J. Math. Soc. Japan
Vol. 68, No. 1 (2016) pp. 91–99
doi: 10.2969/jmsj/06810091

Multipliers of Hardy spaces associated with Laguerre expansions

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(Received June 28, 2013)
(Revised Jan. 18, 2014)

Abstract. The purpose of the paper is to study coefficient multipliers of the Hardy spaces $H^p([0,\infty))$ $(0<p<1)$ associated with Laguerre expansions. As a consequence, a Paley type inequality is obtained.

1. Introduction and results.

A function F analytic in the unit disk $\mathbb{D}$ is said to be in the Hardy space $H^p(\mathbb{D})$, $0<p<\infty$, if $\|F\|_{H^p} := \sup_{0\le r<1} M_p(F;r) < \infty$, where $M_p(F;r) = \{(1/2\pi)\int_{-\pi}^{\pi}|F(re^{i\theta})|^p d\theta\}^{1/p}$.

Denote by ℓ^q the sequence space $\ell^q = \big\{\{a_k\} : \|\{a_k\}\|_q = (\sum_{k=0}^{\infty}|a_k|^q)^{1/q} < \infty\big\}$ for $0<q<\infty$, and ℓ^∞ the set of bounded sequences. A sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier of $H^p(\mathbb{D})$ into the sequence space ℓ^q if $\sum_{n=0}^{\infty}|\lambda_n c_n|^q < \infty$ whenever $f = \sum_{n=0}^{\infty} c_n z^n \in H^p(\mathbb{D})$. For a summary of results on multipliers from $H^p(\mathbb{D})$ to ℓ^q for various p and q, see [**8**]. In particular Duren and Shields ([**3**, Theorem 2(i)]) proved the following theorem: The sequence $\{\lambda_n\}$ is a multiplier of $H^p(\mathbb{D})$ into ℓ^q $(0<p<1,\ p\le q<\infty)$ if and only if $\sum_{n=1}^{N} n^{q/p}|\lambda_n|^q = O(N^q)$.

Among coefficient multipliers of the Hardy spaces, the two important ones are the Hardy inequality and the Paley inequality, namely, for $f(z) = \sum_{n=0}^{\infty} c_n z^n \in H^1(\mathbb{D})$,

$$\sum_{n=1}^{\infty} n^{-1}|c_n| \le c\|f\|_{H^1}, \qquad \text{and} \qquad \sum_{k=1}^{\infty} |c_{2^k}|^2 \le c\|f\|_{H^1}^2,$$

where the constant c is independent of f. In the last two decades, analogs of the Hardy inequality in the context of eigenfunction expansions were studied by several authors (cf. [**1**], [**2**], [**4**], [**9**], [**10**], [**14**]). Comparatively, less generalization of the Paley inequality to eigenfunction expansions is achieved, and a substantial work is the Paley inequality for the Jacobi expansion given in [**6**]. Recently, coefficient multipliers of Hardy spaces associated with generalized Hermite expansions are studied in [**7**]. In this paper, we shall study the coefficient multipliers associated with Laguerre expansions on the space

$$H^p([0,\infty)) = \{f \in H^p(\mathbb{R}) : \operatorname{supp} f \subset [0,\infty)\}, \quad 0<p\le 1.$$

2010 *Mathematics Subject Classification.* Primary 42C10; Secondary 42B30, 42A45.

Key Words and Phrases. Hardy space, multipliers, Laguerre expansion.

Supported by the National Natural Science Foundation of China (No. 11371258), the Beijing Natural Science Foundation (No. 1122011).

If $\alpha > -1$, the Laguerre function $\mathcal{L}_n^{(\alpha)}(x)$ is defined by

$$\mathcal{L}_n^{(\alpha)}(x) = \tau_n^\alpha L_n^{(\alpha)}(x) e^{-x/2} x^{\alpha/2},$$

where $\tau_n^\alpha = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$ and $L_n^{(\alpha)}(x)$ is the Laguerre polynomial determined by the orthogonal relation (see [**13**, (5.1.1)])

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = (\tau_n^\alpha)^{-2} \delta_{mn}.$$

The system $\{\mathcal{L}_n^{(\alpha)}(x)\}_{n=0}^\infty$ is a complete orthonormal system on the interval $[0, +\infty)$ with respect to the Lebesgue measure. For a function $f \in L^p([0,\infty))$, $1 \le p \le \infty$, its Laguerre expansion is

$$f \sim \sum_{n=0}^\infty c_n^{(\alpha)}(f) \mathcal{L}_n^{(\alpha)}(x), \qquad c_n^{(\alpha)}(f) = \int_0^\infty f(t) \mathcal{L}_n^{(\alpha)}(t) dt. \tag{1}$$

We shall give an appropriate definition of the coefficients $c_n^{(\alpha)}(f)$, $n = 0, 1, 2, \ldots$, for $f \in H^p([0,\infty))$, $0 < p < 1$, in Section 2.

Our theorem is stated as follows.

THEOREM 1.1. *Let $\alpha \ge 0$, $\alpha^* = +\infty$ for nonnegative even α and $\alpha^* = \alpha/2 + 1$ otherwise, and let $(\alpha^*)^{-1} < p < 1 \le q < \infty$. If a sequence $\{\lambda_n\}_{n=0}^\infty$ satisfies the condition*

$$\sum_{n=1}^N n^{q/p} |\lambda_n|^q = O(N^q), \tag{2}$$

then for all $f \in H^p([0,\infty))$, the Fourier-Laguerre coefficients $c_n^{(\alpha)}(f)$ are well-defined and satisfy

$$\left(\sum_{n=0}^\infty |\lambda_n c_n^{(\alpha)}(f)|^q \right)^{1/q} \le c \|f\|_{H^p([0,\infty))}, \tag{3}$$

where c is a constant independent of f.

Theorem 1.1 shows that a sequence $\{\lambda_n\}_{n=0}^\infty$ is a multiplier of $H^p([0,\infty))$ into the sequence space ℓ^q associated with Laguerre expansions if (3) holds. It is noted that the condition (2) is equivalent to the condition $\sum_{k=n}^{2n} |\lambda_k|^q = O\left(n^{q(1-1/p)}\right)$. An interesting application of Theorem 1.1 is the Paley type inequality for Laguerre expansions, which is stated in the following corollary.

COROLLARY 1.2. *Let $\alpha \ge 0$, $\alpha^* = \infty$ for nonnegative even α and $\alpha^* = \alpha/2 + 1$ otherwise, and let $(\alpha^*)^{-1} < p < 1$. If $\{n_k\}$ is a Hadamard sequence satisfying $n_{k+1}/n_k \ge \rho > 1$ $(k = 1, 2, \ldots)$, then for all $f \in H^p([0,\infty))$, the coefficients $c_n^{(\alpha)}(f)$ of its Laguerre* *expansion satisfy*

$$\sum_{k=1}^{\infty} n_k^{2(1-p^{-1})} |c_{n_k}^{(\alpha)}(f)|^2 < \infty.$$

Throughout the paper, $A = O(B)$ or $A \lesssim B$ means that $A \le cB$ for some positive constant c independent of variables, functions, k, n, etc., but possibly dependent of some fixed parameters and fixed m.

2. Preliminieries.

We begin by recalling some estimates of the Laguerre functions. There are two lemmas on some sharp estimates of $\mathcal{L}_n^{(\alpha)}(x)$ from [**10**] as follows.

Lemma 2.1. *Let $\alpha \ge 0$. If we set $M = [\alpha/2]$, then for each non-negative integer $m \le M$, the $m-$th derivative $(\mathcal{L}_n^{(\alpha)})^{(m)}(x)$ of $\mathcal{L}_n^{(\alpha)}(x)$ with respect to x satisfies,*

$$|(\mathcal{L}_n^{(\alpha)})^{(m)}(x)| \le C_{\alpha,m} n^m, \quad x \in [0,\infty).$$

Futhermore, if $\alpha/2 = 0,1,2,\dots,$ then for $m = 0,1,2,\dots,$

$$|(\mathcal{L}_n^{(\alpha)})^{(m)}(x)| \le C_{\alpha,m} n^m, \quad x \in [0,\infty).$$

Here $C_{\alpha,m}$ are positive constants independent of n.

Lemma 2.2. *Let $\alpha \ge 0$ and let $\alpha/2$ be not an integer. We put $\alpha/2 = M + \delta$, $0 < \delta < 1$. Then for the $M-$th derivative $(\mathcal{L}_n^{(\alpha)})^{(M)}(x)$ of $\mathcal{L}_n^{(\alpha)}(x)$ with respect to x, we have*

$$\left|(\mathcal{L}_n^{(\alpha)})^{(M)}(x+h) - (\mathcal{L}_n^{(\alpha)})^{(M)}(x)\right| \le C_\alpha n^{\alpha/2} |h|^\delta, \quad x,h \in [0,\infty),$$

where C_α is a positive constant independent of n.

Since $H^1([0,\infty)) \subset L([0,\infty))$, the coefficients $c_n^{(\alpha)}(f)$ for $f \in H^1([0,\infty))$ are well defined by (1). But if $f \in H^p([0,\infty))$ for $0 < p < 1$, we need a new definition for the coefficients $c_n^{(\alpha)}(f)$, which is based on the duality relation of the Hardy space $H^p(\mathbb{R})$ and the Lipschitz space $\Lambda_{p^{-1}-1}(\mathbb{R})$.

There are several equivalent definitions for the Lipschitz space $\Lambda_\delta(\mathbb{R})$ (see [**11**], [**12**], [**15**]). Here is the usual one. For $m \ge 1$ and $m-1 < \delta \le m$, $\Lambda_\delta(\mathbb{R})$ is the set of $(m-1)$-times differentiable functions f satisfying $\|f\|_{\Lambda_\delta} := \|f\|_{L^\infty} + \sup_{x,h} |f^{(m-1)}(x+h) - f^{(m-1)}(x)|/|h|^{\delta+1-m} < \infty$ for $\delta \ne m$, and $\|f\|_{\Lambda_\delta} := \|f\|_{L^\infty} + \sup_{x,h} |f^{(m-1)}(x+h) - 2f^{(m-1)}(x) + f^{(m-1)}(x-h)|/|h| < \infty$ for $\delta = m$. Here we use a unified notation $\Lambda_\delta(\mathbb{R})$ for all $\delta > 0$, without use of Zygmund's notation $\Lambda_\delta^*(\mathbb{R})$ for $\delta = m$.

Lemma 2.3 ([**12**, p. 130] or [**16**]). *If $0 < p < 1$ and $g \in \Lambda_{p^{-1}-1}(\mathbb{R})$, then $\mathcal{L}_g(f) = \int_{\mathbb{R}} f(x)g(x)dx$, initially defined for $f \in L^1(\mathbb{R}) \bigcap H^p(\mathbb{R})$, has a bounded extension to*

$H^p(\mathbb{R})$ satisfying $|\mathcal{L}(f)| \le c\|g\|_{\Lambda_{p^{-1}-1}}\|f\|_{H^p}$, where c is a constant independent of g and f.

Now we extend $\mathcal{L}_n^{(\alpha)}(x)$ to the whole line $\mathbb{R}$ in a suitable way. If $\alpha/2 > 0$ is not an integer, then we define

$$\tilde{\mathcal{L}}_n^{(\alpha)}(x) = \begin{cases} \mathcal{L}_n^{(\alpha)}(x), & \text{for } x > 0; \\ 0, & \text{for } x \le 0. \end{cases} \tag{4}$$

If $\alpha/2 \ge 0$ is an integer, we shall use the function

$$\psi(x) = \begin{cases} 1, & \text{for } x \ge 0; \\ (1 - e^{1/x}) \exp\left(- \dfrac{e^{1/x}}{x+1} \right), & \text{for } -1 < x < 0; \\ 0, & \text{for } x \le -1. \end{cases}$$

It is clear that $\psi(x) \in C(\mathbb{R})$. However, for $k \ge 1$, the $k-$th derivative $\psi^{(k)}(x)$ of $\psi(x)$ satisfies $\lim_{x\to -1+0} \psi^{(k)}(x) = \lim_{x \to 0-0} \psi^{(k)}(x) = 0$ by routine evaluations, which implies that $\psi(x) \in C^\infty(\mathbb{R})$ and $|\psi^{(k)}(x)| \le c$, where c is a constant independent of x.

It follows from the formula (see [**13**, (5.1.6)])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{5}$$

that for every positive integer m, there exists a constant $c_m > 0$ such that for all $n \ge 1$ and $x < 0$, $L_n^{(\alpha)}(x) \ge c_m(n^\alpha + n^{\alpha+m}|x|^m) = c_m n^\alpha(1 + (n|x|)^m)$. This shows that for $x < 0$, $L_n^{(\alpha)}(x)$ increases quite rapidly as $n|x|$ increases, which happens even for small $|x|$ and large n.

In view of the above remark, we define, for even integer $\alpha \ge 0$,

$$\tilde{\mathcal{L}}_n^{(\alpha)}(x) = \psi(nx)\mathcal{L}_n^{(\alpha)}(x). \tag{6}$$

The conclusions in Lemma 2.1 and Lemma 2.2 are valid for $\tilde{\mathcal{L}}_n^{(\alpha)}(x)$ instead of $\mathcal{L}_n^{(\alpha)}(x)$ on the whole line $\mathbb{R}$.

Corollary 2.4. *Let $\alpha \ge 0$ and $M = [\alpha/2]$. Then for $x \in \mathbb{R}$,*

(i) *if $\alpha/2$ is not an integer,*

$$\left|(\tilde{\mathcal{L}}_n^{(\alpha)})^{(m)}(x)\right| \lesssim n^m, \ m \le M; \tag{7}$$

(ii) *if $\alpha/2$ is not an integer,*

$$\left|(\tilde{\mathcal{L}}_n^{(\alpha)})^{(M)}(x+h) - (\tilde{\mathcal{L}}_n^{(\alpha)})^{(M)}(x)\right| \lesssim n^{\alpha/2}|h|^\delta, \ \alpha/2 = M + \delta, \ 0 < \delta < 1; \tag{8}$$

(iii) *if $\alpha/2$ is an integer, (7) is true for all $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.*

Proof. Parts (i) and (ii) are easy consequences of (4) by Lemma 2.1 and Lemma 2.2.

For part (iii), it suffices to evaluate $(\tilde{\mathcal{L}}_n^{(\alpha)})^{(m)}(x)$ for $-n^{-1} \le x \le 0$ by (6). In this case, by Leibniz' rule,

$$(\tilde{\mathcal{L}}_n^{(\alpha)})^{(m)}(x) = \sum_{l=0}^{m} \binom{m}{l} \psi^{(m-l)}(nx) n^{m-l} (\mathcal{L}_n^{(\alpha)})^{(l)}(x). \tag{9}$$

Since $L_n^{(\alpha)}(x)' = -L_{n-1}^{(\alpha+1)}(x)$ (see [**13**, (5.1.14)]),

$$(\mathcal{L}_n^{(\alpha)})^{(l)}(x) = \tau_n^{\alpha} \sum_{\substack{i+j \le l \\ j \le \alpha/2}} c_{l,i,j} e^{-x/2} L_{n-i}^{(\alpha+i)}(x) x^{\alpha/2-j},$$

and from (5), for $-n^{-1} \le x \le 0$ we have

$$0 \le L_n^{(\alpha)}(x) \lesssim n^{\alpha} \sum_{k=0}^{n} \binom{n}{n-k} \frac{n^{-k}}{k!} \lesssim n^{\alpha}(1+n^{-1})^n \lesssim n^{\alpha}.$$

Thus it follows that, for $-n^{-1} \le x \le 0$,

$$\left|(\mathcal{L}_n^{(\alpha)})^{(l)}(x)\right| \lesssim n^{-\alpha/2} \sum_{\substack{i+j \le l \\ j \le \alpha/2}} n^{\alpha+i} |x|^{\alpha/2-j} \lesssim n^l.$$

Substituting this into (9) yields, for $-n^{-1} \le x \le 0$,

$$\left|(\tilde{\mathcal{L}}_n^{(\alpha)})^{(m)}(x)\right| \lesssim \sum_{l=0}^{m} n^{m-l} n^l \lesssim n^m. \qquad \square$$

By Corollary 2.4, $\tilde{\mathcal{L}}_n^{(\alpha)}(x) \in \Lambda_{p^{-1}-1}(\mathbb{R})$ for $0 < p < 1$ in the case $\alpha = 0, 2, 4 \dots$ and $\tilde{\mathcal{L}}_n^{(\alpha)}(x) \in \Lambda_{p^{-1}-1}(\mathbb{R})$ for $p^{-1} - 1 < \alpha/2$ in the case $\alpha \ne 0, 2, 4 \dots$. For $0 < p < 1$, the coefficients $c_n^{(\alpha)}(f)$ of $f \in H^p([0,\infty))$ associated with Laguerre expansions are defined by

$$c_n^{(\alpha)}(f) = \mathcal{L}_{\tilde{\mathcal{L}}_n^{(\alpha)}(x)}(f).$$

We see that the coefficients $c_n^{(\alpha)}(f)$ are independent of the choice of an extension $\tilde{\mathcal{L}}_n^{(\alpha)}(x) \in \Lambda_{p^{-1}-1}(\mathbb{R})$. It is easy to see that the substitute definition of the coefficients $c_n^{(\alpha)}(f)$ is consistent with the previous definition for "good" functions. In fact, $c_n^{(\alpha)}(f) = \int_0^{\infty} f(t)\mathcal{L}_n^{(\alpha)}(t)dt$ for all $f \in H^p([0,\infty)) \cap L^1(\mathbb{R})$. However it is not always meaningful in general for all $H^p([0,\infty))$, $0 < p < 1$, since the functions $\mathcal{L}_n^{(\alpha)}(x)$ are not sufficiently smooth for most of α. Indeed we have

Proposition 2.5. *Let $\alpha \geq 0$. The Fourier-Laguerre coefficients $c_n^{(\alpha)}(f)$ of $f \in H^p([0,\infty))$ are well defined for all $0 < p \leq 1$ if α is a nonnegative even integer and for $(\alpha/2+1)^{-1} < p \leq 1$ otherwise.*

3. Proof of Theorem 1.1.

Now we shall prove Theorem 1.1. Our approach is based on the duality of $H^p(\mathbb{R})$ and $\Lambda_{p^{-1}-1}(\mathbb{R})$.

Proof. We fix a sequence $\{b_n\}_{n=0}^{\infty} \in \ell^{q'}$, $q^{-1}+q'^{-1}=1$, and for $n=1,2,\dots$, let

$$g_n(x) = \sum_{k=0}^{n} \lambda_k b_k \tilde{\mathcal{L}}_k^{(\alpha)}(x). \tag{10}$$

By Lemma 2.3, one has $|\mathcal{L}_{g_n}(f)| \leq c\|g_n\|_{\Lambda_{p^{-1}-1}} \|f\|_{H^p([0,\infty))}$, or equivalently,

$$\left| \sum_{k=0}^{n} \lambda_k b_k c_k^{(\alpha)}(f) \right| \leq c\|g_n\|_{\Lambda_{p^{-1}-1}} \|f\|_{H^p([0,\infty))}.$$

In order to prove (3) it suffices to show that there is a constant c' independent of n and $\{b_k\} \in \ell^{q'}$ such that

$$\|g_n\|_{\Lambda_{p^{-1}-1}} \leq c' \|\{b_k\}\|_{q'}. \tag{11}$$

Once (11) is true, then it follows that

$$\left(\sum_{k=0}^{n} |\lambda_k c_k^{(\alpha)}(f)|^q \right)^{1/q} \leq cc' \|f\|_{H^p([0,\infty))},$$

which proves the theorem by letting $n \to \infty$.

First we consider the case when $m-1 < p^{-1}-1 < m$, $p^{-1}-1 < \alpha/2$. Suppose $x \neq y$ and put $h = y-x$.

From (10) we have

$$\left|g_n^{(m-1)}(x) - g_n^{(m-1)}(y)\right| \leq \sum_{k=0}^{n} |\lambda_k b_k| \left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x) - (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(y) \right|. \tag{12}$$

If $n \leq |h|^{-1}$, we apply Corollary 2.4 (part (i) for $m < \alpha/2$ and part (ii) for $\alpha/2 \leq m < \alpha/2+1$) to get an upper bound of $|g_n^{(m-1)}(x) - g_n^{(m-1)}(y)|$ as a multiple of

$$\sum_{k=0}^{n} |\lambda_k b_k| k^{m-1+\gamma} |h|^\gamma \le |h|^\gamma \|\{b_k\}\|_{q'} \left(\sum_{k=0}^{n} |\lambda_k|^q k^{q(m-1+\gamma)} \right)^{1/q}, \tag{13}$$

where $\gamma = 1$ if $m-1 < p^{-1}-1 < m < \alpha/2$, and $\gamma = \alpha/2+1-m$ if $m-1 < p^{-1}-1 < \alpha/2 \le m$.

Under the condition (2) summing by parts gives $\sum_{k=0}^{n} |\lambda_k|^q k^{q(m-1+\gamma)} = O(n^{q(m+\gamma-1/p)})$. Hence

$$\left(\sum_{k=0}^{n} |\lambda_k|^q k^{q(m-1+\gamma)} \right)^{1/q} \lesssim n^{\gamma+m-p^{-1}} \le |h|^{p^{-1}-m-\gamma}$$

for $n \le |h|^{-1}$. Substituting this into (13) yields

$$\left| g_n^{(m-1)}(x) - g_n^{(m-1)}(y) \right| \lesssim \|\{b_k\}\|_{q'} |h|^{p^{-1}-m}. \tag{14}$$

If $n > |h|^{-1}$, the summation of those terms in (12) for $k \le |h|^{-1}$ has the same bound $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m}$ as above, and the summation of the terms for $|h|^{-1} < k \le n$, in virtue of Corollary 2.4 (parts (i) and (iii)), is dominated by

$$\begin{aligned} &\sum_{|h|^{-1}<k\le n} |\lambda_k b_k| \left(\left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x) \right| + \left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(y) \right| \right) \\ &\lesssim \sum_{|h|^{-1}<k\le n} |\lambda_k b_k| k^{m-1} \le \|\{b_k\}\|_{q'} \left(\sum_{|h|^{-1}<k\le n} |\lambda_k|^q k^{q(m-1)} \right)^{1/q}. \end{aligned} \tag{15}$$

By the condition (2), summing by parts again gives $\sum_{k\ge N} |\lambda_k|^q k^{q(m-1)} = O(N^{q(m-p^{-1})})$. Thus we have

$$\left(\sum_{|h|^{-1}<k\le n} |\lambda_k|^q k^{q(m-1)} \right)^{1/q} \lesssim (|h|^{-1})^{m-p^{-1}} = |h|^{p^{-1}-m}$$

for $n > |h|^{-1}$. Substituting this into (15) yields an upper bound of the summation of the terms in (12) for $|h|^{-1} < k \le n$ as $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m}$. Thus (14) is proved to be true for all n and h, so that (11) is shown whenever $m-1 < p^{-1}-1 < m$, $p^{-1}-1 < \alpha/2$.

Finally we prove (11) for $p^{-1}-1 = m < \alpha/2$. We shall need to evaluate the second order difference of $g_n^{(m-1)}$, that is sufficient by the definition about Λ_δ for $\delta = m$. From (10) it follows, for $h \ne 0$, that $|g_n^{(m-1)}(x+h) - 2g_n^{(m-1)}(x) + g_n^{(m-1)}(x-h)|$ is bounded by

$$\sum_{k=0}^{n} |\lambda_k b_k| \left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x+h) - 2(\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x) + (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x-h) \right|. \tag{16}$$

If $1 \le p^{-1} - 1 = m < \alpha/2 - 1$, this is dominated by $\sum_{k=0}^{n} |\lambda_k b_k| |(\tilde{\mathcal{L}}_k^{(\alpha)})^{(m+1)}(x')| |h|^2$ with some x' between $x-h$ and $x+h$, and furthermore, in virtue of Corollary 2.4 (parts (i) and (iii)), by a multiple of

$$|h|^2 \sum_{k=0}^{n} |\lambda_k b_k| k^{m+1} \le |h|^2 \|\{b_k\}\|_{q'} \left(\sum_{k=0}^{n} |\lambda_k|^q k^{q(m+1)} \right)^{1/q}. \tag{17}$$

Since $m + 1 = p^{-1}$, the condition (2) gives

$$\sum_{k=0}^{n} |\lambda_k|^q k^{q(m+1)} \lesssim n^q \le |h|^{-q}$$

for $n \le |h|^{-1}$. Substituting this into (17) yields, for $n \le |h|^{-1}$,

$$\left| g_n^{(m-1)}(x+h) - 2 g_n^{(m-1)}(x) + g_n^{(m-1)}(x-h) \right| \lesssim \|\{b_k\}\|_{q'} |h|. \tag{18}$$

If α is not nonnegative even and $\alpha/2 - 1 \le p^{-1} - 1 = m < \alpha/2$, we note that

$$\begin{aligned} &\left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x+h) - 2(\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x) + (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m-1)}(x-h) \right| \\ &\quad = \left| (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m)}(x') - (\tilde{\mathcal{L}}_k^{(\alpha)})^{(m)}(x'') \right| |h| \end{aligned}$$

by the mean-value theorem, where x' and x'' lay between $x-h$ and $x+h$, and furthermore, by Corollary 2.4 (ii) this is bounded by

$$c k^{\alpha/2} |h|^{\alpha/2 - m} |h| = c k^{\alpha/2} |h|^{\alpha/2 + 1 - m}.$$

Hence the expression in (16) is dominated by a multiple of $\sum_{k=0}^{n} |\lambda_k b_k| k^{\alpha/2} |h|^{\alpha/2 - m + 1}$, which has the same bound as in (13) with $\gamma = \alpha/2 + 1 - m$, and also the bound $c\|\{b_k\}\|_{q'} |h|^{p^{-1} - m} = c\|\{b_k\}\|_{q'} |h|$ for $n \le |h|^{-1}$ as in (14) since $\alpha/2 + 1 - p^{-1} > 0$. Thus (18) is shown to be true for $n \le |h|^{-1}$.

If $n > |h|^{-1}$, the summation of the terms for $k \le |h|^{-1}$ in (16) has the same bound as in (18), and the summation of those for $|h|^{-1} < k \le n$ is dealt with by the same way as in (15) to obtain its bound $c\|\{b_k\}\|_{q'} |h|^{p^{-1} - m} = c\|\{b_k\}\|_{q'} |h|$. Therefore (18) is verified for all n and h, and hence (11) is proved for $p^{-1} - 1 = m < \alpha/2$.

If α is a nonnegative even integer, the two cases discussed above, i.e. $m - 1 < p^{-1} - 1 < m$ and $p^{-1} - 1 = m$, are true for all $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, without the restriction $p^{-1} - 1 < \alpha/2$.

The proof of Theorem 1.1 is completed. □

Acknowledgments. The authors would like to thank the anonymous referees for their helpful comments and suggestions which have improved the original manuscript.

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