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Functional limit theorems for processes pieced together from excursions

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Dedicated to the memory of Professor Kiyosi Itô

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Abstract. A notion of convergence of excursion measures is introduced. It is proved that convergence of excursion measures implies convergence in law of the processes pieced together from excursions. This result is applied to obtain homogenization theorems of jumping-in extensions for positive self-similar Markov processes, for Walsh diffusions and for the Brownian motion on the Sierpiński gasket.

1. Introduction.

In the previous work [**22**], the author obtained homogenization results of jumping-in extensions for diffusion processes on the half line. The proof was based on the construction of a sample path from excursions using Itô's excursion theory [**11**] and the time-change method. The key to the proof was to prove convergence of time-changed paths of the Brownian excursion based on the results of Fitzsimmons–Yano [**8**].

The aim of this paper is to establish a general limit theorem (Theorem 2.5) which asserts, roughly speaking, that

$$\boldsymbol{n}_n \to \boldsymbol{n}_\infty \text{ implies } X_n \xrightarrow{\text{law}} X_\infty, \tag{1.1}$$

where $\boldsymbol{n}_n$'s are excursion measures and X_n's are the processes pieced together from excursions. For a given Hunt process for which the origin is regular for itself, the excursion measure away from the origin characterizes the law of the Hunt process. Hence it may be natural that (1.1) should hold. But in what sense is "$\boldsymbol{n}_n \to \boldsymbol{n}_\infty$"?

We introduce a notion of convergence of excursion measures as an analogue to Skorokhod's a.s.-convergence realization of weak convergence of probability measures. We roughly say that $\boldsymbol{n}_n \to \boldsymbol{n}_\infty$ if all $\boldsymbol{n}_n$'s can be realized as the pullbacks of a common σ-finite measure, say $\boldsymbol{n}_n = \nu \circ \Phi_n^{-1}$, where Φ_n's are measurable mappings which take values in the functional space of càdlàg paths equipped with the Skorokhod topology and which satisfy $\Phi_n \to \Phi_\infty$, ν-a.e. The key to the proof of (1.1) is to realize X_n's from a common Poisson point process and to construct random time-changes which establish the convergence $\Phi_n \to \Phi_\infty$ in the Skorokhod topology.

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We apply the general theorem to obtain homogenization theorems of jumping-in extensions. Let S be a Borel subset of $\mathbb{R}^d$ containing 0 and $H^0 = \{X, (\mathbb{P}_x^0)_{x\in S}\}$ be a Hunt process stopped upon hitting 0. Let S' be a measurable space and let $\{\boldsymbol{n}_v\}_{v\in S'}$ be a kernel such that for each $v \in S'$ the measure $\boldsymbol{n}_v$ is the excursion measure of an extension of H^0. A jumping-in extension is the process $X_{\rho,j}$ pieced together from excursions corresponding to the excursion measure defined by

$$\boldsymbol{n}_{\rho,j} = \int_{S'} \rho(\mathrm{d}v)\boldsymbol{n}_v + \int_{S\setminus\{0\}} j(\mathrm{d}x)\mathbb{P}_x^0 \tag{1.2}$$

for some finite measure ρ on S' and some σ-finite measure j on $S \setminus \{0\}$. (The excursion measure of any extension of H^0 may admit a representation of the form (1.2); see Itô [**12**, Section 7].) Let $c > 1$ be a fixed constant. For $\gamma > 0$, we define the scaling transformation

$$(\Psi_\gamma w)(t) = c^{-\gamma} w(ct). \tag{1.3}$$

For certain constants $\alpha > 0$ and $\gamma > 0$, we study the following scaled objects:

$$\boldsymbol{n}_{\rho,j}^{(n)} = c^{\gamma n}\boldsymbol{n}_{\rho,j} \circ (\Psi_\alpha^n)^{-1}, \quad X_{\rho,j}^{(n)} = \Psi_\alpha^n X_{\rho,j}. \tag{1.4}$$

We shall provide sufficient conditions for the following two types of convergences:

$$\boldsymbol{n}_{\rho,j}^{(n)} \to \begin{cases} \boldsymbol{n}_{\rho^*,0} & \text{in the } \textbf{jumping-in vanishing case}, \\ \boldsymbol{n}_{0,j^*} & \text{in the } \textbf{jumping-in dominant case} \end{cases} \tag{1.5}$$

for some ρ^* and j^*. Thanks to the general theorem (1.1), the convergence (1.5) leads to the corresponding convergence in law of the scaled process $X_{\rho,j}^{(n)}$, which can be regarded as a homogenization result. In particular, we take up positive self-similar Markov processes, Walsh diffusions, and the Brownian motion on the Sierpiński gasket.

Let us give a remark on earlier works about jumping-in extensions. Jumping-in extensions of diffusion processes were discussed by Feller [**6**] in his study of determination of all possible boundary conditions for the generator of a diffusion process with accessible boundaries. Such processes appear in the study of population genetics; see, e.g., Hutzenthaler–Taylor [**9**]. The sample path construction of the jumping-in extensions was first established by Itô–McKean [**13**] for Brownian motions using time-change method involving an independent Poisson process. Itô [**11**] established his theory of Poisson point process of excursions to construct a sample path by piecing together from excursions produced by a Poisson point process. Yano utilized Itô's method in [**22**] to obtain homogenization results of jumping-in extensions for diffusion processes on the half line and in [**23**] to determine possible jumping-in extensions of diffusion processes on an interval.

Note also that Lambert–Simatos [**15**] proved (1.1) in a certain sense which is different from ours. They gave a general condition for convergence of regenerative processes assuming the convergence of excursions bigger than ε in some given functional, which are called the *ε-big excursions*, for all $\varepsilon > 0$.

This paper is organized as follows. In Section 2, we give basic facts about the piecing procedure of excursions and state the general limit theorem. In Section 3, we state the homogenization theorems for jumping-in extensions in a rather general framework. In Section 4, we discuss three examples of the homogenization theorems: the positive self-similar Markov processes, the Walsh diffusions and the Brownian motion on the Sierpiński gasket. Section 5 and Section 6 are devoted to the proofs of the general limit theorem and the homogenization theorems, respectively.

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2. General limit theorem.

2.1. Notations about excursions.

Let $d \geq 1$ and let $D = D_{\mathbb{R}^d}$ denote the set of all functions $w : [0, \infty) \to \mathbb{R}^d$ which are càdlàg, i.e., right continuous with left limits. We say $w_n \to w$ *uniformly on compacts*, or simply $w_n \to w$ *uc*, if $\sup_{t \in [0,t_0]} |w_n(t) - w(t)| \to 0$ for all $t_0 > 0$. We equip D with the *Skorokhod topology*; we say $w_n \to w$ in D if there exists a sequence of time-changes $\{I_n\}$ of $[0, \infty)$ such that each $I_n : [0, \infty) \to [0, \infty)$ is bijective, continuous and increasing and

$$I_n - I \to 0 \text{ uc and } w_n - w \circ I_n \to 0 \text{ uc}, \tag{2.1}$$

where $I(t) \equiv t$ denotes the identity time-change. It is well-known that D is a Polish space. We write $\mathcal{B}(D)$ for the σ-field generated by all open subsets of D. Let $X = (X(t))_{t \geq 0}$ denote the coordinate process on D, i.e.,

$$X(w)(t) = X(t)(w) = w(t). \tag{2.2}$$

For $x \in \mathbb{R}^d$, we denote the hitting time of x by

$$T_x(w) = \inf\{t > 0 : w(t) = x\}, \tag{2.3}$$

where we adopt the usual convention $\inf \emptyset = \infty$. We denote

$$\|w\| = \sup_{t \geq 0} |w(t)| \quad \text{for } w \in D. \tag{2.4}$$

Paths stopped upon hitting 0 are called *excursions away from* 0. The set of all excursions away from 0 will be denoted by

$$D^0 = \{w \in D : w(t) = w(t \wedge T_0(w)) \text{ for all } t \geq 0\}. \tag{2.5}$$

We write $o \in D^0$ for the path $o(t) \equiv 0$. Note that, for $w \in D^0$, we have $T_0(w) = 0$ if and only if $w = o$.

For $t \in [0, \infty)$, we define the shift operator $\theta_t : D \to D$ by

$$(\theta_t w)(s) = w(t+s), \quad s \geq 0. \tag{2.6}$$

We denote $\mathcal{F}_t^0 = \sigma(X(s) : s \leq t)$ and set $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^0$.

2.2. The process pieced together from excursions.

We denote $\sharp\{\cdot\}$ by the number of elements of the set $\{\cdot\}$. For a σ-finite measure ν on a measurable space E and a measurable functional f on E, we write $\nu[f]$ for $\int_E f \mathrm{d}\nu$ whenever the integral is well-defined.

We first recall the usual notion of a Poisson point process; see, e.g., [**10**, Section I.9] for the basic facts about it. Let ν be a σ-finite measure on a measurable space E. We call $\{(p^{(l)})_{l \in \mathcal{D}(p)}, \mathbb{P}\}$ a *Poisson point process on E with characteristic measure ν* if the random measure N_l for $l \geq 0$ defined by

$$N_l(A) = \sharp\{s \in \mathcal{D}(p) \cap [0,l] : p^{(s)} \in A\}, \quad A \in \mathcal{B}(E) \tag{2.7}$$

satisfies that for any non-negative measurable functional f on E the process $(N_l[f])_{l \geq 0}$ is a Poisson process with intensity $\nu[1 - \mathrm{e}^{-f}]$.

We second introduce an auxiliary notation modifying the usual notation of a Poisson point process. Let $\boldsymbol{n}$ be a σ-finite measure on D such that $\boldsymbol{n}(\{o\}) = 0$. We call $\{p = (p^{(l)})_{l \geq 0}, \mathbb{P}\}$ a *Poisson point process on D outside o with characteristic measure $\boldsymbol{n}$* if $\{(p^{(l)})_{l \in \mathcal{D}(p)}, \mathbb{P}\}$ for $\mathcal{D}(p) = \{l \geq 0 : p^{(l)} \neq o\}$ is a Poisson point process on $D \setminus \{o\}$ with characteristic measure $\boldsymbol{n}|_{D \setminus \{o\}}$. Note that a Poisson point process $(p^{(l)})_{l \in \mathcal{D}(p)}$ on $D \setminus \{o\}$ can always be extended to a Poisson point process on D outside o by putting $p^{(l)} = o$ when $l \notin \mathcal{D}(p)$.

Let $(\boldsymbol{n}, \varsigma)$ be the pair consisting of a σ-finite measure $\boldsymbol{n}$ on D such that $\boldsymbol{n}(\{o\}) = 0$ and a non-negative constant ς. Let $p = (p^{(l)})_{l \geq 0}$ be a Poisson point process on D outside o with characteristic measure $\boldsymbol{n}$. Noting that $p^{(l)} \in D$ for all $l \geq 0$, we have

$$T_0(p^{(l)}) = \inf\{t > 0 : p^{(l)}(t) = 0\}. \tag{2.8}$$

Let $\varsigma \geq 0$ be a constant and for $l \geq 0$ we define

$$\eta(l) = \eta(p, \varsigma; l) = \varsigma l + \sum_{s \leq l} T_0(p^{(s)}). \tag{2.9}$$

We introduce the following conditions on the pair $(\boldsymbol{n}, \varsigma)$:

(N0) $X \in D^0$ and $0 < T_0 < \infty$, $\boldsymbol{n}$-a.e.;
(N1) $\boldsymbol{n}[T_0 \wedge 1] < \infty$;
(N2) either $\varsigma > 0$ or $\boldsymbol{n}(D) = \infty$;
(N3) $\boldsymbol{n}(\|X\| \geq r) < \infty$ for all $r > 0$.

If the conditions (N0) and (N1) are satisfied, we see that $p^{(l)} \in D^0$ for all $l \geq 0$ and that $T_0(p^{(l)}) = 0$ for all but countably many l.

Lemma 2.1. *Suppose that the conditions* (N0) *and* (N1) *are satisfied. Then*

$(\eta(l))_{l\geq 0} = (\eta(p,\varsigma;l))_{l\geq 0}$ *is an increasing Lévy process with Laplace transform*

$$\mathbb{E}[\mathrm{e}^{-\lambda\eta(l)}] = \exp\big\{ - l\varsigma\lambda - l\boldsymbol{n}\big[1 - \mathrm{e}^{-\lambda T_0}\big]\big\}, \quad \lambda \geq 0. \tag{2.10}$$

If, moreover, the condition (N2) *is satisfied, then it is strictly increasing.*

Lemma 2.1 is well-known, and so we omit its proof.

The following proposition enables us to piece a process together from excursions.

Proposition 2.2. *Suppose that the conditions* (N0)–(N2) *are satisfied. Define*

$$L(t) = L(p,\varsigma;t) = \inf\{l \geq 0 : \eta(l) > t\}, \quad t \geq 0 \tag{2.11}$$

and

$$X(t) = X(p,\varsigma;t) = \begin{cases} p^{(l)}(t - \eta(l-)) & \text{if } \eta(l-) \leq t < \eta(l) \text{ for some } l \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Then it holds that

$$\int_0^t 1_{\{X(s)=0\}}\mathrm{d}s = \varsigma L(t). \tag{2.13}$$

If, moreover, the condition (N3) *is satisfied, then the process* $X(p,\varsigma) = (X(p,\varsigma;t))_{t\geq 0}$ *is* D*-valued. If* $\boldsymbol{n}(D) = \infty$*, the converse is also true*: *if the process* $X(p,\varsigma)$ *is* D*-valued then the condition* (N3) *is satisfied.*

The proof of Proposition 2.2 will be given in Section 5.1.

2.3. General limit theorem.

For real-valued measurable functions $f_1, f_2, \dots$ and f_∞ defined on a measure space $(E,\mathcal{E},\nu)$, we say that $f_n \to f_\infty$, *ν-almost uniformly* if for any $\varepsilon > 0$ there exists $A \in \mathcal{E}$ such that $\nu(A) < \varepsilon$ and $\sup_{A^c}|f_n - f_\infty| \to 0$. Imitating the Skorokhod representation of almost sure convergence, we introduce the following notion of convergence.

Definition 2.3. Let $\boldsymbol{n}_1, \boldsymbol{n}_2, \dots$ and $\boldsymbol{n}_\infty$ be σ-finite measures on D. We say that $\boldsymbol{n}_n \to \boldsymbol{n}_\infty$ if there exist a Polish space E, a σ-finite measure ν on E and measurable mappings $\Phi_1, \Phi_2, \dots, \Phi_\infty$ from E to D such that the following conditions hold:

(G1) $\boldsymbol{n}_n = (\nu \circ \Phi_n^{-1})|_{D\setminus\{o\}}$ for $n = 1, 2, \dots$ and ∞;
(G2) $\Phi_n \to \Phi_\infty$ in D, ν-a.e.;
(G3) $\|\Phi_n\| \to \|\Phi_\infty\|$, ν-almost uniformly;
(G4) $T_0(\Phi_n) \to T_0(\Phi_\infty)$, ν-a.e.;
(G5) there exists $N \in \mathbb{N}$ such that $\nu\big[1 \wedge \sup_{n\geq N} T_0(\Phi_n)\big] < \infty$.

We shall see in Lemma 5.4 that Condition (G3) can be replaced by the following:

(G3)′ $\nu\big(\bigcup_{n=1}^\infty \{\|\Phi_n\| \geq r\}\big) < \infty$ for all $r > 0$.

Remark 2.4. Condition (G2) does not imply Condition (G4). This is because the functional $T_0 : D \to [0,\infty]$ is not continuous; for instance, the sequence of functions $w_n \in D_{\mathbb{R}}$ defined by

$$w_n(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 1/n & \text{if } 1 \le t < 2, \\ 0 & \text{if } t \ge 2, \end{cases} \tag{2.14}$$

satisfies that w_n converges to w_∞ in $D_{\mathbb{R}}$, while $T_0(w_n) \equiv 2 \not\to 1 = T_0(w_\infty)$.

Theorem 2.5. *Let $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots$ and $\boldsymbol{n}_\infty$ be σ-finite measures on D. Let $\varsigma_1, \varsigma_2, \ldots$ and ς_∞ be non-negative constants. Suppose that the following conditions hold*:

(A1) *for each $n \in \mathbb{N} \cup \{\infty\}$, the pair $(\boldsymbol{n}_n, \varsigma_n)$ satisfies Conditions* (N0)–(N3);
(A2) $\boldsymbol{n}_n \to \boldsymbol{n}_\infty$;
(A3) $X(T_0-) = 0$, *$\boldsymbol{n}_\infty$-a.e.*;
(A4) $\varsigma_n \to \varsigma_\infty$.

For $n \in \mathbb{N} \cup \{\infty\}$, let p_n be a Poisson point process on D outside o with characteristic measure $\boldsymbol{n}_n$. Denote $\eta_n(l) = \eta(p_n, \varsigma_n; l)$ and $X_n(t) = X(p_n, \varsigma_n; t)$. Then it holds that

$$(X_n, L_n, \eta_n) \xrightarrow{\text{law}} (X_\infty, L_\infty, \eta_\infty) \quad \text{as } n \to \infty, \tag{2.15}$$

where the convergence is in the sense of law on $D \times D_{\mathbb{R}} \times D_{\mathbb{R}}$.

The proof of Theorem 2.5 will be given in Section 5.3.

3. Homogenization theorems.

3.1. Excursion measures.

Let S be a Borel subset of $\mathbb{R}^d$ containing 0 and let $H^0 = \{X, (\mathbb{P}^0_x)_{x \in S}\}$ be a Hunt process stopped upon hitting 0. A Hunt process $H = \{X, (\mathbb{P}_x)_{x\in S}\}$ is called an *extension* of H^0 if the law of the stopped process $X(t \wedge T_0)$ under $\mathbb{P}_x$ coincides with $\mathbb{P}^0_x$ for all $x \in S$. We introduce the following set of conditions for an extension H of H^0:

(B1) H is a conservative Hunt process with values in S;
(B2) the state 0 is regular for itself, i.e., $\mathbb{P}_0(T_0 = 0) = 1$;
(B3) the state 0 is recurrent, i.e., $\mathbb{P}_x(T_0 < \infty) = 1$ for all $x \in S$.

Let H be an extension of H^0 satisfying Conditions (B1)–(B3). Then the following assertions hold:

(i) there exists a positive continuous additive functional $L = (L(t))_{t\ge 0}$ such that $\int_0^\infty 1_{\{X(s)\neq 0\}} \mathrm{d}L(s) = 0$ (the process L is called the *local time of* 0 *for* X);
(ii) if $A = (A(t))_{t \ge 0}$ is a non-negative continuous additive functional such that $\int_0^\infty 1_{\{X(s)\neq 0\}} \mathrm{d}A(s) = 0$, then $A(t) \equiv kL(t)$ for some constant k.

For the proof of these facts, see, e.g., [**4**, Theorem V.3.13].

We fix L for a choice of the local time of 0. We then see that there exists a constant $\varsigma \geq 0$ such that

$$\int_0^t 1_{\{X(s)=0\}} \mathrm{d}s = \varsigma L(t), \quad t \geq 0. \tag{3.1}$$

The constant ς is called the *stagnancy rate*. Denote

$$\eta(l) = \inf\{t \geq 0 : L(t) > l\}, \quad l \geq 0. \tag{3.2}$$

For $l \geq 0$, we define $p^{(l)} = (p^{(l)}(t))_{t\geq 0} \in D$ by

$$p^{(l)}(t) = \begin{cases} X(\eta(l-)+t) & \text{if } 0 \leq t < \eta(l) - \eta(l-), \\ 0 & \text{if } t \geq \eta(l) - \eta(l-). \end{cases} \tag{3.3}$$

The point process $p = (p^{(l)})_{l\geq 0}$ thus obtained will be called the *point process of excursions for* $\{X, \mathbb{P}_0\}$. It is then known (see [**12**, Section 6]) that $\{p, \mathbb{P}_0\}$ is a Poisson point process on D outside o. Its characteristic measure will be denoted by $\boldsymbol{n}$ and called the *excursion measure*. We now see that

$$L = L(p, \varsigma), \quad \eta = \eta(p, \varsigma) \quad \text{and} \quad X = X(p, \varsigma). \tag{3.4}$$

Theorem 3.1 (Itô). *Let* $(\boldsymbol{n}, \varsigma)$ *be as above. Then the following assertions hold*:

(i) $(\boldsymbol{n}, \varsigma)$ *satisfies Conditions* (N0)–(N3);
(ii) *for any* $t \geq 0$, *any* $A \in \mathcal{F}_t$ *and any* $A' \in \mathcal{B}(D)$, *it holds that*

$$\boldsymbol{n}(\{T_0 > t\} \cap A \cap \theta_t^{-1} A') = \boldsymbol{n}\big[\mathbb{P}^0_{X(t)}(A'); \{T_0 > t\} \cap A\big], \tag{3.5}$$

provided that $\boldsymbol{n}(\{T_0 > t\} \cap A) < \infty$.

For the proof of Theorem 3.1, see Itô [**12**, Section 6] and also Salisbury [**19**].

We also have the strong Markov property for $\boldsymbol{n}$ stated as follows.

Theorem 3.2. *For any stopping time* T, *any* $A \in \mathcal{F}_T$ *and any* $A' \in \mathcal{B}(D)$, *it holds that*

$$\boldsymbol{n}(\{T_0 > T\} \cap A \cap \theta_T^{-1} A') = \boldsymbol{n}\big[\mathbb{P}^0_{X(T)}(A'); \{T_0 > T\} \cap A\big], \tag{3.6}$$

provided that $\boldsymbol{n}(\{T_0 > T\} \cap A) < \infty$.

From this theorem we obtain the following corollary.

Corollary 3.3. *For any* $x \neq 0$ *and any* $A \in \mathcal{B}(D)$, *it holds that*

$$\boldsymbol{n}(\{T_x < T_0\} \cap \theta_{T_x}^{-1} A) = \boldsymbol{n}(T_x < T_0)\mathbb{P}^0_x(A). \tag{3.7}$$

PROOF. By Condition (N3), we have

$$\boldsymbol{n}(\{T_x < T_0\}) \leq \boldsymbol{n}(\|X\| \geq |x|) < \infty. \tag{3.8}$$

Hence we may apply Theorem 3.2 for $T = T_x$. Since $X(T_x) = x$, we obtain (3.7). □

3.2. Scaling property.

Let H be an extension of H^0 satisfying Conditions (B1)–(B3). Let $c > 1$ be a fixed constant. For $\gamma > 0$, we define transformations Ψ_γ and $\widehat{\Psi}_\gamma$ of D by

$$(\Psi_\gamma w)(t) = c^{-\gamma} w(ct), \quad (\widehat{\Psi}_\gamma w)(t) = c^{-1} w(c^\gamma t). \tag{3.9}$$

We introduce the following set of conditions:

(S0) $c > 1$, $\alpha > 0$, $0 < \kappa < 1/\alpha$ and $c^{-\alpha} S \subset S$;
(S1) $\{\Psi_\alpha X, \mathbb{P}_x\} \stackrel{\text{law}}{=} \{X, \mathbb{P}_{c^{-\alpha}x}\}$ for all $x \in S$;
(S2) $\{\Psi_{\alpha\kappa} L, \mathbb{P}_0\} \stackrel{\text{law}}{=} \{L, \mathbb{P}_0\}$.

We need the following lemma.

LEMMA 3.4. *Suppose that Conditions* (S0)–(S2) *are satisfied. Then the stagnancy rate of the process* $\{X, \mathbb{P}_0\}$ *is necessarily equal to* 0.

The proof of Lemma 3.4 will be given in Section 6.1.

Condition (S2) is equivalent to the scaling property of the excursion measure as follows.

PROPOSITION 3.5. *Suppose that Conditions* (S0)–(S1) *are satisfied. Then Condition* (S2) *is equivalent to the following condition*:

(S2)′ $\boldsymbol{n} \circ \Psi_\alpha^{-1} = c^{-\alpha\kappa} \boldsymbol{n}$.

3.3. Homogenization theorem for jumping-in extensions.

In addition to Conditions (B1)–(B3), we introduce the following set of conditions:

(B4) excursions leave 0 continuously, i.e., $X(0) = 0$, $\boldsymbol{n}$-a.e.;
(B5) excursions hit 0 continuously, i.e., $X(T_0-) = 0$, $\boldsymbol{n}$-a.e.

If Conditions (B1)–(B5) and (S0)–(S2) are satisfied, then we see, by Condition (S2)′, that it also satisfies

$$\sigma(c^{-\alpha} x) = c^{\alpha\kappa} \sigma(x) \quad \text{for } x \in S. \tag{3.10}$$

Let H^0 be a Hunt process stopped upon hitting 0 and let $c > 1$, $\alpha > 0$ and $0 < \kappa < 1/\alpha$ be fixed. Let S' be a measurable space and let $\{\boldsymbol{n}_v\}_{v \in S'}$ be a kernel on D. We introduce the following condition:

(B) for each $v \in S'$, the measure $\boldsymbol{n}_v$ is the excursion measure of an extension $H_v = \{X, (\mathbb{P}^v_x)_{x \in S}\}$ of H^0 satisfying Conditions (B1)–(B5) and (S0)–(S2).

For a finite measure ρ on S' and a σ-finite measure j on $S \setminus \{0\}$, we define

$$\boldsymbol{n}_{\rho,j}(\mathrm{d}w) = \int_{S'} \rho(\mathrm{d}v)\boldsymbol{n}_v(\mathrm{d}w) + \int_{S\setminus\{0\}} j(\mathrm{d}x)\mathbb{P}^0_x(\mathrm{d}w). \tag{3.11}$$

For a triplet (ρ, j, ς), we introduce the following condition:

(C1) the pair $(\boldsymbol{n}_{\rho,j}, \varsigma)$ satisfies Conditions (N0)–(N3);
(C2) there exists a measurable map $\psi : S \setminus \{0\} \to S'$ such that, for j-a.e. $x \in S \setminus \{0\}$, $\sigma_{\psi(x)}(x) > 0$ and $\psi(c^{-\alpha n}x) = \psi(x)$ for all $n \in \mathbb{N} \cup \{\infty\}$.

For a triplet (ρ, j, ς) satisfying Condition (C), let $\{p_{\rho,j}, \mathbb{P}\}$ be a Poisson point process on D outside o with characteristic measure $\boldsymbol{n}_{\rho,j}$. We write

$$X_{\rho,j,\varsigma}(t) = X(p_{\rho,j}, \varsigma; t), \quad L_{\rho,j,\varsigma}(t) = L(p_{\rho,j}, \varsigma; t), \quad \eta_{\rho,j,\varsigma}(l) = \eta(p_{\rho,j}, \varsigma; l) \tag{3.12}$$

and call $\{X_{\rho,j,\varsigma}, \mathbb{P}\}$ a *jumping-in extension* of the minimal process H^0.

For a scaling exponent $\gamma > 0$, we define

$$\boldsymbol{n}^{(n)}_{\rho,j} = c^{\gamma n}\boldsymbol{n}_{\rho,j} \circ (\Psi^n_\alpha)^{-1}, \quad \varsigma^{(n)} = c^{-(1-\gamma)n}\varsigma \tag{3.13}$$

and

$$X^{(n)}_{\rho,j,\varsigma} = \Psi^n_\alpha X_{\rho,j,\varsigma}, \quad L^{(n)}_{\rho,j,\varsigma} = \Psi^n_\gamma L_{\rho,j,\varsigma}, \quad \eta^{(n)}_{\rho,j,\varsigma} = \widehat{\Psi}^n_\gamma \eta_{\rho,j,\varsigma}. \tag{3.14}$$

Let H^0 be a Hunt process stopped upon hitting 0 and let $c > 1$, $\alpha > 0$ and $0 < \kappa < 1/\alpha$ be constants. Let $\{\boldsymbol{n}_v\}_{v\in S'}$ be a kernel satisfying Condition (B). Denote

$$\sigma_v(x) = \boldsymbol{n}_v(T_x < T_0) \quad \text{for } v \in S'. \tag{3.15}$$

Let (ρ, j, ς) satisfy Conditions (C1)–(C2). In order to handle various examples together, we give the following two auxiliary theorems.

THEOREM 3.6 (Jumping-in vanishing case). *Suppose the following condition:*

(C3) $T_{c^{-\alpha n}x} \to 0$, $\boldsymbol{n}_{\psi(x)}$*-a.e. for j-a.e.* $x \in S \setminus \{0\}$;
(C4) $(\boldsymbol{n}_{\rho^*,0}, 0)$ *satisfies Conditions* (N0)–(N3)*, where ρ^* is the finite measure on S' defined by*

$$\rho^* = \rho + \int_{S\setminus\{0\}} \frac{j(\mathrm{d}x)}{\sigma_{\psi(x)}(x)}\delta_{\psi(x)}, \tag{3.16}$$

where δ denote the Dirac delta. Then, for the scaling exponent $\gamma = \alpha\kappa$, it holds as $n \to \infty$ that

$$\boldsymbol{n}^{(n)}_{\rho,j} \to \boldsymbol{n}_{\rho^*,0}, \quad \varsigma^{(n)} \to 0 \tag{3.17}$$

and

$$\left\{\left(X^{(n)}_{\rho,j,\varsigma}, L^{(n)}_{\rho,j,\varsigma}, \eta^{(n)}_{\rho,j,\varsigma}\right), \mathbb{P}\right\} \xrightarrow{\text{law}} \{(X_{\rho^*,0,0}, L_{\rho^*,0,0}, \eta_{\rho^*,0,0}), \mathbb{P}\}. \tag{3.18}$$

Theorem 3.7 (Jumping-in dominant case). *Suppose the following condition*:

(C5) *there exist a σ-finite measure μ on a measurable space S'', measurable mappings $J_n : S'' \to S \setminus \{0\}$, $J^* : S'' \to S \setminus \{0\}$, and a constant $\beta \in (0,\kappa)$ such that*

$$c^{\alpha\beta n} \int_{S\setminus\{0\}} \frac{j(\mathrm{d}x)}{\sigma_{\psi(x)}(c^{-\alpha n}x)} f(\psi(x), c^{-\alpha n}x) = \int_{S''} \mu(\mathrm{d}y) f(\psi(J^*y), J_n y) \tag{3.19}$$

for all $n \in \mathbb{N}$ and all non-negative measurable function f on $S' \times S$ and

$$T_{J_n y} \to T_{J^* y} \text{ and } X(T_{J^*y}-) = X(T_{J^*y}),\ \boldsymbol{n}_{\psi(J^*y)}\text{-a.e. for } \mu\text{-a.e. } y \in S''; \tag{3.20}$$

(C6) *$(\boldsymbol{n}_{0,j^*}, 0)$ satisfies Conditions* (N0)–(N3), *where j^* is the σ-finite measure on $S\setminus\{0\}$ defined by*

$$j^* = \int_{S''} \mu(\mathrm{d}y) \sigma_{\psi(J^*y)}(J^*y) \delta_{J^*y}. \tag{3.21}$$

Then, for the scaling exponent $\gamma = \alpha\beta$, it holds as $n \to \infty$ that

$$\boldsymbol{n}^{(n)}_{\rho,j} \to \boldsymbol{n}_{0,j^*}, \quad \varsigma^{(n)} \to 0 \tag{3.22}$$

and

$$\left\{\left(X^{(n)}_{\rho,j,\varsigma}, L^{(n)}_{\rho,j,\varsigma}, \eta^{(n)}_{\rho,j,\varsigma}\right), \mathbb{P}\right\} \xrightarrow{\text{law}} \{(X_{0,j^*,0}, L_{0,j^*,0}, \eta_{0,j^*,0}), \mathbb{P}\}. \tag{3.23}$$

4. Examples.

4.1. Positive self-similar Markov processes.

Let α and κ be positive numbers such that $0 < \kappa < 1/\alpha$ and let $c > 1$ be an arbitrary number. Let $\{X, (\mathbb{P}_x)_{x\geq 0}\}$ be a Hunt process with values in $S = [0,\infty)$ such that (B1)–(B5) and (S0)–(S2) hold and

$$\begin{cases} T_{x_n} \to 0,\ \boldsymbol{n}\text{-a.e.} & \text{if } x_n \to 0, \\ T_{x_n} \to T_x,\ \boldsymbol{n}\text{-a.e.} & \text{if } x_n \to x > 0, \end{cases} \tag{4.1}$$

where $\boldsymbol{n}$ denotes the excursion measure away from 0 according to a particular choice of the local time at 0. Let $\mathbb{P}^0_x$ denote the law of $X(t \wedge T_0)$ under $\mathbb{P}_x$. Then Condition (B) is satisfied for $S' = \{0\}$, $H^0 = \{X, (\mathbb{P}^0_x)_{x>0}\}$ and $\boldsymbol{n}_0 = \boldsymbol{n}$.

Such a process can be obtained in the following manner. Let $\{Z, (\mathbb{Q}_z)_{z\in\mathbb{R}}\}$ be a Lévy process which satisfies the following conditions:

(P1) Z drifts to $-\infty$;
(P2) Z is spectrally negative;
(P3) every point is regular for itself;
(P4) every point is accessible.

By (P1) and (P2), it is known (see, e.g., [**14**, Section 8.1]) that there exists a unique constant $\kappa > 0$ such that the following *Cramér condition* is satisfied:

$$\mathbb{Q}_0\big[\exp(\kappa Z(u))\big] = 1 \quad \text{for all } u \geq 0. \tag{4.2}$$

We recall the *Lamperti transformation* following Lamperti [**16**] as follows. Let α be a fixed constant such that $0 < \alpha < 1/\kappa$. Define

$$\tau(u) = \int_0^u \exp(Z(s)/\alpha)\mathrm{d}s \tag{4.3}$$

and

$$Y(t) = \begin{cases} \exp(Z(\tau^{-1}(t))) & \text{for } 0 \leq t < \tau(\infty), \\ 0 & \text{for } t \geq \tau(\infty). \end{cases} \tag{4.4}$$

For $x > 0$, we write $\mathbb{P}^0_x$ the law of Y under $\mathbb{Q}_{\log x}$ and let $H^0 = \{X, (\mathbb{P}^0_x)_{x>0}\}$. By the theorem obtained by Rivero [**17**], [**18**] and Fitzsimmons [**7**] independently, we see, thanks to the Cramér condition (4.2), that there exists a unique α-self-similar recurrent extension of H^0 whose excursions leave 0 continuously, which we will denote by $\{X, (\mathbb{P}_x)_{x\geq 0}\}$. Then we see that (B1)–(B5) and (S0)–(S2) are satisfied. Since $z \mapsto T_{z+}(Z)$ is a subordinator and has no fixed discontinuity, we see that $\mathbb{P}^0_\varepsilon(D \setminus \{T_{x_n} \to T_x\}) = 0$ for any sequence $\{x_n\}$ converging to $x > \varepsilon > 0$. Using Corollary 3.3, we see that $T_{x_n} \to T_x$, $\boldsymbol{n}$-a.e. for any sequence $\{x_n\}$ converging to $x > 0$. Since $\{X, \boldsymbol{n}\}$ has càdlàg paths and has no positive jumps, we further see that $T_{x_n} \to 0$, $\boldsymbol{n}$-a.e. for any sequence $\{x_n\}$ converging to 0. Consequently we have verified that (4.1) is satisfied. We have thus obtained $\{X, (\mathbb{P}_x)_{x\geq 0}\}$ as desired.

Since (3.10) holds for any $c > 0$, we see that

$$\sigma(x) := \boldsymbol{n}(T_x < T_0) = \delta x^{-\kappa} \tag{4.5}$$

for $\delta = \boldsymbol{n}(T_1 < T_0)$. We need the following.

Lemma 4.1. *There exist positive constants c_1 and c_2 such that*

$$c_1(x^\kappa \wedge 1) \leq \mathbb{P}^0_x[T_0 \wedge 1] \leq c_2(x^\kappa \wedge 1) \quad \textit{for all } x > 0. \tag{4.6}$$

Proof. For $x \geq 1$, we see by the scaling property that

$$0 < \mathbb{P}^0_1(T_0 \geq 1) \leq \mathbb{P}^0_1(T_0 \geq x^{-1/\alpha}) = \mathbb{P}^0_x(T_0 \geq 1) \leq \mathbb{P}^0_x[T_0 \wedge 1] \leq 1. \tag{4.7}$$

By Corollary 3.3 and by (4.5), we have

$$\mathbb{P}^0_x[T_0 \wedge 1] = \delta^{-1} x^\kappa \boldsymbol{n}\big[T_0 \circ \theta_{T_x} \wedge 1; T_x < T_0\big]. \tag{4.8}$$

For $0 < x < 1$, we see by (P2) that $\{T_1 < T_0\} \subset \{T_x < T_1 < T_0\}$, $\boldsymbol{n}$-a.e., and hence we obtain

$$0 < \boldsymbol{n}\big[T_0 \circ \theta_{T_1} \wedge 1; T_1 < T_0\big] \le \boldsymbol{n}\big[T_0 \circ \theta_{T_x} \wedge 1; T_x < T_0\big] \le \boldsymbol{n}[T_0 \wedge 1] < \infty. \tag{4.9}$$

The proof is now complete. □

We identify a measure ρ on $S' = \{0\}$ with a positive number $\rho(\{0\})$. By Lemma 4.1, we see that, a pair $(\boldsymbol{n}_{\rho,j}, \varsigma)$ satisfies (N0)–(N3) if and only if the following conditions are satisfied:

(CP1) ρ is a non-negative constant;
(CP2) j satisfies $\int_{(0,\infty)} (x^\kappa \wedge 1) j(\mathrm{d}x) < \infty$;
(CP3) any one of the following holds: $\rho > 0$, $j((0,\infty)) = \infty$ and $\varsigma > 0$.

COROLLARY 4.2 (Jumping-in vanishing case). *Let (ρ, j, ς) satisfy (CP1)–(CP3). Suppose, moreover, that*

$$\rho^* := \rho + \frac{1}{\delta} \int_{(0,\infty)} x^\kappa j(\mathrm{d}x) < \infty. \tag{4.10}$$

Then the same assertions as Theorem 3.6 hold.

Corollary 4.2 is an immediate consequence of Theorem 3.6, and so we omit its proof.

COROLLARY 4.3 (Jumping-in dominant case). *Let (ρ, j, ς) satisfy (CP1)–(CP3). Let $\beta \in (0, \kappa)$ and $j_0 > 0$ be constants. Suppose, moreover, that*

$$j((x,\infty)) \sim j_0 x^{-\beta} \quad \textit{as } x \to \infty. \tag{4.11}$$

Define a σ-finite measure j^* on $(0,\infty)$ by

$$j^*(\mathrm{d}x) = j_0 \beta x^{-\beta-1} \mathrm{d}x. \tag{4.12}$$

Then the same assertions as Theorem 3.7 hold.

PROOF OF COROLLARY 4.3. We define a function $J : (0,\infty) \to (0,\infty)$ by

$$J(y) = \inf \left\{ x > 0 : \frac{1}{\delta} \int_{(0,x]} s^\kappa j(\mathrm{d}s) > y \right\}. \tag{4.13}$$

In the same way as J we define J^* with j being replaced by j^*. Set $S'' = (0,\infty)$, $\mu(\mathrm{d}y) = \mathrm{d}y$ and $J_n y = c^{-\alpha n} J(c^{\alpha(\kappa-\beta)n} y)$. By (4.11), we see that

$$\frac{1}{\delta}\int_{(0,x]} s^\kappa j(\mathrm{d}s) \sim \frac{j_0\beta}{\delta(\kappa-\beta)}x^{\kappa-\beta} = \frac{1}{\delta}\int_{(0,x]} s^\kappa j^*(\mathrm{d}s) \quad \text{as } x\to\infty, \tag{4.14}$$

which shows that (C5) and (C6) are satisfied. We can thus apply Theorem 3.7. □

4.2. Walsh diffusions.

Let us take up the *Walsh diffusions*, which have been first introduced in Walsh [**20**, Epilogue] and developed in Barlow–Pitman–Yor [**1**]. A Walsh diffusion is a diffusion process with values in $\mathbb{R}^2$ whose stopped process starting from $x \neq 0$ and stopped at 0 takes values in the ray $R(x) := \{rx : r \geq 0\}$. In this paper we confine ourselves to the case where the stopped processes are Bessel ones.

Let $0 < \alpha < 1$ and $c > 1$ be fixed. Let $B = \{X, (\mathbb{Q}_r)_{r\geq 0}\}$ denote the $(\mathrm{d}/\mathrm{d}m)(\mathrm{d}/\mathrm{d}x)$-diffusion process for $m(x) = x^{1/\alpha-1}$ which takes values in $[0,\infty)$ and which has 0 as the reflecting boundary. Let $\boldsymbol{n}_B$ denote the excursion measure of B away from 0 according to a particular choice of the local time at 0 which we will denote by L_B. Then (S0)–(S2) holds for $S = [0,\infty)$ and $\kappa = 1$ (see, e.g., [**5**]). Let $\mathbb{Q}^0_r$ denote the law of $X(t\wedge T_0)$ under $\mathbb{Q}_r$. By the same argument as the proof of Lemma 4.1 (see also [**11**, Example 6.1]), we see that

$$c_1(r\wedge 1) \leq \mathbb{Q}_r[T_0\wedge 1] \leq c_2(r\wedge 1) \quad \text{for all } r>0 \tag{4.15}$$

holds for some positive constants c_1 and c_2.

Let $S = \mathbb{R}^2$ and let $S' = S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$. Define $\psi : \mathbb{R}^2\setminus\{0\} \to S^1$ by $\psi(x) = x/|x|$. We define $(\boldsymbol{n}_v)_{v\in S^1}$ and $(\mathbb{P}^0_x)_{x\in\mathbb{R}^2\setminus\{0\}}$ by

$$\boldsymbol{n}_v = \int_{D_{[0,\infty)}} \boldsymbol{n}_B(\mathrm{d}q)\delta_{qv} \quad \text{and} \quad \mathbb{P}^0_x = \int_{D_{[0,\infty)}} \mathbb{Q}^0_{|x|}(\mathrm{d}q)\delta_{q\psi(x)}, \tag{4.16}$$

where for $q = (q(t))_{t\geq 0} \in D_{[0,\infty)}$ we write $qv = (q(t)v)_{t\geq 0} \in D_{\mathbb{R}^2}$. Since $\boldsymbol{n}_B(T_r < T_0) = \delta r^{-1}$ for all $r > 0$ with $\delta = \boldsymbol{n}_B(T_1 < T_0)$, we see that

$$\boldsymbol{n}_{\psi(x)}(T_x < T_0) = \delta|x|^{-1}, \quad x \in \mathbb{R}^2\setminus\{0\}. \tag{4.17}$$

For $v \in S^1$ and for $\{x_n\} \subset R(v)$, we easily see that

$$\begin{cases} T_{x_n} \to 0, \ \boldsymbol{n}_v\text{-a.e.} & \text{if } x_n \to 0, \\ T_{x_n} \to T_x, \ \boldsymbol{n}_v\text{-a.e.} & \text{if } x_n \to x \neq 0. \end{cases} \tag{4.18}$$

For $v \in S^1$ and $x \in R(v)$, we define $\mathbb{P}^v_x = \int_{D_{[0,\infty)}} \mathbb{Q}_{|x|}(\mathrm{d}q)\delta_{qv}$. For $v \in S^1$ and $x \notin R(v)$, we define $\mathbb{P}^v_x$ as the law of the process which is obtained as $X(t\wedge T_0(X)) + \widetilde{X}((t-T_0(X))\vee 0)$, where $\{X, \mathbb{P}^0_x\}$ and $\{\widetilde{X}, \mathbb{P}_0\}$ are independent processes defined on a common probability space. Then $H_v = \{X, (\mathbb{P}^v_x)_{x\in\mathbb{R}^2}\}$ is an extension of $H^0 = \{X, (\mathbb{P}^0_x)_{x\in\mathbb{R}^2}\}$ and $\boldsymbol{n}_v$ is the excursion measure away from 0 of H_v. It is immediate that (B) holds for $\kappa = 1$.

For measures ρ on S^1 and j on $\mathbb{R}^2\setminus\{0\}$, we define $\boldsymbol{n}_{\rho,j}$ on $D_{\mathbb{R}^2}$ by (1.2). By (4.15), we see that a pair $(\boldsymbol{n}_{\rho,j}, \varsigma)$ satisfies (N0)–(N3) if and only if the following conditions are satisfied:

(CW1) ρ is a finite measure on S^1;
(CW2) j satisfies $\int_{\mathbb{R}^2\setminus\{0\}}(|x| \wedge 1)j(\mathrm{d}x) < \infty$;
(CW3) any one of the following holds: $\rho(S^1) > 0$, $j(\mathbb{R}^2 \setminus \{0\}) = \infty$ and $\varsigma > 0$.

We identify $S^1 \times (0,\infty)$ with $\mathbb{R}^2 \setminus \{0\}$ via the bicontinuous bijection $(v,r) \mapsto rv$. A σ-finite measure $j(\mathrm{d}v\mathrm{d}r)$ on $S^1 \times (0,\infty)$ allows at least one disintegration of the form

$$j(\mathrm{d}v\mathrm{d}r) = \rho_j(\mathrm{d}v)j_v(\mathrm{d}r) \tag{4.19}$$

for a finite measure ρ_j on S^1 and a kernel $\{j_v\}_{v\in S^1}$ on $(0,\infty)$. We can obtain such a disintegration, for example, as follows: Take a measurable function $f(v,r)$ which is positive j-a.e. and which satisfies $j[f] = 1$. Then, by conditioning, the probability measure $\overline{j}(\mathrm{d}v\mathrm{d}r) = f(v,r)j(\mathrm{d}v\mathrm{d}r)$ possesses a unique disintegration $\overline{j} = \rho_{\overline{j}}(\mathrm{d}v)\overline{j}_v(\mathrm{d}r)$ with a probability measure $\rho_{\overline{j}}$ and a probability kernel $\{\overline{j}_v\}_{v\in S^1}$. We set $\rho_j = \rho_{\overline{j}}$ and $j_v(\mathrm{d}r) = f(v,r)^{-1}\overline{j}_v(\mathrm{d}r)$ and then we obtain the disintegration (4.19). The disintegration (4.19) is not unique; in fact, for any bounded measurable function π on S^1 with positive values, we obtain another disintegration $j(\mathrm{d}v\mathrm{d}r) = \rho'_j(\mathrm{d}v)j'_v(\mathrm{d}r)$, where $\rho'_j(\mathrm{d}v) = \pi(v)\rho_j(\mathrm{d}v)$ and $j'_v(\mathrm{d}r) = \pi(v)^{-1}j_v(\mathrm{d}r)$.

COROLLARY 4.4 (Jumping-in vanishing case). *Let (ρ, j, ς) satisfy* (CW1)–(CW3). *Let $j(\mathrm{d}v\mathrm{d}r) = \rho_j(\mathrm{d}v)j_v(\mathrm{d}r)$ be a disintegration of j and suppose that*

$$\int_{\mathbb{R}^2\setminus\{0\}} |x|j(\mathrm{d}x) = \int_{S^1} \pi(v)\rho_j(\mathrm{d}v) < \infty, \tag{4.20}$$

where $\pi(v) = \int_{(0,\infty)} rj_v(\mathrm{d}r)$. Define the finite measure ρ^* on S^1 by

$$\rho^*(\mathrm{d}v) = \rho(\mathrm{d}v) + \frac{1}{\delta}\pi(v)\rho_j(\mathrm{d}v). \tag{4.21}$$

Then the same assertions as Theorem 3.6 *hold.*

Corollary 4.4 is an immediate consequence of Theorem 3.6, and so we omit its proof.

COROLLARY 4.5 (Jumping-in dominant case). *Let (ρ, j, ς) satisfy* (CW1)–(CW3). *Let $j(\mathrm{d}v\mathrm{d}r) = \rho_j(\mathrm{d}v)j_v(\mathrm{d}r)$ be a disintegration of j and suppose that there exist a constant $0 < \beta < 1$ and a non-negative measurable function π on S^1 such that*

$$\int_{S^1} \pi(v)\rho_j(\mathrm{d}v) \in (0,\infty) \tag{4.22}$$

and, for any $v \in S^1$,

$$r^\beta j_v((r,\infty)) \to \pi(v) \quad \text{as } r \to \infty. \tag{4.23}$$

Define a σ-finite measure j^* on $\mathbb{R}^2 \setminus \{0\} \simeq S^1 \times (0,\infty)$ by

$$j^*(\mathrm{d}v\mathrm{d}r) = \pi(v)\rho_j(\mathrm{d}v)\beta r^{-\beta-1}\mathrm{d}r. \tag{4.24}$$

Then the same assertions as Theorem 3.7 *hold.*

Proof of Corollary 4.5. Note that j^* admits a disintegration $j^*(\mathrm{d}v\mathrm{d}r) = \rho_{j^*}(\mathrm{d}v)j^*_v(\mathrm{d}r)$ where $\rho_{j^*}(\mathrm{d}v) = \pi(v)\rho_j(\mathrm{d}v)$ and $j^*_v(\mathrm{d}r) = \beta r^{-\beta-1}\mathrm{d}r$.

Let $S'' = S^1 \times (0,\infty)$ and let $\mu(\mathrm{d}v\mathrm{d}r) = \mathrm{d}r$. For $v \in S^1$, we define a function $J^0_v : (0,\infty) \to (0,\infty)$ by

$$J^0_v y = \inf\left\{ r > 0 : \frac{1}{\delta}\int_{(0,r]} s j_v(\mathrm{d}s) > y \right\} \tag{4.25}$$

and define a function $J : S^1 \times (0,\infty) \to \mathbb{R}^2\setminus\{0\}$ by $J(v,y) = (J^0_v y)v$. In the same way as J we define J^* with j_v being replaced by j^*_v. For $n \in \mathbb{N}$, set $J_n(v,y) = c^{-\alpha n}J(v, c^{\alpha(1-\beta)n}y)$. By the assumption (4.23), we have

$$\frac{1}{\delta}\int_{(0,r]} s j_v(\mathrm{d}s) \sim \frac{\pi(v)\beta}{\delta(1-\beta)} r^{1-\beta} = \frac{1}{\delta}\int_{(0,r]} s j^*_v(\mathrm{d}s) \quad \text{as } r \to \infty, \tag{4.26}$$

which shows that (C5) and (C6) are satisfied. We can thus apply Theorem 3.7. □

4.3. The Brownian motion on the Sierpiński gasket.

We take up the Brownian motion on the *Sierpiński gasket*. For its precise definition and several facts which we will utilize later, see Barlow–Perkins [**2**].

Let $S = G$ denote the Sierpiński gasket in $\mathbb{R}^2$ and let $\{X, (\mathbb{P}_x)_{x\in G}\}$ denote the Brownian motion on G. Noting that every point of G is regular for itself, we let L^x_t denote a jointly continuous version of the local time and denote $L(t) = L^0_t$. Then (S0)–(S2) hold for

$$c = 5, \quad \alpha = \frac{\log 2}{\log 5}, \quad \kappa = \frac{\log 5 - \log 3}{\log 2}. \tag{4.27}$$

Let $\boldsymbol{n}$ denote the excursion measure away from 0.

Let $\mathbb{P}^0_x$ denote the law of $X(t \wedge T_0)$ under $\mathbb{P}_x$ and let $H^0 = \{X, (\mathbb{P}^0_x)_{x\in G}\}$. We denote $x = (x_1, x_2) \in \mathbb{R}^2$ and set $G_\pm = \{x \in G : \pm x_1 \geq 0\}$. We write $\mathbb{P}^\pm_x$ for the law of $Y_\pm$ under $\mathbb{P}_x$ and write $H_\pm = \{X, (\mathbb{P}^\pm_x)_{x\in G}\}$, where

$$Y_\pm(t) = \begin{cases} X(t) & \text{for } 0 \leq t \leq T_0, \\ (\pm|X_1(t)|, X_2(t)) & \text{for } t > T_0. \end{cases} \tag{4.28}$$

We then see that $H_\pm$ are extensions of H^0 whose excursion measures away from 0 are

$$\boldsymbol{n}_\pm = \boldsymbol{n}|_{\{X(t)\in G_\pm \text{ for all } t\geq 0\}}. \tag{4.29}$$

Note that $\boldsymbol{n} = \boldsymbol{n}_+ + \boldsymbol{n}_-$. Letting $S' = \{+,-\}$, we see that Condition (B) is satisfied. We identify a measure ρ on S' with the pair $(\rho_+, \rho_-) := (\rho(\{+\}), \rho(\{-\}))$. A pair $(\boldsymbol{n}_{\rho,j}, \varsigma)$ satisfies (N0)–(N3) if and only if the following conditions are satisfied:

(CG1) ρ_+ and ρ_- are non-negative finite constants;
(CG2) j satisfies $\int_{G\setminus\{0\}} j(\mathrm{d}x)\mathbb{P}^0_x[T_0 \wedge 1] < \infty$;
(CG3) any one of the following holds: $\rho_+ > 0$, $\rho_- > 0$, $j(G \setminus \{0\}) = \infty$ and $\varsigma > 0$.

To obtain the homogenization theorem, we need the following.

LEMMA 4.6. *For $\{x_n\} \subset G$ such that $x_n \to x \neq 0$, it holds that $T_{x_n} \to T_x$, $\boldsymbol{n}$-a.e.*

PROOF. Suppose that the following assertion is established:

$$T_{x_n} \to T_x, \quad \mathbb{P}_a\text{-a.e. for all } a \neq 0. \tag{4.30}$$

If $T_x < T_0$, then $L^y_{T_0} > 0$ for all y in some neighborhood of x, so that we have $T_{x_n} \wedge T_0 = T_{x_n} \to T_x = T_x \wedge T_0$ as $n \to \infty$. If $T_x > T_0$, then $L^y_{T_0} = 0$ for all y in some neighborhood of x, so that we have $T_{x_n} \wedge T_0 = T_x \wedge T_0 = T_0$ for large n. We thus see that $\mathbb{P}^0_a(D \setminus \{T_{x_n} \to T_x\}) = 0$ for all $a \neq 0$. By the Markov property of $\boldsymbol{n}$, we obtain the desired result.

Let us now prove (4.30)[1].

1°). Let us prove $T_x \le \liminf T_{x_n}$. Suppose $t_0 := \liminf T_{x_n} < \infty$. Then there exists a subsequence $\{n(k)\}$ such that $T_{n(k)} \to t_0$, so that we have $X(t_0) = \lim X(T_{x_{n(k)}}) = \lim x_{n(k)} = x$. This shows $T_x \le t_0 = \liminf T_{x_n}$.

2°). Let us prove $T_x \ge \limsup T_{x_n}$. Suppose $T_x < T_0$. For $T_x < t_0 < T_0$, we have $L^y_{t_0} > 0$ for all y in some neighborhood of x, so that we have $\limsup T_{x_n} \le t_0$. This shows $\limsup T_{x_n} \le T_x$.

The proof is now complete. □

For a σ-finite measure j on $G \setminus \{0\}$, set $j_+ = j|_{G_+\setminus\{0\}}$ and $j_- = \check{j}|_{G_+\setminus\{0\}}$, where $\check{j}$ is the pullback of j under $(x_1, x_2) \mapsto (-x_1, x_2)$. We define mappings $\phi_1 : G_+ \setminus \{0\} \to [0,1]$ and $\phi_2 : G_+ \setminus \{0\} \to (0,\infty)$ by

$$\phi_1(x_1,x_2) = \frac{2x_2}{\sqrt{3}x_1 + x_2}, \quad \phi_2(x_1,x_2) = x_1 + \frac{1}{\sqrt{3}}x_2, \quad (x_1,x_2) \in G_+. \tag{4.31}$$

We then see that the mapping $\phi = (\phi_1, \phi_2) : G_+ \setminus \{0\} \to [0,1] \times (0,\infty)$ is a measurable injection. For $v \in [0,1]$, we write

$$R(v) = \{\phi_2(x) : x \in G_+ \setminus \{0\} \text{ and } \phi_1(x) = v\}. \tag{4.32}$$

We then see that $c^{-\alpha}R(v) = R(v)$. Since the pullbacks $j_\pm \circ (\phi_1, \phi_2)^{-1}$ are σ-finite measures on $[0,1] \times (0,\infty)$, we may obtain at least one disintegration of the form

[1] This proof is due to N. Kajino.

$$(j_\pm \circ (\phi_1, \phi_2)^{-1})(\mathrm{d}v\mathrm{d}r) = \rho_j^\pm(\mathrm{d}v) j_v^\pm(\mathrm{d}r) \tag{4.33}$$

for finite measures $\rho_j^\pm$ on $[0,1]$ and kernels $\{j_v^\pm\}_{v\in[0.1]}$ on $(0,\infty)$ such that $j_v^\pm((0,\infty)\setminus R(v)) = 0$ for all $v \in [0,1]$.

COROLLARY 4.7 (Jumping-in dominant case). *Let* (ρ, j, ς) *satisfy* (CG1)–(CG3). *Let* $(j_\pm \circ (\phi_1, \phi_2)^{-1})(\mathrm{d}v\mathrm{d}r) = \rho_j^\pm(\mathrm{d}v) j_v^\pm(\mathrm{d}r)$ *be disintegrations and suppose that there exist a constant* $0 < \beta < 1$ *and a* σ*-finite measure* j^* *on* $G \setminus \{0\}$ *with disintegrations* $(j_\pm^* \circ (\phi_1, \phi_2)^{-1})(\mathrm{d}v\mathrm{d}r) = \rho_{j^*}^\pm(\mathrm{d}v) j_v^{*,\pm}(\mathrm{d}r)$ *such that* $j_v^{*,\pm}((0,\infty)\setminus R(v)) = 0$ *for any* $v \in [0,1]$ *and, for any* $v \in [0,1]$ *and any* $r \in R(v)$,

$$c^{-\alpha(\kappa-\beta)n} \int_{(0,c^{\alpha n} r]} \frac{j_v^\pm(\mathrm{d}s)}{\sigma(\phi^{-1}(s,v))} \to \int_{(0,r]} \frac{j^{*,\pm}(\mathrm{d}s)}{\sigma(\phi^{-1}(s,v))} \quad \text{as } n \to \infty. \tag{4.34}$$

Suppose, moreover, that $(0, j^*, 0)$ *satisfies* (N0)–(N3). *Then the same assertions as Theorem* 3.7 *hold.*

PROOF OF COROLLARY 4.7. Let $S'' = \{+,-\} \times [0,1] \times (0,\infty)$ and let $\mu(\{\pm\} \times \mathrm{d}v\mathrm{d}r) = \mathrm{d}r$. For $v \in [0,1]$, we define a function $J_v^{0,\pm} : (0,\infty) \to (0,\infty)$ by

$$J_v^{0,\pm} y = \inf\left\{ r > 0 : \int_{(0,r]} \frac{j_v^\pm(\mathrm{d}s)}{\sigma(\phi^{-1}(s,v))} > y \right\} \tag{4.35}$$

and define a function $J : \{+,-\} \times [0,1] \times (0,\infty) \to G \setminus \{0\}$ by $J(\pm, v, y) = (J_v^{0,\pm} y)v$. In the same way as J we define J^* with $j_v^\pm$ being replaced by $j_v^{*,\pm}$. For $n \in \mathbb{N}$, set $J_n(\pm, v, y) = c^{-\alpha n} J(\pm, v, c^{\alpha(\kappa-\beta)n} y)$. By the assumption (4.34), we see that (C5) and (C6) are satisfied. We can thus apply Theorem 3.7. □

We may expect the following.

CONJECTURE 4.8. *For* $\{x_n\} \subset G$ *such that* $x_n \to 0$, *it holds that* $T_{x_n} \to 0$, $\boldsymbol{n}$*-a.e.*

We do not know whether Conjecture 4.8 is true or not. If Conjecture 4.8 is true, then we can easily obtain the following.

CONJECTURE 4.9 (Jumping-in vanishing case). *Let* (ρ, j, ς) *satisfy* (CG1)–(CG3). *Suppose, moreover, that* $\int_{G\setminus\{0\}} (j(\mathrm{d}x)/\sigma(x)) < \infty$. *Set*

$$\rho_\pm^* := \rho_\pm + \int_{G_\pm \setminus \{0\}} \frac{j(\mathrm{d}x)}{\sigma(x)}. \tag{4.36}$$

Then the same assertions as Theorem 3.6 *hold.*

5. Proof of the general limit theorem.

5.1. Piecing proposition.

Let us prove Proposition 2.2.

Proof of Proposition 2.2. Let us write $\eta(l)$, $L(t)$ and $X(t)$ simply for $\eta(p,\varsigma;t)$, $L(p,\varsigma;t)$ and $X(p,\varsigma;t)$.

We prove (2.13). For $t \geq 0$, we have

$$\int_0^t 1_{\{X(s)=0\}} \mathrm{d}s = t - \int_0^t 1_{\{X(s)\neq 0\}} \mathrm{d}s \tag{5.1}$$

$$= t - \sum_{l<L(t)} \int_{\eta(l-)}^{\eta(l)} 1_{\{X(s)\neq 0\}} \mathrm{d}s - \int_{\eta(L(t)-)}^{t} 1_{\{X(s)\neq 0\}} \mathrm{d}s \tag{5.2}$$

$$= t - \sum_{l<L(t)} T_0(p(l)) - \{t - \eta(L(t)-)\}, \tag{5.3}$$

which is equal to $\varsigma L(t)$ by the definition (2.9). Thus we obtain (2.13).

Let us assume that the condition (N3) is satisfied. By (N3), we see that, for any $n \in \mathbb{N}$, there are at most finitely many $l \leq n$ such that $\|p^{(l)}\| \geq 1/n$. This shows that, if there exists a sequence l_n converging to l such that $p^{(l_n)} \neq o$, then it implies that $\|p^{(l_n)}\| \to 0$. We now let $t_0 \geq 0$ and we prove that $X(t)$ is càdlàg at $t = t_0$. Set $l_0 = L(t_0)$, $t_1 = \eta(l_0-)$ and $t_2 = \eta(l_0)$ so that $t_1 \leq t_0 \leq t_2$. We divide the proof into three cases.

(i) Suppose that $t_1 \leq t_0 < t_2$. We then have $X(t) = p^{(l_0)}(t-t_1)$ for all $t_1 \leq t < t_2$. This shows that $X(t)$ is right continuous at $t = t_0$ and has left limit at $t = t_0$ except when $t_1 = t_0$. If $t_1 = t_0$, we see, by the above remark, that $X(t_0-) = 0$.

(ii) Suppose that $t_1 < t_0 = t_2$. We then have $X(t) = p^{(l_0)}(t-t_1)$ for all $t_1 \leq t < t_0$. This shows that $X(t)$ has left limit at $t = t_0$. Since η is strictly increasing, there is no l such that $\eta(l_0) = \eta(l-)$, and hence $X(t_0) = 0$ by definition of X. If there exists a sequence l_n decreasing to l_0 such that $p^{(l_n)} \neq o$, we have $\|p^{(l_n)}\| \to 0$ by the above remark, and hence we obtain $X(t_0+) = 0$. Otherwise, we have $X(t_0+) = 0$ by definition of X.

(iii) Suppose that $t_1 = t_0 = t_2$. We then easily see that $X(t_0-) = X(t_0) = X(t_0+) = 0$.

Let us assume that $\boldsymbol{n}(D) = \infty$ but that the condition (N3) is not satisfied. We then have $\boldsymbol{n}\{w \in D : \|w\| \geq r_0\} = \infty$ for some $r_0 > 0$. Hence there exists a sequence l_n decreasing to 0 such that $\|p^{(l_n)}\| \geq r_0$ for all n. By (N1), we have $T_0(p^{(l_n)}) \to 0$, and hence we obtain $\limsup_{t\to 0+} \|X(t)\| \geq r_0$. Since we have $X(0) = 0$ by the definition of X, we see that X is not right-continuous.

Therefore we conclude the proof. □

5.2. Useful lemmas.

For the proof of Theorem 2.5, we need two lemmas. The first one is the following.

LEMMA 5.1. *Let $w_1, w_2, \dots, w_\infty \in D^0$. Suppose that $w_n \to w_\infty$ in D and that $T_0(w_n) \to T_0(w_\infty)$. Then one has $\|w_n\| \to \|w_\infty\|$.*

PROOF. By the assumption that $w_n \to w_\infty$ in D, we may take transformations $I_1, I_2, \dots, I_\infty$ of $[0, \infty)$ such that $I_n \to I$ uc and $w_n - w_\infty \circ I_n \to 0$ uc. Let $\varepsilon > 0$. Then we may choose $N \in \mathbb{N}$ so that for any $n \geq N$ we have

$$\sup_{t \leq T_0(w_\infty)+1} |w_n(t) - w_\infty(I_n(t))| < \varepsilon \tag{5.4}$$

and we have

$$T_0(w_n) \leq T_0(w_\infty) + 1 \quad \text{and} \quad \sup_{t \leq T_0(w_\infty)+1} |I_n(t) - t| \leq 1. \tag{5.5}$$

For $n \geq N$, we have

$$\|w_n\| = \sup_{t \leq T_0(w_n)} |w_n(t)| \tag{5.6}$$

$$\leq \sup_{t \leq T_0(w_\infty)+1} |w_\infty(I_n(t))| + \sup_{t \leq T_0(w_\infty)+1} |w_n(t) - w_\infty(I_n(t))| \tag{5.7}$$

$$\leq \|w_\infty\| + \varepsilon, \tag{5.8}$$

and also we have

$$\|w_\infty\| = \sup_{t \leq T_0(w_\infty)} |w_\infty(t)| \tag{5.9}$$

$$= \sup_{t \leq T_0(w_\infty)+1} |w_\infty(I_n(t))| \tag{5.10}$$

$$\leq \sup_{t \leq T_0(w_\infty)+1} |w_n(t)| + \sup_{t \leq T_0(w_\infty)+1} |w_n(t) - w_\infty(I_n(t))| \tag{5.11}$$

$$\leq \|w_n\| + \varepsilon. \tag{5.12}$$

Hence we obtain $|\|w_n\| - \|w_\infty\|| < \varepsilon$. The proof is complete. □

REMARK 5.2. We cannot remove the assumption $T_0(w_n) \to T_0(w_\infty)$ from Lemma 5.1. In fact, if we set

$$w_n(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 1/n & \text{if } 1 \leq t < n, \\ 2 & \text{if } n \leq t < n+1, \\ 0 & \text{if } t \geq n+1, \end{cases} \qquad w_\infty(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1, \end{cases} \tag{5.13}$$

then we have $w_n \to w_\infty$ in D but $\|w_n\| = 2$ and $\|w_\infty\| = 1$.

The following lemma is partly taken from Bartle [**3**].

LEMMA 5.3 (Bartle [**3**]). *Let $f_1, f_2, \dots, f_\infty$ be real-valued functions defined on a measure space $(E, \mathcal{E}, \nu)$. Then the following statements are equivalent*:

(i) $f_n \to f_\infty$, *ν-almost uniformly*;
(ii) $\nu\big(\bigcup_{n=N}^{\infty}\{|f_n - f_\infty| \geq r\}\big) \to 0$ *as* $N \to \infty$ *for all* $r > 0$;
(iii) $f_n \to f_\infty$, *ν-a.e. and for any $r > 0$ there exists N such that* $\nu\big(\bigcup_{n=N}^{\infty}\{|f_n - f_\infty| \geq r\}\big) < \infty$.

The proof can be found in Bartle [**3**, Theorem 1.7], and so we omit it.

LEMMA 5.4. *Let ν be a σ-finite measure on a Polish space E and let $\Phi_1, \Phi_2, \dots, \Phi_\infty$ be measurable mappings from E to D such that we have $\Phi_n \to \Phi_\infty$ in D, ν-a.e. and $T_0(\Phi_n) \to T_0(\Phi_\infty)$, ν-a.e. Suppose that $\nu\big(\|\Phi_n\| \geq r\big) < \infty$ for all $n = 1, 2, \dots, \infty$ and all $r > 0$. Then the following statements are equivalent*:

(i) $\|\Phi_n\| \to \|\Phi_\infty\|$, *$\nu$-almost uniformly*;
(ii) $\nu\big(\bigcup_{n=1}^{\infty}\{\|\Phi_n\| \geq r\}\big) < \infty$ *for all* $r > 0$.

PROOF. For $r > 0$ and $n \in \mathbb{N} \cup \{\infty\}$, we write

$$A_n^r = \big\{\|\Phi_n\| \geq r\big\} \quad \text{and} \quad B_n^r = \big\{\big|\|\Phi_n\| - \|\Phi_\infty\|\big| \geq r\big\}. \tag{5.14}$$

Suppose (i) is satisfied. Let $r > 0$ be fixed. We then see that $\nu\big(\bigcup_{n=N}^{\infty} B_n^{r/2}\big) < \infty$ for some N by Lemma 5.3. Since we have $\nu\big(\bigcup_{n=1}^{N} B_n^{r/2}\big) < \infty$ by the assumption, we obtain $\nu\big(\bigcup_{n=1}^{\infty} B_n^{r/2}\big) < \infty$. Since we have $\bigcup_{n=1}^{\infty} A_n^r \subset \big(\bigcup_{n=1}^{\infty} B_n^{r/2}\big) \cup A_\infty^{r/2}$, we see that (ii) is satisfied.

Suppose (ii) is satisfied. Let $r > 0$ be fixed. We then see that $\nu\big(\bigcup_{n=1}^{\infty} B_n^r\big) \leq \nu\big(\bigcup_{n=1}^{\infty} A_n^{r/2}\big) + \nu\big(A_\infty^{r/2}\big) < \infty$. Note that, by Lemma 5.1, we have $\|\Phi_n\| \to \|\Phi_\infty\|$, ν-a.e. Hence, by Lemma 5.3, we see that (i) is satisfied. □

5.3. General limit theorem.

We now proceed to the proof of Theorem 2.5.

PROOF OF THEOREM 2.5. Let E, ν, $\Phi_1, \Phi_2, \dots$ and Φ_∞ be as in Definition 2.3. Taking a subsequence if necessary, we may take $N = 1$ in Condition (G5) of Definition 2.3.

Let $p = (p^{(l)})_{l \in D(p)}$ be a Poisson point process on E with characteristic measure ν. For $n \in \mathbb{N} \cup \{\infty\}$, we define

$$q_n^{(l)} = \begin{cases} \Phi_n(p^{(l)}) & \text{if } l \in D(p), \\ o & \text{otherwise.} \end{cases} \tag{5.15}$$

We then see that $q_n = (q_n^{(l)})_{l \geq 0}$ is a Poisson point process on D outside o with characteristic measure $(\nu \circ \Phi_n^{-1})|_{D \setminus \{o\}}$, which is equal to $\boldsymbol{n}_n$ by Assumption (A2) and Condition

(G1). Hence q_n is a realization of p_n, so that we may assume without loss of generality that $p_n = q_n$.

For $m \in \mathbb{N}$, we set

$$\Lambda_m = \{l \in [0,m] : \|p_n^{(l)}\| \geq 1/m \text{ for some } n \in \mathbb{N} \cup \{\infty\}\}. \tag{5.16}$$

By Assumption (A2) and Condition (G3)′, we see that $\sharp\Lambda_m < \infty$ a.s. for all $m \in \mathbb{N}$. By this fact and by the assumptions (A1)–(A3), we see that there exists an event Ω^* of probability one such that for any sample point belonging to Ω^* we have the following:

(L1) for any $l \geq 0$, $p_n^{(l)} \to p_\infty^{(l)}$ in D;
(L2) for any $l \geq 0$, $T_0(p_n^{(l)}) \to T_0(p_\infty^{(l)})$ in $[0,\infty]$;
(L3) for any $l \geq 0$, $\tau(l) := \sup_{n \geq 1} T_0(p_n^{(l)})$ satisfies $\sum_{s \leq l} \tau(s) < \infty$;
(L4) $p_\infty^{(l)}(T_0(p_\infty^{(l)})-) = 0$;
(L5) for any $m \in \mathbb{N}$, $\sharp\Lambda_m < \infty$.

In what follows we pick and fix a sample point belonging to Ω^*.

Since we have

$$\eta_n(l) = \varsigma_n l + \sum_{s \leq l} T_0(p_n^{(l)}), \tag{5.17}$$

we see that, for any $l_0 > 0$,

$$G_n(l_0) := \sup_{l \leq l_0} |\eta_n(l) - \eta_\infty(l)| \tag{5.18}$$

$$\leq |\varsigma_n - \varsigma_\infty| l_0 + \sum_{s \leq l_0} \left|T_0(p_n^{(l)}) - T_0(p_\infty^{(l)})\right|. \tag{5.19}$$

By (L2)–(L3) and by (A4), we apply the dominated convergence theorem to see that

$$G_n(l_0) \to 0 \quad \text{as } n \to \infty \text{ for all } l_0 > 0. \tag{5.20}$$

Hence we obtain $\eta_n \to \eta_\infty$ in D. Moreover, since η_∞ is strictly increasing, we use [**21**, Theorem 7.2] to obtain

$$H_n(t_0) := \sup_{t \leq t_0} |L_n(t) - L_\infty(t)| \to 0 \tag{5.21}$$

as $n \to \infty$ for all $t_0 > 0$. Hence we obtain $L_n \to L_\infty$ in D.

It remains to prove that $X_n \to X_\infty$ in D. So we take an arbitrary subsequence and denote it by the same symbol as the original sequence. It suffices to prove that we can extract a further subsequence along which $X_n \to X_\infty$ in D. We divide the proof into several steps.

Step 1: For $l \geq 0$, since $p_n^{(l)} \to p_\infty^{(l)}$ in D, we see that there exist transformations $I_1^{(l)}, I_2^{(l)}, \dots, I_\infty^{(l)}$ of $[0,\infty)$ such that each $I_n^{(l)}$ is bijective, continuous and increasing and that we have

$$F_n^{(l)}(t_0) := \sup_{t \le t_0} \left| I_n^{(l)}(t) - t \right| + \sup_{t \le t_0} \left| p_n^{(l)}(t) - p_\infty^{(l)}(I_n^{(l)}(t)) \right| \xrightarrow[n\to\infty]{} 0 \tag{5.22}$$

for all $t_0 > 0$. For $n \in \mathbb{N}$ and $m \in \mathbb{N}$, we set

$$F_n(l) = F_n^{(l)}(T_0(p_\infty^{(l)}) + 3) \quad \text{for } l \ge 0 \tag{5.23}$$

and set

$$M_{n,m} = \max_{l \in \Lambda_m} \left\{ F_n(l) + G_n(l) + \left| T_0(p_n^{(l)}) - T_0(p_\infty^{(l)}) \right| \right\}. \tag{5.24}$$

By (5.22), (5.20), (L2) and (L5), we see that $M_{n,m} \to 0$ as $n \to \infty$ for all fixed $m \in \mathbb{N}$. Thus we may take a subsequence $\{n_1(n)\}_{n\in\mathbb{N}}$ such that $M_{n_1(n),n} < 1/n$ for all $n \in \mathbb{N}$. Writing p_n simply for $p_{n_1(n)}$, we may assume without loss of generality that $M_n := M_{n,n} < 1/n$ for all $n \in \mathbb{N}$.

Step 2: We construct a transformation I_n of $[0, \infty)$.

We modify the transformation $I_n^{(l)}$ around $t = T_0(p_n^{(l)})$. Note that

$$|T_0(p_n^{(l)}) - T_0(p_\infty^{(l)})| \le M_n < 1/n \tag{5.25}$$

and that

$$T_0(p_n^{(l)}) + 2/n < T_0(p_\infty^{(l)}) + 3/n \le T_0(p_\infty^{(l)}) + 3. \tag{5.26}$$

Since $F_n(l) \le M_n < 1/n$, we have

$$s_n^{(l)} := I_n^{(l)}(T_0(p_n^{(l)}) - 2/n) \le (T_0(p_n^{(l)}) - 2/n) + M_n < T_0(p_\infty^{(l)}), \tag{5.27}$$

$$t_n^{(l)} := I_n^{(l)}(T_0(p_n^{(l)}) + 2/n) \ge (T_0(p_n^{(l)}) + 2/n) - M_n > T_0(p_\infty^{(l)}). \tag{5.28}$$

We define

$$\widetilde{I}_n^{(l)}(t) = \begin{cases} 0 & \text{if } t = 0, \\ I_n^{(l)}(t) & \text{if } |t - T_0(p_n^{(l)})| \ge 2/n, \\ T_0(p_\infty^{(l)}) & \text{if } t = T_0(p_n^{(l)}), \\ \text{linear} & \text{otherwise.} \end{cases} \tag{5.29}$$

Since $s_n^{(l)} < T_0(p_\infty^{(l)}) < t_n^{(l)}$, we see that $\widetilde{I}_n^{(l)}$ is well-defined, bijective, increasing and continuous and that

$$\widetilde{I}_n^{(l)}\big(T_0(p_n^{(l)})\big) = T_0(p_\infty^{(l)}). \tag{5.30}$$

We note that

$$\sup_{t\geq 0}\left|\widetilde{I}_n^{(l)}(t)-I_n^{(l)}(t)\right| = \left|T_0(p_\infty^{(l)})-I_n^{(l)}(T_0(p_n^{(l)}))\right| \tag{5.31}$$

$$\leq \left|T_0(p_\infty^{(l)})-T_0(p_n^{(l)})\right| + M_n < 2/n. \tag{5.32}$$

Denote $l_n=\max\Lambda_n$ if $\Lambda_n\neq\emptyset$ and $l_n=0$ if $\Lambda_n=\emptyset$. We now define

$$I_n(t)=\begin{cases}0 & \text{if } t=0,\\ \eta_\infty(l-)+\widetilde{I}_n^{(l)}(t-\eta_n(l-)) & \text{if } \eta_n(l-)\leq t<\eta_n(l) \text{ for some } l\in\Lambda_n,\\ t-\eta_n(l_n)+\eta_\infty(l_n) & \text{if } t\geq \eta_n(l_n),\\ \text{linear} & \text{otherwise.}\end{cases} \tag{5.33}$$

We then see that I_n is bijective, increasing and continuous.

Step 3: We prove that $I_n\to I$ uc. Let $t_0>0$ be fixed. Since $L_n(t_0)\to L_\infty(t_0)$, there exists N such that $L_n(t_0)\leq L_\infty(t_0)+1$ for all $n\geq N$.

If t is such that $\eta_n(l-)\leq t\leq\eta_n(l)$ for some $l\in\Lambda_n$, we have, for $n\geq N$,

$$|I_n(t)-t|\leq|\eta_\infty(l-)-\eta_n(l-)|+\left|\widetilde{I}_n^{(l)}(t-\eta_n(l-))-(t-\eta_n(l-))\right| \tag{5.34}$$

$$\leq \sup_{u\leq L_n(t_0)}|\eta_\infty(u)-\eta_n(u)|+\sup_{s\leq T_0(p_n^{(l)})}\left|I_n^{(l)}(s)-s\right|+2/n \tag{5.35}$$

$$\leq G_n(L_\infty(t_0)+1)+\sup_{s\leq T_0(p_\infty^{(l)})+1/n}\left|I_n^{(l)}(s)-s\right|+2/n \tag{5.36}$$

$$\leq G_n(L_\infty(t_0)+1)+3/n. \tag{5.37}$$

Otherwise, we have, by linearity,

$$|I_n(t)-t|\leq\sup_{u\leq L_n(t_0)}|\eta_\infty(u)-\eta_n(u)|\leq G_n(L_\infty(t_0)+1). \tag{5.38}$$

Thus we obtain

$$\sup_{t\leq t_0}|I_n(t)-t|\leq G_n(L_\infty(t_0)+1)+3/n\xrightarrow[n\to\infty]{}0. \tag{5.39}$$

This shows that $I_n\to I$ uc.

Step 4: Let $m\in\mathbb{N}$ be fixed. For $n\geq m$ and for $l\in\Lambda_m$ ($\subset\Lambda_n$), we estimate the supremum over $t\geq 0$ of

$$K_n^{(l)}(t):=\left|p_n^{(l)}(t)-p_\infty^{(l)}(\widetilde{I}_n^{(l)}(t))\right|. \tag{5.40}$$

If $t\leq T_0(p_n^{(l)})-2/n$, we have $t\leq T_0(p_\infty^{(l)})+3$, and hence we have

$$K_n^{(l)}(t) = \left|p_n^{(l)}(t) - p_\infty^{(l)}(I_n^{(l)}(t))\right| \le M_n < 1/n. \tag{5.41}$$

If $T_0(p_n^{(l)}) - 2/n < t < T_0(p_n^{(l)})$, we have $t < T_0(p_\infty^{(l)}) + 1/n$ and

$$\widetilde{I}_n^{(l)}(t) \ge I_n^{(l)}(t) - 2/n \ge t - 3/n \ge T_0(p_\infty^{(l)}) - 4/n, \tag{5.42}$$

which yields

$$K_n^{(l)}(t) \le \left|p_n^{(l)}(t) - p_\infty^{(l)}(I_n^{(l)}(t))\right| + \left|p_\infty^{(l)}(I_n^{(l)}(t))\right| + \left|p_\infty^{(l)}(\widetilde{I}_n^{(l)}(t))\right| \tag{5.43}$$

$$\le 1/n + 2 \sup_{s \ge T_0(p_\infty^{(l)}) - 4/n} \left|p_\infty^{(l)}(s)\right|. \tag{5.44}$$

If $t \ge T_0(p_n^{(l)})$, we have $p_n^{(l)}(t) = p_\infty^{(l)}(I_n^{(l)}(t)) = 0$, so that $K_n^{(l)}(t) = 0$.

Therefore we obtain

$$\sup_{t \ge 0} K_n^{(l)}(t) \le 1/n + 2 \sup_{s \ge T_0(p_\infty^{(l)}) - 4/n} \left|p_\infty^{(l)}(s)\right|, \tag{5.45}$$

which converges to 0 by (L4).

We now set

$$\widetilde{F}_n(l) = \sup_{t \le T_0(p_\infty^{(l)}) + 3} K_n^{(l)}(t) \tag{5.46}$$

and

$$\widetilde{M}_{n,m} = \max_{l \in \Lambda_m} \widetilde{F}_n(l). \tag{5.47}$$

We then have $\widetilde{M}_{n,m} \to 0$ as $n \to \infty$ for all fixed $m \in \mathbb{N}$. Hence we may take a subsequence $\{n_2(n)\}_{n \in \mathbb{N}}$ such that $\widetilde{M}_{n_2(n),n} < 1/n$ for all $n \in \mathbb{N}$. Writing p_n simply for $p_{n_2(n)}$, we may assume without loss of generality that $\widetilde{M}_n := \widetilde{M}_{n,n} < 1/n$ for all $n \in \mathbb{N}$.

Step 5: Let $t_0 > 0$. We estimate the supremum over $0 \le t \le t_0$ of

$$K_n(t) := \left|X_n(t) - X_\infty(I_n(t))\right|. \tag{5.48}$$

Taking a subsequence if necessary, we may assume without loss of generality that

$$\sup_{t \le t_0} |I_n(t) - t| < 1/n \quad \text{for all } n \in \mathbb{N}. \tag{5.49}$$

Let N be such that $L_n(t_0) \le L_\infty(t_0) + 1$ for $n \ge N$. We take N large enough to satisfy $N > L_\infty(t_0 + 1) + 1$. Let $n \ge N$.

Case 1: $\eta_n(l-) \le t < \eta_n(l)$ for some $l \in \Lambda_n$. Since $\eta_\infty(l-) \le I_n(t) < \eta_\infty(l)$, we have

$$K_n(t) = \left| p_n^{(l)}(t - \eta_n(l-)) - p_\infty^{(l)}(\widetilde{I}_n^{(l)}(t - \eta_n(l-))) \right| \le \widetilde{M}_n < 1/n. \tag{5.50}$$

Case 2: $\eta_n(l-) \le t < \eta_n(l)$ for some $l \notin \Lambda_n$. Since $n > L_\infty(t_0) + 1 \ge L_n(t_0)$, we have $\eta_n(n) > t_0 \ge t \ge \eta_n(l-)$. Hence we have $n > l$. Since $l \notin \Lambda_n$, we have $\|p_n^{(l)}\| < 1/n$. We now obtain

$$K_n(t) = \left| p_n^{(l)}(t - \eta_n(l-)) - p_\infty^{(l)}(\widetilde{I}_n^{(l)}(t - \eta_n(l-))) \right| \tag{5.51}$$

$$\le \|p_n^{(l)}\| + \|p_\infty^{(l)}\| \le \widetilde{M}_n < 2/n. \tag{5.52}$$

Case 3: there is no $l \ge 0$ such that $\eta_n(l-) \le t < \eta_n(l)$. In this case we have $X_n(t) = 0$. We divide the proof into three subcases.

Case 3-1: there is no $l \ge 0$ such that $\eta_\infty(l-) \le I_n(t) < \eta_\infty(l)$. In this case we have $X_\infty(I_n(t)) = 0$, so that $K_n(t) = 0$.

Case 3-2: $\eta_\infty(l-) \le I_n(t) < \eta_\infty(l)$ for some $l \ge 0$ with $\|p_\infty^{(l)}\| < 1/n$. In this case we have $|X_\infty(I_n(t))| < 1/n$, so that $K_n(t) < 1/n$.

Case 3-3: $\eta_\infty(l-) \le I_n(t) < \eta_\infty(l)$ for some $l \ge 0$ with $\|p_\infty^{(l)}\| \ge 1/n$. In this case, we have

$$l \le L_\infty(I_n(t)) \le L_\infty(t_0 + 1) < n, \tag{5.53}$$

so that we have $l \in \Lambda_n$. Thus we obtain

$$\sup_{t \le t_0} \left| p_n^{(l)}(t) - p_\infty^{(l)}(I_n^{(l)}(t)) \right| \le M_n < 1/n, \tag{5.54}$$

so that

$$\|p_n^{(l)}\| \ge \|p_\infty^{(l)}\| - \sup_{t \le t_0} \left| p_n^{(l)}(t) - p_\infty^{(l)}(I_n^{(l)}(t)) \right| > 1/n - 1/n = 0. \tag{5.55}$$

From this we see that $\eta_n(l-) < \eta_n(l)$ and by the definition (5.33) of I_n we see that $\eta_n(l-) \le s < \eta_n(l)$ implies $\eta_\infty(l-) \le I_n(s) < \eta_\infty(l)$. Thus we obtain $\eta_n(l-) \le t < \eta_n(l)$, which is a contradiction.

From all the arguments above, we obtain

$$\sup_{t \le t_0} K_n(t) \le 2/n. \tag{5.56}$$

This shows that $X_n - X_\infty \circ I_n \to 0$ uc.

The proof is therefore complete. □

6. Proof of the homogenization theorem.

6.1. Scaling property for local time and excursion measure.

Let us prove Lemma 3.4.

Proof of Lemma 3.4. We deal with processes under $\mathbb{P}_0$. Using (S2) and (3.1), we have

$$c^{-\alpha\kappa n} \int_0^{c^n t} 1_{\{X(s)=0\}} \mathrm{d}s \overset{\text{law}}{=} \int_0^t 1_{\{X(s)=0\}} \mathrm{d}s. \tag{6.1}$$

Hence, using (S1) and then using (6.1), we obtain

$$\int_0^t 1_{\{X(s)=0\}} \mathrm{d}s \overset{\text{law}}{=} \int_0^t 1_{\{(\Psi_\alpha^n X)(s)=0\}} \mathrm{d}s \tag{6.2}$$

$$= c^{-n} \int_0^{c^n t} 1_{\{X(s)=0\}} \mathrm{d}s \tag{6.3}$$

$$\overset{\text{law}}{=} c^{-(1-\alpha\kappa)n} \int_0^t 1_{\{X(s)=0\}} \mathrm{d}s. \tag{6.4}$$

Since $0 < \alpha\kappa < 1$ by (S0), the last quantity converges in law to 0 as $n \to \infty$. □

To prove Proposition 3.5 and for the later use, we prove the following lemma.

Lemma 6.1. *Let* $(\boldsymbol{n}, \varsigma)$ *be the pair consisting of a* σ*-finite measure on* D *and a non-negative constant* ς *and suppose that* $(\boldsymbol{n}, \varsigma)$ *satisfies Conditions* (N0)–(N3). *Let* c, α *and* γ *be positive constants. Let* $\{p, \mathbb{P}\}$ *be a Poisson point process on* D *outside* o *with characteristic measure* $\boldsymbol{n}$ *and denote*

$$\widetilde{p}^{(l)} = \Psi_\alpha p^{(c^\gamma l)}, \quad \widetilde{\boldsymbol{n}} = c^\gamma \boldsymbol{n} \circ \Psi_\alpha^{-1}, \quad \widetilde{\varsigma} = c^{-(1-\gamma)}\varsigma. \tag{6.5}$$

Then it holds that the pair $(\widetilde{\boldsymbol{n}}, \widetilde{\varsigma})$ *satisfies* (N0)–(N3), *that* $\{\widetilde{p}, \mathbb{P}\}$ *is a Poisson point process on* D *outside* o *with characteristic measure* $\widetilde{\boldsymbol{n}}$ *and that*

$$X(\widetilde{p}, \widetilde{\varsigma}) = \Psi_\alpha X(p, \varsigma), \quad L(\widetilde{p}, \widetilde{\varsigma}) = \Psi_\gamma L(p, \varsigma), \quad \eta(\widetilde{p}, \widetilde{\varsigma}) = \widehat{\Psi}_\gamma \eta(p, \varsigma). \tag{6.6}$$

Proof. For $l \geq 0$ and $A \in \mathcal{B}(D \setminus \{o\})$, we have

$$\sharp\{s \leq l : \widetilde{p}^{(s)} \in A\} = \sharp\{s \leq l : p^{(c^\gamma s)} \in \Psi_\alpha^{-1} A\} = N_{c^\gamma l}(\Psi_\alpha^{-1} A), \tag{6.7}$$

where we note that $\Psi_\alpha^{-1}\{o\} = \{o\}$. It is now obvious that $\widetilde{p}$ is again a Poisson point process on D outside o whose characteristic measure is equal to $\widetilde{\boldsymbol{n}} = c^\gamma \boldsymbol{n} \circ \Psi_\alpha^{-1}$. Hence we obtain

$$\widehat{\Psi}_\gamma \eta(p,\varsigma;l) = c^{-1}\Big\{\varsigma c^\gamma l + \sum_{s\le c^\gamma l} T_0(p^{(s)})\Big\} \tag{6.8}$$

$$= \widetilde{\varsigma} l + \sum_{s\le l} T_0(p^{(c^\gamma s)})/c \tag{6.9}$$

$$= \widetilde{\varsigma} l + \sum_{s\le l} T_0(\widetilde{p}^{(s)}), \tag{6.10}$$

which yields $\eta(\widetilde{p},\widetilde{\varsigma}) = \widehat{\Psi}_\gamma \eta(p,\varsigma)$. The other identities of (6.6) are now obvious. □

We now prove Proposition 3.5.

Proof of Proposition 3.5. Suppose that (S2)$'$ is satisfied. Let p be a Poisson point process on D outside o with characteristic measure $\boldsymbol{n}$. Denote $\widetilde{p}^{(l)} = \Psi_\alpha p^{(c^{\alpha\kappa} l)}$. By Lemma 6.1 and (S2)$'$, we see that $\widetilde{p}$ is equal in law to p. On one hand, we see that

$$\{X(\widetilde{p},0), L(\widetilde{p},0)\} \overset{\text{law}}{=} \{X(p,0), L(p,0)\}. \tag{6.11}$$

On the other hand, we see that

$$L(\widetilde{p},0;t) = \inf\left\{l\ge 0 : \sum_{s\le l} T_0(\Psi_\alpha p^{(c^{\alpha\kappa} s)}) > t\right\} \tag{6.12}$$

$$= \inf\left\{l\ge 0 : \sum_{s\le l} T_0(p^{(c^{\alpha\kappa} s)}) > ct\right\} \tag{6.13}$$

$$= c^{-\alpha\kappa} L(p,0;ct). \tag{6.14}$$

Since $\{X(p,0), L(p,0)\}$ is identical in law to the pair $\{X, L\}$ of the coordinate process X and its local time of 0 under $\mathbb{P}_0$, we obtain (S2).

Conversely, suppose that (S2) is satisfied. Let us write $\widetilde{X} = \Psi_\alpha X$ and $\widetilde{L} = \Psi_{\alpha\kappa} L$. It is then obvious that for any $t>0$

$$\int_0^t 1_{\{\widetilde{X}(s)\ne 0\}} \mathrm{d}\widetilde{L}(s) = c^{-\alpha\kappa}\int_0^{ct} 1_{\{X(s)\ne 0\}} \mathrm{d}L(s) = 0. \tag{6.15}$$

From this it follows that $\widetilde{L}$ is a choice of the local time of 0 for $\widetilde{X}$, and hence $\{\widetilde{X},\widetilde{L}\} \overset{\text{law}}{=} \{X, kL\}$ for some constant k. Since $\widetilde{L} \overset{\text{law}}{=} L$, we obtain $k=1$. We denote $\widetilde{\eta}$ for the right-continuous inverse of $\widetilde{L}$ and define

$$\widetilde{p}^{(l)}(t) = \begin{cases} \widetilde{X}(\widetilde{\eta}(l-)+t) & \text{if } 0\le t<\widetilde{\eta}(l)-\widetilde{\eta}(l-), \\ o & \text{if } t\ge \widetilde{\eta}(l)-\widetilde{\eta}(l-). \end{cases} \tag{6.16}$$

We then see that the point process $\widetilde{p} = (\widetilde{p}^{(l)})_{l\ge 0}$ is identical in law to the point process of excursions for X, which shows that the characteristic measure of $\widetilde{p}$ is $\boldsymbol{n}$. By Lemma 6.1, we see that the characteristic measure of $\widetilde{p}$ is equal to $c^{\alpha\kappa}\boldsymbol{n} \circ \Psi_\alpha^{-1}$. We thus obtain (S2)′.

The proof is now complete. □

6.2. The jumping-in vanishing case.

We prove Theorem 3.6.

Proof of Theorem 3.6. Set

$$\big(p_{\rho,j}^{(n)}\big)^{(l)} = \Psi_\alpha^n p_{\rho,j}^{(c^{\alpha\kappa n} l)}. \tag{6.17}$$

By Lemma 6.1, we see that the characteristic measure of $p_{\rho,j}^{(n)}$ is $\boldsymbol{n}_{\rho,j}^{(n)}$ and that

$$X_{\rho,j,\varsigma}^{(n)} = X\big(p_{\rho,j}^{(n)}, \varsigma^{(n)}\big), \quad L_{\rho,j,\varsigma}^{(n)} = L\big(p_{\rho,j}^{(n)}, \varsigma^{(n)}\big), \quad \eta_{\rho,j,\varsigma}^{(n)} = \eta\big(p_{\rho,j}^{(n)}, \varsigma^{(n)}\big). \tag{6.18}$$

Set

$$(\boldsymbol{n}_n, \varsigma_n) = \big(\boldsymbol{n}_{\rho,j}^{(n)}, \varsigma^{(n)}\big) \quad \text{and} \quad (\boldsymbol{n}_\infty, \varsigma_\infty) = (\boldsymbol{n}_{\rho^*,0}, 0) \tag{6.19}$$

and we would like to verify that all the assumptions of Theorem 2.5 are satisfied. It is obvious that (A1), (A3) and (A4) are satisfied. We have only to prove that (A2) is satisfied.

By the definition (3.11), we have

$$\boldsymbol{n}_{\rho,j}^{(n)} = c^{\alpha\kappa n}\boldsymbol{n}_{\rho,0} \circ (\Psi_\alpha^n)^{-1} + c^{\alpha\kappa n}\int_{S\setminus\{0\}} j(\mathrm{d}x)\mathbb{P}_x^0 \circ (\Psi_\alpha^n)^{-1}. \tag{6.20}$$

Using (3.11), (S2)′ and (S1), we have

$$\boldsymbol{n}_{\rho,j}^{(n)} = \boldsymbol{n}_{\rho,0} + c^{\alpha\kappa n}\int_{S\setminus\{0\}} j(\mathrm{d}x)\mathbb{P}_{x_n}^0, \tag{6.21}$$

where we write $x_n = c^{-\alpha n}x$. By (3.10) and (C2), we have

$$\boldsymbol{n}_{\psi(x_n)}(T_{x_n} < T_0) = \sigma_{\psi(x)}(x_n) = c^{\alpha\kappa n}\sigma_{\psi(x)}(x) > 0 \tag{6.22}$$

for j-a.e. $x \in S \setminus \{0\}$. Using Corollary 3.3 and formula (3.10), we have, for $A \in \mathcal{B}(D)$,

$$\boldsymbol{n}_{\rho,j}^{(n)}(A) = \boldsymbol{n}_{\rho,0}(A) + c^{\alpha\kappa n}\int_{S\setminus\{0\}} \frac{j(\mathrm{d}x)}{\sigma_{\psi(x_n)}(x_n)} \cdot \boldsymbol{n}_{\psi(x_n)}\big(\{T_{x_n} < T_0\} \cap \theta_{T_{x_n}}^{-1} A\big) \tag{6.23}$$

$$= \boldsymbol{n}_{\rho,0}(A) + \int_{S\setminus\{0\}} \frac{j(\mathrm{d}x)}{\sigma_{\psi(x)}(x)}\boldsymbol{n}_{\psi(x)}\big(\{T_{x_n} < T_0\} \cap \theta_{T_{x_n}}^{-1} A\big). \tag{6.24}$$

Let $E = S \times D$. We define a measure ν on E by

$$\nu(\mathrm{d}x\mathrm{d}w) = \delta_0(\mathrm{d}x)\boldsymbol{n}_{\rho,0}(\mathrm{d}w) + 1_{S\setminus\{0\}}(x)\frac{j(\mathrm{d}x)}{\sigma_{\psi(x)}(x)}\boldsymbol{n}_{\psi(x)}(\mathrm{d}w). \tag{6.25}$$

For $n \in \mathbb{N}$, we define $\Phi_n : E \to D$ by

$$\Phi_n(x,w) = \begin{cases} w & \text{if } x = 0, \\ \theta_{T_{x_n}}(w) & \text{if } T_{x_n}(w) < T_0(w), \\ o & \text{otherwise.} \end{cases} \tag{6.26}$$

For $n = \infty$, we define $\Phi_\infty : E \to D$ by $\Phi_\infty(x,w) = w$. It is obvious by (C4) that (G1) of Definition 2.3 is satisfied. By the definitions of Φ_n's and Φ_∞, we have $T_0(\Phi_n(x,w)) \le T_0(w)$, and hence we see that (G5) is satisfied. Since $\|\Phi_n(x,w)\| \le \|w\|$ in any case, we see that (G3)$'$ is satisfied.

Let us verify that (G2) and (G4) are satisfied. We deal only with ν-a.e. $(x,w) \in E$.

(i) For $x = 0$, we have, for all $n \in \mathbb{N}$,

$$\Phi_n(0,w) = w = \Phi_\infty(0,w), \tag{6.27}$$

so that we have

$$T_0(\Phi_n(0,w)) = T_0(w) = T_0(\Phi_\infty(0,w)). \tag{6.28}$$

(ii) For $x \ne 0$, we have

$$T_0(\Phi_n(x,w)) = T_0(w) - T_{x_n}(w) \to T_0(w) = T_0(\Phi_\infty(x,w)) \quad \boldsymbol{n}_{\psi(x)}\text{-a.e.} \tag{6.29}$$

For $n \in \mathbb{N}$, we define a transformation $I_n : [0,\infty) \to [0,\infty)$ by

$$I_n(t) = \begin{cases} \{1 + nT_{x_n}(w)\}t & \text{if } 0 \le t < 1/n, \\ t + T_{x_n}(w) & \text{if } t \ge 1/n. \end{cases} \tag{6.30}$$

Then we easily see that $I_n \to I$ uc. Since $\Phi_n(x,w)(t) = \Phi_\infty(x,w)(I_n(t))$ for $t \ge 1/n$ and since $w(0) = 0$, we obtain

$$\sup_{t\ge 0}\left|\Phi_n(x,w)(t) - \Phi_\infty(x,w)(I_n(t))\right| \le 2 \sup_{0\le t\le 1/n+T_{x_n}(w)} |w(t)| \to 0. \tag{6.31}$$

This shows that $\Phi_n(x,w) \to \Phi_\infty(x,w)$ in D. We therefore obtain that (G2) and (G4) are satisfied.

The proof is now complete. □

6.3. The jumping-in dominant case.

We prove Theorem 3.7.

Proof of Theorem 3.7. Set

$$\big(p^{(n)}_{\rho,j}\big)^{(l)} = \Psi^n_\alpha p^{(c^{\alpha\beta n} l)}_{\rho,j}. \tag{6.32}$$

By Lemma 6.1, we see that the characteristic measure of $p^{(n)}_{\rho,j}$ coincides with $\boldsymbol{n}^{(n)}_{\rho,j}$ and that

$$X^{(n)}_{\rho,j,\varsigma} = X\big(p^{(n)}_{\rho,j}, \varsigma^{(n)}\big), \quad L^{(n)}_{\rho,j,\varsigma} = L\big(p^{(n)}_{\rho,j}, \varsigma^{(n)}\big), \quad \eta^{(n)}_{\rho,j,\varsigma} = \eta\big(p^{(n)}_{\rho,j}, \varsigma^{(n)}\big). \tag{6.33}$$

Set

$$(\boldsymbol{n}_n, \varsigma_n) = \big(\boldsymbol{n}^{(n)}_{\rho,j}, \varsigma^{(n)}\big) \quad \text{and} \quad (\boldsymbol{n}_\infty, \varsigma_\infty) = (\boldsymbol{n}_{0,j^*}, 0) \tag{6.34}$$

and we would like to verify that all the assumptions of Theorem 2.5 are satisfied. It is obvious that (A1), (A3) and (A4) are satisfied. We have only to prove that (A2) is satisfied.

Using (3.11), (S2)′ and (S1), we have

$$\boldsymbol{n}^{(n)}_{\rho,j} = c^{\alpha\beta n} \boldsymbol{n}_{\rho,0} \circ (\Psi^n_\alpha)^{-1} + c^{\alpha\beta n} \int_{S\setminus\{0\}} j(\mathrm{d}x) \mathbb{P}^0_x \circ (\Psi^n_\alpha)^{-1} \tag{6.35}$$

$$= c^{-\alpha(\kappa-\beta)n} \boldsymbol{n}_{\rho,0} + c^{\alpha\beta n} \int_{S\setminus\{0\}} j(\mathrm{d}x) \mathbb{P}^0_{x_n}, \tag{6.36}$$

where we write $x_n = c^{-\alpha n} x$. Using (3.10), (C2), Corollary 3.3, formula (3.10) and (C5), we have, for $A \in \mathcal{B}(D)$,

$$\boldsymbol{n}^{(n)}_{\rho,j}(A) = c^{-\alpha(\kappa-\beta)n} \boldsymbol{n}_{\rho,0}(A) + c^{\alpha\beta n} \int_{S\setminus\{0\}} \frac{j(\mathrm{d}x)}{\sigma_{\psi(x)}(x_n)} \boldsymbol{n}_{\psi(x)}\big(\{T_{x_n} < T_0\} \cap \theta^{-1}_{T_{x_n}} A\big) \tag{6.37}$$

$$= c^{-\alpha(\kappa-\beta)n} \boldsymbol{n}_{\rho,0}(A) + \int_{S''} \mu(\mathrm{d}y) \boldsymbol{n}_{\psi(J^* y)}\big(\{T_{J_n y} < T_0\} \cap \theta^{-1}_{T_{J_n y}} A\big). \tag{6.38}$$

Let $E = ((0,\infty) \cup S'') \times D$. We define a measure ν on E by

$$\nu(\mathrm{d}y\mathrm{d}w) = 1_{(0,\infty)}(y)\mathrm{d}y \boldsymbol{n}_{\rho,0}(\mathrm{d}w) + 1_{S''}(y)\mu(\mathrm{d}y)\boldsymbol{n}_{\psi(J^* y)}(\mathrm{d}w). \tag{6.39}$$

For $n \in \mathbb{N}$, we define $\Phi_n : E \to D$ by

$$\Phi_n(y,w) = \begin{cases} w & \text{if } y \in (0, c^{-\alpha(\kappa-\beta)n}), \\ \theta_{T_{J_n y}}(w) & \text{if } y \in S'' \text{ and } T_{J_n y}(w) < T_0(w), \\ o & \text{otherwise.} \end{cases} \tag{6.40}$$

For $n = \infty$, we define $\Phi_\infty : E \to D$ by

$$\Phi_\infty(y,w) = \begin{cases} \theta_{T_{J^*y}}(w) & \text{if } y \in S'' \text{ and } T_{J^*y}(w) < T_0(w), \\ o & \text{otherwise.} \end{cases} \tag{6.41}$$

By the above argument and by (C6), we find that (G1) of Definition 2.3 is satisfied. By the definitions of Φ_n's and Φ_∞, we have $T_0(\Phi_n(y,w)) \le T_0(w)$, and hence we see that (G5) is satisfied. Since $\|\Phi_n(y,w)\| \le \|w\|$ in any case, we see that (G3)$'$ is satisfied.

Let us verify that (G2) and (G4) are satisfied. We deal only with ν-a.e. $(y,w) \in E$.

(i) For $y \in (0,\infty)$, we have

$$\Phi_n(y,w) = \begin{cases} w & \text{if } n < \dfrac{1}{\alpha(\kappa-\beta)\log c} \log \dfrac{1}{y}, \\ o & \text{otherwise,} \end{cases} \tag{6.42}$$

since $c > 1$. This shows that $\Phi_n(y,w) = o = \Phi_\infty(y,w)$ for large n.

(ii) For $y \in S''$, we have $T_{J_n y}(w) \to T_{J^*y}(w)$ by (C5). We define transformations $I_n : [0,\infty) \to [0,\infty)$ for $n \in \mathbb{N}$ as follows. Let

$$\tau_n(w) = T_{J_n y}(w) - T_{J^*y}(w) \quad \text{and} \quad \tau_n^\pm(w) = \max\{\pm\tau_n(w), 0\}. \tag{6.43}$$

For $n \in \mathbb{N}$, we define

$$I_n(t) = \begin{cases} \dfrac{1/n + \tau_n^+(w)}{1/n + \tau_n^-(w)} t & \text{if } 0 \le t < 1/n + \tau_n^-(w), \\ t + \tau_n(w) & \text{if } t \ge 1/n + \tau_n^-(w). \end{cases} \tag{6.44}$$

Then we easily see that $I_n \to I$ uc. Let us write

$$T^n_{\min}(w) = \min\{T_{J_n y}(w), T_{J^*y}(w)\}, \quad T^n_{\max}(w) = \max\{T_{J_n y}(w), T_{J^*y}(w)\}. \tag{6.45}$$

Since $\Phi_n(y,w)(t) = \Phi_\infty(y,w)(I_n(t))$ for $t \ge 1/n + \tau_n^-(w)$, we obtain

$$\sup_{t \ge 0} \left| \Phi_n(y,w)(t) - \Phi_\infty(y,w)(I_n(t)) \right| \le 2 \sup_{T^n_{\min}(w) \le t \le 1/n + T^n_{\min}(w)} |w(t)|. \tag{6.46}$$

The last quantity converges to 0 by (C5). This shows that $\Phi_n(y,w) \to \Phi_\infty(y,w)$ in D. We therefore obtain that (G2) and (G4) are satisfied.

The proof is now complete. □

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