d}\Big[\mu\big(|\langle \boldsymbol{a}, V_j(a)\rangle|<\rho^{\beta/2}\big)+\mu\big(\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{\alpha-hld}>\rho^{-\beta/2}\big)\Big].
\end{aligned} \tag{5.20}$$

Here, in the last inequality, we evaluated at $t=0$ and used $r<\alpha$. Note that the set in the first term on the right hand side is already independent of ε and non-random (i.e., either $\emptyset$ or the whole set $\mathcal{W}$).

Recall that, for any q, $E[\|\tilde{J}^{-1}\|_{\alpha-hld}^q+\|\tilde{y}^\varepsilon\|_{\alpha-hld}^q]\le c_1$ for some constant $c_1=c_1(q)$ which is independent of ε. Then, using Chebyshev's inequality, we have

$$\mu\big(\|\langle \boldsymbol{a}, \tilde{J}_\cdot^{-1}V_j(\tilde{y}_\cdot^\varepsilon)\rangle\big\|_{\alpha-hld}>\rho^{-\beta/2}\big)\le c_2\rho^q$$

for some constant $c_2=c_2(q)$ which is independent of ε.

Let us consider the first term on the right hand side of (5.20). By (A1), there exists $c'>0$ such that $\sigma(a)\sigma(a)^*\ge c'\mathrm{Id}_n$ in the form sense. We have

$$n\max_{1\le j\le d}|\langle \boldsymbol{a}, V_j(a)\rangle|^2\ge\sum_{1\le j\le d}|\langle \boldsymbol{a}, V_j(a)\rangle|^2=\langle \boldsymbol{a}, \sigma(a)\sigma(a)^*\boldsymbol{a}\rangle\ge c'>0.$$

Hence, if $\rho^{\beta/2}\le\sqrt{c'/n}$, then $\min_{1\le j\le d}\mu\big(|\langle \boldsymbol{a}, V_j(a)\rangle|<\rho^{\beta/2}\big)=0$ and

$$\mu\big(\langle \boldsymbol{a}, C\boldsymbol{a}\rangle\le\rho\big)\le c_2(q)\rho^q.$$

From this, we can easily see (5.18) holds with $\rho_0(q)=c_2(q+1)^{-1}\wedge(c'/n)^{1/\beta}$. This completes the proof. □

REMARK 5.4. In the above proof, $\hat{\mathcal{H}}$ is another Hilbert space that is unitarily isometric to $\mathcal{H}$. Loosely speaking, it is defined as follows: Denote by $\mathcal{E}$ the set of $\mathbf{R}^d$-valued step functions on $[0,1]$. Let $\hat{\mathcal{H}}$ be the Hilbert space defined as the closure of $\mathcal{E}$ by the inner product

$$\langle I_{[0,s]}v, I_{[0,t]}w\rangle_{\hat{\mathcal{H}}} = R(s,t)\langle v, w\rangle_{\mathbf{R}^d}, \quad (t, s \in [0,1],\, v, w \in \mathbf{R}^d),$$

where we set $R(s,t) = \mathbb{E}[w_s^i w_t^i]$. (For instance, see Section 5.1, Nualart [**16**] or [**1**], [**18**] for more information on $\hat{\mathcal{H}}$.)

6. On-diagonal short time asymptotics.

The aim of this section is to prove Theorem 2.1, namely, on-diagonal short time asymptotic expansion of the density of the solution of the Young SDE (2.1) (or equivalently (5.1)) under the ellipticity assumption (A1).

Let us consider the solution $(y_t) = (y_t(a))$ of Young differential equation (2.1) with an initial condition $y_0 = a \in \mathbf{R}^n$ driven by fBm (w_t) with $H > 1/2$. It is shown in [**18**], [**1**] that, under (A1), the law of the solution has a smooth density $p(t, a, a')$, i.e.,

$$\mathbb{P}\big(y_t(a) \in A\big) = \int_A p(t, a, a')da' \quad (\text{for any Borel set } A \subset \mathbf{R}^n).$$

For $t > 0$, $y_t = y_t(a)$ is $\mathbf{D}_\infty$ and non-degenerate in the sense of Malliavin. By the same argument as in Ikeda and Watanabe [**9**], we have the following expression; $p(t, a, a') = \mathbb{E}[\delta_{a'}(y_t(a))] = {}_{\mathbf{D}_{-\infty}}\langle \delta_{a'}(y_t(a)), \mathbf{1}\rangle_{\mathbf{D}_\infty}$. By the self-similarity of fBm, $(y_{\varepsilon^{1/H}t})_{t\geq 0}$ and $(y_t^\varepsilon)_{t\geq 0}$ have the same law, where y^ε is given by (5.6) with $\gamma = 0$. From this, we see that $p(\varepsilon^{1/H}, a, a') = \mathbb{E}[\delta_{a'}(y_1^\varepsilon(a))]$.

The most important part of the proof is an asymptotic expansion of y_1^ε in $\varepsilon \in (0,1]$ in $\mathbf{D}_\infty$-topology. For that purpose, we introduce the following index set for exponent of ε. Set

$$\Lambda_1 = \left\{ n_1 + \frac{n_2}{H} \;\middle|\; n_1, n_2 \in \mathbf{N} \right\}.$$

We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ the elements of Λ_1 in increasing order. Several smallest elements are explicitly given as follows;

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{H}, \quad \kappa_3 = 2, \quad \kappa_4 = 1 + \frac{1}{H}, \quad \kappa_5 = 3 \wedge \frac{2}{H}, \ldots$$

PROPOSITION 6.1. *The family of Wiener functional y_1^ε $(0 < \varepsilon \leq 1)$ admits the following asymptotic expansion as $\varepsilon \searrow 0$;*

$$y_1^\varepsilon \sim a + \varepsilon f_1 + \varepsilon^{\kappa_2} f_{\kappa_2} + \varepsilon^{\kappa_3} f_{\kappa_3} + \cdots \quad \textit{in } \mathbf{D}_\infty(\mathbf{R}^n)$$

for certain $f_{\kappa_1}, f_{\kappa_2}, \ldots \in \mathbf{D}_\infty(\mathbf{R}^n)$.

PROOF. For $\boldsymbol{j} = (j_1, \ldots, j_m) \in \{0, 1, \ldots, d\}^m$, we set $|\boldsymbol{j}| = m$ and

$$\|\boldsymbol{j}\| = \frac{\sharp\{1 \le k \le m \mid j_k = 0\}}{H} + \sharp\{1 \le k \le m \mid j_k \neq 0\}.$$

We denote by $\mathcal{I}_m$ the totality of such $\boldsymbol{j}$'s with $|\boldsymbol{j}| = m$ and set $\mathcal{I} = \bigcup_{m=1}^{\infty} \mathcal{I}_m$.

We will use the following convention. We set $t = w_t^0$. Then, the ODE for y^ε (that is, (5.6) with $\gamma = 0$) reads;

$$dy_t^\varepsilon = \varepsilon^{1/H} V_0(y_t^\varepsilon) dw_t^0 + \sum_{j=1}^{d} \varepsilon V_j(y_t^\varepsilon) dw_t^j \quad \text{with } y_0^\varepsilon = a. \tag{6.1}$$

It is easy to see that

$$\begin{aligned}
y_1^\varepsilon - a &= \varepsilon^{1/H} \int_0^1 V_0(y_t^\varepsilon) dw_t^0 + \sum_{j=1}^{d} \varepsilon \int_0^1 V_j(y_t^\varepsilon) dw_t^j \\
&= \varepsilon^{1/H} \int_0^1 V_0(a) dw_t^0 + \sum_{j=1}^{d} \varepsilon \int_0^1 V_j(a) dw_t^j \\
&\quad + \varepsilon^{1/H} \int_0^1 \{V_0(y_t^\varepsilon) - V_0(a)\} dw_t^0 + \sum_{j=1}^{d} \varepsilon \int_0^1 \{V_j(y_t^\varepsilon) - V_j(a)\} dw_t^j \\
&= \varepsilon^{1/H} V_0(a) + \sum_{j=1}^{d} \varepsilon V_j(a) w_1^j \\
&\quad + \varepsilon^{1/H} \int_0^1 \int_0^{t_1} \{\varepsilon^{1/H} \hat{V}_0 V_0(y_{t_2}^\varepsilon) dt_2 + \sum_{j'=1}^{d} \varepsilon \hat{V}_{j'} V_0(y_{t_2}^\varepsilon) dw_{t_2}^j\} dw_{t_1}^0 \\
&\quad + \sum_{j=1}^{d} \varepsilon^{1+(1/H)} \int_0^1 \int_0^{t_1} \hat{V}_0 V_j(y_{t_2}^\varepsilon) dw_{t_2}^{j'} dw_{t_1}^j + \sum_{j,j'=1}^{d} \varepsilon^2 \int_0^1 \int_0^{t_1} \hat{V}_{j'} V_j(y_{t_2}^\varepsilon) dw_{t_2}^{j'} dw_{t_1}^j \\
&= \varepsilon^{1/H} V_0(a) + \sum_{j=1}^{d} \varepsilon V_j(a) w_1^j + \sum_{|\boldsymbol{j}|=2} \varepsilon^{\|\boldsymbol{j}\|} \int_0^1 \int_0^{t_1} \hat{V}_{j_2} V_{j_1}(y_{t_2}^\varepsilon) dw_{t_2}^{j_2} dw_{t_1}^{j_1}.
\end{aligned} \tag{6.2}$$

Here, $\hat{V}_i V_j$ denotes a vector field V_i (as a first order differential operator) acting on a $\mathbf{R}^n$-valued function V_j.

Repeating the same argument for the last term on the right hand side of (6.2), we have

$$\begin{aligned}
y_1^\varepsilon - a = \varepsilon^{1/H} V_0(a) + \sum_{j=1}^{d} \varepsilon V_j(a) w_1^j + \sum_{|\boldsymbol{j}|=2} \varepsilon^{\|\boldsymbol{j}\|} \hat{V}_{j_2} V_{j_1}(a) \int_0^1 \int_0^{t_1} dw_{t_2}^{j_2} dw_{t_1}^{j_1} \\
+ \sum_{|\boldsymbol{j}|=3} \varepsilon^{\|\boldsymbol{j}\|} \int_0^1 \int_0^{t_1} \int_0^{t_2} \hat{V}_{j_3} \hat{V}_{j_2} V_{j_1}(y_{t_2}^\varepsilon) dw_{t_3}^{j_3} dw_{t_2}^{j_2} dw_{t_1}^{j_1}.
\end{aligned} \tag{6.3}$$

Here, $\hat{V}_{j_3}\hat{V}_{j_2}V_{j_1} = \hat{V}_{j_3}(\hat{V}_{j_2}V_{j_1})$. In general, we have

$$
\begin{aligned}
y_1^\varepsilon - a = &\sum_{1\le|\boldsymbol{j}|\le n-1} \varepsilon^{\|\boldsymbol{j}\|}\hat{V}_{j_{n-1}}\cdots\hat{V}_{j_2}V_{j_1}(a)\int_0^1\int_0^{t_1}\cdots\int_0^{t_{n-2}} dw_{t_{n-1}}^{j_{n-1}}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1} \\
&+ \sum_{|\boldsymbol{j}|=n}\varepsilon^{\|\boldsymbol{j}\|}\int_0^1\int_0^{t_1}\cdots\int_0^{t_{n-1}}\hat{V}_{j_n}\cdots\hat{V}_{j_2}V_{j_1}(y_{t_n}^\varepsilon)dw_{t_n}^{j_n}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1}.
\end{aligned}
\tag{6.4}
$$

Let us observe the first term. From basic properties of Young integral, we easily see that, for any m, the real-valued functional $\int_0^1\int_0^{t_1}\cdots\int_0^{t_{m-1}} dw_{t_m}^{j_m}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1}$ is in mth (inhomogeneous) Wiener chaos and hence it is in any $\mathbf{D}_{q,k}$ $(1<q<\infty, k\in\mathbf{N})$.

Next we consider the last term in (6.4). We set

$$
Q_\varepsilon(w) = \int_0^1\int_0^{t_1}\cdots\int_0^{t_{n-1}}\hat{V}_{j_n}\cdots\hat{V}_{j_2}V_{j_1}(y_{t_n}^\varepsilon)dw_{t_n}^{j_n}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1}
$$

and will prove $Q_\varepsilon = O(1)$ as $\varepsilon \searrow 0$ in $\mathbf{D}_{q,k}(\mathbf{R}^n)$ for any $1<q<\infty$, $k\in\mathbf{N}$. (For simplicity of notation, we denote $G = \hat{V}_{j_n}\cdots\hat{V}_{j_2}V_{j_1}$ and assume $j_i \neq 0$ for all i. The other case is actually easier.)

Since $\|y^\varepsilon\|_{\alpha-hld}$ is $O(1)$ in any L^q as $\varepsilon\searrow 0$, $Q_\varepsilon(w)$ is $O(1)$ in any L^q, too. Now we estimate the derivatives. For $h\in\mathcal{H}$, we have

$$
\begin{aligned}
D_hQ_\varepsilon(w) = &\int_0^1\cdots\int_0^{t_{n-1}}\nabla G(y_{t_n}^\varepsilon)\langle D_hy_{t_n}^\varepsilon\rangle dw_{t_n}^{j_n}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1} \\
&+ \sum_{l=1}^n\int_0^1\cdots\int_0^{t_{n-1}} G(y_{t_n}^\varepsilon)dw_{t_n}^{j_n}\cdots dh_{t_l}^{j_l}\cdots dw_{t_1}^{j_1}.
\end{aligned}
$$

Hölder norms of y^ε and D_hy^ε were estimated in (5.8)–(5.11). From these, we see that $\|DQ_\varepsilon\|_{\mathcal{H}^*} = O(1)$ in any L^q.

Similarly, $h,k\in\mathcal{H}$, we have

$$
\begin{aligned}
D_kD_hQ_\varepsilon(w) = &\int_0^1\cdots\int_0^{t_{n-1}}\nabla G(y_{t_n}^\varepsilon)\langle D_kD_hy_{t_n}^\varepsilon\rangle dw_{t_n}^{j_n}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1} \\
&+ \int_0^1\cdots\int_0^{t_{n-1}}\nabla G(y_{t_n}^\varepsilon)\langle D_ky_{t_n}^\varepsilon, D_hy_{t_n}^\varepsilon\rangle dw_{t_n}^{j_n}\cdots dw_{t_2}^{j_2}dw_{t_1}^{j_1} \\
&+ \sum_{l=1}^n\int_0^1\cdots\int_0^{t_{n-1}}\nabla G(y_{t_n}^\varepsilon)\langle D_hy_{t_n}^\varepsilon\rangle dw_{t_n}^{j_n}\cdots dk_{t_l}^{j_l}\cdots dw_{t_1}^{j_1} \\
&+ \sum_{l=1}^n\int_0^1\cdots\int_0^{t_{n-1}}\nabla G(y_{t_n}^\varepsilon)\langle D_ky_{t_n}^\varepsilon\rangle dw_{t_n}^{j_n}\cdots dh_{t_l}^{j_l}\cdots dw_{t_1}^{j_1} \\
&+ \sum_{l\neq m}^n\int_0^1\cdots\int_0^{t_{n-1}} G(y_{t_n}^\varepsilon)dw_{t_n}^{j_n}\cdots dh_{t_l}^{j_l}\cdots dh_{t_m}^{j_m}\cdots dw_{t_1}^{j_1}.
\end{aligned}
$$

Hölder norm of $D_k D_h y^\varepsilon$ was estimated in (5.13). Combined with Proposition 4.4, the above implies that $\|D^2 Q_\varepsilon\|_{\mathcal{H}^* \otimes \mathcal{H}^*} = O(1)$ in any L^q. Higher order derivatives can be done in the same way.

Now we prove the proposition. In order to get the asymptotic expansion up to order κ_m (i.e., the remainder is of order κ_{m+1}), it is sufficient (i) to consider the expansion (6.4) with $n-1$ being the smallest integer which is not less than κ_m and (ii) to set

$$f_{\kappa_l}(w) = \sum_{\|\boldsymbol{j}\|=\kappa_l} \hat{V}_{j_n} \cdots \hat{V}_{j_2} V_{j_1}(a) \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} dw_{t_n}^{j_n} \cdots dw_{t_2}^{j_2} dw_{t_1}^{j_1}$$

for all $1 \le l \le m$. □

Before we prove on-diagonal short time kernel asymptotics, we define two more index sets for exponent of ε. Set $\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\}$. Smallest elements of Λ_2 are

$$0, \quad \frac{1}{H} - 1, \quad 1, \quad \frac{1}{H}, \quad \left(3 \wedge \frac{2}{H}\right) - 1, \ldots$$

Next we set $\Lambda_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2\}$. In the sequel, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ stands for all the elements of Λ_3 in increasing order.

Proof of Theorem 2.1. First, note that

$$p(\varepsilon^{1/H}, a, a) = \mathbb{E}[\delta_a(y_1^\varepsilon(a))] = \mathbb{E}\left[\delta_0\left(\varepsilon \frac{y_1^\varepsilon(a) - a}{\varepsilon}\right)\right] = \varepsilon^{-n} \mathbb{E}\left[\delta_0\left(\frac{y_1^\varepsilon(a) - a}{\varepsilon}\right)\right].$$

By Proposition 5.3, $(y_1^\varepsilon(a) - a)/\varepsilon$ is uniformly non-degenerate. It admits asymptotic expansion in $\mathbf{D}_\infty(\mathbf{R}^n)$ as in Proposition 6.1. Then, by Theorem 4.3, the following asymptotic expansion holds in $\tilde{\mathbf{D}}_{-\infty}$ as $\varepsilon \searrow 0$;

$$\delta_0\left(\frac{y_1^\varepsilon(a) - a}{\varepsilon}\right) \sim \phi_0 + \varepsilon^{\nu_1} \phi_{\nu_1} + \varepsilon^{\nu_2} \phi_{\nu_2} + \cdots \quad \text{as } \varepsilon \searrow 0.$$

By taking the generalized expectation and setting $c_{\nu_k} = \mathbb{E}[\phi_{\nu_k}]$, we have

$$p(\varepsilon^{1/H}, a, a) \sim \varepsilon^{-n}\big(c_0 + c_{\nu_1} \varepsilon^{\nu_1} + c_{\nu_2} \varepsilon^{\nu_2} + \cdots\big) \quad \text{as } \varepsilon \searrow 0.$$

Putting $\varepsilon = t^H$, we complete the proof of Theorem 2.1. □

7. Taylor expansion of Itô map around a Cameron–Martin path.

In this section we prove an asymptotic expansion for $\tilde{y}^\varepsilon = I(\varepsilon w + \gamma, \varepsilon^{1/H}\lambda)$, which was defined in (5.6). The base point $\gamma \in \mathcal{H}$ of the expansion is arbitrary, but fixed. First, we prove that $\tilde{y}^\varepsilon$ admits the following expansion in $C^{\alpha-hld}([0,1]; \mathbf{R}^n)$;

$$\tilde{y}^{\varepsilon} \sim \phi^{0} + \varepsilon^{\kappa_1}\phi^{\kappa_1} + \varepsilon^{\kappa_2}\phi^{\kappa_2} + \cdots \quad \text{as } \varepsilon \searrow 0, \quad \left(\kappa_i \in \Lambda_1 = \mathbf{N} + \frac{1}{H}\mathbf{N}\right), \tag{7.1}$$

for some $C^{\alpha-hld}([0,1];\mathbf{R}^n)$-valued Wiener functional $\phi^0, \phi^{\kappa_1}, \phi^{\kappa_2}, \ldots$. Since the Itô map I in the sense of Young integral equation is smooth in Fréchet sense (see [**12**]), this kind expansion holds in deterministic sense. In this paper, however, we need to prove this expansion in L^q-sense.

Before we state the proposition precisely, we now give a heuristic argument to find an explicit form of ϕ^{κ_m}. To find an ODE for ϕ^0 is easy.

$$\begin{aligned} d\tilde{y}^{\varepsilon}_t &= \sigma(\tilde{y}^{\varepsilon}_t)(\varepsilon dw_t + d\gamma_t) + b(\tilde{y}^{\varepsilon}_t)\varepsilon^{1/H}dt && \text{with} \quad \tilde{y}^{\varepsilon}_0 = a, \\ d\phi^0_t &= \sigma(\phi^0_t)d\gamma_t && \text{with} \quad \phi^0_0 = a. \end{aligned} \tag{7.2}$$

Set $\triangle\phi := \tilde{y}^{\varepsilon} - \phi^0$ and put it in the above ODE for $\tilde{y}^{\varepsilon}$. Then we have

$$\begin{aligned} d(\phi^0 + \triangle\phi) &= \sigma(\phi^0 + \triangle\phi)(\varepsilon dw + d\gamma) + b(\phi^0 + \triangle\phi)\varepsilon^{1/H}dt \\ &= \sum_{k=0}^{\infty} \frac{\nabla^k\sigma(\phi^0)}{k!}\langle \underbrace{\triangle\phi, \ldots, \triangle\phi}_{k}; \varepsilon dw + d\gamma\rangle + \sum_{k=0}^{\infty} \frac{\nabla^k b(\phi^0)}{k!}\langle \underbrace{\triangle\phi, \ldots, \triangle\phi}_{k}\rangle \varepsilon^{1/H}dt. \end{aligned}$$

Assume $\triangle\phi$ admits the asymptotic expansion (7.1). Then, by putting it in the above equation and picking up the terms of order ε^{κ_m}, we find an ODE for ϕ^{κ_m}. Note that $\phi^{\kappa_m}_0 = 0$ for all $m \geq 1$.

For $\kappa_m = 1, 1/H, 2$, we can write down the ODEs explicitly as follows;

$$d\phi^1_t - \nabla\sigma(\phi^0_t)\langle\phi^1_t, d\gamma_t\rangle = \sigma(\phi^0_t)dw_t, \tag{7.3}$$

$$d\phi^{1/H}_t - \nabla\sigma(\phi^0_t)\langle\phi^{1/H}_t, d\gamma_t\rangle = b(\phi^0_t)dt, \tag{7.4}$$

$$d\phi^2_t - \nabla\sigma(\phi^0_t)\langle\phi^2_t, d\gamma_t\rangle = \nabla\sigma(\phi^0_t)\langle\phi^1_t, dw_t\rangle + \frac{1}{2}\nabla^2\sigma(\phi^0_t)\langle\phi^1_t, \phi^1_t, d\gamma_t\rangle. \tag{7.5}$$

Note that $\phi^{1/H}$ is independent of w, i.e, non-random with respect to μ.

For $\kappa_m \geq 2$,

$$\begin{aligned} d\phi^{\kappa_m}_t - \nabla\sigma(\phi^0_t)\langle\phi^{\kappa_m}_t, d\gamma_t\rangle &= \sum_{k=1}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m-1} \frac{\nabla^k\sigma(\phi^0_t)}{k!}\langle\phi^{\kappa_{i_1}}_t, \ldots, \phi^{\kappa_{i_k}}_t; dw_t\rangle \\ &\quad + \sum_{k=2}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m} \frac{\nabla^k\sigma(\phi^0_t)}{k!}\langle\phi^{\kappa_{i_1}}_t, \ldots, \phi^{\kappa_{i_k}}_t; d\gamma_t\rangle \\ &\quad + \sum_{k=1}^{\infty} \sum_{\kappa_{i_1}+\cdots+\kappa_{i_k}=\kappa_m-(1/H)} \frac{\nabla^k b(\phi^0_t)}{k!}\langle\phi^{\kappa_{i_1}}_t, \ldots, \phi^{\kappa_{i_k}}_t; dt\rangle. \end{aligned} \tag{7.6}$$

The summations in the first term on the right hand side is taken over all $\kappa_{i_1}, \dots, \kappa_{i_k} \in \Lambda_1 \setminus \{0\}$ such that $\kappa_{i_1} + \cdots + \kappa_{i_k} = \kappa_m - 1$ hold. $\kappa_{i_j} = 0$ is not allowed. So, the sum is actually a finite sum. The second and the third terms should be understood in the same way. An important observation is that the right hand side of (7.6) does not involve ϕ^{κ_m}, but only $\phi^0, \phi^1, \dots, \phi^{\kappa_{m-1}}$. These ODEs have a rigorous meaning. So, we inductively define ϕ^{κ_m} as a unique solution of (7.3)–(7.6).

If the right hand side of (7.3)–(7.6) is denoted by $dQ_t^{\kappa_m}$, then ϕ^{κ_m} can be written explicitly as follows;

$$\phi_T^{\kappa_m} = \tilde{J}(\gamma)_T \int_0^T \tilde{J}(\gamma)_t^{-1} dQ_t^{\kappa_m}, \tag{7.7}$$

where we set $\tilde{J}(\gamma) = J(\gamma, 0) = J(0w + \gamma, 0^{1/H}\lambda)$. See (3.4) for the definition of J.

Define the remainder term $R^{\kappa_{m+1},\varepsilon}$ by

$$R_t^{\kappa_{m+1},\varepsilon} = \tilde{y}_t^\varepsilon - \left(\phi_t^0 + \varepsilon\phi_t^1 + \cdots + \varepsilon^{\kappa_m}\phi_t^{\kappa_m}\right).$$

We will estimate this remainder term in L^q-sense.

Proposition 7.1. *For any $m \in \mathbf{N}$ and $q \in (1,\infty)$, $\|\phi^{\kappa_m}\|_{\alpha-hld} \in L^q(\mu)$ and*

$$\mathbb{E}\left[\|R^{\kappa_{m+1},\varepsilon}\|_{\alpha-hld}^q\right]^{1/q} = O(\varepsilon^{\kappa_{m+1}}) \quad \textit{as } \varepsilon \searrow 0.$$

Proof. From the expression (7.7) and induction, it is easy to see that $\|\phi^{\kappa_m}\|_{\alpha-hld} \in \bigcap_{1<q<\infty} L^q$ for any m. Let us consider $R_t^{1,\varepsilon} = \triangle\phi = \tilde{y}^\varepsilon - \phi^0 = I(\varepsilon w+\gamma, \varepsilon^{1/H}\lambda) - I(\gamma, \mathbf{0})$. Here, I stands for the Itô map and $\mathbf{0}$ stands for one-dimensional constant path staying at 0.

Define $\omega(s,t) = (\|w\|_{\alpha-hld}^p + \|\gamma\|_{\alpha-hld}^p + 1)(t-s)$ with $\alpha = 1/p$. This control function satisfies

$$|(\varepsilon w_t + \gamma_t) - (\varepsilon w_s + \gamma_s)| + |\varepsilon^{1/H}t - \varepsilon^{1/H}s| \le \omega(s,t)^{1/p}$$
$$|\{(\varepsilon w_t + \gamma_t) - (\varepsilon w_s + \gamma_s)\} - \{\gamma_t - \gamma_s\}| + |\varepsilon^{1/H}t - \varepsilon^{1/H}s| \le \varepsilon\omega(s,t)^{1/p}$$

for all $0 \le s \le t \le 1$ and $\varepsilon \in [0,1]$. Hence, by the local Lipschitz continuity of Itô map I,

$$|R_t^{1,\varepsilon} - R_s^{1,\varepsilon}| \le \varepsilon C(1 + \omega(0,1))^{(p-1)/p} \exp(C\omega(0,1))\omega(s,t)^{1/p}$$

for some positive constant C. Since $p < 2$, we can use Fernique's theorem to obtain the desired estimate holds when $\kappa_{m+1} = 1$.

Before we prove the higher order cases, let us observe the concrete expression for several $R^{\kappa_{m+1},\varepsilon}$'s. In the sequel, we write $\kappa_{m+1} =: \kappa_m+$ for simplicity of notation. First we consider $R^{1+,\varepsilon} = R^{1/H,\varepsilon} = \tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1$. A straight forward computation yields;

$$dR_t^{1+,\varepsilon} = \varepsilon\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}dw_t \\ + \big[\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}d\gamma_t - \nabla\sigma(\phi_t^0)\langle\varepsilon\phi_t^1, d\gamma_t\rangle\big] + \varepsilon^{1/H}b(\tilde{y}_t^\varepsilon)dt. \tag{7.8}$$

From this, we immediately have

$$dR_t^{1+,\varepsilon} - \nabla\sigma(\phi_t^0)\langle R_t^{1+,\varepsilon}, d\gamma_t\rangle \\ = \varepsilon\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}dw_t + \frac{1}{2}\int_0^1 d\theta\nabla\sigma(\phi_t^0 + \theta R_t^{1,\varepsilon})\langle R_t^{1,\varepsilon}, R_t^{1,\varepsilon}, d\gamma_t\rangle + \varepsilon^{1/H}b(\tilde{y}_t^\varepsilon)dt \\ (=: dL_t^{1+,\varepsilon}). \tag{7.9}$$

Observe that, on the right hand side, there are only $R^{1,\varepsilon}, \tilde{y}^\varepsilon, \phi^0, \gamma, w$, which are known quantities, but no $R^{1+,\varepsilon}$. Since $R_T^{1+,\varepsilon} = \tilde{J}(\gamma)_T\int_0^T \tilde{J}(\gamma)_t^{-1}dL_t^{1+,\varepsilon}$ as before, it suffices to show that $\|L^{1+,\varepsilon}\|_{\alpha-hld} = O(\varepsilon^{1/H})$ for any L^q.

Since $\|\varepsilon^{1/H}\int_0^\cdot b(\tilde{y}_t^\varepsilon)dt\|_{\alpha-hld} \le C\varepsilon^{1/H}\|\tilde{y}^\varepsilon\|_{\alpha-hld}$, the third term of $L^{1+,\varepsilon}$ is $O(\varepsilon^{1/H})$ in any L^q. Similarly, $\varepsilon\|\int_0^\cdot\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}dw_t\|_{\alpha-hld} \le C\varepsilon\|R^{1,\varepsilon}\|_{\alpha-hld}\|w\|_{\alpha-hld}$, the first term of $L^{1+,\varepsilon}$ is $O(\varepsilon^2)$ in any L^q. For any θ, $\|\nabla\sigma(\phi_\cdot^0 + \theta R_\cdot^{1,\varepsilon})\|_{\alpha-hld} \le C(\|\phi^0\|_{\alpha-hld} + \|R^{1,\varepsilon}\|_{\alpha-hld})$. Hence, we have

$$\Big\|\int_0^\cdot\int_0^1 d\theta\nabla\sigma(\phi_t^0 + \theta R_t^{1,\varepsilon})\langle R_t^{1,\varepsilon}, R_t^{1,\varepsilon}, d\gamma_t\rangle\Big\|_{\alpha-hld} \\ \le C(\|\phi^0\|_{\alpha-hld} + \|R^{1,\varepsilon}\|_{\alpha-hld})\|R^{1,\varepsilon}\|_{\alpha-hld}^2.$$

We see from the above inequality that the second term of $L^{1+,\varepsilon}$ is $O(\varepsilon^2)$ in any L^q and hence $\|L^{1+,\varepsilon}\|_{\alpha-hld} = O(\varepsilon^{1/H})$ in any L^q. Thus, we have obtained the desired estimate for $R^{1+,\varepsilon} = R^{1/H,\varepsilon}$.

The estimate for $R^{(1/H)+,\varepsilon} = \tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1 - \varepsilon^{1/H}\phi^{1/H}$ can easily be obtained as follows. We can immediately see from (7.5) and (7.9) that

$$dR_t^{(1/H)+,\varepsilon} - \nabla\sigma(\phi_t^0)\langle R_t^{(1/H)+,\varepsilon}, d\gamma_t\rangle \\ = \varepsilon\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}dw_t + \frac{1}{2}\int_0^1 d\theta\nabla\sigma(\phi_t^0 + \theta R_t^{1,\varepsilon})\langle R_t^{1,\varepsilon}, R_t^{1,\varepsilon}, d\gamma_t\rangle \\ + \varepsilon^{1/H}\{b(\tilde{y}_t^\varepsilon) - b(\phi_t^0)\}dt \quad (=: dL_t^{(1/H)+,\varepsilon}). \tag{7.10}$$

Notice that we have essentially shown that $\|L^{(1/H)+,\varepsilon}\|_{\alpha-hld} = O(\varepsilon^2)$ in any L^q. Thus, we have obtained the desired estimate for $R^{(1/H)+,\varepsilon} = R^{2,\varepsilon}$.

Next, we will estimate $R^{2+,\varepsilon} = \tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1 - \varepsilon^{1/H}\phi^{1/H} - \varepsilon^2\phi^2$. From (7.4), (7.5), and (7.8), we see that

$$dR_t^{2+,\varepsilon} = \big[\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}\varepsilon dw_t - \nabla\sigma(\phi_t^0)\langle\varepsilon\phi_t^1, \varepsilon dw_t\rangle\big] \\ + \big[\{\sigma(\tilde{y}_t^\varepsilon) - \sigma(\phi_t^0)\}d\gamma_t - \nabla\sigma(\phi_t^0)\langle\varepsilon\phi_t^1 + \varepsilon^{1/H}\phi^{1/H} + \varepsilon^2\phi^2, d\gamma_t\rangle\big]$$

$$-\frac{1}{2}\nabla^2\sigma(\phi_t^0)\langle\varepsilon\phi_t^1,\varepsilon\phi_t^1,d\gamma_t\rangle+\varepsilon^{1/H}\{b(\tilde{y}_t^\varepsilon)-b(\phi_t^0)\}dt. \tag{7.11}$$

The second term on the right hand side is equal to

$$\nabla\sigma(\phi_t^0)\langle dR_t^{2+,\varepsilon},d\gamma_t\rangle+\frac{1}{2}\nabla^2\sigma(\phi_t^0)\langle R_t^{1,\varepsilon},R_t^{1,\varepsilon},d\gamma_t\rangle$$
$$+\int_0^1\frac{(1-\theta)^2d\theta}{2!}\nabla^3\sigma(\phi_t^0+\theta R_t^{1,\varepsilon})\langle R_t^{1,\varepsilon},R_t^{1,\varepsilon},R_t^{1,\varepsilon},d\gamma_t\rangle.$$

Hence, (7.11) is equivalent to the following;

$$\begin{aligned}dR_t^{2+,\varepsilon}-\nabla\sigma(\phi_t^0)\langle dR_t^{2+,\varepsilon},d\gamma_t\rangle&=\big[\{\sigma(\tilde{y}_t^\varepsilon)-\sigma(\phi_t^0)\}\varepsilon dw_t-\nabla\sigma(\phi_t^0)\langle\varepsilon\phi_t^1,\varepsilon dw_t\rangle\big]\\&\quad+\frac{1}{2}\big[\nabla^2\sigma(\phi_t^0)\langle R_t^{1,\varepsilon},R_t^{1,\varepsilon},d\gamma_t\rangle-\nabla^2\sigma(\phi_t^0)\langle\varepsilon\phi_t^1,\varepsilon\phi_t^1,d\gamma_t\rangle\big]\\&\quad+\int_0^1\frac{(1-\theta)^2d\theta}{2!}\nabla^3\sigma(\phi_t^0+\theta R_t^{1,\varepsilon})\langle R_t^{1,\varepsilon},R_t^{1,\varepsilon},R_t^{1,\varepsilon},d\gamma_t\rangle\\&\quad+\varepsilon^{1/H}\{b(\tilde{y}_t^\varepsilon)-b(\phi_t^0)\}dt\quad(=:dL_t^{2+,\varepsilon}).\end{aligned}\tag{7.12}$$

Then, $R_T^{2+,\varepsilon}=\tilde{J}(\gamma)_T\int_0^T\tilde{J}(\gamma)_t^{-1}dL_t^{2+,\varepsilon}$.

Let us observe the right hand side of (7.12). There are no $R^{2+,\varepsilon}$ or ϕ^2. By the assumption of induction, we may only use the relation $R^{2,\varepsilon}=R^{(1/H)+,\varepsilon}=\tilde{y}^\varepsilon-\phi^0-\varepsilon\phi^1-\varepsilon^{1/H}\phi^{1/H}$ and the estimates of $R^{\kappa,\varepsilon}$ for $\kappa=1,1/H,2$ (and of ϕ^κ's). In the same way as above, by using the Taylor expansion, we can prove that $\|L^{2+,\varepsilon}\|_{\alpha-hld}=O(\varepsilon^{1+(1/H)})$ in any L^q. Cancellation of the terms of order ≤ 2 on the right hand side is no mystery because of the way ϕ^κ's are defined. Thus, we have obtained the desired estimate for $R^{2+,\varepsilon}=R^{1+(1/H),\varepsilon}$.

Higher order remainder terms can be dealt with in a similar way. We give a sketch of proof. There exists

$$L_t^{\kappa_{m+1},\varepsilon}=L^{\kappa_{m+1},\varepsilon}[\phi^0,\ldots,\phi^{\kappa_{m-1}};R_t^{1,\varepsilon},\ldots,R^{\kappa_m,\varepsilon};w,\gamma]_t$$

such that $dR_t^{\kappa_{m+1},\varepsilon}-\nabla\sigma(\phi_t^0)\langle dR_t^{\kappa_{m+1},\varepsilon},d\gamma_t\rangle=dL_t^{\kappa_{m+1},\varepsilon}$. Due to cancellation $\|L^{\kappa_{m+1},\varepsilon}\|_{\alpha-hld}=O(\varepsilon^{\kappa_{m+1}})$ holds in any L^q. This proves the assertion. □

The next proposition shows that, when evaluated at $t=1$, Eq. (7.1) gives an asymptotic expansion in $\mathbf{D}_\infty(\mathbf{R}^n)$.

Proposition 7.2. *We have the following asymptotic expansion in* $\mathbf{D}_\infty(\mathbf{R}^n)$.

$$\tilde{y}_1^\varepsilon\sim\phi_1^0+\varepsilon^{\kappa_1}\phi_1^{\kappa_1}+\varepsilon^{\kappa_2}\phi_1^{\kappa_2}+\cdots\quad\textit{as }\varepsilon\searrow 0. \tag{7.13}$$

Here, $0=\kappa_0<\kappa_1<\kappa_2<\cdots$ *are all the elements of* $\Lambda_1=\mathbf{N}+\mathbf{N}/H$ *in increasing order.*

Proof. By using induction and basic properties of Young integral, we can easily see that $\phi_1^{\kappa_m}$ is in $[\kappa_m]$-th inhomogeneous Wiener chaos for each t and m. In particular, $\phi_1^{\kappa_m} \in \mathbf{D}_\infty$. If $k \geq [\kappa_m]+1$, then $D^k R_1^{\kappa_{m+1},\varepsilon} = D^k \tilde{y}_1^\varepsilon$. From Proposition 5.1, this is $O(\varepsilon^k)$, and hence $O(\varepsilon^{\kappa_{m+1}})$ in any L^q. A stronger version of Meyer's equivalence (e.g., Theorem 4.6, [**20**]) implies that $R_1^{\kappa_{m+1},\varepsilon}$ is $O(\varepsilon^{\kappa_{m+1}})$ in $\mathbf{D}_{q,k}$ for any q and sufficiently large k. Since $\mathbf{D}_{q,k}$-norm is increasing in k, the proof is completed. □

We now recall the following Taylor expansion of Itô map around γ in the deterministic sense.

Lemma 7.3. (i) *For each m, there exists $c = c(\kappa_m)$ such that*

$$\|\phi^{\kappa_m}\|_{\alpha-hld} \leq c(1+\|w\|_{\alpha-hld})^{\kappa_m} \quad \textit{for all } w \in C_0^{\alpha-hld}([0,1],\mathbf{R}^d).$$

(ii) *For each m and $r>0$, there exists $c' = c'(\kappa_m, r)$ such that*

$$\|R^{\kappa_{m+1},\varepsilon}\|_{\alpha-hld} \leq c'(\varepsilon + \|\varepsilon w\|_{\alpha-hld})^{\kappa_{m+1}}, \quad \textit{if } \|\varepsilon w\|_{\alpha-hld} \leq r.$$

Proof. This is immediate since $\tilde{y}^\varepsilon = I(\varepsilon w + \gamma, \varepsilon^{1/H}\lambda)$ and Itô map I is Fréchet smooth by Li and Lyons's result [**12**]. It is also possible to prove this lemma by using the explicit expression of $R^{\kappa_{m+1},\varepsilon}$ and mathematical induction as in the proof of Proposition 7.1 above. □

8. Off-diagonal short time asymptotics.

In this section we prove the short time asymptotics of kernel function $p_t(a,a')$ when $a \neq a'$. We basically follow Watanabe [**21**]. In this paper, however, we can localize around the energy minimizing path in the abstract Wiener space since Itô map is continuous in our setting. This makes the proof slightly simpler.

8.1. Localization around energy minimizing path.

For $\gamma \in \mathcal{H}$, let $\phi^0 = \phi^0(\gamma)$ be a unique solution of (7.2), which starts at $a \in \mathbf{R}^n$. Set, for $a \neq a'$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

We only consider the case that $K_a^{a'}$ is not empty. For example, if (A1) is satisfied for any a, then $K_a^{a'}$ is not empty for any a'. From the Schilder-type large deviation theory, it is easy to see that $\inf\{\|\gamma\|_\mathcal{H} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_\mathcal{H} \mid \gamma \in K_a^{a'}\}$.

We continue to assume (A1). Now we introduce another assumption;

(A2): $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathcal{H}$-norm exists uniquely.

In the sequel, $\bar{\gamma}$ denotes the minimizer in Assumption (A2) and we use the results of the previous section for this $\bar{\gamma}$.

Note that (i) the mapping $\gamma \in \mathcal{H} \hookrightarrow \mathcal{W} \mapsto \phi_1^0(\gamma) \in \mathbf{R}^n$ is Fréchet differentiable and (ii) its Jacobian is a surjective linear mapping from $\mathcal{H}$ to $\mathbf{R}^n$ for any γ, because there exists a positive constant $c = c(\gamma)$ such that

$$\big(\langle D\phi_1^{0,i}(\gamma), D\phi_1^{0,j}(\gamma)\rangle_{\mathcal{H}^*}\big)_{1\le i,j\le n} \ge c\cdot \mathrm{Id}_n. \tag{8.1}$$

This can be shown in the same way as in the proof of Proposition 5.3. (Actually, it is easier since γ is non-random and fixed here.)

Therefore, by the Lagrange multiplier method, there exists $\bar\nu = (\bar\nu_1, \ldots, \bar\nu_n) \in \mathbf{R}^n$ uniquely such that the map

$$\mathcal{H} \times \mathbf{R}^n \ni (\gamma, \nu) \mapsto \frac{1}{2}\|\gamma\|^2_{\mathcal{H}} - \langle \nu, \phi_1^0(\gamma) - a'\rangle_{\mathbf{R}^n} \in \mathbf{R} \tag{8.2}$$

attains extremum at $(\bar\gamma, \bar\nu)$. By differentiating in the direction of $k \in \mathcal{H}$, we have

$$\langle\bar\gamma, k\rangle_{\mathcal{H}} = \langle\bar\nu, D_k\phi_1^0(\bar\gamma)\rangle_{\mathbf{R}^n} = \Big\langle\bar\nu, \tilde J(\bar\gamma)_1 \int_0^1 \tilde J(\bar\gamma)_t^{-1}\sigma(\phi_t^0(\bar\gamma))dk_t\Big\rangle_{\mathbf{R}^n}. \tag{8.3}$$

Here, the definition of $\tilde J(\bar\gamma)$ was given just below (7.7) and the integral on the right hand side is Young integral. Hence, $\langle\bar\gamma, \cdot\,\rangle_{\mathcal{H}}$ extends to a continuous linear functional on $\mathcal{W}$.

Let us introduce Besov-type norms. In the context of Malliavin calculus, these norms are often more useful than Hölder norms and p-variation norms since (a power of) these norms become $\mathbf{D}_\infty$-functionals. For $m > 0$, $0 < \theta < 1$, and $x \in C_0([0,1], \mathbf{R}^d)$, we set

$$\|x\|_{m,\theta-B} := \bigg(\iint_{0\le s\le t\le 1} \frac{|x_t - x_s|^m}{|t-s|^{2+m\theta}} dsdt\bigg)^{1/m}.$$

and $C_0^{m,\theta-B}([0,1], \mathbf{R}^d) = \{x \in C_0([0,1], \mathbf{R}^d) \mid \|x\|_{m,\theta-B} < \infty\}$. It is known that $\|x\|_{\theta-hld} \le c\|x\|_{m,\theta-B}$ for some constant $c = c_{m,\theta} > 0$. Hence, $C_0^{m,\theta-B}([0,1], \mathbf{R}^d) \subset C_0^{\theta-hld}([0,1], \mathbf{R}^d)$.

Let (w_t) be fBm with Hurst parameter $H \in (1/2, 1)$ and let $\alpha(= 1/p) < H$ as before. Since $\mathbb{E}[|w_t - w_s|^2] = d|t-s|^{2H}$, we can easily see $\mathbb{E}[\|x\|^m_{m,\alpha-B}] < \infty$ if $m > 1/(H-\alpha)$. Therefore, the law of fBm, $\mu = \mu^H$, is supported in $C_0^{m,\alpha-B}([0,1], \mathbf{R}^d)$ if $m > 1/(H-\alpha)$. Set $\mathcal{W}_B$ to be the closure of Cameron–Martin space $\mathcal{H} = \mathcal{H}^H$ in $C_0^{m,\alpha-B}([0,1], \mathbf{R}^d)$. Then, $(\mathcal{W}_B, \mathcal{H}, \mu)$ is also an abstract Wiener space.

Now we recall Schilder-type large deviation principle for scaled Gaussian measures. For $\varepsilon > 0$, let μ_ε be the law of the process $(\varepsilon w_t)_{0\le t\le 1}$. This is a measure on $\mathcal{W}_B$. Set $\mathcal{I}(w) = \|w\|^2_{\mathcal{H}}/2$ (if $w \in \mathcal{H}$) and $\mathcal{I}(w) = \infty$ (otherwise). It is well-known that $\mathcal{I} : \mathcal{W}_B \to [0, \infty]$ is lower semicontinuous and that $\mathcal{I}$ is good, i.e., the level set $\{w \mid \mathcal{I}(w) \le r\}$ is compact in $\mathcal{W}_B$ for any $r \in [0, \infty)$.

The family $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies large deviation principle as $\varepsilon \searrow 0$ with a good rate function $\mathcal{I}$, that is, for any measurable set $A \subset \mathcal{W}_B$

$$-\inf_{w\in A^\circ}\mathcal{I}(w) \le \liminf_{\varepsilon\searrow 0}\varepsilon^2\log\mu_\varepsilon(A^\circ) \le \limsup_{\varepsilon\searrow 0}\varepsilon^2\log\mu_\varepsilon(\bar A) \le -\inf_{w\in\bar A}\mathcal{I}(w). \tag{8.4}$$

Next, set $\hat{\mu}_\varepsilon = \mu_\varepsilon \otimes \delta_{\varepsilon^{1/H}\lambda}$, where λ is a one-dimensional path defined by $\lambda_t = t$ and $\otimes$ stands for the product of probability measures. In other words, $\hat{\mu}_\varepsilon$ is the law of the $(d+1)$-dimensional process $(\varepsilon w_t, \varepsilon^{1/H} t)_{0 \le t \le 1}$ under μ. This measure is supported on $\mathcal{W}_B \oplus \mathbf{R}\langle\lambda\rangle \subset C_0^{m,\alpha-B}([0,1];\mathbf{R}^{d+1})$. Define $\hat{\mathcal{I}}(w;l) = \|w\|_{\mathcal{H}}^2/2$ (if $w \in \mathcal{H}$ and $l_t \equiv 0$) and $\hat{\mathcal{I}}(w,l) = \infty$ (otherwise). Here, l is a one-dimensional path.

From (8.4) we can easily show that $\{\hat{\mu}_\varepsilon\}_{\varepsilon>0}$ satisfies large deviation principle as $\varepsilon \searrow 0$ with a good rate function $\hat{\mathcal{I}}$, that is, for any measurable set $A \subset \mathcal{W}_B \oplus \mathbf{R}\langle\lambda\rangle$,

$$- \inf_{w \in A^\circ} \hat{\mathcal{I}}(w) \le \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \hat{\mu}_\varepsilon(A^\circ) \le \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \hat{\mu}_\varepsilon(\bar{A}) \le - \inf_{w \in \bar{A}} \hat{\mathcal{I}}(w). \tag{8.5}$$

We will use (8.5) in Lemma 8.1 below to show that only a neighborhood of the minimizer $\bar{\gamma}$ contributes to the asymptotic expansion.

From now on, we will fix an even integer $m > 0$ such that $m > 1/(H-\alpha)$. Then, it is easy to check $\|w\|_{m,\alpha-B}^m \in \mathbf{D}_\infty$. In fact, this functional is an element of mth inhomogeneous Wiener chaos, i.e., $D^{m+1}\|w\|_{m,\alpha-B}^m = 0$.

Now we introduce a cut-off function. Let $\psi : \mathbf{R} \to [0,1]$ be a smooth function such that $\psi(u) = 1$ if $|u| \le 1/2$ and $\psi(u) = 0$ if $|u| \ge 1$. For each $\eta > 0$ and $\varepsilon > 0$, we set

$$\chi_\eta(\varepsilon, w) = \psi\left(\frac{1}{\eta^m} \|\varepsilon w - \bar{\gamma}\|_{m,\alpha-B}^m \right).$$

We can easily see that $\chi_\eta(\varepsilon, \cdot\,) \in \mathbf{D}_\infty$. Shifting by $\bar{\gamma}/\varepsilon$, we have

$$\chi_\eta\left(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon} \right) = \psi\left(\frac{\varepsilon^m}{\eta^m} \|w\|_{m,\alpha-B}^m \right).$$

It is easy to see from Taylor expansion for ψ that, for any $\eta > 0$ and any $M \in \mathbf{N}$, the following asymptotics holds;

$$\chi_\eta\left(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon} \right) = 1 + O(\varepsilon^M) \quad \text{in } \mathbf{D}_\infty \text{ as } \varepsilon \searrow 0. \tag{8.6}$$

The following lemma states that only the paths sufficiently close to the minimizer $\bar{\gamma}$ contribute to the asymptotics.

Lemma 8.1. *Assume* (A1) *and* (A2). *Then, for any $\eta > 0$, there exists $c = c_\eta > 0$ such that*

$$0 \le \mathbb{E}\big[(1 - \chi_\eta(\varepsilon, w)) \cdot \delta_{a'}(y_1^\varepsilon)\big] = O\left(\exp\left\{ - \frac{\|\bar{\gamma}\|_{\mathcal{H}}^2 + c}{2\varepsilon^2} \right\} \right) \quad \textit{as } \varepsilon \searrow 0.$$

Proof. We take $\eta' > 0$ arbitrarily and we will fix it for a while. It is obvious that

$$0 \le \mathbb{E}\big[(1 - \chi_\eta(\varepsilon, w)) \cdot \delta_{a'}(y_1^\varepsilon)\big] = \mathbb{E}\left[(1 - \chi_\eta(\varepsilon, w)) \psi\left(\frac{|y_1^\varepsilon - a'|^2}{\eta'^2} \right) \cdot \delta_{a'}(y_1^\varepsilon) \right]. \tag{8.7}$$

Set $g(u) = u \vee 0$ for $u \in \mathbf{R}$. Then, in the sense of distributional derivative, $g''(u) = \delta_0$. Take a bounded continuous function $C : \mathbf{R}^n \to \mathbf{R}$ such that $C(u_1, \ldots, u_n) = g(u_1 - a'_1) g(u_2 - a'_2) \cdots g(u_n - a'_n)$ if $|u - a'| \le 2\eta'$. Then, the right hand side of (8.7) is equal to

$$\mathbb{E}\Big[(1-\psi)\Big(\frac{1}{\eta^m}\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B}\Big) \cdot \psi\Big(\frac{|y^\varepsilon_1 - a'|^2}{\eta'^2}\Big) \cdot (\partial^2_1 \cdots \partial^2_n C)(y^\varepsilon_1)\Big]. \tag{8.8}$$

Now, we use integration by parts for (generalized) Wiener functionals as in pp. 6–7, [**21**] to see that (8.8) is equal to a finite sum of the following form;

$$\sum_{j,k} \mathbb{E}\Big[F_{j,k}(\varepsilon, w) \cdot (1-\psi)^{(j)}\Big(\frac{1}{\eta^m}\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B}\Big) \cdot \psi^{(k)}\Big(\frac{|y^\varepsilon_1 - a'|^2}{\eta'^2}\Big) \cdot C(y^\varepsilon_1)\Big]. \tag{8.9}$$

Here, $F_{j,k}(\varepsilon, w)$ is a polynomial in components of (i) y^ε_1 and its derivatives, (ii) $\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B}$ and its derivatives, (iii) $\tau(\varepsilon)$, which is Malliavin covariance matrix of y^ε_1, and its derivatives, and (iv) $\kappa(\varepsilon) := \tau(\varepsilon)^{-1}$. Note that the derivatives of $\kappa(\varepsilon)$ do not appear.

From Proposition 5.3, there exists $r' > 0$ such that $|\kappa^{ij}(\varepsilon)| = O(\varepsilon^{-r'})$ in L^q as $\varepsilon \searrow 0$ for all $1 < q < \infty$. (Recall a well-known formula to obtain the inverse matrix A^{-1} with the adjugate matrix of A divided by $\det A$.) Therefore, there exists $r > 0$ such that $|F_{j,k}(\varepsilon)| = O(\varepsilon^{-r})$ in L^q as $\varepsilon \searrow 0$ for all $1 < q < \infty$.

By Hölder's inequality, (8.9) is dominated by

$$\begin{aligned} &\frac{c}{\varepsilon^r} \sum_{j,k} \mathbb{E}\Big[\Big|(1-\psi)^{(j)}\Big(\frac{1}{\eta^m}\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B}\Big)\Big|^{q'} \Big|\psi^{(k)}\Big(\frac{|y^\varepsilon_1 - a'|^2}{\eta'^2}\Big)\Big|^{q'}\Big]^{1/q'} \\ &\le \frac{c}{\varepsilon^r} \mu\Big[\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B} \ge \frac{\eta^m}{2},\ |y^\varepsilon_1 - a'| \le \eta'\Big]^{1/q'}. \end{aligned} \tag{8.10}$$

Here, $1/q + 1/q' = 1$ and $c = c(q, q', \eta, \eta')$ is a positive constant, which may change from line to line.

Since we may let $q' \searrow 1$ after taking $\limsup$, we obtain the following;

$$\begin{aligned} &\limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{E}\big[(1 - \chi_\eta(\varepsilon, w)) \cdot \delta_{a'}(y^\varepsilon_1)\big] \\ &\quad \le \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \mu\Big[\|\varepsilon w - \bar{\gamma}\|^m_{m,\alpha-B} \ge \frac{\eta^m}{2},\ |y^\varepsilon_1 - a'| \le \eta'\Big] \\ &\quad = \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \hat{\mu}^\varepsilon\Big[\Big\{(w, l) \in \mathcal{W}_B \oplus \mathbf{R}\langle\lambda\rangle \ \Big|\ \|w - \bar{\gamma}\|^m_{m,\alpha-B} \ge \frac{\eta^m}{2},\ |I(w, l)_1 - a'| \le \eta'\Big\}\Big] \\ &\quad \le -\inf\Big\{\frac{\|\gamma\|^2_{\mathcal{H}}}{2} \ \Big|\ \|\gamma - \bar{\gamma}\|^m_{m,\alpha-B} \ge \frac{\eta^m}{2},\ |\phi^0[\gamma]_1 - a'| \le \eta'\Big\}. \end{aligned} \tag{8.11}$$

Here, I denotes the Itô map corresponding to ODE (5.1) and we have used the large deviation for the last inequality. (Note that continuity of Itô map is used.) Recall that $\phi^0[\gamma] = I(\gamma, \mathbf{0})$ is given by ODE (7.2).

Now let η' tend to 0. As η' decreases, the right hand side of (8.11) decreases. The proof is finished if the limit is strictly smaller than $-\|\bar{\gamma}\|_{\mathcal{H}}^2/2$. Assume otherwise. Then, there exists $\{\gamma_k\}_{k=1}^{\infty} \subset \mathcal{H}$ such that

$$\|\gamma_k - \bar{\gamma}\|_{m,\alpha-B}^m \geq \frac{\eta^m}{2}, \quad |\phi^0[\gamma_k]_1 - a'| \leq \frac{1}{k}, \text{ and, } \liminf_{k\to\infty}\Big(-\frac{\|\gamma_k\|_{\mathcal{H}}^2}{2}\Big) \geq -\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2}.$$

In particular, $\{\gamma_k\}$ is bounded in $\mathcal{H}$ and, hence, precompact in $\mathcal{W}_B$. Let γ_∞ be any limit point. For simplicity, a subsequence that converges to γ_∞ is again denoted by $\{\gamma_k\}$. Since $\gamma \mapsto \phi^0[\gamma]_1$ is continuous with respect to the topology of $\mathcal{W}_B$, we see that $\phi^0[\gamma_\infty]_1 = a'$ holds. Also, we have $\|\gamma_\infty - \bar{\gamma}\|_{m,\alpha-B}^m \geq \eta^m/2$. So, $\gamma_\infty \neq \bar{\gamma}$. From the lower semicontinuity of the rate function, we see that $\gamma_\infty \in \mathcal{H}$ and $\|\gamma_\infty\|_{\mathcal{H}}^2/2 \leq \|\bar{\gamma}\|_{\mathcal{H}}^2/2$. This clearly contradicts Assumption (A2). □

8.2. Integrability lemmas.

In this subsection, we prove a few lemmas for integrability of Wiener functionals of exponential type which will be used in the short time asymptotic expansion.

Throughout this subsection we assume (A2). Let $\bar{\gamma}$ be as in (A2) and let ϕ^{κ_j} and $R^{\kappa_j+,\varepsilon} = R^{\kappa_{j+1},\varepsilon}$ $(j = 0, 1, 2, \ldots)$ be as in Section 7 with $\gamma = \bar{\gamma}$. First we consider

$$\begin{aligned}\frac{R^{2+,\varepsilon}}{\varepsilon^2} &= \frac{1}{\varepsilon^2}(\tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1 - \varepsilon^{1/H}\phi^{1/H} - \varepsilon^2\phi^2) \\ &= \varepsilon^{\kappa_4-2}\phi^{\kappa_4} + \varepsilon^{\kappa_5-2}\phi^{\kappa_5} + \cdots .\end{aligned}$$

Here, $\kappa_4 = 1 + (1/H)$ and $\kappa_5 = 3 \wedge (2/H)$.

Lemma 8.2. *Assume* (A2). *For any $M > 0$, there exists $\eta > 0$ such that*

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\big[\exp\big(M\langle\bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2\big)I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}}\big] < \infty.$$

Proof. By Lemma 7.3, if $\|\varepsilon w\|_{\alpha-hld} \leq 1$, then there exists a constant $c_1, c_2 > 0$ such that

$$\|R^{2+,\varepsilon}\|_{\alpha-hld} \leq c_1(\varepsilon + \|\varepsilon w\|_{\alpha-hld})^{1+(1/H)} \leq c_2(\varepsilon + \|\varepsilon w\|_{m,\alpha-B})^{1+(1/H)}.$$

Hence, if $\|\varepsilon w\|_{m,\alpha-B} \leq \eta \leq 1$, then

$$\|R^{2+,\varepsilon}\|_{\alpha-hld}/\varepsilon^2 \leq c_2(1 + \|w\|_{m,\alpha-B})^2(\varepsilon + \eta)^{(1/H)-1}.$$

Recall that, by Fernique's theorem, there exists a positive constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta(1 + \|w\|_{m,\alpha-B})^2)] < \infty$. Take $0 < \eta \leq 1$ so that $M|\bar{\nu}|c_2(2\eta)^{(1/H)-1} \leq \beta$. Then, we see that

$$\sup_{0<\varepsilon\leq\eta} \mathbb{E}\big[\exp\big(M\langle\bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2\big)I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}}\big] < \infty.$$

Note that, if $\|\varepsilon w\|_{m,\alpha-B} \leq \eta$ and $\eta \leq \varepsilon \leq 1$, then $\|R^{2+,\varepsilon}\|_{\alpha-hld}/\varepsilon^2$ is bounded. This completes the proof. □

Next we consider

$$\frac{R^{1+,\varepsilon}}{\varepsilon} = \frac{1}{\varepsilon}(\tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1) = \varepsilon^{(1/H)-1}\phi^{1/H} + \varepsilon^1\phi^2 + \cdots.$$

Lemma 8.3. *Assume* (A2). *For any $M > 0$, there exists $\eta > 0$ such that*

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\big[\exp\big(M\|R^{1+,\varepsilon}\|^2_{\alpha-hld}/\varepsilon^2\big)I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}}\big] < \infty.$$

Proof. By Lemma 7.3, if $\|\varepsilon w\|_{\alpha-hld} \leq 1$, then there exists a constant $c_1 > 0$ such that

$$\|R^{1+,\varepsilon}\|_{\alpha-hld} \leq c_1(\varepsilon + \|\varepsilon w\|_{\alpha-hld})^{1/H} \leq c_2(\varepsilon + \|\varepsilon w\|_{m,\alpha-B})^{1/H}.$$

Hence, if $\|\varepsilon w\|_{m,\alpha-B} \leq \eta \leq 1$, then

$$\|R^{2+,\varepsilon}\|^2_{\alpha-hld}/\varepsilon^2 \leq c_2(1 + \|w\|_{m,\alpha-B})^2(\varepsilon + \eta)^{(2/H)-2}.$$

Then, we can prove the lemma in the same way as in Lemma 8.2. □

From now on we assume (A1) and (A2). In addition, we introduce the following assumption;

(A3)': $\quad\mathbb{E}[\exp(\langle\bar{\nu}, \phi_1^2\rangle) \mid \phi_1^1 = 0] < \infty.$

For all $1 \leq j \leq n$, $\phi_1^{1,j} \in \mathcal{W}_B^* \subset \mathcal{H}^*$. When we regard $\phi_1^{1,j}$ as an element of $\mathcal{H}$ by Riesz isometry, we write $\sharp\phi_1^{1,j} \in \mathcal{H} \subset \mathcal{W}_B$. We have an orthogonal decomposition $\mathcal{H} = \ker\phi_1^1 \oplus (\ker\phi_1^1)^\perp$. We denote by π the orthogonal projection from $\mathcal{H}$ onto $\ker\phi_1^1$. Note that $(\ker\phi_1^1)^\perp$ is an n-dimensional linear subspace spanned by $\{\sharp\phi_1^{1,1}, \ldots, \sharp\phi_1^{1,n}\}$. Since $\dim(\ker\phi_1^1)^\perp < \infty$, the abstract Wiener space splits into two; $\mathcal{W}_B = \overline{\ker\phi_1^1}^{\|\cdot\|_{m,\alpha-B}} \oplus (\ker\phi_1^1)^\perp$. The projection π naturally extends to the one from $\mathcal{W}_B$ onto $\overline{\ker\phi_1^1}^{\|\cdot\|_{m,\alpha-B}}$, which is again denoted by the same symbol. There exist Gaussian measures μ_1 and μ_2 such that $(\overline{\ker\phi_1^1}^{\|\cdot\|_{m,\alpha-B}}, \ker\phi_1^1, \mu_1)$ and $((\ker\phi_1^1)^\perp, (\ker\phi_1^1)^\perp, \mu_2)$ are abstract Wiener spaces. Naturally, $\mu_1 = \pi_*\mu$, $\mu_2 = \pi_*^\perp\mu$ and $\mu = \mu_1 \times \mu_2$ (the product measure). One may think μ_1 is the definition of the conditional measure $\mathbb{P}[\,\cdot \mid \phi_1^1 = 0]$ in (A3)' above.

Therefore, (A3)' is equivalent to the following;

$$\mathbb{E}[\exp(\langle\bar{\nu}, \phi_1^2 \circ \pi\rangle)] < \infty. \tag{8.12}$$

Set

$$\psi(w,w') = \frac{1}{2}\tilde{J}(\bar{\gamma})_1 \int_0^1 \tilde{J}(\bar{\gamma})_t^{-1}\{\nabla\sigma(\phi_t^0)\langle\phi_t^1(w'), dw_t\rangle + \nabla\sigma(\phi_t^0)\langle\phi_t^1(w), dw'_t\rangle\}$$
$$+ \frac{1}{2}\tilde{J}(\bar{\gamma})_1 \int_0^1 \tilde{J}(\bar{\gamma})_t^{-1}\nabla^2\sigma(\phi_t^0)\langle\phi_t^1(w), \phi_t^1(w'), d\bar{\gamma}_t\rangle, \tag{8.13}$$

where $\phi_T^1(w) = \tilde{J}(\bar{\gamma})_T \int_0^T \tilde{J}(\bar{\gamma})_t^{-1}\sigma(\phi_t^0)dw_t$. Then, ψ is a bounded bilinear mapping on $\mathcal{W}_B$ and so is $\psi\langle\pi\cdot,\pi\cdot\rangle$. Clearly, $\psi(w,w) = \phi_1^2(w)$ and $\psi(\pi w, \pi w) = \phi_1^2(\pi w)$. By Goodman's theorem (see Theorem 4.6, p. 83, [**10**]), restricted on $\mathcal{H}\times\mathcal{H}$, $\langle\bar{\nu}, \psi\langle\pi\cdot,\pi\cdot\rangle\rangle$ is of trace class and, in particular, Hilbert–Schmidt. The corresponding trace class operator on $\mathcal{H}$ and corresponding element of the second Wiener chaos are denoted by A and Ξ_A, respectively. Then, $\langle\bar{\nu}, \phi_1^2(\pi w)\rangle = \Xi_A(w) + \mathrm{Tr}(A)$. Hence, (8.12) is equivalent to $\mathbb{E}[\exp(\Xi_A)] < \infty$, which in turn is equivalent to $\sup\mathrm{Spec}(A) < 1/2$. Since the inequality is strict, there exists $r > 1$ such that $\sup\mathrm{Spec}(rA) < 1/2$. This implies $\mathbb{E}[\exp(\Xi_{rA})] = \mathbb{E}[\exp(r\Xi_A)] < \infty$. Summing it up, we have seen that (A3)' is equivalent to the following;

$$\mathbb{E}[\exp(r\langle\bar{\nu}, \phi_1^2\circ\pi\rangle)] < \infty \quad \text{for some } r > 1. \tag{8.14}$$

Let us check here that (A3) and (A3)' are equivalent under (A1), (A2).

PROPOSITION 8.4. *Under* (A1) *and* (A2)*, the two conditions* (A3) *and* (A3)' *are equivalent.*

PROOF. As is explained above, $(A3)'$ is equivalent to $\sup\mathrm{Spec}(A) < 1/2$. Keep in mind that the only accumulation point of $\mathrm{Spec}(A)$ is 0, since A is of trace class. Let $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ be a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$ as in (A3). Then, a straight forward calculation shows that

$$\begin{aligned}
\frac{d^2}{du^2}\bigg|_{u=0} \frac{\|f(u)\|_{\mathcal{H}}^2}{2} &= \frac{d^2}{du^2}\bigg|_{u=0}\left(\frac{\|f(u)\|_{\mathcal{H}}^2}{2} - \langle\bar{\nu}, \phi_1^0(f_u) - a'\rangle\right) \\
&= \|f'(0)\|_{\mathcal{H}}^2 + \langle f''(0), \bar{\gamma}\rangle_{\mathcal{H}} - \big\langle\bar{\nu}, D\phi_1^0(\bar{\gamma})\langle f''(0)\rangle\big\rangle - \big\langle\bar{\nu}, D^2\phi_1^0(\bar{\gamma})\langle f'(0), f'(0)\rangle\big\rangle \\
&= \|f'(0)\|_{\mathcal{H}}^2 - \big\langle\bar{\nu}, D^2\phi_1^0(\bar{\gamma})\langle\pi f'(0), \pi f'(0)\rangle\big\rangle \\
&= \|f'(0)\|_{\mathcal{H}}^2 - 2\big\langle\bar{\nu}, \psi\langle\pi f'(0), \pi f'(0)\rangle\big\rangle,
\end{aligned} \tag{8.15}$$

where we used (8.2)–(8.3) and the fact that $f'(0)$ is tangent to the submanifold $K_a^{a'}$. Since $f'(0)$ can be any non-zero element in $\mathrm{Im}\,\pi$, $\sup\mathrm{Spec}(A) < 1/2$ is equivalent to that right hand side of (8.15) is strictly positive, that is (A3). □

The following is a key technical lemma. It states that, restricted on a sufficiently small subset, $\exp(\langle\bar{\nu}, R_1^{2,\varepsilon}\rangle/\varepsilon^2) \in \cup_{1<q<\infty} L^q$ uniformly in ε.

LEMMA 8.5. *Assume* (A1), (A2) *and* (A3). *Then, there exists* $r_1 > 1$ *and* $\eta > 0$ *such that*

$$\sup_{0<\varepsilon\le 1} \mathbb{E}\big[\exp\big(r_1\langle\bar{\nu}, R_1^{2,\varepsilon}\rangle/\varepsilon^2\big) I_{\{\|\varepsilon w\|_{m,\alpha-B}\le\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\le\eta_1\}}\big] < \infty$$

for any $\eta_1 > 0$.

Proof. By Lemma 8.2 and the relation $R_1^{2,\varepsilon}/\varepsilon^2 = \phi_1^2 + R_1^{2+,\varepsilon}/\varepsilon^2$, it is sufficient to show that

$$\sup_{0<\varepsilon\le 1} \mathbb{E}\big[\exp\big(r_1\langle\bar{\nu}, \phi_1^2\rangle\big) I_{\{\|\varepsilon w\|_{m,\alpha-B}\le\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\le\eta_1\}}\big] < \infty. \tag{8.16}$$

We give an explicit expression for the projection π. Set $C_{jj'} = \langle\phi_1^{1,j}, \phi_1^{1,j'}\rangle_{\mathcal{H}^*}$ and $C = (C_{jj'})_{1\le j,j'\le n} \in GL(n, \mathbf{R})$. The components of its inverse is denoted by $C^{-1} = (D_{jj'})_{1\le j,j'\le n}$. By straight forward calculation, $\pi : \mathcal{H} \to \ker\phi_1^1$ is given by

$$\pi h = h - \sum_{j,j'} {}_{\mathcal{H}^*}\langle\phi_1^{1,j}, h\rangle_{\mathcal{H}} D_{jj'} \cdot {}^\sharp\phi_1^{1,j'}.$$

From this, it is easy to see that $\pi : \mathcal{W}_B \to \overline{\ker\phi_1^1}$ is given by

$$\pi w = w - \sum_{j,j'} \phi_1^{1,j}(w) D_{jj'} \cdot {}^\sharp\phi_1^{1,j'}. \tag{8.17}$$

Then, we have

$$\begin{aligned}\phi_1^2(w) = \psi\langle w, w\rangle &= \phi_1^2(\pi w) + 2\sum_{j,j'} \phi_1^{1,j}(w) D_{jj'} \cdot \psi\langle w, {}^\sharp\phi_1^{1,j'}\rangle \\ &\quad + \sum_{j,j',k,k'} \phi_1^{1,j}(w)\phi_1^{1,k}(w) D_{jj'}D_{kk'} \cdot \psi\langle {}^\sharp\phi_1^{1,j'}, {}^\sharp\phi_1^{1,k'}\rangle =: J_1 + J_2 + J_3. \end{aligned} \tag{8.18}$$

Exponential integrability of the first term J_1 on the right hand side of (8.18) is given in (8.14). So, we estimate the second term J_2. Since $\varepsilon\phi_1^1(w) = R_1^{1+,\varepsilon}(w) - R_1^{1,\varepsilon}(w)$,

$$\begin{aligned}\big|\phi_1^{1,j}(w)\psi\langle w, {}^\sharp\phi_1^{1,j'}\rangle\big| &\le c_1\bigg\{\bigg|\frac{R_1^{1+,\varepsilon}(w)}{\varepsilon}\bigg| + \bigg|\frac{R_1^{1,\varepsilon}(w)}{\varepsilon}\bigg|\bigg\}\|w\|_{m,\alpha-B} \\ &\le c_1\bigg\{\bigg|\frac{c' R_1^{1+,\varepsilon}(w)}{\varepsilon}\bigg|^2 + \frac{\|w\|_{m,\alpha-B}^2}{4c'^2}\bigg\} + c_1\bigg|\frac{R_1^{1,\varepsilon}(w)}{\varepsilon}\bigg|\|w\|_{m,\alpha-B}\end{aligned}$$

for any $c' > 0$.

Set $c_2 = 2c_1 n^2 \sup_{j,j'}|D_{j,j'}|$ and let $M > 0$. Then, by Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\big[e^{M|J_2|} I_{\{\|\varepsilon w\|_{m,\alpha-B}\le\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\le\eta_1\}}\big] \\ &\quad\le \mathbb{E}\big[\exp\big(3Mc_2c'^2|R_1^{1+,\varepsilon}/\varepsilon|^2\big) I_{\{\|\varepsilon w\|_{m,\alpha-B}\le\eta\}}\big]^{1/3} \\ &\qquad\times \mathbb{E}\big[e^{3Mc_2\|w\|_{m,\alpha-B}^2/(4c')}\big]^{1/3} \mathbb{E}\big[e^{3Mc_2\eta_1\|w\|_{m,\alpha-B}}\big]^{1/3}.\end{aligned}$$

For any $M > 0$ and $\eta_1 > 0$, the third factor is integrable. If c' is chosen sufficiently large, then the second factor is also integrable by Fernique's theorem. By Lemma 8.3, there exists $\eta > 0$ such that $\sup_\varepsilon$ of the first factor is finite and, hence,

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\big[e^{M|J_2|} I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\leq\eta_1\}}\big] < \infty. \tag{8.19}$$

Since $\phi_1^{1,j}(w)\phi_1^{1,k}(w) = \varepsilon^{-1}\{R_1^{1+,\varepsilon}(w)^j - R_1^{1,\varepsilon}(w)^j\}\phi_1^{1,k}(w)$, we can deal with J_3 in the same way. For any $M > 0$ and $\eta_1 > 0$, there exists $\eta > 0$ such that

$$\sup_{0<\varepsilon\leq 1} \mathbb{E}\big[e^{M|J_3|} I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\leq\eta_1\}}\big] < \infty. \tag{8.20}$$

Let $r > 1$ be as in (8.14). Set $r_1 = (1 + r)/2 > 1$, $q = 2r/(1 + r) > 1$, and $1/q + 1/q' = 1$. Then, from Hölder's inequality and (8.14), (8.18)–(8.20), we can easily see that

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(r_1\langle\bar{\nu}, \phi_1^2\rangle\big) I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\leq\eta_1\}}\big] \\
&\quad\leq \mathbb{E}\big[\exp\big(r\langle\bar{\nu}, \phi_1^2\circ\pi\rangle\big)\big]^{1/q} \prod_{i=1}^{2} \mathbb{E}\big[e^{2q' r_1|\bar{\nu}||J_i|} I_{\{\|\varepsilon w\|_{m,\alpha-B}\leq\eta\}} I_{\{|R_1^{1,\varepsilon}/\varepsilon|\leq\eta_1\}}\big]^{1/(2q')}.
\end{aligned}$$

From this, (8.16) is immediate. This completes the proof. □

8.3. Proof of off-diagonal short time asymptotics.

In this subsection we prove Theorem 2.2, namely, off-diagonal short time asymptotics of the density of the solution $(y_t) = (y_t(a))$ of Young ODE (5.1) driven by fBm (w_t) with $1/2 < H < 1$ under Assumptions (A1)–(A3).

First, let us calculate the kernel $p(t, a, a')$. Take $\eta > 0$ as in Lemma 8.5. Then, we see

$$\begin{aligned}
p(\varepsilon^{1/H}, a, a') &= \mathbb{E}\big[\delta_{a'}(y_1^\varepsilon)\big] \\
&= \mathbb{E}\big[\delta_{a'}(y_1^\varepsilon)\chi_\eta(\varepsilon, w)\big] + \mathbb{E}\big[\delta_{a'}(y_1^\varepsilon)\{1 - \chi_\eta(\varepsilon, w)\}\big] =: I_1 + I_2.
\end{aligned}$$

As we have shown in Lemma 8.1, the second term I_2 on the right hand side does not contribute to the asymptotic expansion. So, we have only to calculate the first term I_1. By Cameron–Martin formula,

$$I_1 = \mathbb{E}\bigg[\exp\bigg(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2} - \frac{1}{\varepsilon}\langle\bar{\gamma}, w\rangle\bigg)\delta_{a'}(\tilde{y}_1^\varepsilon)\chi_\eta\bigg(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}\bigg)\bigg].$$

Recall that $\langle\bar{\gamma}, w\rangle = \langle\bar{\nu}, \phi_1^1(w)\rangle$ for all w. Hence, noting that $\phi^{1/H}$ is non-random, we have

$$
\begin{aligned}
I_1 &= \exp\Big(-\frac{\|\bar\gamma\|_{\mathcal H}^2}{2\varepsilon^2}\Big)\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}\langle\bar\nu,\phi_1^1\rangle\Big)\delta_{a'}(a'+\varepsilon\phi_1^1+\varepsilon^{1/H}\phi_1^{1/H}+R_1^{2,\varepsilon})\chi_\eta\Big(\varepsilon,w+\frac{\bar\gamma}{\varepsilon}\Big)\Big] \\
&= \frac{1}{\varepsilon^n}\exp\Big(-\frac{\|\bar\gamma\|_{\mathcal H}^2}{2\varepsilon^2}\Big) \\
&\quad\times\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}\langle\bar\nu,\phi_1^1\rangle\Big)\delta_0(\phi_1^1+\varepsilon^{(1/H)-1}\phi_1^{1/H}+\varepsilon^{-1}R_1^{2,\varepsilon})\chi_\eta\Big(\varepsilon,w+\frac{\bar\gamma}{\varepsilon}\Big)\Big] \\
&= \frac{1}{\varepsilon^n}\exp\Big(-\frac{\|\bar\gamma\|_{\mathcal H}^2}{2\varepsilon^2}+\frac{\langle\bar\nu,\phi_1^{1/H}\rangle}{\varepsilon^{2-(1/H)}}\Big) \\
&\quad\times\mathbb{E}\Big[\exp\Big(\langle\bar\nu,R_1^{2,\varepsilon}\rangle/\varepsilon^2\Big)\delta_0(\phi_1^1+\varepsilon^{(1/H)-1}\phi_1^{1/H}+\varepsilon^{-1}R_1^{2,\varepsilon})\chi_\eta\Big(\varepsilon,w+\frac{\bar\gamma}{\varepsilon}\Big)\Big] \\
&= \frac{1}{\varepsilon^n}\exp\Big(-\frac{\|\bar\gamma\|_{\mathcal H}^2}{2\varepsilon^2}+\frac{\langle\bar\nu,\phi_1^{1/H}\rangle}{\varepsilon^{2-(1/H)}}\Big)\mathbb{E}\Big[F(\varepsilon,w)\delta_0\Big(\frac{\tilde y_1^\varepsilon-a'}{\varepsilon}\Big)\Big],
\end{aligned}
$$

where

$$
F(\varepsilon,w)=\exp\big(\varepsilon^{-2}\langle\bar\nu,R_1^{2,\varepsilon}\rangle\big)\chi_\eta\Big(\varepsilon,w+\frac{\bar\gamma}{\varepsilon}\Big)\psi\Big(\frac{1}{\eta_1^2}\Big|\frac{\tilde y_1^\varepsilon-a'}{\varepsilon}\Big|^2\Big) \tag{8.21}
$$

for any positive constant η_1. It is easy to see that (i) $\chi_\eta(\varepsilon,w+\bar\gamma/\varepsilon)$ and its derivatives vanish outside $\{\|\varepsilon w\|_{m,\alpha-B}\le\eta\}$ and (ii) $\psi\big(\eta_1^{-2}\big|(\tilde y_1^\varepsilon-a')/\varepsilon\big|^2\big)$ and its derivatives vanish outside $\{|R_1^{1,\varepsilon}/\varepsilon|\le\eta_1\}$. Hence, by Lemma 8.5, $F(\varepsilon,w)\in\tilde{\mathbf{D}}_\infty$ and $F(\varepsilon,w)=O(1)$ with respect to that topology. Roughly speaking, since $\delta_0((\tilde y_1^\varepsilon-a')/\varepsilon)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_{-\infty}$, the problem reduces to whether $F(\varepsilon,w)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_\infty$.

Lemma 8.6. *Assume* (A1)–(A3). *For any $M\in\mathbf{N}$, we have*

$$
\mathbb{E}\Big[F(\varepsilon,w)\delta_0\Big(\frac{\tilde y_1^\varepsilon-a'}{\varepsilon}\Big)\Big]=\mathbb{E}\Big[F(\varepsilon,w)\psi(|\phi_1^1/\eta_1|^2)\delta_0\Big(\frac{\tilde y_1^\varepsilon-a'}{\varepsilon}\Big)\Big]+O(\varepsilon^M)
$$

as $\varepsilon\searrow 0$.

Proof. By using Taylor expansion for ψ, we see that, for given M, there exist $m\in\mathbf{N}$ and $G_j(\varepsilon,w)\in\mathbf{D}_\infty$ $(1\le j\le m)$ such that

$$
\psi\Big(\frac{1}{\eta_1^2}\Big|\frac{\tilde y_1^\varepsilon-a'}{\varepsilon}\Big|^2\Big)=\psi\Big(\Big|\frac{\phi_1^1}{\eta_1}\Big|^2\Big)+\sum_{j=1}^m\psi^{(j)}\Big(\Big|\frac{\phi_1^1}{\eta_1}\Big|^2\Big)G_j(\varepsilon,w)+O(\varepsilon^M) \tag{8.22}
$$

in $\mathbf{D}_\infty$ as $\varepsilon\searrow 0$. $G_j(\varepsilon,w)=O(1)$, but its explicit form is not important. Note that $\psi^{(j)}(|\phi_1^1/\eta_1|^2)T(\phi_1^1)=0$ if $j\ge 1$ and $\mathrm{supp}(T)\subset\{a\in\mathbf{R}^n\mid |a|<\eta_1/2\}$.

By Theorem 4.3 and Proposition 5.3, $\delta_0((\tilde y_1^\varepsilon-a')/\varepsilon)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_{-\infty}$ as follows. As before, we set $\{0=\nu_0<\nu_1<\nu_2<\cdots\}$ to be all the elements of Λ_3 in increasing order. For given M, let $l \in \mathbf{N}$ be the smallest integer such that $M \le \nu_{l+1}$. Then, for some $\Phi_{\nu_j} \in \tilde{\mathbf{D}}_{-\infty}$ $(1 \le j \le l)$, it holds that

$$\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon) = \delta_0(\phi_1^1) + \varepsilon^{\nu_1}\Phi_{\nu_1} + \cdots + \varepsilon^{\nu_l}\Phi_{\nu_l} + O(\varepsilon^{\nu_{l+1}}) \tag{8.23}$$

in $\tilde{\mathbf{D}}_{-\infty}$ as $\varepsilon \searrow 0$. Here, Φ_{ν_j} is a finite linear combination of terms of the form

$$\partial^\alpha \delta_0(\phi_1^1) \times \{\text{a polynomial of the components of } \phi_1^{\kappa_i}\text{ 's}\}.$$

Hence, $\psi^{(j')}(|\phi_1^1/\eta_1|^2)\Phi_{\nu_j}$ vanish for all j, j'.

Now, using (8.22) and (8.23), we prove the lemma.

$$\begin{aligned}
&\mathbb{E}\big[F(\varepsilon, w)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\big] \\
&\quad = \mathbb{E}\bigg[F(\varepsilon, w)\psi\bigg(\frac{1}{\eta_1^2}\bigg|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\bigg|^2\bigg)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\bigg] \\
&\quad = \mathbb{E}\big[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\big] \\
&\qquad + \mathbb{E}\bigg[F(\varepsilon, w)\bigg(\sum_{j=1}^m \psi^{(j)}\bigg(\bigg|\frac{\phi_1^1}{\eta_1}\bigg|^2\bigg)G_j(\varepsilon, w)\bigg)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\bigg] + O(\varepsilon^M) \\
&\quad = \mathbb{E}\big[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\big] \\
&\qquad + \mathbb{E}\bigg[F(\varepsilon, w)\bigg(\sum_{j=1}^m \psi^{(j)}\bigg(\bigg|\frac{\phi_1^1}{\eta_1}\bigg|^2\bigg)G_j(\varepsilon, w)\bigg)\big(\delta_0(\phi_1^1) + \cdots + \varepsilon^{\nu_l}\Phi_{\nu_l}\big)\bigg] + O(\varepsilon^M) \\
&\quad = \mathbb{E}\big[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)\big] + O(\varepsilon^M).
\end{aligned}$$

Thus, we have shown the lemma. □

Set $\Lambda_2' = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1, 1/H\}\} = \{0 < H^{-1} - 1 < (3 \wedge 2H^{-1}) - 2 < \cdots\}$. Next we set $\Lambda_3' = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \ldots, a_m \in \Lambda_2'\}$. In the following lemma, $\{0 = \rho_0 < \rho_1 < \rho_2 < \cdots\}$ stands for all the elements of Λ_3' in increasing order.

Lemma 8.7. *Assume* (A1)–(A3) *and let* $F(\varepsilon, w) \in \tilde{\mathbf{D}}_\infty$ *as in* (8.21). *Then, for every* $k = 1, 2, 3, \ldots$,

$$\begin{aligned}
&F(\varepsilon, w)\psi(|\phi_1^1(w)/\eta_1|^2) \\
&\quad = \exp\big(\langle \bar{\nu}, \phi_1^2(w)\rangle\big)\psi(|\phi_1^1(w)/\eta_1|^2)^2\{1 + \varepsilon^{\rho_1}\gamma_{\rho_1}(w) + \cdots + \varepsilon^{\rho_k}\gamma_{\rho_k}(w)\} + F_{k+1}(\varepsilon, w),
\end{aligned}$$

where $F_{k+1}(\varepsilon, w) \in \tilde{\mathbf{D}}_\infty$ *satisfies that*

$$F_{k+1}(\varepsilon, w)T(\phi_1^1) = O(\varepsilon^{\rho_{k+1}}) \quad \text{in } \mathbf{D}_{-\infty} \text{ as } \varepsilon \searrow 0$$

for any $T \in \mathcal{S}'(\mathbf{R}^n)$ *with* $\mathrm{supp}(T) \subset \{a \in \mathbf{R}^n \mid |a| \le \eta_1/2\}$. *Moreover,* $\gamma_{\rho_j} \in \mathbf{D}_\infty$

$(j=1,2,\ldots)$ *are determined by the following formal expansion* $(\kappa_4 = H^{-1}+1)$;

$$\sum_{m=0}^{\infty} \frac{\langle \bar{\nu}, R_1^{2+,\varepsilon}/\varepsilon^2\rangle^m}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!}\bigl\{\varepsilon^{\kappa_4-2}\langle \bar{\nu}, \phi_1^{\kappa_4}\rangle + \varepsilon^{\kappa_5-2}\langle \bar{\nu}, \phi_1^{\kappa_5}\rangle + \cdots\bigr\}^m$$
$$= 1 + \varepsilon^{\rho_1}\gamma_{\rho_1} + \varepsilon^{\rho_2}\gamma_{\rho_2} + \cdots .$$

Proof. Let $r_1 > 1$ be as in Lemma 8.5. First we show that, for any $\eta_1 > 0$,

$$\mathbb{E}\bigl[\exp\bigl(r_1\langle \bar{\nu}, \phi_1^2\rangle\bigr) I_{\{|\phi_1^1|\le \eta_1\}}\bigr] < \infty. \tag{8.24}$$

We can choose a subsequence $\{\varepsilon_k\}$ such that, as $k\to\infty$, $\varepsilon_k \searrow 0$ and $R_1^{1,\varepsilon_k}/\varepsilon_k \to \phi_1^1$ a.s. To prove (8.24), we apply Fatou's lemma to (8.16) with η_1 replaced by $2\eta_1$.

$$\infty > \liminf_{k\to\infty} \mathbb{E}\bigl[\exp\bigl(r_1\langle \bar{\nu}, \phi_1^2\rangle\bigr) I_{\{\|\varepsilon_k w\|_{m,\alpha-B}\le \eta\}} I_{\{|R_1^{1,\varepsilon_k}/\varepsilon_k|\le 2\eta_1\}}\bigr]$$
$$\ge \mathbb{E}\bigl[\exp\bigl(r_1\langle \bar{\nu}, \phi_1^2\rangle\bigr) \liminf_{k\to\infty} I_{\{|R_1^{1,\varepsilon_k}/\varepsilon_k|\le 2\eta_1\}}\bigr] \ge \mathbb{E}\bigl[\exp\bigl(r_1\langle \bar{\nu}, \phi_1^2\rangle\bigr) I_{\{|\phi_1^1|\le \eta_1\}}\bigr].$$

From (8.24), it is easy to check that $\exp\bigl(\langle \bar{\nu}, \phi_1^2(w)\rangle\bigr)\psi(|\phi_1^1(w)/\eta_1|^2) \in \tilde{\mathbf{D}}_\infty$.

Now we expand $\exp(\langle \bar{\nu}, R_1^{2,\varepsilon}\rangle/\varepsilon^2) = \exp(\langle \bar{\nu}, \phi_1^2(w)\rangle)\exp(\langle \bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2)$ in ε. Set $Q_{l+1} : \mathbf{R} \to \mathbf{R}$ by

$$Q_{l+1}(u) = e^u - \left(1 + u + \frac{u^2}{2!} + \cdots + \frac{u^l}{l!}\right) = u^{l+1}\int_0^1 \frac{(1-\theta)^l}{l!} e^{\theta u} d\theta \quad (u \in \mathbf{R}).$$

We will prove that, for sufficiently large $l \in \mathbf{N}$, as $\varepsilon \searrow 0$,

$$e^{\langle \bar{\nu}, \phi_1^2\rangle} Q_{l+1}(\langle \bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2)\chi_\eta\left(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}\right)\psi(|\phi_1^1(w)/\eta_1|^2) = O(\varepsilon^{\rho_{k+1}}) \quad \text{in } \tilde{\mathbf{D}}_\infty. \tag{8.25}$$

Note that $\chi_\eta(\varepsilon, w+\bar{\gamma}/\varepsilon) = O(1)$ in $\mathbf{D}_\infty$ as $\varepsilon \searrow 0$ by (8.6). By Proposition 7.2, $R_1^{2+,\varepsilon}/\varepsilon^2 = O(\varepsilon^{(1/H)-1})$ in $\mathbf{D}_\infty$. So, if $l+1 \ge \rho_{k+1}/\{(1/H)-1\}$, then $(\langle \bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2)^{l+1} = O(\varepsilon^{\rho_{k+1}})$ in $\mathbf{D}_\infty$. Therefore, in order to verify (8.25), it is sufficient to show that, as $\varepsilon \searrow 0$,

$$\int_0^1 (1-\theta)^l e^{\langle \bar{\nu}, \phi_1^2 + \theta R_1^{2+,\varepsilon}/\varepsilon^2\rangle} d\theta \cdot \chi_\eta\left(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}\right)\psi(|\phi_1^1(w)/\eta_1|^2) = O(1) \quad \text{in } \tilde{\mathbf{D}}_\infty. \tag{8.26}$$

To verify the integrability of this Wiener functional, note that $e^{\theta u} \le 1 + e^u$ for all $u \in \mathbf{R}$ and $0 \le \theta \le 1$. This implies that the first factor on the left hand side of (8.26) is dominated by $e^{\langle \bar{\nu}, \phi_1^2\rangle} + e^{\langle \bar{\nu}, R_1^{2,\varepsilon}\rangle/\varepsilon^2}$. From Lemma 8.5 and (8.24), we see that the left hand side of (8.26) is $O(1)$ in any L^q $(1 < q < \infty)$. In the same way, the Malliavin derivatives of the left hand side of (8.26) are $O(1)$ in any L^q.

It is easy to see that, as $\varepsilon \searrow 0$,

$$\sum_{k=0}^{l} \frac{\{\langle \bar{\nu}, R_1^{2+,\varepsilon}\rangle/\varepsilon^2\}^k}{k!} = 1 + \varepsilon^{\rho_1}\gamma_{\rho_1} + \cdots + \varepsilon^{\rho_k}\gamma_{\rho_k} + O(\varepsilon^{\rho_{k+1}}) \quad \text{in } \mathbf{D}_\infty. \tag{8.27}$$

From this and (8.6), we see that

$$\begin{aligned} &F(\varepsilon, w)\psi(|\phi_1^1(w)/\eta_1|^2) \\ &\quad = \exp\bigl(\langle \bar{\nu}, \phi_1^2(w)\bigr)\psi(|\phi_1^1(w)/\eta_1|^2)\psi\left(\frac{1}{\eta_1^2}\left|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right|^2\right)\{1 + \varepsilon^{\rho_1}\gamma_{\rho_1}(w) + \cdots + \varepsilon^{\rho_k}\gamma_{\rho_k}(w)\} \\ &\qquad + O(\varepsilon^{\rho_{k+1}}) \quad \text{in } \tilde{\mathbf{D}}_\infty. \end{aligned}$$

Using (8.22), we finish the proof. □

Proof of Theorem 2.2. Here we prove our main theorem in this paper. We set

$$\Lambda_4 = \Lambda_3 + \Lambda_3' = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda_3'\}.$$

We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ all the elements of Λ_4 in increasing order. There is no mystery why this index set appears in the short time expansion of the kernel because, very formally speaking, the problem reduces to finding asymptotic behavior of $\mathbb{E}[\exp(\langle \bar{\nu}, R_1^{2,\varepsilon}\rangle/\varepsilon^2)\cdot\delta_0(R_1^{1,\varepsilon}/\varepsilon)]$, as we have seen. Now, by (8.21), Lemma 8.6, Lemma 8.7, and (8.23), we can easily prove Theorem 2.2. (First, expand the Watanabe distribution by (8.23), then expand F by Lemma 8.7.) □

9. Sufficient condition for (A2) and (A3).

In this final section we give a sufficient condition for our main result (Theorem 2.2) on the off-diagonal asymptotics and compare it with a preceding result by Baudoin and Ouyang (Theorem 1.2, [**2**]), which is probably the only paper on this kind of problem.

Proposition 9.1. *Assume* (A1) *at the starting point* $a \in \mathbf{R}^n$. *If* a' *is sufficiently near* a, *then* (A2) *and* (A3) *are satisfied and, in particular, Theorem* 2.2 *holds for such* a'.

In the latter half of this section, we will prove this proposition in a rather general setting so that the same argument applies to a wider class of Gaussian processes. (To obtain Proposition 9.1, just set $F = \phi_1^0$ and $x = a'$ in Proposition 9.4.)

Before doing so, we first recall the result in [**2**] and compare. They set $n = d$ and assume (A1) for *any* starting point $a \in \mathbf{R}^d$ and, moreover, the following assumption (H):

(H): There exist smooth and bounded real-valued functions ω_{ij}^l such that

$$\omega_{ij}^l = -\omega_{il}^j \quad \text{and} \quad [V_i, V_j] = \sum_{i=1}^d \omega_{ij}^l V_l \quad \text{for all } 1 \le i, j, l \le d.$$

Note that V_0 does not appear in this condition. Under (A1) for any a, $\sigma(a)\sigma(a)^*$ is a $d \times d$ positive symmetric matrix, where $\sigma(a) = [V_1(a), \ldots, V_d(a)]$ as before. As a result, a Riemannian metric tensor $(g_{ij}(a))_{1 \le i,j \le d}$ is defined on $\mathbf{R}^d$ by $g^{ij}(a) = [\sigma(a)\sigma(a)^*]^{ij}$. The distance with respect to this Riemannian structure is denoted by $d(a, a')$. In terms of Riemannian geometry, (H) is equivalent to the condition that $\nabla^{LC}_X Y = [X, Y]$ for all smooth vector fields X, Y, where ∇^{LC} is the Levi–Civita connection for this metric. From this, one can guess that this assumption may not be very mild.

They proved short time kernel asymptotics under these assumptions when a and a' are sufficiently near. The following is Theorem 1.2, [**2**] (Notations are adjusted):

THEOREM 9.2. *Assume that* $n = d$, $V_0 \equiv 0$, (H), *and* (A1) *for any starting point* $a \in \mathbf{R}^d$. *Then, in a neighborhood* U *of* a, *we have*

$$p(t, a, a') = \frac{1}{t^{Hn}} \exp\Big(-\frac{d(a, a')^2}{2t^{2H}} \Big)$$
$$\times \Big(\sum_{i=0}^{N} \alpha_{2i}(a, a') t^{2iH} + r_{N+1}(t, a, a') t^{2(N+1)H} \Big), \quad a' \in U$$

near $t = 0$ *for any* $N = 1, 2, \ldots$. *Moreover,* U *can be chosen so that* α_{2i} *are smooth on* $U \times U$ *and for all multi-indices* β, β'

$$\sup_{t \le t_0} \sup_{a, a' \in U \times U} \big| \partial_a^{\beta} \partial_{a'}^{\beta'} r_{N+1}(t, a, a') \big| < \infty, \quad (\textit{for some } t_0 > 0).$$

Now we compare the two results. The most important issue is of course whether the asymptotic expansion holds or not. Concerning this point, we observe (i)–(ii) below;

(i) The conditions on the dimension ($n = d$), and on vector fields ($V_0 \equiv 0$ and (H)) in [**2**] are much stronger than ours. Moreover, the ellipticity condition (A1) is assumed at any a in [**2**]. So we believe that our result is "basically" better than Theorem 1.2, [**2**].

(ii) In our paper we did not give a quantitative estimate of how near a and a' should be in order for the asymptotics to hold (neither in [**2**]). Therefore, we could not say our result completely includes Theorem 1.2, [**2**].

The following (iii) may not be a major issue, but Theorem 1.2, [**2**] is better than ours concerning this point.

(iii) In Theorem 9.2, or Theorem 1.2, [**2**], they proved smoothness of the coefficient and gave an uniform estimate of (derivatives of) the remainder terms. However, we did not.

REMARK 9.3. If we assume (A1) everywhere, then a Riemannian structure on $\mathbf{R}^n$ is naturally induced as we explained above. If the case of the usual stochastic analysis (i.e., $H = 1/2$), (A2) and (A3) have a geometric meaning. (See Remark 3.2, [**21**], which was originally in [**15**], [**4**].) First, (A2) means that there is a unique shortest geodesics between a and a'. Second, (A3) or (A3)' means that these two points are not conjugate along the geodesics. So, Assumptions (A1)–(A3) are very mild and cover a lot of examples.

It seems natural to guess from this that, in our case (i.e., $1/2 < H < 1$), too, Assumptions (A1)–(A3) are not bad. At this moment, however, the author is not aware of a nice example except Proposition 9.1.

For the rest of this section, we discuss in a general setting. Our goal here is to prove a generalized version of Proposition 9.1. The key is the implicit function theorem.

Let $\mathcal{H}$ be a real separable Hilbert space and let $F : \mathcal{H} \to \mathbf{R}^n$ be a Fréchet smooth map such that $F(0) = a$ and the tangent map $DF(h) : \mathcal{H} \to \mathbf{R}^n$ is surjective at any $h \in \mathcal{H}$. Necessarily, F is a surjection onto a certain neighborhood of a in $\mathbf{R}^n$. By a well-known application of the inverse/implicit function theorem, $F^{-1}(x) \subset \mathcal{H}$ is a Hilbert submanifold for any $x \in \mathbf{R}^n$ if it is not empty. We define

$$d(a, x) = \inf\{\|h\|_{\mathcal{H}} \mid h \in F^{-1}(x)\}.$$

If x is sufficiently near a, then $d(a, x) < \infty$.

Proposition 9.4. *Let the notations be as above. Furthermore, we assume that, for any x sufficiently near a, the minimum in the definition of $d(a, x)$ above is actually attained. Then, for any x sufficiently near a, we have the following*;

(i) *There exists a unique $h_x \in F^{-1}(x)$ such that $d(a, x) = \|h_x\|_{\mathcal{H}}$.*
(ii) *The mapping $x \mapsto d(a, x)^2$ is smooth.*
(iii) *The Hessian of $F^{-1}(x) \ni h \mapsto \|h\|^2_{\mathcal{H}}/2$ at h_x is non-degenerate in the sense in* (A3).

Proof. Set $\mathcal{K} = \ker DF(0)$. This is a closed linear subspace in $\mathcal{H}$ which is tangent to $F^{-1}(a)$ at 0. We denote by $\hat{D}$ and $\hat{D}^{\perp}$ the gradient operator on $\mathcal{K}$ and $\mathcal{K}^{\perp}$, respectively. Then, $D = \hat{D} + \hat{D}^{\perp}$. We often write $h = (k, l)$, where k and l are the orthogonal projections onto $\mathcal{K}$ and $\mathcal{K}^{\perp}$, respectively.

Consider the following function $G : \mathcal{K} \times \mathcal{K}^{\perp} \times \mathbf{R}^n (= \mathcal{H} \times \mathbf{R}^n) \to \mathbf{R}^n$ defined by $G(k, l; x) = F(k, l) - x$. Then, $G(0, 0; a) = 0$. By the assumption, $(\hat{D}^{\perp}G)(0, 0; a) = (\hat{D}^{\perp}F)(0, 0)$ is a linear isomorphism from $\mathcal{K}^{\perp}$ to $\mathbf{R}^n$.

Hence, we can use the implicit function theorem near $(0, 0; a)$ to have the following; There exist open neighborhoods $V \subset \mathbf{R}^n$ of a, $W \subset \mathcal{K}$ of $0 \in \mathcal{K}$, and $U \subset \mathcal{K}^{\perp}$ of $0 \in \mathcal{K}^{\perp}$ such that a unique implicit function $l = l(k; x)$ for $G = 0$ from $W \times V$ to U exists. Moreover, l is smooth. Therefore, if $F^{-1}(x) \cap (W \times U) \neq \emptyset$, any element of the set is of the form $(k, l(k; x))$ for some $k \in W$. Note that $l(0; a) = 0$ and $\hat{D}l(0; a) = 0 \in L(\mathcal{K}, \mathcal{K}^{\perp})$ since $F^{-1}(a)$ and $\mathcal{K}$ are tangent at $0 \in \mathcal{H}$.

Next, consider $(k, x) \to \|(k, l(k; x))\|^2_{\mathcal{H}}/2 = (\|k\|^2 + \|l(k; x)\|^2)/2$. Take $\hat{D}$ of this function and we get

$$\hat{G}(k, x) := \langle k, \cdot \rangle_{\mathcal{K}} + \langle l(k; x), \hat{D}l(k; x)\rangle_{\mathcal{K}^{\perp}},$$

which is a smooth map from $W \times V$ to $\mathcal{K}^*$. Note that $\hat{G}(0, a) = 0$ and

$$\hat{D}\hat{G}(k,x) = \langle \cdot\,,\,\cdot \rangle_{\mathcal{K}} + \langle l(k;x), \hat{D}^2 l(k;x)\rangle_{\mathcal{K}^\perp} + \langle \hat{D}l(k;x), \hat{D}l(k;x)\rangle_{\mathcal{K}^\perp}.$$

This takes values in $L(\mathcal{K},\mathcal{K}^*) = L^{(2)}(\mathcal{K}\times\mathcal{K};\mathbf{R})$, where the latter space is the space of bounded bilinear maps from $\mathcal{K}\times\mathcal{K}$ to $\mathbf{R}$). Since $\hat{D}\hat{G}(0,a) = \langle \cdot\,,\,\cdot\rangle_{\mathcal{K}}$, which is clearly a linear isomorphism when regarded as an element of $L(\mathcal{K},\mathcal{K}^*)$, we can use the implicit function theorem again. If we retake V and W smaller, then there exists a unique implicit function $k = k(x)$ for $\hat{G} = 0$ from V to W. Moreover, k is smooth in x.

Take $r > 0$ small enough so that the open $\mathcal{H}$-ball B_r of radius r centered at $0 \in \mathcal{H}$ is contained in $W \times U$. Assume $F^{-1}(x) \cap B_r \neq \emptyset$. Then, the minimum is the definition of $d(x,a)$ must be achieved inside B_r. That point can be written as $(k_0, l(k_0,x))$ in a unique way. Any point of $F^{-1}(x)$ near $(k_0, l(k_0,x))$ can also be expressed using the implicit function like this. As a result, this point must be a critical point of $k \mapsto \|(k, l(k;x))\|^2_{\mathcal{H}}/2$ and hence $\hat{G}(k_0,x) = 0$. Therefore, such k_0 must be unique, namely, $k_0 = k(x)$. Note that $k(a) = 0$. Thus, we have seen $h_x = (k(x), l(k(x),x))$ and shown (i) and (ii).

We now show (iii). Let $f : (-\varepsilon_0,\varepsilon_0) \to F^{-1}(x)$ such that $f(0) = k(x)$ and $f'(0) \neq 0$. Then, $(d/du)^2|_{u=0}\|f(u)\|^2_{\mathcal{H}}/2$ depends only on $f'(0)$, i.e., $f''(0)$ is irrelevant. (We can check this by using the Lagrange multiplier method in the same way as in (8.15) in the proof of Proposition 8.4.) So, we have only to consider $f(u) = (k(x)+u\xi; l(k(x)+u\xi;x))$ for any non-zero $\xi \in \mathcal{K}$. By straight forward computation, we have

$$\left(\frac{d}{du}\right)^2\bigg|_{u=0} \frac{\|f(u)\|^2_{\mathcal{H}}}{2} = \|\xi\|^2 + \big(l(k(x);x), (\hat{D}_\xi)^2 l(k(x);x)\big) + \|\hat{D}_\xi l(k(x);x)\|^2.$$

By the smoothness of l and k, the right hand side is larger than $\|\xi\|^2/2$ if x is sufficiently near a. This proves (iii). □

ACKNOWLEDGEMENT. The author thanks an anonymous referee for suggesting how to improve Section 9.

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