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Conformal invariants defined by harmonic functions on Riemann surfaces

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Abstract. In this paper, we consider conformal invariants defined by various spaces of harmonic functions on Riemann surfaces. The Harnack distance is a typical one. We give sharp inequalities comparing those invariants with the hyperbolic metric on the Riemann surface and we determine when equalities hold. We also describe the Harnack distance in terms of the Martin compactification and discuss some properties of the distance.

1. Introduction.

Let R be an open Riemann surface. Among many conformal invariants on R, we are interested in quantities derived from some spaces of harmonic functions on R, such as spaces of positive harmonic functions, bounded harmonic functions and harmonic Hardy spaces. The classical Harnack distance is one of the important quantities. We compare those invariants with the hyperbolic distance.

On the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the hyperbolic metric $\lambda_{\mathbb{D}}(z)|dz|$ is defined by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1-|z|^2}.$$

We assume that the Riemann surface R admits the hyperbolic metric, namely, the universal covering of R is conformally equivalent to the unit disk $\mathbb{D}$, and the hyperbolic metric $\lambda_R(z)|dz|$ on R is defined by the relation $\pi^*(\lambda_R(w)|dw|) = \lambda_{\mathbb{D}}(z)|dz|$ for $w = \pi(z)$, where $\pi : \mathbb{D} \to R$ is a universal covering map and π^* stands for the pull back operation for metrics. For two points a, b in R, the hyperbolic distance $d_R(a, b)$ is defined by the standard manner by $\lambda_R(z)|dz|$.

In [**4**], Herron and Minda consider conformal invariants defined by the spaces of bounded harmonic functions and positive harmonic functions on R, and compare them with the hyperbolic distance on R. We shall give alternative proofs of their results. The method we will take also works to show similar properties of conformal invariants for harmonic Hardy spaces (Theorems 4.1 and 4.2).

It is rather known that the Martin compactification is a useful tool in studying positive harmonic functions. We describe the Harnack distance, which is defined by positive

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harmonic functions on a Riemann surface, in terms of the Martin compactification of the Riemann surface. Using this description, we compute the Harnack distance on the complement of a closed polar set on a Riemann surface. The result is obtained by Herron [**3**] for the Harnack distance in a domain of the Euclidean space. We also discuss the completeness of the distance.

2. Preliminaries.

2.1. Spaces of harmonic functions and invariants.

Throughout this paper, we consider a Riemann surface R which is open and admits Green's functions. Therefore, the universal covering of R must be conformally equivalent to the unit disk. We consider the following spaces of harmonic functions on R:

$$HP(R) = \{u : u \text{ is harmonic and positive on } R\},$$
$$HB(R) = \{u : u \text{ is bounded harmonic on } R\},$$

and for $p \in (1, +\infty)$,

$$h^p(R) = \{u : u \text{ is harmonic and } |u|^p \text{ has a harmonic majorant on } R\}.$$

We call $h^p(R)$ *the harmonic Hardy space* of rank p. We consider the norm $\|u\|_{h^p}$ called *the Hardy norm* for $u \in h^p(R)$ by

$$\|u\|_{h^p} = (L.H.M.|u|^p(a_0))^{1/p},$$

where $L.H.M.|u|^p$ is the least harmonic majorant and $a_0 \in R$ is a base point. It is known that both $(HB(R), \|\cdot\|_\infty)$ and $(h^p(R), \|\cdot\|_{h^p})$ are Banach spaces.

Since the norm $\|\cdot\|_{h^p}$ depends on choice of the base point a_0, we call it the Hardy norm with respect to the base point a_0 if we need to count the dependance of a_0.

For two points $a, b \in R$, we define

$$d_H^R(a,b) = \sup\left\{\left|\log \frac{u(a)}{u(b)}\right| : u \in HP(R)\right\}.$$

The quantity $d_H^R(\cdot,\cdot)$ is called *the Harnack* (*pseudo-*)*distance* on R.

For $HB(R)$ and $h^p(R)$, we define

$$\rho_B^R(a,b) = \sup\{|u(a) - u(b)|/\|u\|_\infty : u \in HB(R) \setminus \{0\}\}$$

and

$$\rho_p^R(a,b) = \sup\{|u(a) - u(b)|/\|u\|_{h^p} : u \in h^p(R) \setminus \{0\}\}.$$

We also consider conformal metrics corresponding to above spaces as follows:

$$\beta_H^R(z)|dz| = \sup\left\{\frac{|u_x - iu_y|}{u(z)}|dz| : u \in HP(R)\right\},$$

$$\beta_B^R(z)|dz| = \sup\left\{\frac{|u_x - iu_y|}{\|u\|_\infty}|dz| : u \in HB(R)\setminus\{0\}\right\},$$

and

$$\beta_p^R(z)|dz| = \sup\left\{\frac{|u_x - iu_y|}{\|u\|_{h^p}}|dz| : u \in h^p(R)\setminus\{0\}\right\}.$$

The metric $\beta_B^R(z)|dz|$ is introduced by Oikawa [**6**] and it is called *the harmonic metric.*

Those quantities are defined by taking the supremums. However, using normal family arguments, we verify that there are harmonic functions in their spaces such that they attain their supremums.

Herron and Minda [**4**] compare $d_H^R, \rho_B^R, \beta_H^R(z)|dz|$ and $\beta_B^R(z)|dz|$ with the hyperbolic distance and the hyperbolic metric on R. In the later section, we shall give alternative proofs of their results. We see that our method works to obtain similar estimates for ρ_p^R and $\beta_p^R(z)|dz|$ as well.

2.2. The Martin compactification.

Let $g^R(\cdot, q)$ be Green's function on R with pole at $q \in R$. Take a base point $a_0 \in R$, and consider a function

$$u_q(\cdot) = \frac{g^R(\cdot, q)}{g^R(a_0, q)}$$

for $q \in R$. The function u_q is in $HP(R \setminus \{q\})$ and $u_q(a_0) = 1$. Thus, a family $\{u_{q_n}\}_{n=1}^\infty$ becomes a normal family on any compact subset of R when the sequence $\{q_n\}_{n=1}^\infty$ tends to the ideal boundary of R. We may add a boundary point q as the limit of $\{q_n\}_{n=1}^\infty$ if $k_q(\cdot) := \lim_{n\to\infty} u_{q_n}(\cdot)$ exists and it belongs to $HP(R)$. The positive harmonic function k_q is called *the Martin kernel* with pole at q.

All such boundary points together with R is called *the Martin compactification* of R with base point a_0 and it is denoted by R_M^*. It is known that R_M^* is a metrizable compactification of R. The set $\Delta := R_M^* \setminus R$ is called *the Martin boundary.* A boundary point $q \in \Delta$ is called *minimal* if the Martin kernel $k_q(\cdot)$ is minimal, where $u \in HP(R)$ is minimal if $0 < v \leq u$ ($v \in HP(R)$) implies $v = cu$ for some constant c. The set of minimal points is called *the minimal boundary* and it is denoted by Δ_1.

It is known that the Martin compactification does not depend on the choice of the base point. Indeed, for two Martin compactifications R_1^* and R_2^* of R with different base points, the identity map $\iota : R \to R$ extends to a homeomorphism ι^* from R_1^* onto R_2^*. Furthermore, minimal points of R_1^* are mapped those of R_2^* via ι^*. As for the fundamental facts of the Martin compactification, one may refer Constantinescu and Cornea [**2**].

The minimal boundary Δ_1 plays an important role in $HP(R)$.

THEOREM 2.1. *For each $u \in HP(R)$, there exists a unique measure μ on Δ_1 such* *that*

$$u(p) = \int_{\Delta_1} k_q(p) d\mu(q). \tag{2.1}$$

It is known that the Martin compactification $\mathbb{D}^*_M$ of the unit disk $\mathbb{D} = \{|z| < 1\}$ is identified with the Euclidean closure $\overline{\mathbb{D}} = \{|z| \le 1\}$ and k_q is the Poisson kernel at $q \in \partial\mathbb{D}$. Hence, Theorem 2.1 is a generalization of the classical Herglotz theorem (cf. [**1**, 6.14 Corollary]) for $HP(\mathbb{D})$.

3. Invariants for $HP(R)$ and $HB(R)$.

This section devotes to give alternative proofs of results by Herron and Minda [**4**] and our proofs give an idea used in the later sections. Some computations in this section are also found in [**1**] for the same quantities in the unit ball of the Euclidean space.

Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ so that $\mathbb{D}/\Gamma = R$ and $\pi : \mathbb{D} \to R$ the universal covering map. For every function f on R, the function $\tilde{f} := f \circ \pi$ is a Γ-automorphic function for Γ, that is,

$$\tilde{f} \circ \gamma = \tilde{f} \tag{3.1}$$

holds for any $\gamma \in \Gamma$. Conversely, if (3.1) holds for any $\gamma \in \Gamma$, then $\tilde{f}$ defines a function f on R with $\tilde{f} = f \circ \pi$. Therefore, $HP(R)_\Gamma$, the space of Γ-automorphic positive harmonic functions on $\mathbb{D}$ is identified with $HP(R)$. Similarly, $HB(R)_\Gamma$, the space of Γ-automorphic bounded harmonic functions on $\mathbb{D}$, is identified with $HB(R)$.

Now, we show the following theorem:

Theorem 3.1. *Let R be an open Riemann surface and a, b two distinct points on R. Then,*

$$d_H^R(a, b) \le d_R(a, b) \tag{3.2}$$

and

$$\rho_B^R(a, b) \le \frac{8}{\pi} \arctan\left(\tanh \frac{d_R(a, b)}{4}\right) \tag{3.3}$$

hold, where $d_R(\cdot, \cdot)$ is the hyperbolic distance on R.

Furthermore, if $R = \mathbb{D}$, then equalities hold in (3.2) *and* (3.3) *for any points $a, b \in \mathbb{D}$. More precisely, let $u_{a,b} \in HP(\mathbb{D})$ denote a function which gives the equality of* (3.2). *Then, $u_{a,b}(z) = cP(\alpha; z)$ or $cP(\beta; z)$ for some positive constant c, where $\alpha, \beta \in \partial\mathbb{D}$ are the endpoints of the hyperbolic geodesic passing through a, b; let $v_{a,b} \in HB(\mathbb{D})$ $\|v_{a,b}\|_\infty = 1$ denote a function which gives the equality of* (3.3). *Then, up to sign, $v_{a,b}$ has boundary value 1 on I_+ and -1 on I_-, where $I_\pm$ are the connected components of $\partial\mathbb{D} \setminus \{\alpha', \beta'\}$ for the endpoints α', β' of the hyperbolic bisector between a and b.*

PROOF. Take $\tilde{a}, \tilde{b} \in \mathbb{D}$ such that $\pi(\tilde{a}) = a$, $\pi(\tilde{b}) = b$ and $d_R(a, b) = d_{\mathbb{D}}(\tilde{a}, \tilde{b})$. Using the above identifications $HP(R) \simeq HP(\mathbb{D})_\Gamma$ and $HB(R) \simeq HB(\mathbb{D})_\Gamma$, we see that

$$d_H^R(a, b) = \sup\left\{\left|\log \frac{u(\tilde{a})}{u(\tilde{b})}\right| : u \in HP(\mathbb{D})_\Gamma\right\},$$

and

$$\rho_B^R(a, b) = \sup\left\{|u(\tilde{a}) - u(\tilde{b})|/\|u\|_\infty : u \in HB(\mathbb{D})_\Gamma \setminus \{0\}\right\}.$$

Obviously, $HP(\mathbb{D})_\Gamma \subset HP(\mathbb{D})$ and $HB(\mathbb{D})_\Gamma \subset HB(\mathbb{D})$. We have

$$\sup\left\{\left|\log \frac{u(\tilde{a})}{u(\tilde{b})}\right| : u \in HP(\mathbb{D})_\Gamma\right\} \le d_H^{\mathbb{D}}(\tilde{a}, \tilde{b}), \tag{3.4}$$

and

$$\sup\left\{|u(\tilde{a}) - u(\tilde{b})|/\|u\|_\infty : u \in HB(\mathbb{D})_\Gamma \setminus \{0\}\right\} \le \rho_B^{\mathbb{D}}(\tilde{a}, \tilde{b}). \tag{3.5}$$

Hence, it suffices to show that

$$d_H^{\mathbb{D}}(\tilde{a}, \tilde{b}) = d_{\mathbb{D}}(\tilde{a}, \tilde{b}) \tag{3.6}$$

and

$$\rho_B^{\mathbb{D}}(\tilde{a}, \tilde{b}) = \frac{8}{\pi} \arctan\left(\tanh \frac{d_{\mathbb{D}}(\tilde{a}, \tilde{b})}{4}\right). \tag{3.7}$$

To show (3.6), we use the Herglotz theorem for positive harmonic functions.

We may assume that $\tilde{a} = 0$ and $\tilde{b} = r \in (0, 1)$. Let u be in $HP(\mathbb{D})$. It follows from the Herglotz theorem that there exists a unique positive measure μ on $\partial\mathbb{D}$ such that

$$u(z) = \int_{\partial\mathbb{D}} P(e^{i\theta}; z) d\mu \quad (z \in \mathbb{D}), \tag{3.8}$$

where $P(e^{i\theta}; z)$ is the Poisson kernel. We may also assume that $u(0) = 1$. Then, from (3.8), we obtain

$$1 = u(0) = \int_{\partial\mathbb{D}} d\mu.$$

Thus, u is represented by a probability measure in (3.8) and immediately we obtain

$$e^{-d_{\mathbb{D}}(0,r)} = \frac{1-r}{1+r} \le u(r) \le \frac{1+r}{1-r} = e^{d_{\mathbb{D}}(0,r)}.$$

Indeed, both equalities hold for Dirac measures $\delta_{\{-1\}}$ and $\delta_{\{1\}}$ with support at -1 and 1, respectively. In other words, for $u = P(-1;\cdot)$ and $u = P(1;\cdot)$, the equality (3.6) holds and only those functions give the equality (3.6) under the condition that $u(0) = 1$. Since $\{-1, 1\}$ is the set of the end points of the geodesic passing through 0 and r, we have shown the statement for d_H^R.

To prove (3.7), we use other integral representation for bounded harmonic functions. Fatou's theorem (cf. [**1**, Chapter 6]) guarantees us that every bounded harmonic function u has non-tangential limits $u^*(e^{i\theta})$ almost everywhere on $\partial\mathbb{D}$ and u is represented by the Poisson integral of u^*, namely,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})P(e^{i\theta}; z)d\theta. \tag{3.9}$$

In showing (3.7), we may assume that $\tilde{a} = -r$, $\tilde{b} = r$ for some $r \in (0, 1)$ and $\|u\|_\infty = \|u^*\|_\infty = 1$. Then, we have

$$\begin{aligned} |u(r) - u(-r)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} u^*(e^{i\theta})\big(P(e^{i\theta}; r) - P(e^{i\theta}; -r)\big)d\theta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |u^*(e^{i\theta})||(P(e^{i\theta}; r) - P(e^{i\theta}; -r)|d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |(P(e^{i\theta}; r) - P(e^{i\theta}; -r)|d\theta. \end{aligned}$$

Therefore, we have

$$\rho_B^{\mathbb{D}}(r, -r) \le \frac{1}{2\pi}\int_0^{2\pi} |(P(e^{i\theta}; r) - P(e^{i\theta}; -r)|d\theta.$$

On the other hand,

$$P(e^{i\theta}; r) - P(e^{i\theta}; -r) = \frac{4r(1 - r^2)\cos\theta}{(1 + r^2 - 2r\cos\theta)(1 + r^2 + 2r\cos\theta)}$$

and we see that $P(e^{i\theta}; r) - P(e^{i\theta}; -r) > 0$ for $\theta \in (0, \pi/2) \cup (3\pi/2, 2\pi)$ and < 0 for $\theta \in (\pi/2, 3\pi/2)$. We define $u_0^* \in L^\infty(\partial\mathbb{D})$ by setting $u_0^*(e^{i\theta}) = 1$ for $\theta \in (0, \pi/2) \cup (3\pi/2, 2\pi)$ and $= -1$ for $\theta \in (\pi/2, 3\pi/2)$. Then the Poisson integral u_0 of u_0^*,

$$u_0(z) = \frac{1}{2\pi}\int_0^{2\pi} u_0^*(e^{i\theta})P(e^{i\theta}; z)d\theta$$

is in $HB(\mathbb{D})$ with $\|u\|_\infty = 1$. Furthermore, we have

$$u_0^*(r) - u_0^*(-r) = \frac{1}{2\pi}\int_0^{2\pi} |(P(e^{i\theta}; r) - P(e^{i\theta}; -r)|d\theta.$$

Therefore, we obtain

$$\rho_B^{\mathbb{D}}(r,-r) = \frac{1}{2\pi}\int_0^{2\pi} |(P(e^{i\theta};r) - P(e^{i\theta};-r)|d\theta.$$

Since it is not hard to verify that

$$\frac{1}{2\pi}\int_0^{2\pi} |(P(e^{i\theta};r) - P(e^{i\theta};-r)|d\theta = \frac{8}{\pi}\arctan\left(\tanh\frac{d_{\mathbb{D}}(r,-r)}{4}\right),$$

we obtain (3.7) and we verify that only $\pm u_0$ gives the equality (3.7) among functions in $u \in HB(\mathbb{D})$ with $\|u\|_\infty = 1$. Since $\{i,-i\}$ is the set of the end points of the bisector between $-r$ and r, we verify that the proof is completed. □

Using this theorem, we may determine Riemann surfaces where equalities hold for some points $a, b \in R$ in (3.6) and (3.7).

Theorem 3.2. *Suppose that R is not a simply connected Riemann surface. Then,*

(1) *there exist distinct points a, b in R such that $d_H^R(a,b) = d_R(a,b)$ if and only if R is* (*conformally equivalent to*) *the punctured disk $\mathbb{D}^*$, and a, b lie on the same radius of $\mathbb{D}^*$;*
(2) *there exist distinct points a, b in R such that $\rho_B^R(a,b) = (8/\pi)\arctan(\tanh(d_R(a,b)/4))$ if and only if R is* (*conformally equivalent to*) *an annulus $A(k) := \{0 < k < |z| < 1\}$ and a, b are symmetric with respect to the core curve $\{|z| = \sqrt{k}\}$ of $A(k)$.*

Proof. Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ which represents R and $\pi : \mathbb{D} \to R = \mathbb{D}/\Gamma$ be the universal covering map.

Proof of (1). We may assume that $\pi(0) = a$ and $\pi(r) = b$ for some $r \in (0,1)$, and $d_{\mathbb{D}}(0,r) = d_R(a,b)$. We take $u \in HP(R)$ satisfying $u(a) = 1$ and $\log(u(a)/u(b)) = d_H^R(a,b)$. Since $\tilde{u} = u \circ \pi \in HP(\mathbb{D})$, we have

$$d_R(a,b) = d_H^R(a,b) = \left|\log\frac{u(\pi(0))}{u(\pi(r))}\right| \le d_H^{\mathbb{D}}(0,r) = d_R(a,b).$$

Thus, we conclude $|\log(\tilde{u}(0)/\tilde{u}(r))| = d_H^{\mathbb{D}}(0,r)$. Since $\tilde{u}(0) = u(a) = 1$, it follows from Theorem 3.1 that $\tilde{u}(z) = P(1;z)$ or $\tilde{u}(z) = P(-1;z)$. We may assume $\tilde{u}(z) = P(1;z)$. Since $\tilde{u}$ belongs to $HP(\mathbb{D})_\Gamma$, we verify that Γ is a parabolic cyclic group fixing $z = 1$. Indeed, $z = 1$ is a unique singularity of $P(1;z)$. Hence, every element of Γ has to fix the point because $P(1;z)$ is automorphic for Γ.

Therefore, $R = \mathbb{D}/\Gamma$ is the punctured disk $\mathbb{D}^*$, and $a = \pi(0), b = \pi(r)$ lie on the same radius in $\mathbb{D}^*$ because the positive real axis in $\mathbb{D}$ is mapped a radius in $\mathbb{D}^*$ via the universal covering map π.

Proof of (2). We may assume that $\pi(-r) = a$ and $\pi(r) = b$ for some $r \in (0,1)$, and $d_{\mathbb{D}}(-r,r) = d_R(a,b)$. We take $u \in HB(R)$ with $\|u\|_\infty = 1$ such that $|u(b) - u(a)| = \rho_B^R(a,b)$. Since $\tilde{u} = u \circ \pi \in HB(\mathbb{D})$ with $\|\tilde{u}\|_\infty = 1$, we have

$$\rho_B^R(a,b) = \frac{8}{\pi}\arctan\left(\tanh\frac{d_{\mathbb{D}}(-r,r)}{4}\right) = |\tilde{u}(r) - \tilde{u}(-r)|$$
$$\le \rho_B^{\mathbb{D}}(r,-r) = \frac{8}{\pi}\arctan\left(\tanh\frac{d_{\mathbb{D}}(-r,r)}{4}\right).$$

Thus, we conclude that $|\tilde{u}(r) - \tilde{u}(-r)| = \rho_B^{\mathbb{D}}(r,-r)$ and therefore, $\tilde{u} = u_0$ or $= -u_0$, where $u_0 \in HB(\mathbb{D})$ is the function obtained in the proof of Theorem 3.1. Since $\tilde{u}$ belongs to $HB(\mathbb{D})_\Gamma$, we verify that Γ is a hyperbolic cyclic group fixing $\pm i$. Hence, $R = \mathbb{D}/\Gamma$ is an annulus, and $a = \pi(-r)$, $b = \pi(r)$ are symmetric with respect to $\pi(\{iy : -1 < y < 1\})$ which is the core curve of the annulus. □

We may show the similar results for conformal metrics by using the same ideas of the proofs of Theorems 3.1 and 3.2.

Theorem 3.3. *Let R be a hyperbolic Riemann surface. Then,*

$$\beta_H^R(z)|dz| \le \lambda_R(z)|dz| \tag{3.10}$$

and

$$\beta_B^R(z)|dz| \le \frac{2}{\pi}\lambda_R(z)|dz| \tag{3.11}$$

hold at every point in R, where $\lambda_R(z)|dz|$ is the hyperbolic metric on R. If $R = \mathbb{D}$, then the equalities hold in (3.10) and (3.11) at every point in $\mathbb{D}$. More precisely, if $u_a \in HP(\mathbb{D})$ gives the equality in (3.10) at $a \in \mathbb{D}$, then $u_a(z) = cP(\alpha; z)$ for some $c > 0$ and $\alpha \in \partial\mathbb{D}$; if $v_a \in HB(\mathbb{D})$ with $\|v_a\|_\infty = 1$ gives the equality in (3.11), then $v_a(z) = \pm v_{a,b}(z)$, where $v_{a,b}$ is the function given in Theorem 3.1 for any $b \in \mathbb{D}$ $(a \ne b)$.

Proof. The proof is done by the same idea as in Theorem 3.1. Exactly the same reasons as in (3.4) and (3.5), we obtain

$$\pi^*(\beta_H^R(w)|dw|) \le \beta_H^{\mathbb{D}}(z)|dz|,$$

and

$$\pi^*(\beta_B^R(w)|dw|) \le \beta_B^R(z)|dz|,$$

where $R \ni w = \pi(z)$ is a universal covering map as before. Hence, it suffices to show that $\beta_H^{\mathbb{D}}(z)|dz| = \lambda_{\mathbb{D}}(z)|dz|$ and $\beta_B^{\mathbb{D}}(z)|dz| = (2/\pi)\lambda_{\mathbb{D}}(z)|dz|$ at $z = 0$.

Let u be a function in $HP(\mathbb{D})$ with $u(0) = 1$. It follows from the Herglotz theorem that there exists a probability measure μ on $\partial\mathbb{D}$ such that

$$u(z) = \int_{\partial\mathbb{D}} P(e^{i\theta}; z)d\mu$$

holds on $\mathbb{D}$. Thus,

$$|u_x(0) - iu_y(0)| = 2\left|\int_{\partial\mathbb{D}} e^{-i\theta} d\mu\right| = 2\left|\int_{\partial\mathbb{D}} \cos\theta d\mu - i\int_{\partial\mathbb{D}} \sin\theta d\mu\right|$$
$$= 2\sqrt{\left(\int_{\partial\mathbb{D}} \cos\theta d\mu\right)^2 + \left(\int_{\partial\mathbb{D}} \sin\theta d\mu\right)^2} \leq 2.$$

From Schwartz' inequality,

$$\left(\int_{\partial\mathbb{D}} \cos\theta d\mu\right)^2 \leq \left(\int_{\partial\mathbb{D}} d\mu\right)\left(\int_{\partial\mathbb{D}} \cos^2\theta d\mu\right) = \int_{\partial\mathbb{D}} \cos^2\theta d\mu$$

and

$$\left(\int_{\partial\mathbb{D}} \sin\theta d\mu\right)^2 \leq \left(\int_{\partial\mathbb{D}} d\mu\right)\left(\int_{\partial\mathbb{D}} \sin^2\theta d\mu\right) = \int_{\partial\mathbb{D}} \sin^2\theta d\mu.$$

Since the equalities hold only when both $\cos\theta$ and $\sin\theta$ are constants on the support of μ, μ must be a Dirac measure δ_α for some $\alpha \in \partial\mathbb{D}$. Indeed, for $u(z) = P(\alpha; z) = \int_{\partial\mathbb{D}} P(e^{i\theta;z}) d\delta_\alpha$, $|u_x(0) - iu_y(0)| = 2$. Therefore, $P(\alpha; z)$ gives $\beta_H^{\mathbb{D}}(z)|dz|$ at $z = 0$. Since $\lambda_{\mathbb{D}}(z)|dz| = 2|dz|$ at $z = 0$, we have shown that $\beta_H^{\mathbb{D}}(z)|dz| = \lambda_{\mathbb{D}}(z)|dz|$ at $z = 0$.

Next, we consider $\beta_B^{\mathbb{D}}(z)|dz|$ at $z = 0$. Let u be a function in $HB(\mathbb{D})$ with $\|u\|_\infty = 1$. We consider

$$|u_x(0) - iu_y(0)| = |\text{grad } u|(0) = \max_{\varphi}\left|\frac{d}{dr}u(re^{i\varphi})\bigg|_{r=0}\right|. \tag{3.12}$$

Since

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})P(e^{i\theta}; z)d\theta,$$

for $u^* \in L^\infty(\partial\mathbb{D})$ with $\|u^*\|_\infty = 1$, we have

$$\left|\frac{d}{dr}u(r)\bigg|_{r=0}\right| = \left|\frac{1}{\pi}\int_0^{2\pi} u^*(e^{i\theta})\cos\theta d\theta\right| \leq \frac{1}{\pi}\int_0^{2\pi} |\cos\theta| d\theta = \frac{4}{\pi},$$

and the equality holds only for $u^* = \pm v_{-r,r}$, the same function in Theorem 3.1 for some $r \in (0, 1)$. Thus, if $\varphi = 0$ gives the maximum in (3.12), we conclude that $\beta_B^{\mathbb{D}}(z)|dz| = (2/\pi)\lambda_{\mathbb{D}}(z)|dz|$ at $z = 0$ and $\pm v_{-r,r}$ gives the equality. The same argument works for any $\varphi \in [0, 2\pi)$ and we verify that the proof is completed. □

Equalities in (3.10) and (3.11) hold at every point on R if the Riemann surface is simply connected. If R is not simply connected, then we may show similar results to Theorem 3.2.

Theorem 3.4. *Suppose that R is not a simply connected Riemann surface. Then,*

(1) *there exists a point α in R such that $\beta^R_H(z)|dz| = \lambda_R(z)|dz|$ at α if and only if R is conformally equivalent to the punctured disk*;
(2) *there exists a point α in R such that $\beta^R_B(z)|dz| = (2/\pi)\lambda_R(z)|dz|$ at α if and only if R is* (*conformally*) *equivalent to an annulus $A(k) := \{0 < k < |z| < 1\}$ and α lies on the core curve $\{|z| = \sqrt{k}\}$ of $A(k)$.*

Proof. The proof is done by the same manner as that of Theorem 3.2. So, we may leave it for the reader. □

4. Invariants for $h^p(R)$.

In this section, we compare ρ^R_p and $\beta^R_p(z)|dz|$ $(1 < p < \infty)$ with the hyperbolic metric of R.

We note the following the decreasing property of those quantities.

Lemma 4.1. *Let $f : X \ni z \mapsto f(z) = w \in Y$ be a holomorphic map between two Riemann surfaces X and Y. Then,*

$$\rho^Y_p(f(a), f(b)) \le \rho^X_p(a, b),$$

and

$$f^*(\beta^Y_p(w)|dw|) \le \beta^X_p(z)|dz|,$$

where $\|u\|_{h^p} = (L.H.M.|u|^p(a_0))^{1/p}$ for $u \in h^p(X)$ and $\|u'\|_{h^p} = (L.H.M.|u'|^p(f(a_0)))^{1/p}$ for $u' \in h^p(Y)$.

Proof. For $u \in h^p(Y)$, we denote by v the least harmonic majorant of $|u|^p$ and by $\tilde{v}$ the least harmonic majorant of $|u \circ f|^p$. Since $|u|^p \circ f \le v \circ f$ and $v \circ f$ is harmonic, we see that $\tilde{v} \le v \circ f$. Thus, we obtain $\|u \circ f\|_{h^p} \le \|u\|_{h^p}$ for $u \in h^p(Y)$. Hence, we have

$$\frac{|u \circ f(b) - u \circ f(a)|}{\|u\|_{h^p}} \le \frac{|u \circ f(b) - u \circ f(a)|}{\|u \circ f\|_{h^p}} \le \rho^X_p(a, b),$$

and we see that $\rho^Y_p(f(a), f(b)) \le \rho^X_p(a, b)$ because $u \circ f \in h^p(X)$ for $u \in h^p(Y)$.

The same argument yields that $f^*(\beta^Y_p(w)|dw|) \le \beta^X_p(z)|dz|$. □

Let $\pi : \mathbb{D} \to R$ be a universal covering. From Lemma 4.1, we have

$$\rho^R_p(a, b) \le \rho^{\mathbb{D}}_p(\tilde{a}, \tilde{b}),$$

for $\tilde{a}, \tilde{b} \in \mathbb{D}$ with $\pi(\tilde{a}) = a$, $\pi(\tilde{b}) = b$ and $d_R(a, b) = d_{\mathbb{D}}(\tilde{a}, \tilde{b})$.

We compute $\rho^{\mathbb{D}}_p(a, b)$ for $a, b \in \mathbb{D}$. We may assume that $a = -r, b = r$ $(0 < r < 1)$. Let u be a function in $h^p(\mathbb{D})$ with $\|u\|_{h^p} = 1$. It is well known (cf. [**1**]) that there exists a function $u^* \in L^p(\partial\mathbb{D})$ such that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})P(e^{i\theta};z)d\theta. \tag{4.1}$$

We also see that

$$1 = \|u\|_{h^p} = (L.H.M.|u|^p(a_0))^{1/p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |u^*(e^{i\theta})|^p P(e^{i\theta};a_0)d\theta\right)^{1/p}.$$

By Hölder's inequality, we have

$$\begin{aligned}|u(r) - u(-r)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})(P(e^{i\theta};r) - P(e^{i\theta};-r))d\theta\right| \\ &\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta};r) - P(e^{i\theta};-r)|^q P(e^{i\theta};a_0)^{1-q}d\theta\right)^{1/q} \qquad (4.2)\end{aligned}$$

where $q = p/(p-1)$. Moreover, the equality is achieved if and only if $u^*P_r = |P_r|^q\|P_r\|_{q,a_0}^{-q/p}$ almost everywhere on $\partial\mathbb{D}$, where

$$P_r(\cdot) = (P(\cdot;r) - P(\cdot;-r))P(\cdot;a_0)^{-1},$$

and $\|\cdot\|_{q,a_0}$ is the L^q-norm on $\partial\mathbb{D}$ with respect to $d\theta_{a_0} := P(e^{i\theta};a_0)d\theta$. Therefore, $\rho_p^{\mathbb{D}}(r,-r)$ is equal to the right hand side of (4.2).

If $a_0 = 0$, the midpoint between $-r$ and r, then $P_r(\cdot) = P(\cdot;r) - P(\cdot;-r)$ and $d\theta_{a_0} = d\theta$. We also see that

$$P_r(z) = r^2\mathrm{Re}\left(\frac{z}{1-r^2z^2}\right)$$

for $z \in \partial\mathbb{D}$. Since $r = \tanh(d_{\mathbb{D}}(a,b)/4)$ and $\rho_p^{\mathbb{D}}$ is a conformal invariant, we have

$$\rho_p^{\mathbb{D}}(a,b) = \|v_{d_{\mathbb{D}}(a,b)}\|_{h^q}, \tag{4.3}$$

where

$$v_d(z) = (\tanh(d/4))^2\mathrm{Re}\left(\frac{z}{1-(\tanh(d/4))^2z^2}\right) \quad (z \in \mathbb{D})$$

and the base point a_0 for the Hardy norm is the midpoint between a and b.

When the base point a_0 is not the midpoint between a and b, $\rho_p^{\mathbb{D}}(a,b)$ depends on hyperbolic distances among the three points, a_0, a and b. We may take $a_0 = 0$. Then, the same method as (4.2) shows

$$|u(b)-u(a)| \le \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta};b)-P(e^{i\theta};a)\right|^q d\theta\right)^{1/q}$$

for $u \in h^p(\mathbb{D})$ with $\|u\|_{h^p}=1$. Putting

$$P_{a,b}(z)=\mathrm{Re}\biggl(\frac{2(b-a)\bar{z}}{(1-a\bar{z})(1-b\bar{z})}\biggr),$$

we see that $P_{a,b}(z)$ is harmonic on $\overline{\mathbb{D}}$ and $P_{a,b}(\cdot)=P(\cdot;b)-P(\cdot;a)$ on $\partial\mathbb{D}$. Therefore, for $a,b\in\mathbb{D}$, we obtain

$$\rho_p^{\mathbb{D}}(a,b)=\|P_{a,b}\|_{h^q} \tag{4.4}$$

for the Hardy norm with respect to the origin. Although it is hard to describe the L^q-norm of $P_{a,b}$ by using hyperbolic distances among a_0, a and b, we may express $\rho_p^{\mathbb{D}}(a,b)$ by means of the hyperbolic distance between a and b if $a_0=a$. When $a_0=a$, we may assume that $a_0=a=0$ and $b=r\in(0,1)$. Then,

$$P_{0,r}(z)=\mathrm{Re}\biggl(\frac{2rz}{1-rz}\biggr).$$

Hence, we have

$$\rho_p^{\mathbb{D}}(a,b)=\|\tilde{v}_{d_{\mathbb{D}}(a,b)}\|_{h^q}, \tag{4.5}$$

where

$$\tilde{v}_d(z)=\mathrm{Re}\biggl(\frac{2\tanh(d/2)z}{1-\tanh(d/2)z}\biggr) \quad (z\in\mathbb{D}),$$

and the base point for the Hardy norm is a.

Now, we give the main results of this section.

Theorem 4.1. *Let R be an open Riemann surface and $h^p(R)$ the harmonic Hardy space of rank p $(1<p<\infty)$ with the Hardy norm $\|\cdot\|_{h^p}$ with respect to $a_0\in R$. Then,*

(1) *if a_0 is the midpoint of the shortest geodesic between a and b, then $\rho_p^R(a,b)\le \|v_{d_R(a,b)}\|_{h^q}$;*
(2) *$\rho_p^R(a_0,a)\le\|\tilde{v}_{d_R(a_0,a)}\|_{h^q}$;*
(3) *$\rho_p^R(a,b)\le\|P_{\alpha,\beta}\|_{h^q}$, where $\alpha,\beta\in\mathbb{D}$ are points satisfying conditions: $\pi(\alpha)=a$ and $\pi(\beta)=b$ for a universal covering $\pi:\mathbb{D}\to R$ with $\pi(0)=a_0$.*

Moreover, one of the equalities in the above inequalities holds if and only if R is conformally equivalent to the unit disk.

Proof. All statements are proved by the same argument. It follows from Lemma 4.1 that $\rho_p^R(a,b)\le\rho_p^{\mathbb{D}}(\tilde{a},\tilde{b})$, where $\tilde{a},\tilde{b}\in\mathbb{D}$ are lifts of a,b via $\pi:\mathbb{D}\to R$ with $d_R(a,b)=$

$d_{\mathbb{D}}(\tilde{a},\tilde{b})$. Since we have already calculated $\rho_p^{\mathbb{D}}(\cdot,\cdot)$, we obtain immediately the above inequality in each case. Furthermore, it is easy to see that v_d, $\tilde{v}_d$ and $P_{a,b}$ are not automorphic for any non-trivial Fuchsian group. Thus, we conclude that R is conformally equivalent to the unit disk if an equality holds. □

We also have a similar result for $\beta_p^R(z)|dz|$ as follows.

THEOREM 4.2. *Let R be an open Riemann surface and $h^p(R)$ the harmonic Hardy space of rank p $(1<p<\infty)$ with the Hardy norm $\|\cdot\|_{h^p}$ with respect to $a_0 \in R$. Then,*

$$\beta_p^R(z)|dz| \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^q d\theta\right)^{1/q} \lambda_R(z)|dz| \quad (p^{-1}+q^{-1}=1) \tag{4.6}$$

at a_0. Moreover, the equality holds if and only if R is conformally equivalent to the unit disk.

PROOF. From Lemma 4.1 we have

$$\pi^*(\beta_p^R(w)|dw|) \leq \beta_p^{\mathbb{D}}(z)|dz| \quad (w=\pi(z)) \tag{4.7}$$

for a universal covering map $\pi : \mathbb{D} \to R$.

We consider $\beta_p^{\mathbb{D}}(z)|dz|$ at the base point $a_0 \in \mathbb{D}$ of the Hardy norm of $h^p(\mathbb{D})$. We may assume that $a_0 = 0$. Let u be a function in $h^p(\mathbb{D})$ with $\|u\|_{h^p} = 1$. Then,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})P(e^{i\theta};z)d\theta \tag{4.8}$$

for some $u^* \in L^p(\partial\mathbb{D})$ with $\|u^*\|_p = 1$ (cf. [**1**]). Hence, we have

$$\begin{aligned}
\left|\frac{d}{dr}u(r)\Big|_{r=0}\right| &= \left|\frac{1}{\pi}\int_0^{2\pi} u^*(e^{i\theta})\cos\theta d\theta\right| \\
&\leq 2\left(\frac{1}{2\pi}\int_0^{2\pi} |u^*(e^{i\theta})|^p d\theta\right)^{1/p}\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^q d\theta\right)^{1/q} \\
&= 2\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^q d\theta\right)^{1/q},
\end{aligned} \tag{4.9}$$

and the equality holds only if $u^*(e^{i\theta}) = |\cos\theta|^q\cos^{-1}\theta\|\cos\theta\|_q^{q-1}$ almost everywhere. We consider

$$|u_x(0) - iu_y(0)| = |\text{grad } u|(0) = \max_{\varphi}\left|\frac{d}{dr}u(re^{i\varphi})\Big|_{r=0}\right|.$$

Thus, if $\varphi = 0$ gives the maximum, then

$$\beta_p^{\mathbb{D}}(z)|dz| = 2\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^q d\theta\right)^{1/q} |dz|$$

at $z = 0$. Since $\lambda_{\mathbb{D}}(z)|dz| = 2|dz|$ at $z = 0$, we have

$$\beta_p^{\mathbb{D}}(z)|dz| = \left(\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^q d\theta\right)^{1/q} \lambda_{\mathbb{D}}(z)|dz|.$$

By the same argument, we obtain the same conclusion when other angle φ gives the maximum. Hence, from (4.7) we verify that the desired inequality (4.6) holds. Obviously, $u^*(e^{i\theta}) = |\cos\theta|^q \cos^{-1}\theta \|\cos\theta\|_q^{q-1}$ which gives the equality is not automorphic for any non-trivial Fuchsian group. Thus, we see that the equality holds in (4.6) if and only if the Riemann surface R is simply connected. □

5. The Martin compactification and the Harnack distance.

In this section, we consider the Harnack distance $d_H^R(p_1, p_2)$ $(p_1, p_2 \in R)$ from view point of the Martin compactification by using an idea in Section 3.

Let R_M^* be the Martin compactification of R with base point a_0. We denote by $HP_0(R)$ the set of positive harmonic functions u on R with $u(p_1) = 1$. Then,

$$d_H^R(p_1, p_2) = \sup\{|\log u(p_2)| : u \in HP_0(R)\}.$$

It follows from Theorem 2.1 that there exists a measure μ on Δ_1 such that

$$u(p) = \int_{\Delta_1} k_q(p)d\mu(q)$$

for $u \in HP_0(R)$. Since $u(p_1) = 1$, we have

$$1 = u(p_1) = \int_{\Delta_1} k_q(p_1)d\mu(q).$$

Hence, $d\nu = k_q(p_1)d\mu$ is a probability measure on Δ_1 and

$$u(p) = \int_{\Delta_1} \frac{k_q(p)}{k_q(p_1)} d\nu(q).$$

Conversely, if we put

$$h_\nu(p) = \int_{\Delta_1} \frac{k_q(p)}{k_q(p_1)} d\nu(q)$$

for a probability measure $d\nu$ on Δ_1, then h_ν is in $HP_0(R)$. Therefore, we verify that

$$d_H^R(p_1,p_2)=\sup\left\{\left|\log\int_{\Delta_1}\frac{k_q(p_2)}{k_q(p_1)}d\nu(q)\right| : d\nu \text{ is a probability measure on } \Delta_1\right\}.$$

Since $k_q(a_0)=1$, we obtain:

PROPOSITION 5.1. *Let R_M^* be the Martin compactification of R with base point a_0. Then*

$$d_H^R(p_1,p_2)=\sup_{q\in\Delta_1}\left|\log\frac{k_q(p_2)}{k_q(p_1)}\right| \tag{5.1}$$

and

$$d_H^R(a_0,p)=\sup_{q\in\Delta_1}|\log k_q(p)|. \tag{5.2}$$

A subset E of R is called *polar* if for every $p\in E$ there exist a neighborhood U of p and a positive superharmonic function s on U such that $s|_E=+\infty$. Then, we have the following;

THEOREM 5.1. *Let E be a closed polar subset of R. Then, for $p_1,p_2\in R_E:=R\setminus E$*

$$d_H^{R_E}(p_1,p_2)=\max\left\{d_H^R(p_1,p_2),\sup_{q\in E}\left|\log\frac{g^R(p_1,q)}{g^R(p_2,q)}\right|\right\}, \tag{5.3}$$

where $g^R(\cdot,p)$ is Green's function of R with pole at p.

REMARK 5.1. Using a different method, Herron [**3**] shows the same result on subdomains of the Euclidean space. It is not hard to see that our proof below works on the Euclidean space.

PROOF. For a point $p\in R_E$, we consider Green's function $g^{R_E}(\cdot,p)$ of R_E with pole at p. From the minimality of Green's functions, we see that $g^{R_E}(\cdot,p)\le g^R(\cdot,p)$ on R_E. On the other hand, since E is polar, $g^{R_E}(\cdot,p)$ is extended to a harmonic function to any point of E. Thus, we have $g^R(\cdot,p)\le g^{R_E}(\cdot,p)$ on R because of the minimality of Green's functions again. So, we conclude that $g^{R_E}(\cdot,p)=g^R(\cdot,p)$.

Since the Martin compactification is defined by using Green's functions, we verify that the Martin compactification $(R_E)_M^*$ of R_R is identified with the Martin compactification R_M^* of R. More precisely, the inclusion map $\iota: R_E\hookrightarrow R$ extends to a homeomorphism from $(R_E)_M^*$ onto R_M^* and $\lim_{p\to q}\iota(p)=q$ for every $q\in E$. We also see that the set of minimal points of $\partial(R_E)_M^*\setminus E$ corresponds exactly to that of R_M^*. Furthermore, every point $q\in E$ is a minimal point as a boundary point of $(R_E)_M^*$. Indeed, if a sequence $\{p_n\}_{n=1}^\infty$ in R_E converges to $q\in E$, then obviously

$$\lim_{n\to\infty}\frac{g^R(z,p_n)}{g^R(a_0,p_n)}=\frac{g^R(z,q)}{g^R(a_0,q)}=: u_q(z)\in HP(R_E).$$

If $v \in HP(R_E)$ is less than u_q, then it is extended to $R \setminus \{q\}$ as a positive harmonic function. Hence, by the minimality of Green's functions, we conclude $v = cu_q$ for some constant $c > 0$ and u_q is a minimal function on R_E.

Let k_q and $k_{q,E}$ denote Martin kernels on R^*_M and $(R_E)^*_M$, respectively. We also denote by Δ_1 and $\Delta_{1,E}$ the sets of minimal points of R^*_M and on $(R_E)^*_M$, respectively. From (5.1), we have

$$d_H^{R_E}(p_1, p_2) = \sup_{q \in \Delta_{1,E}} \left| \log \frac{k_{q,E}(p_1)}{k_{q,E}(p_2)} \right|.$$

Since $\Delta_{1,E} = (\Delta_{1,E} \setminus E) \cup E$,

$$d_H^{R_E}(p_1, p_2) = \max \left\{ \sup_{q \in \Delta_{1,E} \setminus E} \left| \log \frac{k_{q,E}(p_1)}{k_{q,E}(p_2)} \right|, \sup_{q \in E} \left| \log \frac{k_{q,E}(p_1)}{k_{q,E}(p_2)} \right| \right\}.$$

As we have already seen that $\Delta_{1,E} \setminus E$ is identified with Δ_1,

$$\sup_{q \in \Delta_{1,E} \setminus E} \left| \log \frac{k_{q,E}(p_1)}{k_{q,E}(p_2)} \right| = \sup_{q \in \Delta_1} \left| \log \frac{k_q(p_1)}{k_q(p_2)} \right| = d_H^R(p_1, p_2).$$

Obviously, for $q \in E$

$$k_{q,E}(p) = \frac{g^R(p, q)}{g^R(a_0, q)} \quad (p \in R_E),$$

and we have

$$\frac{k_{q,E}(p_1)}{k_{q,E}(p_2)} = \frac{g^R(p_1, q)}{g^R(p_2, q)}.$$

Thus, we verify that (5.3) is true. □

It is an important problem to find a condition under which the Harnack pseudo distance becomes a distance on a Riemann surface R (see [**7**] for a discussion). We assume that the Harnack pseudo distance is a distance on R. Then, we have the following:

Proposition 5.2. *If the Harnack pseudo distance d_H^R is a distance on R, then Δ_1 consists of more than two points.*

Proof. If Δ_1 consists of one point, then it follows from Theorem 2.1 that $HP(R)$ does not contain non-constant functions. Hence, $d_H^R(p, q) = 0$ for any $p, q \in R$ and d_H^R is not a distance.

If Δ_1 consists of two points, then Theorem 2.1 implies that there exists a non-constant function $u_1 \in HP(R)$ such that every $u \in HP(R)$ is represented by

$$u = c_0 + c_1 u_1$$

for some constants $c_0, c_1 \geq 0$. We may take a constant $\alpha > 0$ such that $L_\alpha := \{p \in R : u_1(p) = \alpha\}$ is not empty. In fact, L_α contains an analytic arc. Then, for any distinct points $p, q \in L_\alpha$ we have

$$u(p) = c_0 + c_1 u_1(p) = c_0 + c_1\alpha = u(q).$$

Thus, we see that $d_H^R(p, q) = 0$ and d_H^R is not a distance.

Therefore, we conclude that Δ_1 contains more than two points if d_H^R is a distance. □

We consider conditions of R for the Harnack distance to be complete.

An open Riemann surface R is called *regular* for Green's functions if for any $\varepsilon > 0$, $\{p \in R : g^R(p, q) \geq \varepsilon\}$ is compact on R.

THEOREM 5.2. *Let E be a non-empty closed polar subset of R. If R is regular for Green's functions, then the Harnack distance $d_H^{R_E}$ on $R_E = R \setminus E$ is complete.*

PROOF. Let $\{p_n\}_{n=1}^\infty$ be a Cauchy sequence with respect to $d_H^{R_E}$. Then, there exists a constance $M > 0$ such that $d_H^{R_E}(p_1, p_n) < M$ for any $n \in \mathbb{N}$. It follows from Theorem 5.1 that

$$\sup_{q \in E} \left| \log \frac{g^R(p_n, q)}{g^R(p_1, q)} \right| < M,$$

for any $n \in \mathbb{N}$. If $p_n \to q \in E$, then $g^R(p_n, q) \to \infty$ and we obtain $g^R(p_n, q)/g^R(p_1, q) \to \infty$. This is a contradiction. Next, assume that there exists a subsequence $\{p_{n_k}\}_{k=1}^\infty$ of $\{p_n\}_{n=1}^\infty$ such that it tends to the ideal boundary of R. Then, from the regularity of R for Green's functions, we see that $g^R(p_{n_k}, q) \to 0$. Hence, we have

$$\left| \log \frac{g^R(p_{n_k}, q)}{g^R(p_1, q)} \right| \to \infty.$$

This is also a contradiction.

We have seen that the Cauchy sequence $\{p_n\}_{n=1}^\infty$ lies in a compact subset of R and every accumulation point is not in E. Let p', p'' be accumulation points of the sequence. Then, we see that $d_H^{R_E}(p', p'') = 0$. Since d_H^R is a distance and $d_H^{R_E} \geq d_H^R$ on $R_E \times R_E$, we conclude that $p' = p''$ and $\{p_n\}_{n=1}^\infty$ converges to $p' = p'' \in R_E$. Thus, we verify that the Harnack distance on R_E is complete. □

REMARK 5.2. The above argument shows that a Cauchy sequence $\{p_n\}_{n=1}^\infty$ with respect to $d_H^{R_E}$ is in a compact subset of R_E even if d_H^R is not a distance.

Now, we introduce a property of the Martin compactification (cf. [**5**]): A Riemann surface R is said to have *Picard existence property* (PEP) if every Martin kernel k_q extends to $\Delta \setminus \{q\}$ and vanishes there.

THEOREM 5.3. *If a Riemann surface R has PEP, then the Harnack distance d_H^R*

is complete on R.

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence with respect to d_H^R. We may show that the sequence is contained in a compact subset of R. If not, by taking a subsequence, we may assume that p_n tends to a point q_0 in the Martin boundary Δ of R as $n \to \infty$. Since d_H^R is a distance on R, Δ_1 consists on more than two points (Proposition 5.2). Therefore, we may take a point $q_1 \in \Delta_1$ other than q_0. Since R has PEP, $\lim_{n\to\infty} k_{q_1}(p_n) = 0$. Thus, we conclude that $d_H^R(p_1, p_n) \to \infty$. It is absurd because $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to d_H^R. Hence, $\{p_n\}_{n=1}^{\infty}$ is contained in a compact subset of R. Applying the same argument as the proof of Theorem 5.2, we may show that $\{p_n\}_{n=1}^{\infty}$ converges to a point in R. □

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