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# Generalized coderivations of bicomodules

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**Abstract.** We introduce a generalized coderivation from a bicomodule to a bicomodule over corings, which is a generalization of a coderivation. For each $(\mathcal{D}, \mathcal{C})$-bicomodule $N$ over corings $\mathcal{C}$ and $\mathcal{D}$, we construct the universal generalized coderivation $\upsilon_N : \mathcal{U}(N) \to N$ such that every generalized coderivation from a $(\mathcal{D}, \mathcal{C})$-bicomodule $M$ to $N$ is uniquely expressed as $\upsilon_N \circ f$ with some $(\mathcal{D}, \mathcal{C})$-bicomodule map $f : M \to \mathcal{U}(N)$. $\mathcal{U}(N)$ is isomorphic to the cotensor product of $N$ and $\mathcal{U}(\mathcal{D} \otimes_R \mathcal{C})$. We show that a coring $\mathcal{C}$ is coseparable if and only if, for any coring $\mathcal{D}$, all generalized coderivations from a $(\mathcal{D}, \mathcal{C})$-bicomodule to a $(\mathcal{D}, \mathcal{C})$-bicomodule are inner.

## 1. Introduction.

A coderivation of a coalgebra was introduced by Doi [**3**] and Nakajima [**8**]. This notion was extended by Guzman [**5**] to a cointegration from a bicomodule to another bicomodule over corings. Recently, a generalized coderivation of a coalgebra was introduced by Nakajima [**10**], which is a dual notion of a generalized derivation of an algebra defined by Nakajima [**9**]. In [**6**] the author of this paper extended a generalized derivation to a map from a bimodule to a bimodule. Dualizing this notion, we can extend the definition of a generalized coderivation to a map from a bicomodule to a bicomodule over corings.

In this paper, we investigate this new generalized coderivation. The definition is given in Section 2. In Section 3, we construct a universal generalized coderivation. For each $(\mathcal{D}, \mathcal{C})$-bicomodule $N$ over corings $\mathcal{C}$ and $\mathcal{D}$, there exists a $(\mathcal{D}, \mathcal{C})$-bicomodule $\mathcal{U}(N)$ and a generalized coderivation $\upsilon_N : \mathcal{U}(N) \to N$ such that every generalized coderivation from a $(\mathcal{D}, \mathcal{C})$-bicomodule $M$ to $N$ is uniquely expressed as $\upsilon_N \circ f$ with some $(\mathcal{D}, \mathcal{C})$-bicomodule map $f : M \to \mathcal{U}(N)$. Moreover, in Section 4, we show that $\mathcal{U}(N)$ is isomorphic to $N \Box_{\mathcal{D}^{\mathrm{cop}} \otimes_R \mathcal{C}} \mathcal{U}(\mathcal{D} \otimes_R \mathcal{C})$ as $(\mathcal{D}, \mathcal{C})$-bicomodule. Finally, in Section 5, we characterize a coseparable coring. A coderivation was introduced in the context of cohomology theory of coalgebras in [**3**], and it was proved that a coalgebra is coseparable if and only if all coderivations are inner. This result was extended in [**5**] for cointegrations. We prove a corresponding result for our generalized coderivations.

Throughout this paper, $R$ denotes a commutative ring with an identity element, every algebra is an associative $R$-algebra with an identity element, and every module is unitary. Every coring has a counit and every comodule is counitary. Notations are based on [**2**]. For an $R$-algebra $A$, the category of right $A$-modules is denoted by $\mathbf{M}_A$. For $R$-algebras $A$ and $B$, the category of $(B, A)$-bimodules on which right and left actions

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of $R$ coincide is denoted by ${}_B\mathbf{M}_A$. If $X$, $Y \in {}_B\mathbf{M}_A$, then the set of all $(B, A)$-bimodule maps from $X$ to $Y$ is denoted by ${}_B\mathrm{Hom}_A(X, Y)$. For a coring $\mathcal{C}$, its coproduct is denoted by $\Delta_{\mathcal{C}}$, its counit is denoted by $\varepsilon_{\mathcal{C}}$, and the category of right $\mathcal{C}$-comodules is denoted by $\mathbf{M}^{\mathcal{C}}$. For corings $\mathcal{C}$ and $\mathcal{D}$, the category of $(\mathcal{D}, \mathcal{C})$-bicomodules is denoted by ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. For $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, the right and left coactions on $M$ are denoted by $\rho^M$ and ${}^M\rho$, respectively, and we set ${}^M\rho^M = ({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M$. If $\mathcal{C}$ is an $A$-coring and $\mathcal{D}$ is a $B$-coring, then, for $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, ${}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M, N)$, ${}_B\mathrm{Hom}^{\mathcal{C}}(M, N)$, and ${}^{\mathcal{D}}\mathrm{Hom}_A(M, N)$ denote the set of all $(\mathcal{D}, \mathcal{C})$-bicomodule maps, the set of all right $\mathcal{C}$-comodule left $B$-module maps, and the set of all left $\mathcal{D}$-comodule right $A$-module maps from $M$ to $N$, respectively. The identity map of a set $X$ is denoted by $I_X$.

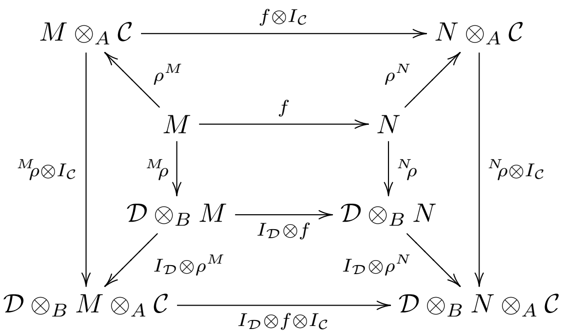
## 2. Definition of generalized coderivations.

In this section, $A$ and $B$ will represent $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring.

DEFINITION 2.1. For each $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, we shall define an $R$-linear map

$$Q_{M,N} : {}_B\mathrm{Hom}_A(M, N) \to {}_B\mathrm{Hom}_A(M,\ \mathcal{D} \otimes_B N \otimes_A \mathcal{C}).$$

For $f \in {}_B\mathrm{Hom}_A(M, N)$, we can consider the following diagram:




Using maps appeared in this diagram, we set

$$\begin{aligned} Q_{M,N}(f) = {} & (I_{\mathcal{D}} \otimes \rho^N) \circ {}^N\rho \circ f - (I_{\mathcal{D}} \otimes \rho^N) \circ (I_{\mathcal{D}} \otimes f) \circ {}^M\rho \\ & - ({}^N\rho \otimes I_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M + (I_{\mathcal{D}} \otimes f \otimes I_{\mathcal{C}}) \circ ({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M. \end{aligned}$$

In other words, using the maps $M \to \mathcal{D} \otimes_B N \otimes_A \mathcal{C}$ appeared in the above diagram, we set

$$\begin{aligned} Q_{M,N}(f) = {} & (\text{a map through } f) - (\text{a map through } I_{\mathcal{D}} \otimes f) \\ & - (\text{a map through } f \otimes I_{\mathcal{C}}) + (\text{a map through } I_{\mathcal{D}} \otimes f \otimes I_{\mathcal{C}}). \end{aligned}$$

If $f$ is a $(\mathcal{D}, \mathcal{C})$-bicomodule map, then the above diagram is commutative. Hence we get the next

Lemma 2.2. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, $\mathcal{D}$ a $B$-coring, $L$, $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, and $f \in {}_B\mathrm{Hom}_A(M, N)$. Then*

(1) $Q_{L,N}(f \circ g) = Q_{M,N}(f) \circ g$ *for all* $g \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(L, M)$.
(2) $Q_{M,L}(h \circ f) = (I_{\mathcal{D}} \otimes h \otimes I_{\mathcal{C}}) \circ Q_{M,N}(f)$ *for all* $h \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(N, L)$.

The next is an immediate consequence of Lemma 2.2.

Corollary 2.3. *$Q$ is a natural transformation.*

Definition 2.4. We define the functor

$$ {}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}} : \left({}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}\right)^{op} \times {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_R $$

as the kernel of the natural transformation $Q$, i.e., ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}$ is the subfunctor of ${}_B\mathrm{Hom}_A : \left({}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}\right)^{op} \times {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_R$ determined by ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N) = \mathrm{Ker}\, Q_{M,N}$ for $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. An element of ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N)$ is called a *generalized coderivation.*

Theorem 2.5. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. Let $M = \coprod_{i \in I} M_i$ be a coproduct in ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ with the structure maps $\iota_i : M_i \to M$ $(i \in I)$ and $N = \prod_{j \in J} N_j$ a finite product in ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ with the structure maps $\pi_j : N \to N_j$ $(j \in J)$. Then, the $R$-linear map*

$$ {}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N) \ni f \mapsto (\pi_j \circ f \circ \iota_i) \in \prod_{(i,j) \in I \times J} {}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M_i,\, N_j) $$

*is an isomorphism.*

Proof. It is well-known that the $R$-linear map

$$ {}_B\mathrm{Hom}_A(M, N) \ni f \mapsto (\pi_j \circ f \circ \iota_i) \in \prod_{(i,j) \in I \times J} {}_B\mathrm{Hom}_A(M_i,\, N_j) $$

is an isomorphism. Let $f \in {}_B\mathrm{Hom}_A(M, N)$. Then, by Lemma 2.2, we have

$$ Q_{M_i,N_j}(\pi_j \circ f \circ \iota_i) = (I_{\mathcal{D}} \otimes \pi_j \otimes I_{\mathcal{C}}) \circ Q_{M,N}(f) \circ \iota_i $$

for all $i \in I$ and $j \in J$. Since $I_{\mathcal{D}} \otimes \pi_j \otimes I_{\mathcal{C}}$ $(j \in J)$ are the structure maps of the finite product $\mathcal{D} \otimes_B N \otimes_A \mathcal{C} = \prod_{j \in J} \mathcal{D} \otimes_B N_j \otimes_A \mathcal{C}$, $Q_{M,N}(f) = 0$ is equivalent to $Q_{M_i,N_j}(\pi_j \circ f \circ \iota_i) = 0$ for all $i \in I$ and $j \in J$. Hence, we get the assertion. □

Definition 2.6. For each $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, we set

$$ {}^{\mathcal{D}}\mathrm{GInCoder}^{\mathcal{C}}(M, N) = {}_B\mathrm{Hom}^{\mathcal{C}}(M, N) + {}^{\mathcal{D}}\mathrm{Hom}_A(M, N). $$

An element of ${}^{\mathcal{D}}\mathrm{GInCoder}^{\mathcal{C}}(M, N)$ is called a *generalized inner coderivation.*

We can easily see the next

Lemma 2.7. *${}^{\mathcal{D}}\mathrm{GInCoder}^{\mathcal{C}}(M,N)$ 's determine a subfunctor of ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}$.*

We shall show that our generalized coderivation is a generalization of a generalized coderivation introduced in Nakajima [**10**].

Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. According to [**5**], a map $f$ in ${}_A\mathrm{Hom}_A(M,\mathcal{C})$ is called a *coderivation* if $\Delta_{\mathcal{C}} \circ f = (f \otimes I_{\mathcal{C}}) \circ \rho^M + (I_{\mathcal{C}} \otimes f) \circ {}^M\rho$. A map $f$ in ${}_A\mathrm{Hom}_A(M,\mathcal{C})$ is called a *Nakajima's generalized coderivation* if $\Delta_{\mathcal{C}} \circ f - (f \otimes I_{\mathcal{C}}) \circ \rho^M - (I_{\mathcal{C}} \otimes f) \circ {}^M\rho$ is a $(\mathcal{C},\mathcal{C})$-bicomodule map.

Theorem 2.8. *Let $A$ be an $R$-algebra, $\mathcal{C}$ an $A$-coring, $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$, and $f \in {}_A\mathrm{Hom}_A(M,\mathcal{C})$. Then $f \in {}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M,\mathcal{C})$ if and only if $f$ is a Nakajima's generalized coderivation.*

Proof. We set

$$h = \Delta_{\mathcal{C}} \circ f - (f \otimes I_{\mathcal{C}}) \circ \rho^M - (I_{\mathcal{C}} \otimes f) \circ {}^M\rho.$$

Then we see that

$$\begin{aligned}(I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ h = {} & (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} \circ f - (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M \\ & - (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ (I_{\mathcal{C}} \otimes f) \circ {}^M\rho \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(h \otimes I_{\mathcal{C}}) \circ \rho^M \\ &\quad = (\Delta_{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M - (f \otimes I_{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (\rho^M \otimes I_{\mathcal{C}}) \circ \rho^M \\ &\qquad - (I_{\mathcal{C}} \otimes f \otimes I_{\mathcal{C}}) \circ ({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M. \end{aligned} \tag{2.2}$$

By definition we have

$$\begin{aligned} Q_{M,\mathcal{C}}(f) = {} & (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} \circ f - (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ (I_{\mathcal{C}} \otimes f) \circ {}^M\rho \\ & - (\Delta_{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M + (I_{\mathcal{C}} \otimes f \otimes I_{\mathcal{C}}) \circ ({}^M\rho \otimes I_{\mathcal{C}}) \circ \rho^M. \end{aligned} \tag{2.3}$$

The commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\rho^M} & M \otimes_A \mathcal{C} & \xrightarrow{f \otimes I_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ {\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle I_M \otimes \Delta_{\mathcal{C}}} & & \downarrow{\scriptstyle I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}} \\ M \otimes_A \mathcal{C} & \xrightarrow[\rho^M \otimes I_{\mathcal{C}}]{} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow[f \otimes I_{\mathcal{C}} \otimes I_{\mathcal{C}}]{} & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array}$$

shows that

$$(I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = (f \otimes I_{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (\rho^M \otimes I_{\mathcal{C}}) \circ \rho^M. \tag{2.4}$$

Combining the equations (2.1), (2.2), (2.3), and (2.4), we get

$$Q_{M,\mathcal{C}}(f) = (I_{\mathcal{C}} \otimes \Delta_{\mathcal{C}}) \circ h - (h \otimes I_{\mathcal{C}}) \circ \rho^M.$$

Hence $f \in {}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M, \mathcal{C})$ if and only if $h$ is a right $\mathcal{C}$-comodule map. Similarly $f \in {}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M, \mathcal{C})$ if and only if $h$ is a left $\mathcal{C}$-comodule map. □

Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$. We consider well-known $R$-isomorphisms

$$\mathfrak{R}_M : {}_A\mathrm{Hom}_A(M, A) \to {}_A\mathrm{Hom}^{\mathcal{C}}(M, \mathcal{C}) \quad \text{and}$$
$$\mathfrak{L}_M : {}_A\mathrm{Hom}_A(M, A) \to {}^{\mathcal{C}}\mathrm{Hom}_A(M, \mathcal{C}).$$

For $\xi \in {}_A\mathrm{Hom}_A(M, A)$, $\mathfrak{R}_M(\xi)$ is the composition map

$$M \xrightarrow{\rho^M} M \otimes_A \mathcal{C} \xrightarrow{\xi \otimes I_{\mathcal{C}}} A \otimes_A \mathcal{C} \xrightarrow{\text{canonical isom.}} \mathcal{C}$$

and $\mathfrak{L}_M(\xi)$ is the composition map

$$M \xrightarrow{{}^M\rho} \mathcal{C} \otimes_A M \xrightarrow{I_{\mathcal{C}} \otimes \xi} \mathcal{C} \otimes_A A \xrightarrow{\text{canonical isom.}} \mathcal{C}.$$

Usually $\mathfrak{R}_M(\xi)$ and $\mathfrak{L}_M(\xi)$ are represented by $(\xi \otimes I_{\mathcal{C}}) \circ \rho^M$ and $(I_{\mathcal{C}} \otimes \xi) \circ {}^M\rho$, respectively. According to [**5**], a map of the form $\mathfrak{R}_M(\xi) - \mathfrak{L}_M(\xi)$ with some $\xi \in {}_A\mathrm{Hom}_A(M, A)$ is called an *inner coderivation*. Obviously every inner coderivation is a generalized inner coderivation.

## 3. Universal generalized coderivation.

In this section, we construct the universal coderivations. We will use the following notations. Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, $\mathcal{D}$ a $B$-coring, and $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. We denote by $\varepsilon^M$ the composition map

$$M \otimes_A \mathcal{C} \xrightarrow{I_M \otimes \varepsilon_{\mathcal{C}}} M \otimes_A A \xrightarrow{\text{canonical isom.}} M.$$

Similarly, we denote by ${}^M\varepsilon$ the composition map

$$\mathcal{D} \otimes_B M \xrightarrow{\varepsilon_{\mathcal{D}} \otimes I_M} B \otimes_B M \xrightarrow{\text{canonical isom.}} M.$$

We denote by ${}^M\varepsilon^M$ the composition map

$$\mathcal{D} \otimes_B M \otimes_A \mathcal{C} \xrightarrow{\varepsilon_{\mathcal{D}} \otimes I_M \otimes \varepsilon_{\mathcal{C}}} B \otimes_B M \otimes_A A \xrightarrow{\text{canonical isom.}} M.$$

Usually, $\varepsilon^M$, ${}^M\varepsilon$, and ${}^M\varepsilon^M$ are represented by $I_M \otimes \varepsilon_{\mathcal{C}}$, $\varepsilon_{\mathcal{D}} \otimes I_M$, and $\varepsilon_{\mathcal{D}} \otimes I_M \otimes \varepsilon_{\mathcal{C}}$, respectively. We set

$$e^M = \rho^M \circ \varepsilon^M, \quad {}^M e = {}^M\rho \circ {}^M\varepsilon, \quad \text{and} \quad {}^M e^M = {}^M\rho^M \circ {}^M\varepsilon^M.$$

DEFINITION 3.1. Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. We define the natural transformation $E : \mathcal{D} \otimes_B (\ ) \otimes_A \mathcal{C} \to \mathcal{D} \otimes_B (\ ) \otimes_A \mathcal{C}$ of $(B, A)$-bimodule maps by setting

$$E_M = I_{\mathcal{D}\otimes_B M \otimes_A \mathcal{C}} - I_{\mathcal{D}} \otimes e^M - {}^M e \otimes I_{\mathcal{C}} + {}^M e^M$$

for every $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. We define the functor $\mathcal{U} : {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}} \to {}_B\mathbf{M}_A$ as the kernel of the natural transformation $E$. For each $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, let

$$\upsilon_M : \mathcal{U}(M) \to M$$

denote the restriction map of ${}^M\varepsilon^M$ to $\mathcal{U}(M)$.

By definition, for any $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ and $f \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M, N)$, the diagram of $(B, A)$-bimodule maps

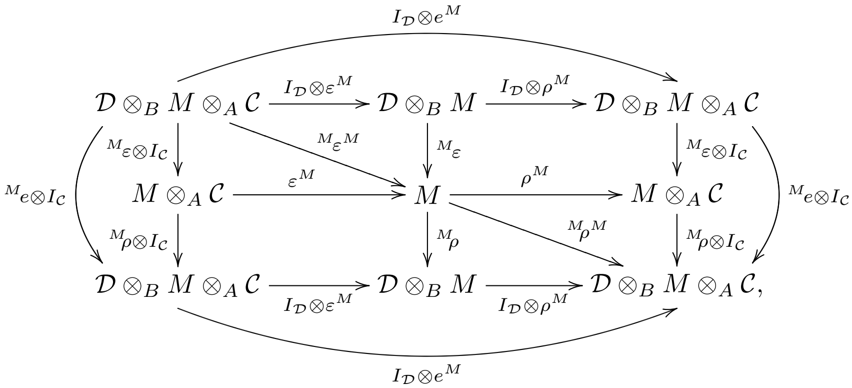
$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{U}(M) & \xrightarrow{\text{inclusion}} & \mathcal{D} \otimes_B M \otimes_A \mathcal{C} & \xrightarrow{E_M} & \mathcal{D} \otimes_B M \otimes_A \mathcal{C} \\
 & & \Big\downarrow{\scriptstyle \mathcal{U}(f)} & & \Big\downarrow{\scriptstyle I_{\mathcal{D}}\otimes f \otimes I_{\mathcal{C}}} & & \Big\downarrow{\scriptstyle I_{\mathcal{D}}\otimes f \otimes I_{\mathcal{C}}} \\
0 & \longrightarrow & \mathcal{U}(N) & \xrightarrow[\text{inclusion}]{} & \mathcal{D} \otimes_B N \otimes_A \mathcal{C} & \xrightarrow[E_N]{} & \mathcal{D} \otimes_B N \otimes_A \mathcal{C}
\end{array}$$

is commutative and two rows are exact.

LEMMA 3.2. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. For every $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, there hold the following.*

(1) ${}^M e \otimes I_{\mathcal{C}}$ *and* $I_{\mathcal{D}} \otimes e^M$ *are commuting idempotents in the endomorphism ring* ${}_B\mathrm{Hom}_A(\mathcal{D} \otimes_B M \otimes_A \mathcal{C},\ \mathcal{D} \otimes_B M \otimes_A \mathcal{C})$ *and* ${}^M e^M = ({}^M e \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes e^M)$ *holds.*
(2) $E_M = (I_{\mathcal{D}\otimes_B M\otimes_A \mathcal{C}} - {}^M e \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}\otimes_B M\otimes_A \mathcal{C}} - I_{\mathcal{D}} \otimes e^M)$.
(3) $E_M \circ E_M = E_M$.
(4) $\mathcal{U}(M)$ *is a direct summand of* $\mathcal{D} \otimes_B M \otimes_A \mathcal{C}$ *as a* $(B, A)$*-bimodule.*
(5) $E_M = Q_{\mathcal{D}\otimes_B M\otimes_A \mathcal{C},\, M}({}^M\varepsilon^M)$.

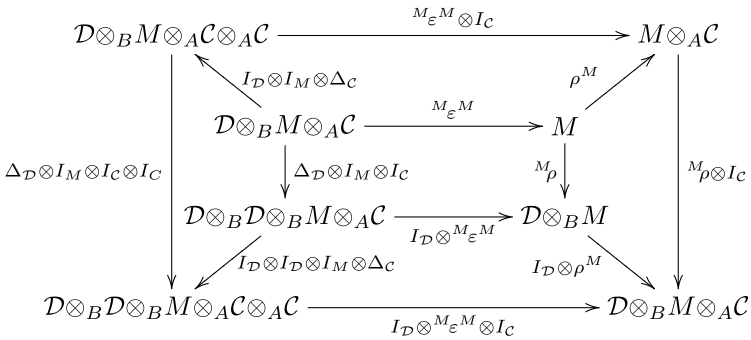
PROOF. (1) Since $\varepsilon^M \circ \rho^M = I_M$, ${}^M\varepsilon \circ {}^M\rho = I_M$, and ${}^M\varepsilon^M \circ {}^M\rho^M = I_M$, we have $e^M \circ e^M = e^M$, ${}^M e \circ {}^M e = {}^M e$, and ${}^M e^M \circ {}^M e^M = {}^M e^M$. In the commutative diagram




the commutativity of the outer rectangle and the diagonal shows that

$$ {}^Me^M = ({}^Me\otimes I_{\mathcal{C}})\circ(I_{\mathcal{D}}\otimes e^M) = (I_{\mathcal{D}}\otimes e^M)\circ({}^Me\otimes I_{\mathcal{C}}). $$

(2) is immediate from (1) and the definition of $E_M$.
(3) is clear by (1) and (2).
(4) is clear by (3) and the definition of $\mathcal{U}(M)$.
(5) We consider the following diagram.




Then we see that

$$ (I_{\mathcal{D}}\otimes\rho^M)\circ{}^M\rho\circ{}^M\varepsilon^M = {}^M\rho^M\circ{}^M\varepsilon^M = {}^Me^M, $$

$$ \begin{aligned}(I_{\mathcal{D}}\otimes\rho^M)&\circ(I_{\mathcal{D}}\otimes{}^M\varepsilon^M)\circ(\Delta_{\mathcal{D}}\otimes I_M\otimes I_{\mathcal{C}})\\ &= (I_{\mathcal{D}}\otimes\rho^M)\circ(I_{\mathcal{D}}\otimes\varepsilon^M) = I_{\mathcal{D}}\otimes e^M,\end{aligned} $$

$$ \begin{aligned}({}^M\rho\otimes I_{\mathcal{C}})&\circ({}^M\varepsilon^M\otimes I_{\mathcal{C}})\circ(I_{\mathcal{D}}\otimes I_M\otimes\Delta_{\mathcal{C}})\\ &= ({}^M\rho\otimes I_{\mathcal{C}})\circ({}^M\varepsilon\otimes I_{\mathcal{C}}) = {}^Me\otimes I_{\mathcal{C}},\ \text{and}\end{aligned} $$

$$ (I_{\mathcal{D}}\otimes{}^M\varepsilon^M\otimes I_{\mathcal{C}})\circ(\Delta_{\mathcal{D}}\otimes I_M\otimes I_{\mathcal{C}}\otimes I_{\mathcal{C}})\circ(I_{\mathcal{D}}\otimes I_M\otimes\Delta_{\mathcal{C}}) = I_{D\otimes_B M\otimes_A\mathcal{C}}. $$

Combining these equations, we get the assertion. □

In Definition 3.1, $\mathcal{U}$ is a functor from ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ to ${}_B\mathbf{M}_A$. The next theorem shows that $\mathcal{U}$ is a functor from ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ to ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$.

THEOREM 3.3. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. Then $\mathcal{U}$ is a subfunctor of the functor $\mathcal{D} \otimes_B (\ ) \otimes_A \mathcal{C} : {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}} \to {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, and $\upsilon_M$'s determine the natural transformation $\upsilon : \mathcal{U} \to I_{{}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}}$.*

To prove Theorem 3.3, we use the well-known fact that every bicomodule can be viewed as a one-sided comodule. Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. Consider the coring $\mathcal{F} = \mathcal{D}^{cop} \otimes_R \mathcal{C}$ over the $R$-algebra $\Lambda = B^{op} \otimes_R A$, where $B^{op}$ is the opposite algebra of $B$ and $\mathcal{D}^{cop}$ is the opposite $B^{op}$-coring of $\mathcal{D}$ defined in [**4**, Opposite coring 1.7]. Then, by [**4**, Proposition 1.8], a $(\mathcal{D}, \mathcal{C})$-bicomodule is no other than a right $\mathcal{F}$-comodule. Actually, for $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, the right coaction $\rho^M_{\mathcal{F}}$ of $\mathcal{F}$ on $M$ is the composition map

$$M \xrightarrow{{}^M\rho^M} \mathcal{D} \otimes_B M \otimes_A \mathcal{C} \xrightarrow{t} M \otimes_\Lambda \mathcal{F},$$

where $t$ is defined by $t(d \otimes m \otimes c) = m \otimes (d \otimes c)$. Similarly, for $N \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{D}}$, the left coaction ${}^N\rho_{\mathcal{F}}$ of $\mathcal{F}$ on $N$ is given by the composition map

$$N \xrightarrow{{}^N\rho^N} \mathcal{C} \otimes_A N \otimes_B \mathcal{D} \xrightarrow{t'} \mathcal{F} \otimes_\Lambda N,$$

where $t'$ is defined by $t'(c \otimes n \otimes d) = (d \otimes c) \otimes n$.

We prepare an easy lemma.

LEMMA 3.4. *Let $\Lambda$ be an $R$-algebra, $\mathcal{F}$ a $\Lambda$-coring, and $h : M \to N$ a morphism in $\mathbf{M}^{\mathcal{F}}$. Let $N'$ be an $\mathcal{F}$-subcomodule of $N$ with $h(M) \subseteq N'$. If $N'$ is an $\mathcal{F}$-pure $\Lambda$-submodule of $N$, then the map $h' : M \ni x \mapsto h(x) \in N'$ is an $\mathcal{F}$-comodule map.*

PROOF. Let $\iota : N' \to N$ denote the inclusion map. Then, in the diagram

$$\begin{array}{ccccc} M & & \xrightarrow{\quad\rho^M\quad} & & M \otimes_\Lambda \mathcal{F} \\ & \searrow^{h} & & \swarrow^{h \otimes I_{\mathcal{F}}} & \\ {\scriptstyle h'}\downarrow & & N \xrightarrow{\rho^N} N \otimes_\Lambda \mathcal{F} & & \downarrow{\scriptstyle h' \otimes I_{\mathcal{F}}} \\ & \nearrow_{\iota} & & \nwarrow_{\iota \otimes I_{\mathcal{F}}} & \\ N' & & \xrightarrow[\quad\rho^{N'}\quad]{} & & N' \otimes_\Lambda \mathcal{F} \end{array}$$

all subdiagrams except the outer rectangle are commutative. Since $\iota \otimes I_{\mathcal{F}}$ is an injective map, the outer rectangle is commutative. □

PROOF OF THEOREM 3.3. Let $M \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. As a map $\mathcal{D} \otimes_B M \otimes_A \mathcal{C} \to \mathcal{D} \otimes_B M \otimes_A \mathcal{C} \otimes_A \mathcal{C}$, we see that

$$(I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ ({}^M e \otimes I_{\mathcal{C}}) = {}^M e \otimes \Delta_{\mathcal{C}} = ({}^M e \otimes I_{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}).$$

Since $E_M \circ ({}^M e \otimes I_{\mathcal{C}}) = 0$ by (1) and (2) of Lemma 3.2, we have

$$(E_M \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ ({}^M e \otimes I_{\mathcal{C}}) = 0. \tag{3.1}$$

By composing (3.1) with $I_{\mathcal{D}} \otimes e^M$ on the right, and using Lemma 3.2 (1), we get

$$(E_M \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ {}^M e^M = 0. \tag{3.2}$$

On the other hand, as a map $M \otimes_A \mathcal{C} \to M \otimes_A \mathcal{C} \otimes_A \mathcal{C}$, we see that

$$(e^M \otimes I_{\mathcal{C}}) \circ (I_M \otimes \Delta_{\mathcal{C}}) = (\rho^M \otimes I_{\mathcal{C}}) \circ (\varepsilon^M \otimes I_{\mathcal{C}}) \circ (I_M \otimes \Delta_C) = \rho^M \otimes I_{\mathcal{C}} \tag{3.3}$$

and

$$(\rho^M \otimes I_{\mathcal{C}}) \circ e^M = (\rho^M \otimes I_{\mathcal{C}}) \circ \rho^M \circ \varepsilon^M = (I_M \otimes \Delta_{\mathcal{C}}) \circ \rho^M \circ \varepsilon^M = (I_M \otimes \Delta_{\mathcal{C}}) \circ e^M. \tag{3.4}$$

Combining the equations (3.3) and (3.4), we have

$$(e^M \otimes I_{\mathcal{C}}) \circ (I_M \otimes \Delta_{\mathcal{C}}) \circ e^M = (I_M \otimes \Delta_{\mathcal{C}}) \circ e^M,$$

and hence

$$(I_{M \otimes_A \mathcal{C} \otimes_A \mathcal{C}} - e^M \otimes I_{\mathcal{C}}) \circ (I_M \otimes \Delta_{\mathcal{C}}) \circ e^M = 0.$$

It follows that

$$((I_{\mathcal{D} \otimes_B M \otimes_A \mathcal{C}} - I_{\mathcal{D}} \otimes e^M) \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes e^M) = 0.$$

By (1) and (2) of Lemma 3.2, we have

$$(E_M \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes e^M) = 0. \tag{3.5}$$

By the equations (3.1), (3.2), and (3.5), we have

$$(E_M \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}}) \circ (I_{\mathcal{D} \otimes_B M \otimes_A \mathcal{C}} - E_M) = 0. \tag{3.6}$$

Noting (3) and (4) of Lemma 3.2, the equation (3.6) means that $(I_{\mathcal{D}} \otimes I_M \otimes \Delta_{\mathcal{C}})(\mathcal{U}(M))$ is contained in $\mathcal{U}(M) \otimes_A \mathcal{C}$. Hence, $\mathcal{U}(M)$ is a right $\mathcal{C}$-subcomodule of $\mathcal{D} \otimes_B M \otimes_A \mathcal{C}$. Similarly, $\mathcal{U}(M)$ is a left $\mathcal{D}$-subcomodule.

Let $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ and $f \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M, N)$. Let $h$ denote the composition map

$$\mathcal{U}(M) \xrightarrow{\text{inclusion}} D \otimes_B M \otimes_A \mathcal{C} \xrightarrow{I_{\mathcal{D}} \otimes f \otimes I_{\mathcal{C}}} D \otimes_B N \otimes_A \mathcal{C}$$

of $(\mathcal{D},\mathcal{C})$-bicomodule maps. We consider the coring $\mathcal{F}=\mathcal{D}^{cop}\otimes_R\mathcal{C}$ over the $R$-algebra $\Lambda=B^{op}\otimes_R A$. Then, $h$ is an $\mathcal{F}$-comodule map and $\mathcal{U}(N)$ is a pure $\Lambda$-submodule of $\mathcal{D}\otimes_B N\otimes_A\mathcal{C}$ by Lemma 3.2 (4). Therefore, by Lemma 3.4, $\mathcal{U}(f)$ is an $\mathcal{F}$-comodule map. Hence $\mathcal{U}$ is a functor from ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$ to ${}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. □

Theorem 3.5. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, $\mathcal{D}$ a $B$-coring, and $M$, $N\in{}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. Then the $R$-linear map*

$$ {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{U}(N))\ni f\mapsto \upsilon_N\circ f\in{}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M,N)$$

*is a natural isomorphism. In particular, $\upsilon_N$ belongs to ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(\mathcal{U}(N),\,N)$.*

Proof. As is well-known, the $R$-linear map

$$ {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M,\ \mathcal{D}\otimes_B N\otimes_A\mathcal{C})\ni f\mapsto {}^{N}\varepsilon^{N}\circ f\in{}_B\mathrm{Hom}_A(M,N)$$

is an isomorphism with the inverse map $g\mapsto(I_{\mathcal{D}}\otimes g\otimes I_{\mathcal{C}})\circ{}^{M}\rho^{M}$. Let $f\in{}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(M,\ \mathcal{D}\otimes_B N\otimes_A\mathcal{C})$. By Lemma 2.2 (1) and Lemma 3.2 (5), we have $Q_{M,N}({}^{N}\varepsilon^{N}\circ f)=E_N\circ f$. Therefore, ${}^{N}\varepsilon^{N}\circ f\in{}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M,N)$ is equivalent to $f(M)\subseteq\operatorname{Ker}E_N=\mathcal{U}(N)$. Noting Lemma 3.4, we get the assertion. □

## 4. A property of the functor $\mathcal{U}$.

Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. Consider the coring $\mathcal{F}=\mathcal{D}^{cop}\otimes_R\mathcal{C}$ over the $R$-algebra $\Lambda=B^{op}\otimes_R A$. As usual, $V=\mathcal{D}\otimes_B\mathcal{D}\otimes_R\mathcal{C}\otimes_A\mathcal{C}$ is a $(\mathcal{D},\mathcal{C})$-bicomodule, and hence $V$ is a right $\mathcal{F}$-comodule. We can consider $V$ as an $(A,B)$-bimodule, with left action of $A$ on the first $\mathcal{C}$ factor, and right action of $B$ on the second $\mathcal{D}$ factor. As such, it is $(A,B)$-isomorphic to $V^o=\mathcal{C}\otimes_A\mathcal{C}\otimes_R\mathcal{D}\otimes_B\mathcal{D}$ via the twist map $V\ni d\otimes d'\otimes c\otimes c'\mapsto c\otimes c'\otimes d\otimes d'\in V^o$. We can transfer the $(\mathcal{C},\mathcal{D})$-bicomodule structure of $V^o$ to $V$, making it into a left $\mathcal{F}$-comodule. It is clear that $V$ is an $(\mathcal{F},\mathcal{F})$-bicomodule. Under these notations, we have the next

Lemma 4.1. *$\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ is an $(\mathcal{F},\mathcal{F})$-sub-bicomodule of $V$.*

Proof. By Theorem 3.3, $\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ is a $(\mathcal{D},\mathcal{C})$-sub-bicomodule of $V$, i.e., $\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ is a right $\mathcal{F}$-subcomodule of $V$. We use two maps

$$e^{\mathcal{C}}:\mathcal{C}\otimes_A\mathcal{C}\ni x\otimes y\mapsto\Delta_{\mathcal{C}}(x)\varepsilon_{\mathcal{C}}(y)\in\mathcal{C}\otimes_A\mathcal{C}\quad\text{and}$$
$${}^{\mathcal{D}}e:\mathcal{D}\otimes_B\mathcal{D}\ni x\otimes y\mapsto\varepsilon_{\mathcal{D}}(x)\Delta_{\mathcal{D}}(y)\in\mathcal{D}\otimes_B\mathcal{D}$$

defined at the first part of the previous section. By Lemma 3.2 (2), we have

$$E_{\mathcal{D}\otimes_R\mathcal{C}}=(I_V-{}^{\mathcal{D}}e\otimes I_{\mathcal{C}}\otimes I_{\mathcal{C}})\circ(I_V-I_{\mathcal{D}}\otimes I_{\mathcal{D}}\otimes e^{\mathcal{C}}).$$

Since $e^{\mathcal{C}}$ is a left $\mathcal{C}$-comodule map and ${}^{\mathcal{D}}e$ is a right $\mathcal{D}$-comodule map, $E_{\mathcal{D}\otimes_R\mathcal{C}}$ is a $(\mathcal{C},\mathcal{D})$-

bicomodule map. By Lemma 3.2 (4) and [**5**, Proposition 1.1 2], $\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ is a $(\mathcal{C},\mathcal{D})$-sub-bicomodule of $V$, i.e., $\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ is a left $\mathcal{F}$-subcomodule of $V$. $\square$

THEOREM 4.2. *Let $A$ and $B$ be $R$-algebras, $\mathcal{C}$ an $A$-coring, and $\mathcal{D}$ a $B$-coring. Consider the coring $\mathcal{F}=\mathcal{D}^{cop}\otimes_R\mathcal{C}$ over the $R$-algebra $\Lambda=B^{op}\otimes_R A$, where $B^{op}$ is the opposite algebra of $B$ and $\mathcal{D}^{cop}$ is the opposite $B^{op}$-coring of $\mathcal{D}$. Then, for every $M\in{}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$, $\mathcal{U}(M)$ is isomorphic to $M\,\square_{\mathcal{F}}\,\mathcal{U}(\mathcal{D}\otimes_R\mathcal{C})$ as a right $\mathcal{F}$-comodule.*

PROOF. We set $M_1=\mathcal{D}\otimes_B M\otimes_A\mathcal{C}$ and $M_2=\mathcal{D}\otimes_B\mathcal{D}\otimes_B M\otimes_A\mathcal{C}\otimes_A\mathcal{C}$, and define the $(\mathcal{D},\mathcal{C})$-bicomodule map $\omega: M_1\to M_2$ by setting $\omega=I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}-\Delta_{\mathcal{D}}\otimes I_M\otimes\Delta_{\mathcal{C}}$. We consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{U}(M) & \xrightarrow{\iota_M} & \mathcal{D}\otimes_B M\otimes_A\mathcal{C} & \xrightarrow{E_M} & \mathcal{D}\otimes_B M\otimes_A\mathcal{C} \\
 & & \downarrow{\scriptstyle \mathcal{U}({}^{M}\rho^{M})} & & \downarrow{\scriptstyle I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}} & & \downarrow{\scriptstyle I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}} \\
0 & \longrightarrow & \mathcal{U}(M_1) & \xrightarrow{\iota_{M_1}} & \mathcal{D}\otimes_B M_1\otimes_A\mathcal{C} & \xrightarrow{E_{M_1}} & \mathcal{D}\otimes_B M_1\otimes_A\mathcal{C} \\
 & & \downarrow{\scriptstyle \mathcal{U}(\omega)} & & \downarrow{\scriptstyle I_{\mathcal{D}}\otimes\omega\otimes I_{\mathcal{C}}} & & \downarrow{\scriptstyle I_{\mathcal{D}}\otimes\omega\otimes I_{\mathcal{C}}} \\
0 & \longrightarrow & \mathcal{U}(M_2) & \xrightarrow[\iota_{M_2}]{} & \mathcal{D}\otimes_B M_2\otimes_A\mathcal{C} & \xrightarrow[E_{M_2}]{} & \mathcal{D}\otimes_B M_2\otimes_A\mathcal{C}
\end{array} \tag{4.1}$$

of $(B,A)$-bimodule maps, where $\iota_M$, $\iota_{M_1}$, and $\iota_{M_2}$ are inclusion maps. By definition, all rows are exact. Since ${}^{M}\rho^{M}$ is a section in ${}_B\mathbf{M}_A$, $I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}$ is also a section. The $(\mathcal{D},\mathcal{C})$-bicomodule structure of $M$ yields the commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{{}^{M}\rho^{M}} & M_1 \\
\downarrow{\scriptstyle {}^{M}\rho^{M}} & & \downarrow{\scriptstyle \Delta_{\mathcal{D}}\otimes I_M\otimes\Delta_{\mathcal{C}}} \\
M_1 & \xrightarrow[I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}]{} & M_2.
\end{array}$$

It follows that $\omega\circ{}^{M}\rho^{M}=0$. Hence we have

$$\mathrm{Im}\,(I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}})\subseteq\mathrm{Ker}\,(I_{\mathcal{D}}\otimes\omega\otimes I_{\mathcal{C}}). \tag{4.2}$$

It is easy to see that the diagram

$$\begin{array}{ccc}
M_1 & \xrightarrow{I_{\mathcal{D}}\otimes{}^{M}\rho^{M}\otimes I_{\mathcal{C}}} & M_2 \\
\downarrow{\scriptstyle {}^{M}\varepsilon^{M}} & & \downarrow{\scriptstyle {}^{\mathcal{D}}\varepsilon\otimes I_M\otimes\varepsilon^{\mathcal{C}}} \\
M & \xrightarrow[{}^{M}\rho^{M}]{} & M_1
\end{array}$$

is commutative. Since $({}^{\mathcal{D}}\varepsilon\otimes I_M\otimes\varepsilon^{\mathcal{C}})\circ(\Delta_{\mathcal{D}}\otimes I_M\otimes\Delta_{\mathcal{C}})=I_{M_1}$, we have

$$({}^{\mathcal{D}}\varepsilon \otimes I_M \otimes \varepsilon^{\mathcal{C}}) \circ \omega = {}^{M}\rho^{M} \circ {}^{M}\varepsilon^{M} - I_{M_1}.$$

It follows that

$$(I_{\mathcal{D}} \otimes {}^{\mathcal{D}}\varepsilon \otimes I_M \otimes \varepsilon^{\mathcal{C}} \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes \omega \otimes I_{\mathcal{C}}) = (I_{\mathcal{D}} \otimes {}^{M}\rho^{M} \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes {}^{M}\varepsilon^{M} \otimes I_{\mathcal{C}}) - I_{M_2}.$$

This yields that

$$\operatorname{Ker}(I_{\mathcal{D}} \otimes \omega \otimes I_{\mathcal{C}}) \subseteq \operatorname{Im}(I_{\mathcal{D}} \otimes {}^{M}\rho^{M} \otimes I_{\mathcal{C}}). \tag{4.3}$$

By the equations (4.2) and (4.3), we have

$$\operatorname{Im}(I_{\mathcal{D}} \otimes {}^{M}\rho^{M} \otimes I_{\mathcal{C}}) = \operatorname{Ker}(I_{\mathcal{D}} \otimes \omega \otimes I_{\mathcal{C}}).$$

Therefore the middle column and the right column of the diagram (4.1) are exact. Hence the left column of (4.1) is also exact.

By Lemma 4.1, $U = \mathcal{U}(\mathcal{D} \otimes_R \mathcal{C})$ is an $(\mathcal{F}, \mathcal{F})$-sub-bicomodule of $V = \mathcal{D} \otimes_B \mathcal{D} \otimes_R \mathcal{C} \otimes_A \mathcal{C}$. The right $\mathcal{F}$-comodule isomorphism $T_1 : \mathcal{D} \otimes_B M_1 \otimes_A \mathcal{C} \to M \otimes_\Lambda V$ defined by $T_1(d \otimes d' \otimes m \otimes c \otimes c') = m \otimes (d \otimes d' \otimes c \otimes c')$ yields the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}(M_1) & \xrightarrow{\iota_{M_1}} & \mathcal{D} \otimes_B M_1 \otimes_A \mathcal{C} & \xrightarrow{E_{M_1}} & \mathcal{D} \otimes_B M_1 \otimes_A \mathcal{C} \\
 & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle T_1} & & \Big\downarrow{\scriptstyle T_1} \\
0 & \longrightarrow & M \otimes_\Lambda U & \xrightarrow[I_M \otimes \iota]{} & M \otimes_\Lambda V & \xrightarrow[I_M \otimes E_{\mathcal{D} \otimes_R \mathcal{C}}]{} & M \otimes_\Lambda V
\end{array}$$

with commutative right square, where $\iota : U \to V$ is the inclusion map. Since top row is exact in $\mathbf{M}_\Lambda$ and $\iota$ is a section in $\mathbf{M}_\Lambda$ by Lemma 3.2 (4), the bottom row is exact in $\mathbf{M}_\Lambda$. Therefore, there exists a right $\Lambda$-module isomorphism $\varphi : \mathcal{U}(M_1) \to M \otimes_\Lambda U$ such that the left square is commutative. We consider the right $\mathcal{F}$-comodule map

$$h = T_1 \circ \iota_{M_1} : \mathcal{U}(M_1) \to M \otimes_\Lambda V.$$

By Lemma 3.2 (4), $M \otimes_\Lambda U$ is isomorphic to a pure $\Lambda$-submodule of $M \otimes_\Lambda V$. Therefore, by Lemma 3.4, $\varphi$ is a right $\mathcal{F}$-comodule map. Similarly, the right $\mathcal{F}$-comodule isomorphism $T_2 : \mathcal{D} \otimes_B M_2 \otimes_A \mathcal{C} \to M \otimes_\Lambda \mathcal{F} \otimes_\Lambda V$ defined by

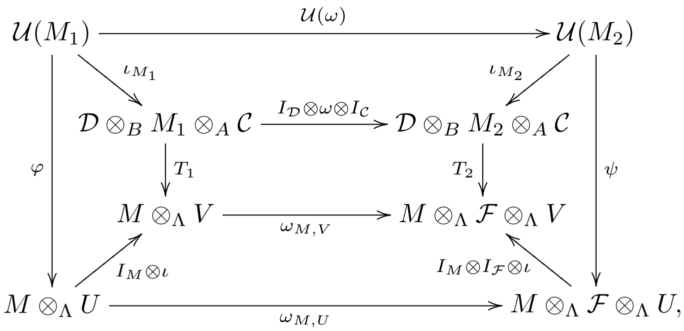
$$T_2(d \otimes d' \otimes d'' \otimes m \otimes c \otimes c' \otimes c'') = m \otimes (d'' \otimes c) \otimes (d \otimes d' \otimes c' \otimes c'')$$

yields the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}(M_2) & \xrightarrow{\iota_{M_2}} & \mathcal{D} \otimes_B M_2 \otimes_A \mathcal{C} & \xrightarrow{E_{M_2}} & \mathcal{D} \otimes_B M_2 \otimes_A \mathcal{C} \\
 & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle T_2} & & \Big\downarrow{\scriptstyle T_2} \\
0 & \longrightarrow & M \otimes_\Lambda \mathcal{F} \otimes_\Lambda U & \xrightarrow[I_M \otimes I_{\mathcal{F}} \otimes \iota]{} & M \otimes_\Lambda \mathcal{F} \otimes_\Lambda V & \xrightarrow[I_M \otimes I_{\mathcal{F}} \otimes E_{\mathcal{D} \otimes_R \mathcal{C}}]{} & M \otimes_\Lambda \mathcal{F} \otimes_\Lambda V
\end{array}$$

with exact rows in $\mathbf{M}_\Lambda$ and with a right $\mathcal{F}$-comodule isomorphism $\psi : \mathcal{U}(M_2) \to M \otimes_\Lambda \mathcal{F} \otimes_\Lambda U$.

Next, we define right $\Lambda$-module maps $\omega_{M,V} : M \otimes_\Lambda V \to M \otimes_\Lambda \mathcal{F} \otimes_\Lambda V$ and $\omega_{M,U} : M \otimes_\Lambda U \to M \otimes_\Lambda \mathcal{F} \otimes_\Lambda U$ by setting $\omega_{M,V} = \rho^M_{\mathcal{F}} \otimes I_V - I_M \otimes {}^V\rho_{\mathcal{F}}$ and $\omega_{M,U} = \rho^M_{\mathcal{F}} \otimes I_U - I_M \otimes {}^U\rho_{\mathcal{F}}$. Then in the diagram




all the subdiagrams except the outer rectangle are commutative. Since $\iota$ is a section in $\mathbf{M}_\Lambda$ by Lemma 3.2 (4), $I_M \otimes I_{\mathcal{F}} \otimes \iota$ is also a section. Hence the outer rectangle is commutative. Thus we get the exact sequence

$$0 \longrightarrow \mathcal{U}(M) \xrightarrow{\varphi \circ \mathcal{U}({}^M\rho^M)} M \otimes_\Lambda U \xrightarrow{\omega_{M,U}} M \otimes_\Lambda \mathcal{F} \otimes_\Lambda U$$

in $\mathbf{M}_\Lambda$. Since $\varphi \circ \mathcal{U}\big({}^M\rho^M\big)$ is a right $\mathcal{F}$-comodule map, $\mathcal{U}(M)$ is isomorphic to $M \,\square_{\mathcal{F}}\, U$ as a right $\mathcal{F}$-comodule. $\square$

## 5. Coseparable corings.

According to [**7**] and [**5**], an $A$-coring $\mathcal{C}$ is said to be *coseparable* if the coproduct $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ of $\mathcal{C}$ splits as a $(\mathcal{C}, \mathcal{C})$-bicomodule map.

THEOREM 5.1. *Let $A$ be an $R$-algebra and $\mathcal{C}$ an $A$-coring. Then the following conditions are equivalent*:

(1) *$\mathcal{C}$ is a coseparable $A$-coring.*
(2) ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}} = {}^{\mathcal{D}}\mathrm{GInCoder}^{\mathcal{C}}$ *for any $R$-algebra $B$ and any $B$-coring $\mathcal{D}$.*
(3) ${}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M, M) = {}^{\mathcal{C}}\mathrm{GInCoder}^{\mathcal{C}}(M, M)$ *for all $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$.*
(4) ${}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M, \mathcal{C}) = {}^{\mathcal{C}}\mathrm{GInCoder}^{\mathcal{C}}(M, \mathcal{C})$ *for all $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$.*

PROOF. (1) $\Rightarrow$ (2). We use the separability of the forgetful functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$ which was proved in [**1**, Corollary 3.6]. Let $B$ be an $R$-algebra, $\mathcal{D}$ a $B$-coring, and $M$, $N \in {}^{\mathcal{D}}\mathbf{M}^{\mathcal{C}}$. By [**5**, Corollary 1.3], there exists a $(\mathcal{D}, \mathcal{C})$-bicomodule map

$$\nu : N \otimes_A \mathcal{C} \to N$$

such that $\nu \circ \rho^N = I_N$. We define the $R$-linear map

$$\Phi : {}_B\mathrm{Hom}_A(M, N) \to {}_B\mathrm{Hom}^{\mathcal{C}}(M, N)$$

by setting $\Phi(f) = \nu \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M$ for $f \in {}_B\mathrm{Hom}_A(M, N)$. For any $f \in {}_B\mathrm{Hom}^{\mathcal{C}}(M, N)$, we see that

$$\Phi(f) = \nu \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = \nu \circ \rho^N \circ f = f.$$

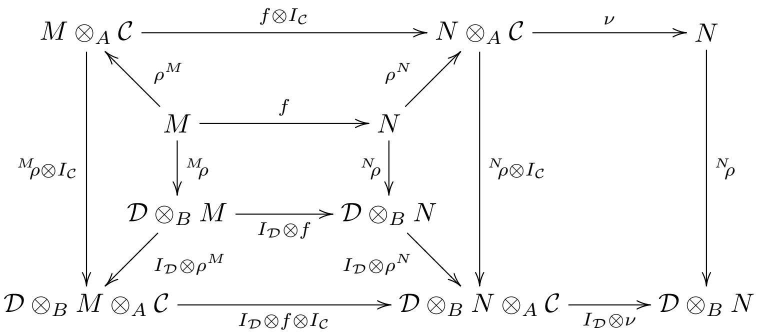
Hence, we have

$${}_B\mathrm{Hom}_A(M, N) = {}_B\mathrm{Hom}^{\mathcal{C}}(M, N) \oplus \mathrm{Ker}\,\Phi.$$

It follows that

$${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N) = {}_B\mathrm{Hom}^{\mathcal{C}}(M, N) \oplus \big({}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N) \cap \mathrm{Ker}\,\Phi\big). \tag{5.1}$$

For any $f \in {}_B\mathrm{Hom}_A(M, N)$, we consider the following diagram.




We can see the following.

$$(I_{\mathcal{D}} \otimes \nu) \circ (I_{\mathcal{D}} \otimes \rho^N) \circ {}^N\!\rho \circ f = {}^N\!\rho \circ f$$

$$(I_{\mathcal{D}} \otimes \nu) \circ (I_{\mathcal{D}} \otimes \rho^N) \circ (I_{\mathcal{D}} \otimes f) \circ {}^M\!\rho = (I_{\mathcal{D}} \otimes f) \circ {}^M\!\rho$$

$$(I_{\mathcal{D}} \otimes \nu) \circ ({}^N\!\rho \otimes I_{\mathcal{C}}) \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = {}^N\!\rho \circ \nu \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = {}^N\!\rho \circ \Phi(f)$$

$$(I_{\mathcal{D}} \otimes \nu) \circ (I_{\mathcal{D}} \otimes f \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{D}} \otimes \rho^M) \circ {}^M\!\rho = \big(I_{\mathcal{D}} \otimes \Phi(f)\big) \circ {}^M\!\rho$$

Combining these equations, we get

$$\begin{aligned}(I_{\mathcal{D}} \otimes \nu) \circ Q_{M,N}(f) \\ = {}^N\!\rho \circ f - (I_{\mathcal{D}} \otimes f) \circ {}^M\!\rho - {}^N\!\rho \circ \Phi(f) + \big(I_{\mathcal{D}} \otimes \Phi(f)\big) \circ {}^M\!\rho.\end{aligned}$$

If $f$ belongs to ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M, N) \cap \mathrm{Ker}\,\Phi$, then we have ${}^N\!\rho \circ f = (I_{\mathcal{D}} \otimes f) \circ {}^M\!\rho$, and hence $f \in {}^{\mathcal{D}}\mathrm{Hom}_A(M,N)$. By the equation (5.1), we conclude that ${}^{\mathcal{D}}\mathrm{GCoder}^{\mathcal{C}}(M,N) = {}^{\mathcal{D}}\mathrm{GInCoder}^{\mathcal{C}}(M,N)$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and $f \in {}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M,\mathcal{C})$. Set $N = M \oplus \mathcal{C}$ and define $\tilde{f} \in {}_A\mathrm{Hom}_A(N,N)$ by setting $\tilde{f}(m,c) = \big(0, f(m)\big)$ for $m \in M$ and $c \in \mathcal{C}$. By Theorem 2.5, $\tilde{f}$ belongs to ${}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(N,N)$. By the assumption, $\tilde{f}$ can be written as $\tilde{f} = g + h$ with some $g \in {}_A\mathrm{Hom}^{\mathcal{C}}(N,N)$ and $h \in {}^{\mathcal{C}}\mathrm{Hom}_A(N,N)$. Let $\iota : M \to N$ denote the injection map and $\pi : N \to \mathcal{C}$ the projection map. Then, we have $f = \pi \circ g \circ \iota + \pi \circ h \circ \iota$. Since $\pi \circ g \circ \iota \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$ and $\pi \circ h \circ \iota \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$, $f$ is a generalized inner coderivation.

(4) $\Rightarrow$ (1). Let $M \in {}^{\mathcal{C}}\mathbf{M}^{\mathcal{C}}$ and $f : M \to \mathcal{C}$ be a coderivation. Let $\varepsilon_{\mathcal{C}}^{(2)}$ denote the composition map

$$\mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\varepsilon_{\mathcal{C}} \otimes \varepsilon_{\mathcal{C}}} A \otimes_A A \xrightarrow{\text{canonical isom.}} A.$$

We can see that $\varepsilon_{\mathcal{C}}^{(2)} \circ \Delta_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$, $\varepsilon_{\mathcal{C}}^{(2)} \circ (f \otimes I_{\mathcal{C}}) \circ \rho^M = \varepsilon_{\mathcal{C}} \circ f$, and $\varepsilon_{\mathcal{C}}^{(2)} \circ (I_{\mathcal{C}} \otimes f) \circ {}^M\rho = \varepsilon_{\mathcal{C}} \circ f$. It follows that $\varepsilon_{\mathcal{C}}^{(2)} \circ \big(\Delta_{\mathcal{C}} \circ f - (f \otimes I_{\mathcal{C}}) \circ \rho^M - (I_{\mathcal{C}} \otimes f) \circ {}^M\rho\big) = -\varepsilon_{\mathcal{C}} \circ f$. Hence $\varepsilon_{\mathcal{C}} \circ f = 0$. Since $f$ belongs to ${}^{\mathcal{C}}\mathrm{GCoder}^{\mathcal{C}}(M,\mathcal{C})$ by Theorem 2.8, there exist $g \in {}_A\mathrm{Hom}^{\mathcal{C}}(M,\mathcal{C})$ and $h \in {}^{\mathcal{C}}\mathrm{Hom}_A(M,\mathcal{C})$ such that $f = g + h$. Then we have

$$f = \mathfrak{R}_M \circ (\mathfrak{R}_M)^{-1}(g) + \mathfrak{L}_M \circ (\mathfrak{L}_M)^{-1}(h) = \mathfrak{R}_M(\varepsilon_{\mathcal{C}} \circ g) + \mathfrak{L}_M(\varepsilon_{\mathcal{C}} \circ h).$$

Since $\varepsilon_{\mathcal{C}} \circ g + \varepsilon_{\mathcal{C}} \circ h = \varepsilon_{\mathcal{C}} \circ f = 0$, we have $f = \mathfrak{R}_M(\varepsilon_{\mathcal{C}} \circ g) - \mathfrak{L}_M(\varepsilon_{\mathcal{C}} \circ g)$. Hence, $f$ is an inner coderivation. By [**5**, Theorem 3.10], $\mathcal{C}$ is a coseparable $A$-coring. □

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