J. Math. Soc. Japan
Vol. 67, No. 4 (2015) pp. 1631–1669
doi: 10.2969/jmsj/06741631

Minkowski content of the intersection of a Schramm-Loewner evolution (SLE) curve with the real line

By Gregory F. Lawler

Dedicated to the memory of Kiyosi Itô

(Received Nov. 4, 2014)
(Revised Mar. 13, 2015)

Abstract. The Schramm-Loewner evolution (SLE) is a probability measure on random fractal curves that arise as scaling limits of two-dimensional statistical physics systems. In this paper we survey some results about the Hausdorff dimension and Minkowski content of SLE_κ paths and then extend the recent work on Minkowski content to the intersection of an SLE path with the real line.

1. Introduction.

The Schramm-Loewner evolution (SLE) is a family of curves introduced by Oded Schramm [**21**] as a candidate for the scaling limit of discrete statistical physics models in two dimensions. We will consider chordal SLE_κ in simply connected domains in this paper for $0 < \kappa < 8$. (SLE_κ for $\kappa \geq 8$ is also interesting but it has two-dimensional plane-filling paths for which the questions discussed in this paper are not relevant.) This is a probability measure on curves connecting two distinct boundary points; by conformal invariance, it suffices to define it in the upper half plane $\mathbb{H}$ with boundary points 0 and ∞. There is a one-to-one relationship between $\kappa \in (0, 8)$ and the fractal dimension $d_\kappa \in (1, 2)$; indeed, $d_\kappa = 1 + (\kappa/8)$. In this paper, we first review a number of recent results about the fractal behavior of the paths. Then we will use the ideas to prove a new result about the Minkowski content of the "real points" of an SLE curve.

We will parametrize SLE_κ in a way that is most convenient for doing the stochastic calculus computations. Throughout this paper, we let $0 < \kappa < 8$ and $a = 2/\kappa \in (1/4, \infty)$. Then chordal SLE_κ from 0 to ∞ can be defined as the random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ such that the following holds. We will write γ_t for $\gamma[0, t]$ and let H_t be the unbounded component of $\mathbb{H} \setminus \gamma_t$. Let $g_t : H_t \to \mathbb{H}$ be the conformal transformation with $g_t(z) - z = o(1)$, as $z \to \infty$. Then g_t satisfies the Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1}$$

where $U_t = -W_t$ is a standard (real) Brownian motion. For every $z \in \mathbb{C} \setminus \{0\}$, the solution to (1) exists up to time $T_z \in (0, \infty]$ satisfying $g_t(\overline{z}) = \overline{g_t(z)}$ and $T_{\overline{z}} = T_z$. Then

---

2010 *Mathematics Subject Classification.* Primary 60J67.
Key Words and Phrases. Schramm-Loewner evolution, Hausdorff dimension, Minkowski content.
This research was supported by National Science Foundation grant DMS-0907143.

$$H_t = \{z \in \mathbb{H} : T_z > t\}.$$

Under our parametrization,

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

More generally, SLE_κ connecting distinct boundary points in a simply connected domain is obtained by transforming this measure by a conformal map. This is considered as a measure on paths modulo reparametrization.

The definition of γ as given is indirect. Instead of specifying the random dynamics of the curve γ, an equation is given for the flow away from the curve induced by the random conformal maps. If $z \in \mathbb{C} \setminus \{0\}$ and $Z_t = Z_t(z) = g_t(z) - U_t = g_t(z) + dW_t$, then the Loewner equation (1) becomes a stochastic differential equation

$$dZ_t = \frac{a}{Z_t}\, dt + dW_t.$$

This allows us to use the powerful techniques of Itô calculus to analyze g_t and Z_t. Understanding the curve γ is trickier because it is the moving boundary of the domain of a conformal map (equivalently, the boundary of the image of the inverse map). We will discuss properties of γ in this paper. Although $\gamma(t)$ itself does not satisfy an SDE, the analysis makes heavy use of traditional stochastic analysis on quantities such as Z_t.

Rohde and Schramm showed [**20**] that such a curve γ exists. Moreover, if $0 < \kappa \leq 4$, the curve is simple with $\gamma(0, \infty) \subset \mathbb{H}$ while if $4 \leq \kappa < 8$, the curve has double points and hits the real line. They also gave strong evidence, as well as a rigorous upper bound, that the fractal dimension of the path is $d = d_\kappa = 1 + (\kappa/8) = 1 + (1/4a)$. The basic idea is as follows. Suppose that the curves have fractal dimension d. Then we would expect that approximately ϵ^{-d} disks of radius ϵ would be needed to cover the intersection of γ with a fixed disk. This would indicate that the probability that a given disk of radius ϵ intersects γ should decay like ϵ^{2-d} as $\epsilon \to 0$. This leads to the following conjecture: there exists a function $G(z)$ such that

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathbb{P}\{\mathrm{dist}(z, \gamma) \leq \epsilon\} = G(z). \tag{2}$$

Using Itô's formula and scale invariance of SLE_κ, Rohde and Schramm showed that this would only be possible if $d = 1 + (\kappa/8)$ and

$$G(z) = \tilde{c}\, [\mathrm{Im}(z)]^{d-2}\, [\sin \arg(z)]^{4a-1}.$$

This function is often called the *chordal* SLE_κ *Green's function.* More generally, we define the Green's function for SLE_κ in simply connected D from w_1 to w_2 by

$$G_D(z; w_1, w_2) = |F'(z)|^{2-d}\, G(F(z)),$$

where $F : D \to \mathbb{H}$ is a conformal transformation with $F(w_1) = 0, F(w_2) = \infty$. We can also write

$$G_D(z; w_1, w_2) = \tilde{c}\, 2^{2-d}\, \mathrm{crad}_D(z)^{d-2}\, S(z; w_1, w_2)^{4a-1},$$

where crad denotes conformal radius, that is, $\mathrm{crad}_D(\zeta) = |f'(0)|$, where $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = \zeta$. Also, S denotes the conformally invariant quantity "sine of the angle of z with respect to w_1, w_2", that is, $\sin \arg F(z)$, with F as above. The Koebe 1/4-theorem [**5**, Theorem 2.3] and the Schwarz lemma imply that

$$\mathrm{dist}(\zeta, \partial D) \le \mathrm{crad}_D(\zeta) \le 4\ \mathrm{dist}(\zeta, \partial D).$$

While Rohde and Schramm did not establish the limit (2), it has since been established, first with distance replaced with conformal radius [**10**, Theorem 3.11]

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathbb{P}\{\mathrm{crad}(z, \mathbb{H} \setminus \gamma) \le \epsilon\} = G^*(z), \tag{3}$$

where $G^*(z) = c^*\, [\mathrm{Im}(z)]^{d-2}\, [\sin \arg(z)]^{4a-1}$ for an explicit $c^* = c^*_\kappa$, and later in [**14**] there is a proof of (2) with $G(z) = \tilde{c}\, [\mathrm{Im}(z)]^{d-2}\, [\sin \arg(z)]^{4a-1}$ without an explicit expression for $\tilde{c} = \tilde{c}_\kappa$.

There exists a standard technique for proving Hausdorff dimension of random curves. In order to apply it for SLE_κ one needs "second moment estimates", that is, estimation of

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon, \mathrm{dist}(w, \gamma) < \epsilon\}.$$

In analogy with (2), we might conjecture that there is a function $G(z, w)$ such that

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon, \mathrm{dist}(w, \gamma) < \epsilon\} \sim G(z, w)\, \epsilon^{2(2-d)}, \quad \epsilon \downarrow 0, \tag{4}$$

and a little thought will lead one to guess that $G(z, w) \asymp G(z)\, |z - w|^{d-2}$ as $w \to z$. This two-point estimate turns out to be tricky. For example, suppose that $\tau = \inf\{t : \mathrm{dist}(z, \gamma) < \epsilon\}$ and suppose that $\mathrm{dist}(w, \gamma_\tau) > \epsilon$. Then the conditional probability $\mathbb{P}\{\mathrm{dist}(w, \gamma) < \epsilon \mid \gamma_\tau\}$ depends very strongly on the curve γ_τ. Indeed there is no nontrivial uniform upper bound on this conditional probability. Beffara [**4**] was the first to show that for z, w not too close to the real line

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon, \mathrm{dist}(w, \gamma) < \epsilon\} \asymp \epsilon^{2(2-d)}\, |z - w|^{d-2},$$

and proved that the Hausdorff dimension of the paths is d with probability one. An alternative approach to this theorem with extensions to exceptional points of the path using the "reverse Loewner flow" can be found in [**9**], [**11**]. Beffara's two-point estimate was refined in [**16**] where it is shown that there exists a function $G(z, w)$ such that the "conformal radius" version of (4) holds,

$$\mathbb{P}\{\mathrm{crad}_{\mathbb{H}\setminus\gamma}(z) < \epsilon, \mathrm{crad}_{\mathbb{H}\setminus\gamma}(w) < \epsilon\} \sim G^*(z,w)\,\epsilon^{2(2-d)}, \quad \epsilon \downarrow 0.$$

The function $G^*(z,w)$ was not given explicitly. However, it was described in terms of a process called *two-sided radial* SLE_κ (*from* 0 *to* ∞ *through* z *stopped when it reaches* z). Two-sided radial SLE_κ is the limiting distribution of SLE_κ conditioned so that $\mathrm{crad}_{\mathbb{H}\setminus\gamma}(z) < \epsilon$. It can be defined precisely in terms of a particular local martingale and Girsanov's theorem. Roughly speaking, if a path is going to get very close to both z and w it is going to first get very close to one of them and then will get very close to the other. What the path does not do (proving this is one of the hardest steps in these estimates!) is to go back and forth getting close to z, then close to w, then even closer to z, then even closer to w, etc. It is proved that

$$\begin{aligned}\lim_{\epsilon\downarrow 0} \epsilon^{2(d-2)}\,&\mathbb{P}\{\mathrm{crad}_{\mathbb{H}\setminus\gamma}(z) < \epsilon, \mathrm{crad}_{\mathbb{H}\setminus\gamma}(w) < \epsilon\}\\ &= G^*(z)\,\mathbb{E}^*_z\big[G^*_{H_{T_z}}(w)\big] + G^*(w)\,\mathbb{E}^*_w\big[G^*_{H_{T_w}}(z)\big].\end{aligned} \tag{5}$$

We read the right hand side as the (normalized) "probability to go through z times the conditional probability to go through w afterwards given that the path goes through z" plus the analogous term with z and w switched. Here $\mathbb{E}^*_z$ denotes the measure of two-sided radial going through z and T_z is the time that it reaches z. The process reaches z with probability one in this new measure which shows that the measure is singular with respect to chordal SLE which does not hit points. Using the results in [**14**], we can derive (4) with $G(z,w) = c\,G^*(z,w)$. While the function $G(z,w)$ is not known explicitly, one can give an explicit function $\phi(z,w)$ such that $G(z,w) \asymp \phi(z,w)$; in particular, one can show that $G(z,w) \asymp G(z)\,|z-w|^{d-2}$ as $w \to z$.

The technique for establishing lower bounds for Hausdorff dimension is to find a measure supported on the curve that is "d-dimensional"; such measures are sometimes called Frostman measures. The one and two point estimates suffice for showing tightness of a sequence of approximating measures, and then a subsequential limit is taken. This subsequential limit establishes the Hausdorff dimension, but it would be nice to be able to show that there is a limit measure. The measure would be the *d-dimensional Minkowski content* given for a subset $V \subset \mathbb{C}$ by

$$\mathrm{Cont}_d[V] = \lim_{\epsilon\downarrow 0} \epsilon^{d-2}\,\mathrm{Area}\{z : \mathrm{dist}(z,V) < \epsilon\},$$

provided that the limit exists. Suppose that γ is an SLE_κ path in $\mathbb{C}$ and that $\mathrm{Cont}_d[\gamma_t]$ exists for all t such that $t \mapsto \mathrm{Cont}_d[\gamma_t]$ is continuous and strictly increasing. Then the curve could be reparametrized so that $\mathrm{Cont}_d[\gamma_t] = t$ for all t. This is called the *natural parametrization* of SLE_κ and is conjectured (but has not been proved) to be the scaling limit of the discrete paths given by the number of steps of a path.

Establishing the existence of the limit is nontrival for SLE. Suppose for a moment that it did exist. Let V be a disk in $\mathbb{C}$ and let $Y_t = \mathrm{Cont}(\gamma_t \cap V)$, $Y_\infty = \lim_{t\to\infty} Y_t = \mathrm{Cont}(\gamma \cap V)$. Using (2) and (4), we see that

$$\mathbb{E}[Y] = \int_V G(z)\, dA(z), \quad \mathbb{E}[Y^2] = \int_V \int_V G(z,w)\, dA(w)\, dA(z),$$

where dA denotes integration with respect to area. Using the Markov property, we see that

$$\mathbb{E}[Y \mid \gamma_t] = Y_t + \mathbb{E}[\mathrm{Cont}(\gamma[t,\infty) \cap V) \mid \gamma_t] = Y_t + L_t, \tag{6}$$

where

$$L_t = \int_{V \setminus \gamma_t} G_{H_t}(z; \gamma(t), \infty)\, dA(z).$$

This gives a characterization of Y_t as the unique increasing process such that $Y_t + L_t$ is a martingale. In [**15**], [**17**], the natural parametrization was constructed using this Doob–Meyer decomposition of the supermartingale L_t. More recently, in [**14**] the Minkowski content for SLE_κ was established: with probability one, $\mathrm{Cont}[\gamma_t]$ exists and gives a continuous, strictly increasing function of t. We will adapt the idea of that paper to prove a (somewhat easier) boundary version of this result. Before discussing the boundary questions, we make a few remarks about the Minkowski content.

- Implicit in the assumption (6) is the additivity of the Minkowski content on the path. A necessary property of SLE_κ in order for the proof to work is that the fractal dimension of the set of double points of the path is strictly less than d. For $\kappa \le 4$, this is obvious since the paths are simple, but it is also true for $4 < \kappa < 8$. See [**18**] for more information on the dimension of the set of double points.
- We could also define the 2-dimensional Minkowski content of a Brownian path in $\mathbb{R}^d$. If $d \le 2$, this turns out to be zero, but for $d \ge 3$ it is nonzero and in fact grows linearly in time. (This follows, e.g., from estimates in [**8**, p.252].) One needs $d \ge 3$ for the double points of Brownian motion to have strictly smaller dimension than two.
- The Hausdorff d-dimensional measure of SLE_κ paths is zero [**19**]. This is typical for random fractals. Roughly speaking, the Hausdorff measure is obtained by finding optimal coverings by balls of radius less than or equal to ϵ while the Minkowski content covers by balls of radius exactly ϵ. For deterministic self-similar fractals, this often does not make a significant difference, but for realizations of random fractals one can find better coverings by allowing the radius to vary.
- Similarly, the Hausdorff 2-dimensional measure of Brownian paths is zero. For Brownian motion, one can find the gauge function such that the Hausdorff measure with the gauge function is nontrivial. (See, e.g., [**23**].) The main part of the analysis is to find the upper tail for the amount of time that a Brownian motion spends in a disk. This is a relatively straightforward eigenvalue computation.
- We do not know if there is such a gauge function for the SLE_κ paths. To start to analyze this, we would need the upper tail for the occupation measure of a disk. This is much more difficult for SLE than for Brownian motion since the distribution of the future depends on the past and we do not know what an SLE path looks like given that it has spent a long time in a disk.

We now consider the equivalent boundary questions. Suppose $x > 0$. Find the exponent β and function $\tilde{G}(x)$ such that

$$\mathbb{P}\{\mathrm{dist}(x,\gamma) \leq \epsilon\} \sim \hat{G}(x)\,\epsilon^{\beta}, \quad \epsilon \downarrow 0.$$

Scaling implies that $\mathbb{P}\{\mathrm{dist}(x,\gamma) \leq \epsilon\, x\} = \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \epsilon\}$ and hence the function $\hat{G}(x)$ would have to be $\hat{c}\, x^{-\beta}$ for some $\hat{c}$. For $4 < \kappa < 8$ we could consider a similar quantity

$$\mathbb{P}\{\gamma \cap [x-\epsilon, x+\epsilon] \neq \emptyset\}.$$

It has been known for a while, see, e.g., [**1**], that

$$\beta = 4a - 1 = \frac{8}{\kappa} - 1, \tag{7}$$

and that $\mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \epsilon\} \asymp \epsilon^{\beta}$. However, we do not believe that the limit has been shown to exist. This is the first step to giving the Minkowski content of the set and is the first theorem of this paper.

THEOREM 1. *If $0 < \kappa < 8$, there exists $\hat{c} = \hat{c}_\kappa \in (0,\infty)$ such that if $x \in \mathbb{R} \setminus \{0\}$ and γ is an* SLE_κ *path from 0 to ∞, then*

$$\mathbb{P}\{\mathrm{dist}(x,\gamma) \leq \epsilon\,|x|\} = \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \epsilon\} \sim \hat{c}\,\epsilon^{\beta}, \quad \epsilon \downarrow 0, \tag{8}$$

In fact, there exist $c < \infty$, $0 < \alpha \leq 1$ such that for all $\epsilon > 0$,

$$\left|\epsilon^{-\beta}\,\mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \epsilon\} - \hat{c}\right| \leq c\,\epsilon^{\alpha}. \tag{9}$$

Here $\beta = (8/\kappa) - 1$.

It will be easier to get a "conformal radius" version of this result first. Since x is a boundary point of H_t, we cannot talk of the conformal radius of H_t with respect to 1. Instead, we will reflect our domain and let

$$D_t = H_t \cup \{\overline{z} : z \in H_t\} \cup \{x > 0 : T_x > t\}. \tag{10}$$

If $t < T_x$, then x is an interior point of D_t and we can let

$$\Upsilon_t(x) = \frac{1}{4}\,\mathrm{crad}_{D_t}(x).$$

The factor $1/4$ is put in so that $\Upsilon_0(x) = x$. The conformal radius is closely related to the distance to the boundary; indeed, the Koebe 1/4 theorem and the Schwarz lemma imply that

$$\frac{1}{4}\,\mathrm{dist}(x,\partial D_t) \leq \Upsilon_t(x) \leq \mathrm{dist}(x,\partial D_t). \tag{11}$$

We write $\Upsilon_t = \Upsilon_t(1)$. As a preliminary step in proving this theorem, we will show that

$$\mathbb{P}\{\Upsilon_\infty \leq \epsilon\} \sim c'\,\epsilon^\beta, \tag{12}$$

for an explicit value of c'.

In [**2**], Alberts and Sheffield used the relation $\mathbb{P}\{\gamma \cap [1-\epsilon, 1+\epsilon] \neq \emptyset\} \asymp \epsilon^\beta$ to study the dimension of $\gamma \cap \mathbb{R}$ for $4 < \kappa < 8$. (Actually, in this case, the exact probability is known.) The expected number of intervals of length ϵ needed to cover $\gamma \cap [1,2]$ is $\epsilon^{\beta-1}$ which predicts a fractal dimension of $1-\beta = 2-(8/\kappa)$. They used the standard method to prove that the Hausdorff dimension is indeed $1-\beta$. The main step is to get a two-point estimate. In analogy to the interior point situation, we would expect that there is a function $\tilde{G}(x,y)$ for $x < y$, such that

$$\mathbb{P}\{\mathrm{dist}(x,\gamma) < \epsilon, \mathrm{dist}(y,\gamma) < \epsilon\} \sim \tilde{G}(x,y)\,\epsilon^{-2\beta},$$

and scaling would imply that

$$\tilde{G}(r, r(1+x)) = r^{-2\beta}\tilde{G}(1, 1+x) =: \phi(x).$$

Also, a heuristic estimate would give $\phi(x) \asymp x^{-\beta}$. Alberts and Sheffield showed that $\mathbb{P}\{\mathrm{dist}(1,\gamma) < \epsilon, \mathrm{dist}(1+x,\gamma) < \epsilon\} \asymp x^{-\beta}\,\epsilon^{2\beta}$, which allowed them to prove the result. The function ϕ was found by Schramm and Zhou [**22**] to be a constant times $x^{-\beta}\,h(x)$ where $h = h_\kappa$ is the hypergeometric function

$$h(x) = \frac{\Gamma(2a)\,\Gamma(8a-1)}{\Gamma(4a)\,\Gamma(6a-1)}\,{}_2F_1(2a, 1-4a; 4a; x). \tag{13}$$

The constant is put in so that $h(1) = 1$ and hence

$$h(0) = \frac{\Gamma(2a)\,\Gamma(8a-1)}{\Gamma(4a)\,\Gamma(6a-1)}.$$

THEOREM 2. *If $0 < \kappa < 8$, and $x > 0$, then*

$$\lim_{\epsilon,\delta\downarrow 0} \epsilon^{-\beta}\,\delta^{-\beta}\mathbb{P}\{\Upsilon_\infty(1) \leq \epsilon, \Upsilon_\infty(1+x) \leq \delta\} = (c')^2\,x^{-\beta}\,h\left(\frac{x}{1+x}\right), \tag{14}$$

$$\lim_{\epsilon,\delta\downarrow 0} \epsilon^{-\beta}\,\delta^{-\beta}\mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \epsilon, \mathrm{dist}(1+x,\gamma) \leq \delta\} = \hat{c}^2\,x^{-\beta}\,h\left(\frac{x}{1+x}\right), \tag{15}$$

where $\hat{c}$ is as in Theorem 1, *c' is given in* (12), *and h is given in* (13).

We will also give another expression for the limit that is analogous to (5),

$$\lim_{\epsilon,\delta\downarrow 0} \epsilon^{-\beta}\,\delta^{-\beta}\mathbb{P}\{\Upsilon_\infty(1) \leq \epsilon, \Upsilon_\infty(1+x) \leq \delta\} = (c')^2\,\mathbb{E}^*\left[\Upsilon_{T_1}(1+x)^{-\beta}\right], \tag{16}$$

where $\mathbb{E}^*$ denotes a measure called *two-sided chordal from* 0 *to* ∞ *through* 1 (*stopped when it reaches* 1).

One of the reasons to establish (9) is to establish the existence of the Minkowski content of $\gamma \cap \mathbb{R}$ for $4 < \kappa < 8$. The boundary analogues of the results in [**15**], [**17**] were carried out in [**3**]. However, the existence of the content was left open. For $y_1 < x < y_2$, there are two possible ways to define the content $\mathrm{Cont}_{1-\beta}(\gamma \cap [y_1, y_2])$,

$$Y[y_1, y_2] = \lim_{\epsilon \downarrow 0} \epsilon^{\beta-1}\, l\{x \in [y_1, y_2] : \mathrm{dist}(x, \gamma) \le \epsilon\}, \tag{17}$$

$$\tilde{Y}[y_1, y_2] = \lim_{\epsilon \downarrow 0} \epsilon^{\beta-1}\, l\{x \in [y_1, y_2] : \mathrm{dist}(x, \gamma \cap \mathbb{R}) \le \epsilon\},$$

where l denotes length. We will use the first, but our argument would work equally well for the second and $\tilde{Y}[y_1, y_2] = c\, Y[y_1, y_2]$ for some constant c.

THEOREM 3. *For every* $0 < y_1 < y_2 < \infty$, *the limit* (17) *exists with probability one. Moreover if* $Y = Y[y_1, y_2]$, *then*

$$\mathbb{E}[Y] = \int_{y_1}^{y_2} \tilde{G}(x)\, dx = \frac{\tilde{c}\,(y_2 - y_1)^{1-\beta}}{1-\beta}, \quad \mathbb{E}[Y^2] = 2\int_{y_1}^{y_2}\int_{x}^{y_2} \tilde{G}(x, y)\, dy\, dx.$$

The next two sections are dedicated to proving these theorems. The basic outline is close to that in [**14**], but is actually easier in the boundary case. We will rely very little on results of previous papers. Our hope is that this paper can be read as an introduction to the proof techniques used in this area.

We finish with two final sections. In Section 4, we discuss a particular one-dimensional SDE. This equation arises in the work here, and we discuss one way to establish the facts that we need. An important fact that we need is that the equation satisfies a form of a parabolic Harnack inequality. What this means in this simple example is that within one unit of time, independent of the starting position, the density of the processes is within multiplicative constants of the invariant density. The basic idea is fundamental to many of our arguments, and since this is not familiar to some probabilists, we explain how we can get this for many equations using known results about the Bessel process and Girsanov's theorem.

The final section proves some analogous results in the context of what are sometimes called $\mathrm{SLE}(\kappa, \rho)$ processes. These processes are obtained from Girsanov's theorem by tilting by an appropriate local martingale. We give some examples of how to get results; the goal here is to demonstrate a proof technique that has become standard in this area.

I would like to thank Dapeng Zhan and an anonymous referee for comments on this paper.

2. Proof of Theorem 1.

We will prove the stronger result (9). We fix $0 < \kappa < 8$ and let $a = 2/\kappa \in (1/4, \infty)$. All constants may depend on κ. We assume that $\gamma(t)$ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$ with corresponding conformal maps g_t satisfying (1) where $U_t = -W_t$ is a standard real

Brownian motion. We consider $g_t(z)$ defined for $z \in \mathbb{C} \setminus \{0\}$ up to time T_z. Let $T = T_1$ and $X_t = g_t(1) - U_t$ which satisfies

$$dX_t = \frac{a}{X_t}\, dt + dW_t, \quad 0 \le t < T, \tag{18}$$

with $X_0 = 1$. Let D_t be the reflected domain as in (10), so that

$$\mathbb{C} \setminus D_t = (-\infty, 0] \cup \{z \in \mathbb{C} : T_z \le t\},$$

and $\Upsilon_t = \mathrm{crad}_{D_t}(1)/4$. Let x_t denote the rightmost point of $[\mathbb{C} \setminus D_t] \cap \mathbb{R}$ and let

$$g(x_t) = \inf\{g(x) : x > 0 : T_x > t\}, \quad O_t = g_t(x_t) - U_t.$$

In the case $\kappa \le 4$, we view x_t as the prime end corresponding to the "positive" side of 0. The Loewner equation implies that

$$dO_t = \frac{a}{O_t}\, dt + dW_t.$$

If $4 < \kappa < 8$, we must interpret this as a reflected Bessel process. We also set

$$Y_t = X_t - O_t, \quad J_t = \frac{Y_t}{X_t}.$$

The scaling property of conformal radius implies that

$$\Upsilon_t = \frac{1}{4}\, \mathrm{crad}_{D_t}(1) = \frac{\mathrm{crad}_{g_t(D_t)}(X_t)}{4\, g_t'(1)} = \frac{X_t - O_t}{g_t'(1)} = \frac{Y_t}{g_t'(1)} = J_t\, \frac{X_t}{g_t'(1)}.$$

Using the Loewner equation (1) we see that

$$\partial_t Y_t = \frac{a}{X_t} - \frac{a}{O_t} = -\frac{a\, Y_t}{X_t\, O_t} = -\frac{a\, Y_t}{X_t^2\,(1 - J_t)}, \quad \text{if } O_t > 0, \tag{19}$$

$$\partial_t g_t'(1) = -\frac{a\, g_t'(1)}{X_t^2},$$

$$\partial_t \Upsilon_t = a\, \Upsilon_t \left[\frac{1}{X_t^2} - \frac{1}{O_t\, X_t}\right] = -a\, \Upsilon_t\, \frac{Y_t}{X_t^2\,(X_t - Y_t)} = -a\, \Upsilon_t\, \frac{J_t}{X_t^2\,(1 - J_t)}. \tag{20}$$

Here we write dX_t to denote a stochastic differential and $\partial_t Y_t$ to denote an actual derivative. Using Itô's formula, we can see that

$$M_t = g_t'(1)^{4a-1}\, X_t^{1-4a} = g_t'(1)^{\beta}\, X_t^{-\beta}, \quad t < T,$$

is a local martingale satisfying

$$dM_t = \frac{1-4a}{X_t}\, M_t\, dW_t, \quad M_0 = 1. \tag{21}$$

We will let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations with respect to the measure obtained by tilting by the local martingale M_t. To be precise, if $\epsilon > 0$, $\tau < T$ is a stopping time such that $\mathrm{dist}(1, \gamma_\tau) \geq \epsilon$, and Z is a random variable depending only on γ_τ, then

$$\mathbb{E}^*[Z] = \mathbb{E}[Z\, M_\tau].$$

By the Girsanov theorem and (21), there is a standard Brownian motion B_t with respect to $\mathbb{P}^*$ such that

$$dW_t = \frac{1-4a}{X_t}\, dt + dB_t, \quad t < T.$$

The process γ under the measure $\mathbb{P}^*$ is an example of what is called an SLE(κ, ρ) process. This particular case is called *two-sided chordal* (*from* 0 *to* ∞ *in* $\mathbb{H}$ *through* 1 *stopped when it reaches* 1) in [**17**].

Note that

$$dX_t = \frac{1-3a}{X_t}\, dt + dB_t.$$

Using Itô's formula and the product rule, we see that

$$dJ_t = \frac{J_t}{X_t^2}\left(1 - a - \frac{a}{1-J_t}\right) dt - \frac{J_t}{X_t}\, dW_t \tag{22}$$

$$= \frac{J_t}{X_t^2}\left(3a - \frac{a}{1-J_t}\right) dt - \frac{J_t}{X_t}\, dB_t. \tag{23}$$

This equation becomes nicer if we use a "radial parametrization" in which $\log \Upsilon_t$ decays linearly. Define the stopping times

$$\sigma(t) = \inf\{s : \Upsilon_s = e^{-at}\}. \tag{24}$$

On the event $\{\sigma(t) < \infty\}$ we define $\hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-at}$, $\hat{X}_t = X_{\sigma(t)}$, $\hat{J}_t = J_{\sigma(t)}$, $\hat{g}_t = g_{\sigma(t)}$, $\hat{D}_t = D_{\sigma(t)}$ and

$$\hat{M}_t = M_{\sigma(t)} = |\hat{g}_t'(z)|^\beta\, \hat{X}_t^{-\beta} = \hat{\Upsilon}_t^{-\beta}\, [\hat{J}_t\, X_t]^\beta\, X_t^{-\beta} = e^{at\beta}\, \hat{J}_t^\beta.$$

By the chain rule and (20), we see that

$$-a\, \hat{\Upsilon}_t = -a\, e^{-at} = \partial_t \hat{\Upsilon}_t = \dot{\sigma}(t)\, \frac{-a\, \hat{J}_t}{\hat{X}_t^2\, (1 - \hat{J}_t)}\, \hat{\Upsilon}_t,$$

and hence

$$\dot{\sigma}(t) = \frac{\hat{X}_t^2\,(1-\hat{J}_t)}{\hat{J}_t}.$$

Using this we can change time in the equation (23) to get

$$d\hat{J}_t = \big[2a - 3a\,\hat{J}_t\big]\,dt + \sqrt{\hat{J}_t(1-\hat{J}_t)}\,d\hat{B}_t, \tag{25}$$

for a standard Brownian motion $\hat{B}_t$. It is convenient to change variables in (25). If we write $\hat{J}_t = (1-\cos\hat{\Theta}_t)/2$, then Itô's formula shows that $\hat{\Theta}_t$, $0 < \hat{\Theta}_t \leq \pi$ satisfies

$$d\hat{\Theta}_t = \phi(\hat{\Theta}_t)\,dt + d\hat{B}_t, \quad \text{if } 0 < \hat{\Theta}_t < \pi, \tag{26}$$

where

$$\phi(\theta) = \left(3a - \frac{1}{2}\right)\cot\theta + \frac{a}{\sin\theta}.$$

Note that as $\theta \downarrow 0$,

$$\phi(\theta) = \frac{4a-1/2}{\theta}\,[1+O(\theta^2)], \quad \phi(\pi-\theta) = \frac{1/2-2a}{\theta}\,[1+O(\theta^2)]. \tag{27}$$

By comparison with the Bessel process we can see that a process satisfying (26) does not reach the origin for all $\kappa < 8$, does not reach π if $0 < \kappa \leq 4$; but does reach π if $4 < \kappa < 8$. In the latter case it is reflected in the same way that the Bessel process is reflected.

Using this equation we can see that $\mathbb{P}^*\{\sigma(t) < \infty\} = 1$. Indeed, since $\hat{\Theta}_t$ does not reach zero in finite time, neither does $\hat{X}_t$. This implies that in the tilted measure $\Upsilon_T = 0$ and hence $\operatorname{dist}(\gamma_T, 1) = 0$. Below we discuss the stronger fact, that with $\mathbb{P}^*$-probability one

$$\lim_{t \uparrow T} \gamma(t) = 1.$$

Although M_t in the original half-plane capacity parametrization is only a local martingale, the process $\hat{M}_t$ is an actual martingale; indeed, it is bounded on every compact time interval.

The invariant probability density for (26) is

$$\psi(\theta) = c\,(\sin\theta)^{4a-1}\,(1-\cos\theta)^{2a},$$

where c is chosen to make this a probability density. There are various ways to see this. One way, as described in Section 4, is to see that (26) can be obtained by starting with a Brownian motion and weighting locally by Φ, where in this case,

$$\Phi(x) = [\sin x]^{2a-1/2}\,[1-\cos x]^a.$$

The invariant density is $\psi(\theta) = c\,\Phi^2$ where c is a normalizing constant. From this we get the invariant density for $\hat{J}_t$ is a beta distribution

$$h(x) = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a)}\, x^{4a-1}\,(1-x)^{2a-1},$$

and

$$\int_0^1 x^{1-4a}\,h(x)\,dx = \frac{1}{2a}\,\frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a)} = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)}.$$

Let $\hat{\psi}_t(\theta_0,\theta)$ denote the density (as a function of θ) of $\hat{\Theta}_t$ given $\hat{\Theta}_0 = \theta_0$. Standard techniques (see Section 4 for more details) show that there exists c_1, c_2, α such that

$$c_1\,\psi(\theta) \le \hat{\psi}_t(\theta_0,\theta) \le c_2\,\psi(\theta), \quad t \ge 1, \tag{28}$$

$$\hat{\psi}_t(\theta_0,\theta) = \psi(\theta)\,[1 + O(e^{-\alpha t})]. \tag{29}$$

Indeed, for (28), we can use properties of the Bessel process and (27) to show that our process is absolutely continuous with respect to a Bessel process near the endpoints. With this, we can get (29) either by computing an eigenvalue or by a coupling argument. It follows that for any positive function f,

$$\mathbb{E}^*\big[f(\hat{\Theta}_t) \mid \hat{\Theta}_0 = \theta_0\big] = \int_0^\pi f(\theta)\,\hat{\psi}_t(\theta_0,\theta)\,d\theta = [1 + O(e^{-\alpha t})]\int_0^\pi f(\theta)\,\psi(\theta)\,d\theta. \tag{30}$$

Using $f(\theta) = [1-\cos\theta]/2$, we see that

$$\mathbb{E}^*\big[\hat{J}_t \mid \hat{\Theta}_0 = \theta_0\big] = \frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)} + O(e^{-\alpha t}).$$

Let $\tilde{\Theta}_t = \hat{\Theta}_{t/a}$ so that $\tilde{\Upsilon}_t = e^{-t}$, and let $\tilde{\sigma}(t) = \sigma(t/a) = \inf\{s : \Upsilon_s = e^{-t}\}$. Let $\psi_t(\theta_0,\theta') = \hat{\psi}_{t/a}(y,\theta')$ denote the transition probability for $\tilde{\Theta}_t$ assuming $\tilde{\Theta}_0 = y$. This also satisfies (28)–(30) (with a different α). We emphasize that ψ describes the evolution with respect to the tilted measure $\mathbb{P}^*$, that is,

$$\mathbb{P}^*\{\tilde{\Theta}_t \in V \mid \tilde{\Theta}_0 = y\} = \int_V \psi_t(y,\theta)\,d\theta.$$

Let $\phi_t(y,\theta)$ denote the density of $\tilde{\Theta}_t$ with respect to the measure $\mathbb{P}$ *restricted* to the event $\{\tilde{\sigma}(t) < \infty\}$ assuming $\tilde{\Theta}_0 = y$, that is,

$$\mathbb{P}\{\tilde{\sigma}(t) < \infty; \tilde{\Theta}_t \in V \mid \tilde{\Theta}_0 = y\} = \int_V \phi_t(y,\theta)\,d\theta.$$

We can write ϕ_t in terms of ψ_t and the Radon-Nikodym derivative between the measures $\mathbb{P}, \mathbb{P}^*$. Let $F(x) = [1 - \cos x]/2$ so that $\tilde{J}_t = F(\tilde{\Theta}_t)$, $\tilde{M}_t = e^{\beta t}\, F(\tilde{\Theta}_t)^\beta$. If $\tilde{\Theta}_0 = y$, then

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{\tilde{M}_t}{\tilde{M}_0} = F(y)^{-\beta}\, e^{\beta t}\, F(\Theta_t)^\beta,$$

and hence

$$\begin{aligned}\phi_t(y,\theta) &= F(y)^\beta\, e^{-\beta t}\, \psi_t(y,\theta)\, F(\theta)^{-\beta} \\ &= F(y)^\beta\, e^{-\beta t}\, \psi(\theta)\, F(\theta)^{-\beta}\, [1 + O(e^{-\alpha t})].\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}\{\tilde{\sigma}(t) < \infty\} &= \mathbb{E}[1\{\tilde{\sigma}(t) < \infty\}] \\ &= e^{-\beta t}\, \mathbb{E}\big[\tilde{M}_t\, F(\tilde{\Theta}_t)^{-\beta}; \tilde{\sigma}(t) < \infty\big] \\ &= e^{-\beta t}\, \mathbb{E}^*\big[F(\tilde{\Theta}_t)^{-\beta}\big] \qquad (31) \\ &= e^{-\beta t}\left[\frac{\Gamma(6a)}{\Gamma(4a)\,\Gamma(2a+1)} + O(e^{-\alpha t})\right]. \qquad (32)\end{aligned}$$

In the third equality we used $\mathbb{P}^*\{\tilde{\sigma}(t) < \infty\} = 1$. Using the Koebe 1/4-theorem, we can see that (32) implies that

$$\mathbb{P}\{\mathrm{dist}(1,\gamma) \le \epsilon\} \asymp \epsilon^\beta. \qquad (33)$$

Let

$$q(\epsilon) = \int_0^\pi \mathbb{P}\{\mathrm{dist}(1,\gamma) \le \epsilon\, F(\theta)\}\, \psi(\theta)\, F(\theta)^{-\beta}\, d\theta,$$

$$c_+ = \limsup_{\epsilon \downarrow 0} \epsilon^{-\beta}\, q(\epsilon).$$

The estimate (33) implies that $0 < c_+ < \infty$. We will now show that

$$\lim_{\epsilon \downarrow 0} \epsilon^{-\beta}\, q(\epsilon) = c_+. \qquad (34)$$

Note that $\tilde{g}_t'(1) = e^t\, \tilde{X}_t\, \tilde{J}_t$. Let

$$h_t(z) = \frac{\tilde{g}_t(z) - \tilde{U}_t}{\tilde{X}_t},$$

so that $h_t : \tilde{D}_t \to \mathbb{C}\backslash(-\infty, \tilde{J}_t]$ with $h_t(1) = 1$. Recall that $h_t'(1) = \tilde{J}_t\, e^t$ and $\mathrm{dist}(1, \partial\tilde{D}_t) \asymp e^{-t}$. Distortion estimates (see (35)) imply that for every $u > 0$, there exists $s_0 = s_0(u) <$

∞ such that for all t and all $s \geq s_0$, the image under h_t of the disk of radius $e^{-(t+s)}$ about 1 is contained in a disk of radius $\tilde{J}_t\, e^{-(s-u)}$ about 1 and contains a disk of radius $\tilde{J}_t\, e^{-(s+u)}$ about 1. Therefore, on the event $\{\tilde{\sigma}(t) < \infty\}$,

$$\mathbb{P}\big\{\,\mathrm{dist}(1,\gamma) \leq \tilde{J}_t\, e^{-(s+u)} \mid \gamma_{\tilde{\sigma}(t)}\big\} \leq \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq e^{-(t+s)} \mid \gamma_{\tilde{\sigma}(t)}\}$$
$$\leq \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq \tilde{J}_t\, e^{-(s-u)} \mid \gamma_{\tilde{\sigma}(t)}\}.$$

Let $p(s) = e^{s\beta}\, q(e^{-s})$. Then, for all $s \geq s_0$, and all $t \geq 0$,

$$\begin{aligned}
&e^{-\beta(t+s)}\, p(t+s)\\
&\quad = \int_0^\pi \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq F(x)\, e^{-(t+s)}\}\, \psi(x)\, F(x)^{-\beta}\, dx\\
&\quad \geq \int_0^\pi \psi(x)\, F(x)^{-\beta} \left[\int_0^\pi \phi_t(x,y)\, \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq F(y)\, e^{-(s+u)}\}\, dy\right] dx\\
&\quad = e^{-\beta t} \int_0^\pi \int_0^\pi F(y)^{-\beta}\, \psi(x)\, \psi_t(x,y)\, \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq F(y)\, e^{-(s+u)}\}\, dy\, dx\\
&\quad = e^{-\beta t} \int_0^\pi F(y)^{-\beta} \left[\int_0^\pi \psi(x)\, \psi_t(x,y)\, dx\right] \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq F(y)\, e^{-(s+u)}\}\, dy\\
&\quad = e^{-\beta t} \int_0^\pi F(y)^{-\beta}\, \psi(y)\, \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq F(y)\, e^{-(s+u)}\}\, dx\\
&\quad = e^{-\beta t}\, q(e^{-(s+u)}) = e^{-\beta(t+s+u)}\, p(s+u).
\end{aligned}$$

Hence for all $s \geq s_0$ and all t,

$$p(s+t) \geq e^{-\beta u}\, p(s+u)$$

which implies that

$$\liminf_{t\to\infty} p(t) \geq e^{-\beta u} \limsup_{t\to\infty} p(t) = e^{-\beta u}\, c_+,$$

and since this holds for all $u > 0$, we get (34).

To finish the argument, note that if $s \geq s_0(u)$, then

$$\begin{aligned}
&\mathbb{P}\{\mathrm{dist}(1,\gamma) \leq e^{-(t+s)}\}\\
&\quad \geq \int_0^\pi \phi_t(\pi,\theta)\, \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq e^{-(s+u)}\, F(\theta)\}\, d\theta\\
&\quad = e^{-\beta t}\, \psi(\pi) \int_0^\pi \psi_t(\pi,\theta)\, \mathbb{P}\{\mathrm{dist}(1,\gamma) \leq e^{-(s+u)}\, F(\theta)\}\, F(\theta)^{-\beta}\, d\theta
\end{aligned}$$

$$= e^{-\beta t}\,\psi(\pi)\,[1 - O(e^{-\alpha t})] \int_0^{\pi} \psi(\theta)\ \mathbb{P}\{\mathrm{dist}(1,\gamma) \le e^{-(s+u)}\,F(\theta)\}\,F(\theta)^{-\beta}\,d\theta$$
$$= e^{-\beta t}\,q(e^{-(s+u)})\,\psi(\pi)\,[1 - O(e^{-\alpha t})].$$

Similarly,

$$\mathbb{P}\{\mathrm{dist}(1,\gamma) \le e^{-(t+s)}\} \le e^{-\beta t}\,q(e^{-(s-u)})\,\psi(\pi)\,[1 + O(e^{-\alpha t})].$$

2.1. Two-point estimates.

We will consider two-point estimates and, among other things, will prove Theorem 2. As in [**16**], the starting point for two-point estimates is a very good one-point estimate. The next proposition is an almost immediate corollary of (9), but it will be useful to state it in this form. The proof uses classical distortion theorems (see, e.g., [**5**, Theorem 2.5]). We need the following: there exists $c_0 < \infty$ such that if $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal transformation with $f(0) = 0$, then

$$|f(z) - f'(z)\,z| \le c_0\,|f'(z)|\,|z|^2, \quad |z| \le 1/2. \tag{35}$$

An explicit c_0 can be given, but we only need that the constant c_0 is uniform over all such f.

PROPOSITION 2.1. *If γ is an* SLE_κ *path, $\epsilon > 0$, and ρ is a stopping time with* $\mathrm{dist}(\gamma_\rho, 1) \ge 2\epsilon$, *then*

$$\mathbb{P}\{\mathrm{dist}(\gamma, 1) \le \epsilon \mid \gamma_\rho\} = \hat{c}\,\epsilon^{\beta}\,(J_\rho/\Upsilon_\rho)^{\beta}\,[1 + O((J_\rho/\Upsilon_\rho)^{\alpha})], \tag{36}$$

where $\hat{c}, \alpha$ are as in (9). *In particular, there exists $c < \infty$, such that for all such ϵ, γ,*

$$\mathbb{P}\{\mathrm{dist}(\gamma, 1) \le \epsilon \mid \gamma_\rho\} \le c\left(\frac{\epsilon}{\mathrm{dist}(\gamma_\rho, 1)}\right)^{\beta}. \tag{37}$$

PROOF. We view g_t as a conformal transformation of the reflected domain D_t and let $\delta = g_t'(1)$. Let $\mathcal{B}(z, r)$ denote the closed disk of radius r about z. The distortion estimate (35) implies that

$$\mathcal{B}(g_\rho(1), \delta\epsilon[1 - c_0\epsilon]) \subset g_\rho[\mathcal{B}(1,\epsilon)] \subset \mathcal{B}(g_\rho(1), \delta\epsilon[1 + c_0\epsilon]).$$

Therefore, (8) implies that

$$\begin{aligned}\mathbb{P}\{\mathrm{dist}(\gamma, 1) \le \epsilon \mid \gamma_\rho\} &= \hat{c}\,\epsilon^{\beta}\,(\delta/X_\rho)^{\beta}\,[1 + O((\delta/X_\rho)^{\alpha})] \\ &= \hat{c}\,\epsilon^{\beta}\,(J_\rho/\Upsilon_\rho)^{\beta}\,[1 + O((J_\rho/\Upsilon_\rho)^{\alpha})].\end{aligned}$$ □

It is useful to phrase the boundary estimate in terms of a conformally invariant quantity, *excursion measure*. There are various ways to define this measure. If D is a domain and V_1, V_2 are disjoint analytic curves on ∂D, we define

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \partial_{\mathbf{n}} \phi(z)\, |dz|,$$

where $\partial_{\mathbf{n}}$ denotes derivative in the inward normal direction and ϕ is the harmonic function on D with boundary value 1 on V_2 and zero elsewhere. It is well known that $\mathcal{E}_D(V_1, V_2) = \mathcal{E}_D(V_2, V_1)$, and that this is a conformal invariant,

$$\mathcal{E}_D(V_1, V_2) = \mathcal{E}_{f(D)}(f(V_1), f(V_2)).$$

It is not hard to show that if η is a simple curve in $\overline{\mathbb{H}}$ that intersects the positive real line, then

$$\mathcal{E}_{\mathbb{H}\setminus\eta}(\eta, (-\infty, 0]) \wedge 1 \asymp \frac{\operatorname{diam}(\eta)}{\operatorname{dist}(\eta, (-\infty, 0])} \wedge 1.$$

Therefore, using (8) (actually, we need only the up-to-constants version from [**1**]), we can see that

$$\mathbb{P}\{\gamma \cap \eta \neq \emptyset\} \leq c\big[\mathcal{E}_{\mathbb{H}\setminus\eta}(\eta, (-\infty, 0])\big]^{\beta}. \tag{38}$$

In applying this estimate, we use the following easy estimate. Suppose $\eta_1, \eta_2 : [0,1] \to \overline{\mathbb{H}}$ are two simple curves with $\eta_1(0), \eta_2(0) \in \mathbb{R}$ and $\operatorname{diam}(\eta_1) + \operatorname{diam}(\eta_2) \leq \operatorname{dist}(\eta_1, \eta_2)$. Then if D is the connected component of $\mathbb{H} \setminus (\eta_1 \cup \eta_2)$ containing η_1 and η_2 on its boundary,

$$\mathcal{E}_D(\eta_1, \eta_2) \asymp \frac{\operatorname{diam}(\eta_1)\, \operatorname{diam}(\eta_2)}{\operatorname{dist}(\eta_1, \eta_2)^2}. \tag{39}$$

LEMMA 2.2. *There exists $c < \infty$ such that the following holds. Suppose $\eta_1 : (0, \infty) \to \mathbb{H}$ is a simple curve with $\eta_1(0+) = 0$ and $\eta_1(t) \to \infty$ as $t \to \infty$. Let $\eta : (0, 1) \to \mathbb{H}$ be a curve with $\eta(0+) \in (0, \infty)$ and $\eta \cap \eta_1 = \emptyset$. Let D denote the connected component of $\mathbb{H} \setminus (\eta_1 \cup \eta)$ whose boundary includes both η_1 and η. Then if γ is an* SLE$_\kappa$ *curve from* 0 *to* ∞ *in* $\mathbb{H}$,

$$\mathbb{P}\{\gamma \cap \eta \neq \emptyset\} \leq c\, \mathcal{E}_D(\eta, \eta_1)^{\beta}. \tag{40}$$

PROOF. Since η_1 disconnects η from $(-\infty, 0)$ in $\mathbb{H}$, monotonicity of the excursion measure implies that $\mathcal{E}_D(\eta, \eta_1) \geq \mathcal{E}_D(\eta, (-\infty, 0])$ and we can use (38). $\square$

The next lemma is the "up-to-constants" two-point estimate. Similar estimates were given in [**2**] and [**22**] but for completeness sake we give a proof here. This estimate is significantly easier than the analogous estimate for two interior points. The topology of the boundary is such that once the SLE curve gets near $1 + x$ it is unlikely to get close to 1 in the future.

LEMMA 2.3. *There exists $c < \infty$ such that if $x > 0$, then*

$$\mathbb{P}\{\mathrm{dist}(\gamma, 1) < \epsilon, \mathrm{dist}(\gamma, 1+x) < \delta\} \le c\, x^{-\beta}\, \epsilon^{\beta}\, \delta^{\beta}. \tag{41}$$

PROOF. Without loss of generality we will assume that $\epsilon \le 1/10$ and $\delta \le x/10$. Indeed, if either of these does not hold, then the result follows directly from (8). Without loss of generality we may assume that $\delta = 2^{-n}$ for some integer n. Let r be the integer with $2^{-r} \le x < 2^{-r+1}$. Let $\sigma = \inf\{t : \mathrm{dist}(1, \gamma_t) = \epsilon\}$ and let $\tau_k = \inf\{t : \mathrm{dist}(1+x, \gamma_t) = 2^{-k}\}$. Using (8) and (37), we get

$$\begin{aligned}\mathbb{P}\{\sigma < \tau_{r+4} < \tau_n < \infty\} &\le \mathbb{P}\{\sigma < \infty\}\, \mathbb{P}\{\tau_n < \infty \mid \sigma < \tau_{r+4}\} \\ &\le c\, \epsilon^{\beta}\, [2^{-n}/2^{-(r+4)}]^{\beta} \\ &\le c\, \epsilon^{\beta}\, x^{-\beta}\, \delta^{\beta}.\end{aligned}$$

We will show that

$$\mathbb{P}\{\tau_n < \sigma < \infty\} \le c\, \epsilon^{\beta}\, x^{-\beta}\, \delta^{\beta}\, 2^{-\beta n}, \tag{42}$$

and if $r + 4 \le k < n - 1$,

$$\mathbb{P}\{\tau_k < \sigma < \tau_{k+1} < \infty; \tau_n < \infty\} \le c\, \epsilon^{\beta}\, x^{-\beta}\, \delta^{\beta}\, 2^{-\beta(k-r)}. \tag{43}$$

The result follows by summing over k.

Let C denote the half circle in $\mathbb{H}$ of radius ϵ about 1 and η_k the half circle in $\mathbb{H}$ of radius 2^{-k} about $1 + x$. Let $H = H_{\tau_k}$. The circle η_k separates C from one-side of the curve γ_{τ_k}. Therefore, (40) and monotonicity of excursion measure imply that if D_k is the component of $\mathbb{H} \setminus (C \cup \eta_k)$ containing C and η_k on its boundary,

$$\mathbb{P}\{\sigma < \infty \mid \tau_k < \sigma\} \le c\, \mathcal{E}_{D_k}(C, \eta_k)^{\beta}.$$

Using (39), we see that

$$\mathcal{E}_{D_k}(C, \eta_k) \le \frac{\mathrm{diam}(C)\ \mathrm{diam}(\eta_k)}{\mathrm{dist}(C, \eta_k)^2} \le c\, \epsilon\, 2^{-k}\, x^{-2}.$$

Therefore,

$$\mathbb{P}\{\sigma < \infty \mid \tau_k < \sigma\} \le c\, \epsilon^{\beta}\, 2^{-k\beta}\, x^{-2\beta} \le c\, \epsilon^{\beta}\, x^{-\beta}\, 2^{-\beta(k-r)}. \tag{44}$$

The one-point estimates (8) and (37) give

$$\mathbb{P}\{\tau_k < \sigma\} \le \mathbb{P}\{\tau_k < \infty\} \le c\, 2^{-\beta k}, \tag{45}$$

$$\mathbb{P}\{\tau_n < \infty \mid \tau_k < \sigma < \tau_{k+1}\} \le c\, 2^{-\beta(n-k)}. \tag{46}$$

The estimates (44)–(46) imply (42) and (43). □

In the preceding proof, we split the events $\{\sigma < \infty, \tau_n < \infty\}$ into a union of event like those on the left-hand side of (43). If we had used just the one-point estimate to estimate the quantities, the right-hand side would not have had the $2^{-\beta k}$ term and the estimate would not be good enough. We had to do a more delicate argument showing that as k gets larger, these probabilities decrease, that is, the SLE_κ does not want to get close to $1+x$, and then close to 1 afterwards. The topology here was such that the proof was not too difficult. It is more challenging in the case of two interior points, see [**4**], [**16**].

Before proving Theorem 2, we will prove a continuity result for two-sided chordal SLE_κ (from 0 to ∞ going through 1 stopped at 1). Recall that this is the solution to the Loewner equation (1) where the driving function $U_t = -W_t$ satisfies

$$dW_t = \frac{1-4a}{X_t}\,dt + dB_t,$$

and B_t is a $\mathbb{P}^*$-Brownian motion. In other words,

$$dX_t = \frac{1-3a}{X_t}\,dt + dB_t,$$

which is valid up to time $T = T_1 < \infty$. In Section 2 we saw that with $\mathbb{P}^*$-probability one, $\mathrm{dist}(\gamma_T, 1) = 0$. The next proposition shows that the path is continuous at the endpoint. This is an extension of results in [**6**], [**12**] where radial and two-sided radial SLE are studied, and is related to [**7**] where boundary questions are studied.

PROPOSITION 2.4. *With $\mathbb{P}^*$-probability one,*

$$\lim_{t\uparrow T} \gamma(t) = 1.$$

In fact, if C_s denotes the half circle in $\mathbb{H}$ centered at 1 of radius e^{-s}, and $\sigma_s = \inf\{t : \mathrm{dist}(\gamma, 1) = e^{-s}\}$, then there exists c', ξ such that

$$\mathbb{P}^*\{\gamma[\sigma_s, T] \cap C_{s/2} \neq \emptyset\} \le c\, e^{-\xi s}.$$

PROOF. If $r < s$, then $H_{\sigma_s} \cap C_r$ contains a (finite or countably infinite) collection of subarcs that are crosscuts of H_{σ_s}. We will focus on the two crosscuts that hit the real line. Let $l^-_{s,r}, l^+_{s,r}$ denote the closure of the subarcs whose endpoints include $1 - e^{-r}$ and $1 + e^{-r}$, respectively, let $l_{s,r} = l^-_{s,r} \cup l^+_{s,r}$ and

$$\lambda^{\pm}_{s,r} = \inf\{t \ge \sigma_s : \gamma(t) \in l^{\pm}_{s,r}\}, \quad \lambda_{s,r} = \lambda^+_{s,r} \wedge \lambda^-_{s,r}.$$

We claim there exists $c < \infty$ such that for all $0 < r < s$, if $\sigma_s < \infty$, then

$$\mathbb{P}^*\{\lambda_{s,r} < \infty \mid \gamma_{\sigma_s}\} \le c\, e^{\beta(r-s)}. \tag{47}$$

We will first show that

$$\mathbb{P}^*\{\lambda_{s,r} < \sigma_{s+1} \mid \gamma_{\sigma_s}\} \leq c\, e^{\beta(r-s)}. \tag{48}$$

We may assume $s > r+2$. Let $\lambda = \lambda^-_{s,s-2} < \lambda^-_{s,r}$. Then

$$\mathbb{P}^*\{\lambda_{s,r} < \sigma_{s+1}\} = \mathbb{P}^*\{\lambda < \sigma_{s+1}\}\, \mathbb{P}^*\{\lambda_{s,r} < \sigma_{s+1} \mid \lambda < \sigma_{s+1}\}.$$

We claim that $J_\lambda \asymp 1$. To see this, let ∂^1_t denote the part of ∂H_t that is sent to $[U_t, g_t(1)]$ and let $\partial^2_t = \{x \in \partial^1_t \cap \mathbb{R} : T_x > t\}$ be the part of ∂^1_t that is not on the hull at time t. Then using conformal invariance, we can see that

$$J_t = \lim_{y \to \infty} \frac{\mathrm{hm}_{H_t}(iy, \partial^2_t)}{\mathrm{hm}_{H_t}(iy, \partial^1_t)},$$

where hm denotes harmonic measure. At time λ consider the subarc η of C_{s-1} starting at $1-e^{-(s-1)}$ stopping when it reaches γ_λ. Let l denote a line starting at 1 towards $\gamma(\sigma_s)$ stopped when it reaches γ_λ. Note that the length of l is at most e^{-s}; the length of η is at most $\pi\, e^{-s+1}$ and $\mathrm{dist}(l,\eta) \geq \mathrm{dist}(C_s, C_{s-1}) = e^{-s}$. Hence there exists an absolute c such that the excursion measure between l and η in H_λ is less than c. If y is large, then any path starting at iy that leaves H_λ at $\partial^1_\lambda \setminus \partial^2_\lambda$ must go through l and then through η. Let R be the component of $H_\lambda \setminus (l \cup \eta)$ that includes both l as η on its boundary and consider it as a conformal rectangle with two of its sides being l, η. We know that $\mathcal{E}_R(l,\eta) \leq c$ for some absolute c. By mapping to a standard rectangle for which the harmonic measure can be given explicitly (in terms of an integral or an infinite series), we can see there exists $\delta = \delta(c) > 0$ such that for all $z \in R$ sufficiently close to l,

$$\mathrm{hm}_R(z, I) \leq \delta\, \mathrm{hm}_R(z,\eta), \quad I = [1-e^{-s}, 1].$$

Therefore, for all y sufficiently large,

$$\mathrm{hm}_{H_\lambda}(iy, I) \geq \delta\, \mathrm{hm}_{H_\lambda}(iy, \partial^1),$$

which establishes the claim.

Hence, on the event $\{\lambda < \sigma_{s+1} < \infty\}$ we have $M_{\sigma_{s+1}} \leq c\, M_\lambda$. This implies that

$$\mathbb{P}^*\{\lambda^-_{s,r} < \sigma_{s+1} \mid \lambda < \sigma_{s+1}\} \leq c\,\mathbb{P}\{\lambda^-_{s,r} < \sigma_{s+1} \mid \lambda < \sigma_{s+1}\} \leq \mathbb{P}\{\lambda^-_{s,r} < \infty \mid \lambda < \sigma_{s+1}\}.$$

On the event $\lambda^-_{s,r} < \sigma_{s+1}$, the arc $l^-_{s,s-2}$ separates $l^-_{s,r}$ from one side of the curve γ_λ. If D denotes the connected component of $H_\lambda \setminus (l^-_{s,s-2} \cup l^-_{s,r})$, then (38) and (39) imply that

$$\mathcal{E}_D(l^-_{s,s-2}, l^-_{s,r}) \leq c\, e^{-(s-r)},$$

and hence,

$$\mathbb{P}^*\{\lambda^-_{s,r} < \infty \mid \lambda < \sigma_{s+1}\} \le c\, e^{-\beta(s-r)}.$$

For the other side, let $\lambda^+ = \lambda^+_{s,r}$ and write $H = H_{\lambda^+}$, $\partial^j = \partial^j_{\lambda^+}$. We claim that if $\lambda^+ < \infty$, then $J_{\lambda^+} \le c\, e^{-\beta(s-r)}\, J_{\sigma_s}$. To see this, let η_0 denote the circular arc in $H \cap C_{s-r}$ with endpoints $1+e^{-\beta r}$ and $\gamma(\lambda^+)$, let η_1 denote the circular arc in $H \cap C_{s-1}$ that includes $1+e^{-\beta(s-1)}$, and let η_2 be the circular arc in $H \cap C_s$ that includes $1+e^{-\beta s}$. Note that for y large, any Brownian path starting at iy that leaves H at ∂^2 must go through the arcs η_0, η_1, η_2 in order before reaching ∂^2. Let R_1 be the connected component of $H \setminus (\eta_0 \cup \eta_1)$ that contains both η_0 and η_1 on its boundary. We view this as a conformal rectangle with η_0, η_1 two of the edges. The other two edges are $[1+e^{1-s}, 1+e^{r-s}]$ and a subset of ∂^1. We have $\mathcal{E}_{R_1}(\eta_0, \eta_1) \le \mathcal{E}_{\mathbb{H}\setminus(C_{s-1}\cup C_{s-r})}(C_{s-1}, C_{s-r}) \le c\, e^{r-s}$. Hence, using the estimate for rectangles, we see for every $z \in \eta_0$, the probability that a Brownian motion starting at z hits η_2 before leaving H, is less than $c\, e^{r-s}$ times the probability that it leaves H at ∂^1.

Given that the Brownian motion reaches η^2 before leaving H, the probability that it leaves H at ∂^2 is bounded above by the supremum of this probability over $z \in \partial^2$. This probability is only larger if we replace H with H_{σ_s} and by the (boundary) Harnack principle this is bounded above by a constant times the probability that a Brownian motion leaves H at ∂^2 given that it reaches $l^+_{s,s}$. But this last probability is bounded above by J_s. Combining all these estimates gives the claim.

Once we have the claim, we see that if $\lambda^+ < \sigma_{s+1}$, then $M_{\lambda^+} \le c\, e^{\beta(r-s)}\, M_{\sigma_s}$, and hence

$$\mathbb{P}^*\{\lambda^+_{s,r} \le \sigma_{s+1} \mid \sigma_s\} \le c\, e^{-\beta(s-r)}.$$

This finishes the proof of (48). By iterating this argument, we see that

$$\mathbb{P}^*\{\lambda_{s+k,r} < \sigma_{s+k+1} \mid \lambda_{s+k,r} > \sigma_{s+k}\} \le c\ e^{\beta(r-s-k)},$$

and by summing over k we get (47).

To finish the argument, for each positive integer s let $K(s)$ be the largest positive integer k such that $\gamma[\sigma_{s-k}, \sigma_k] \cap C_{s-k-1} = \emptyset$. By looking at a particular event, it is not difficult to show that for every r there exists $\delta_r > 0$ such that

$$\mathbb{P}^*\{K(s+r+2) \ge r \mid \gamma_{\sigma_s}\} = \mathbb{P}^*\{\gamma[\sigma_{s+2}, \sigma_{s+r+2}] \cap C_s = \emptyset \mid \gamma_{\sigma_s}\} \ge \delta_r. \tag{49}$$

If $\gamma[\sigma_{s+2}, \sigma_{s+r+2}] \cap C_s = \emptyset$, then any curve from $\gamma(\sigma_{s+r+2})$ to C_s staying in $H_{\sigma_{s+r+s}}$ must go though $l_{s+r+2,s+2}$. Hence (47) implies that

$$\mathbb{P}^*\{K(s+1) \ge r+1 \mid K(s) \ge r\} \ge 1 - c\, e^{-\beta s}. \tag{50}$$

The inequalities (49) and (50) imply that there exists c', ξ such that

$$\mathbb{P}\{K(s) \le s/2\} \le c'\, e^{-\xi s}.$$

See, e.g., [**12**, Lemma 4.5]. The value of ξ depends on the values of δ_r and so is difficult to estimate; however, we do get existence. $\Box$

Proof of Theorem 2. We fix $0 < \kappa < 8$ and $x > 0$. All constants may depend on κ and x. Let $X_t = g_t(1) - U_t, Z_t = g_t(1+x) - U_t$, and $O_t = g_t(x_t) - U_t$. Let

$$J_t = \frac{X_t - O_t}{X_t}, \quad K_t = \frac{Z_t - O_t}{Z_t},$$

$$\Upsilon_t = \Upsilon_t(1) = \frac{X_t - O_t}{g_t'(1)}, \quad \Psi_t = \Upsilon_t(1+x) = \frac{Z_t - O_t}{g_t'(1+x)}.$$

As before, let $T = T_1 = \inf\{t : X_t = 1\}$, let M_t be the local martingale

$$M_t = \Upsilon_t^{-\beta} \, J_t^{\beta},$$

and let $\mathbb{P}^*, \mathbb{E}^*$ denote the measures obtained by tilting by M_t.

Let

$$\sigma_\epsilon = \inf\{t : \Upsilon_t = \epsilon\}, \quad \tau_\delta = \inf\{t : \Psi_t = \delta\}.$$

For fixed u, let us write

$$\{\sigma_\epsilon < \infty, \tau_\delta < \infty\} = \{\sigma_\epsilon \le \tau_u < \tau_\delta < \infty\} \cup \{\tau_u < \sigma_\epsilon < \infty, \tau_\delta < \infty\}.$$

The Koebe 1/4-theorem (11) implies that $\operatorname{dist}(1+x, \gamma_{\tau_u}) \le u$, Arguing as in (43), we see that

$$\mathbb{P}\{\tau_u < \sigma_\epsilon < \infty, \tau_\delta < \infty\} \le c \, \epsilon^\beta \, \delta^\beta \, u^\beta.$$

(The $x^{-\beta}$ and $2^{r\beta}$ terms in (43) have been included in the x-dependent constant c.) Hence,

$$\limsup_{\epsilon,\delta \downarrow 0} \epsilon^{-\beta} \, \delta^{-\beta} \, \mathbb{P}\{\tau_u < \sigma_\epsilon < \infty; \tau_\delta < \infty\} \le c \, u^\beta. \tag{51}$$

For fixed u and $\delta < u/2$, ϵ sufficiently small, we use (36) to get

$$\begin{aligned}
\mathbb{P}\{\sigma_\epsilon \le \tau_u < \tau_\delta < \infty\} &= \mathbb{E}[1\{\sigma_\epsilon < \tau_u < \tau_\delta < \infty\}] \\
&= \mathbb{E}[1\{\sigma_\epsilon < \tau_u\} \, \mathbb{P}\{\tau_\delta < \infty \mid \gamma_{\sigma_\epsilon}\}] \\
&= c' \, \delta^\beta \, [1 + O((\delta/u)^\alpha)] \, \mathbb{E}[1\{\sigma_\epsilon \le \tau_u\} \, (K_{\sigma_\epsilon}/\Psi_{\sigma_\epsilon})^\beta] \\
&= (c')^2 \, \epsilon^\beta \, \delta^\beta \, [1 + O(\delta^\alpha) + O(\epsilon^\alpha)] \, \mathbb{E}^*[(K_{\sigma_\epsilon}/\Psi_{\sigma_\epsilon})^\beta \, 1\{\sigma_\epsilon \le \tau_u\}].
\end{aligned}$$

Using the last lemma, we can see that with $\mathbb{P}^*$-probability one,

$$\lim_{\epsilon\downarrow 0}(K_{\sigma_\epsilon}/\Psi_{\sigma_\epsilon})^\beta\, 1\{\sigma_\epsilon \le \tau_u\} = (K_T/\Psi_T)^\beta\, 1\{T \le \tau_u\} = \Psi_T^{-\beta}\, 1\{T \le \tau_u\}.$$

Here we have used the fact that γ is continuous at time T with $\gamma(T) = 1$ and hence $O_T = X_T = 0$. By the dominated convergence theorem,

$$\lim_{\epsilon,\delta\downarrow 0} \epsilon^{-\beta}\,\delta^{-\beta}\,\mathbb{P}\{\sigma_\epsilon \le \tau_u < \tau_\delta < \infty\} = (c')^2\,\mathbb{E}^*\big[\Psi_T^{-\beta}; T \le \tau_u\big]. \tag{52}$$

Combining (51) and (52) and using the monotone convergence theorem, we get (16).

We are left with computing $\mathbb{E}^*[\Psi_T^{-\beta}]$. Let

$$Q_t = \left[\frac{g_t'(Z_0)\,(Z_0 - X_0)}{Z_t - X_t}\right]^\beta, \qquad f(s) = \mathbb{E}[Q_T \mid X_0 = s\, Z_0],$$

where $\mathbb{E}$ denotes expectation with respect to two-sided chordal SLE going through X_0. Scaling shows that $f(s)$ is independent of X_0. Note that $\mathbb{E}^*[\Psi_T^{-\beta}] = x^{-\beta}\, f(1/(1+x))$. The right-hand side is easier to compute because we do not need to keep track of O_t. Note that (41) implies that f is uniformly bounded on $(0,1)$. Itô's formula shows that

$$d\left[\frac{X_t}{Z_t}\right] = \frac{X_t}{Z_t}\left[\left(\frac{1-3a}{X_t^2} + \frac{1-a}{Z_t^2} + \frac{4a-2}{X_t Z_t}\right)dt + \left(\frac{1}{X_t} - \frac{1}{Z_t}\right)dB_t\right].$$

We now change time choosing

$$\dot\zeta(t) = \frac{X_t^2\, Z_t^2}{(Z_t - X_t)^2} = \frac{X_t^2}{(1 - X_t/Z_t)^2}.$$

If we set $R_t = X_{\zeta(t)}/Z_{\zeta(t)}$, we get

$$dR_t = R_t\left[\left(\frac{1-3a}{(1-R_t)^2} + \frac{(1-a)\,R_t^2}{(1-R_t)^2} + \frac{(4a-2)\,R_t}{(1-R_t)^2}\right)dt + dW_t\right],$$

for a standard Brownian motion W_t. Also,

$$\partial_t Q_t = Q_t\left[\frac{a(1-4a)}{Z_t^2} + \frac{a(4a-1)}{Z_t X_t}\right].$$

Therefore, if $\tilde Q_t = \tilde Q_{\zeta(t)}$,

$$\partial_t \tilde Q_t = \tilde Q_t\,\frac{a(1-4a)R_t^2 + a(4a-1)\,R_t}{(1-R_t)^2} = \tilde Q_t\,\frac{a(4a-1)R_t\,(1-R_t)}{(1-R_t)^2}.$$

Note that

$$\mathbb{E}[Q_T \mid \gamma_{\zeta(t)}] = \tilde Q_t\, f(R_t).$$

The left-hand side is a martingale and

$$d[\tilde{Q}_t\, f(R_t)] = f(R_t)\, d\tilde{Q}_t + \tilde{Q}_t \left[f'(R_t)\, dR_t + \frac{1}{2}\, f''(R_t)\, d\langle R\rangle_t \right].$$

Setting $\phi(s) = f(1-s)$, using Itô's formula again, and setting the dt term equal to zero gives the equation

$$x\,(1-x)\,\phi''(x) + [4a - 2(1-a)x]\,\phi'(x) + 2a\,(4a-1)\,\phi(x) = 0,$$

which is the hypergeometric equation with bounded solution $\phi(x) = {}_2F_1(2a, 1-4a, 4a, x)$. This gives (14).

Similarly, let $\rho_{\epsilon,u} = \sigma_\epsilon \wedge \tau_u$. Since $M_{t\wedge\rho_{\epsilon,u}}\, N_{t\wedge\rho_{\epsilon,u}}$ is a bounded martingale,

$$\mathbb{E}[M_0 N_0] = \mathbb{E}[M_{t\wedge\rho_{\epsilon,u}}\, N_{\sigma_\epsilon\wedge\tau_u}] = \mathbb{E}^*[N_{\sigma_\epsilon\wedge\tau_u}].$$

Arguing as in Lemma 2.3, we see that

$$\mathbb{P}^*\{\tau_u < \sigma_\epsilon\} \le c\, u^{2\beta},$$

and hence,

$$\mathbb{E}^*[N_{\sigma_\epsilon\wedge\tau_u}; \tau_u < \sigma_\epsilon] \le c\, u^{\beta}.$$

Also, as in the previous paragraph,

$$\lim_{\epsilon\downarrow 0} \mathbb{E}^*[N_{\sigma_\epsilon\wedge\tau_u}; \sigma_\epsilon < \tau_u] = \mathbb{E}^*[N_T; T < \tau_u] = \mathbb{E}^*\left[\Psi_T^{-\beta}; T < \tau_u\right].$$

Combining the last two equalities, we see that

$$\lim_{\epsilon\downarrow 0, \delta\downarrow 0} \mathbb{E}^*[N_{\sigma_\epsilon\wedge\tau_\delta}] = \mathbb{E}^*\left[\Psi_T^{-\beta}; T < \tau_u\right].$$

But the left-hand side also equals $N_0 = x^{-\beta}\, h(x/(1+x))$. Combining this with (16) we get (14). □

3. Proof of Theorem 3.

For this section we will assume that $4 < \kappa < 8$ and hence that $\gamma \cap \mathbb{R}$ is a nonempty strict subset of $\mathbb{R}$. We will prove the result with $y_1 = 1$, $y_2 = 2$, but the same ideas work for all $y_1 < y_2$. For $x \in \mathbb{R}$ and $s > 0$, let

$$J_s(x) = e^{\beta s}\, 1\{\mathrm{dist}(x, \gamma) \le e^{-s}\},$$

$$J_s = \int_1^2 J_s(x)\, dx.$$

By (37), we have

$$\mathbb{E}[J_s(x)] = \hat{c}\, x^{-\beta} + O(e^{-\alpha s}),$$
$$\mathbb{E}[J_s] = \int_1^2 \mathbb{E}[J_s(x)]\, dx = \frac{(2^{1-\beta} - 1)\, \hat{c}}{1-\beta} + O(e^{-\alpha s}),$$
$$\mathbb{E}[J_s(x) - J_t(x)] = O(e^{-\alpha(s \wedge t)}), \quad \mathbb{E}[J_s - J_t] = O(e^{-\alpha(s\wedge t)}).$$

The key estimate is the following lemma which is not as sharp as one could prove, but it is good enough for our purposes. We will first show how to derive Theorem 3 from Lemma 3.1, and then we will prove the lemma. Let

$$Q_{s,r}(x) = J_{s+r}(x) - J_s(x), \quad Q_{s,r} = J_{s+r} - J_s.$$

LEMMA 3.1. *There exist $0 < c, \lambda < \infty$ such that if $x \geq 1 + e^{-\lambda s}$ and $0 \leq r \leq 1$, then*

$$|\mathbb{E}[Q_{s,r}(1)\, Q_{s,r}(x)]| \leq c\, e^{-\lambda s}.$$

We will actually prove this with $\lambda = \alpha/8$ but this is not the optimal value; we assume $\lambda < 1$. Using scaling, we see that Lemma 3.1 implies that if $y \geq 1, x \geq y + e^{-s\lambda}$, then

$$|\mathbb{E}[Q_{s,r}(y)\, Q_{s,r}(x)]| \leq c\, e^{-\lambda s}.$$

If $x \leq y + e^{-\lambda s}$ we will use the crude estimate

$$|Q_{s,r}(y)| \leq e^{\beta(s+r)}\, 1\{J_s(y) \neq 0\},$$

and (41) to conclude that

$$|\mathbb{E}[Q_{s,r}(y)\, Q_{s,r}(x)]| \leq e^{2\beta(s+r)}\, \mathbb{P}\{J_s(y) \neq 0, J_s(x) \neq 0\} \leq c\, |x-y|^{-\beta}.$$

Since $\beta < 1$, we see that

$$\mathbb{E}[Q_{s,r}^2] = 2 \int_1^2 \int_y^2 \mathbb{E}[Q_{s,r}(y)\, Q_{s,r}(x)]\, dx\, dy = O(e^{-\xi s}), \quad \xi = (1-\beta)\, \lambda.$$

This is the key estimate and the rest of the argument uses standard techniques. If k is a nonnegative integer, $u = 2^{-k}$, and $Y_n = J_{nu}$, then for $m > n$,

$$\|Y_n - Y_m\|_2 \leq \sum_{j=n+1}^{m} \|Y_j - Y_{j-1}\|_2 \leq c \sum_{j=n+1}^{\infty} e^{-\xi u j/2} \leq c\, e^{-\xi u n/2}.$$

Therefore, the sequence $\{Y_n\}$ is a Cauchy sequence in L^2 and converges to a limit random variable J. The limit does not depend on k, and since $\mathbb{E}[(J_{s+r} - J_s)^2] \leq c\, e^{-\xi s}$ for $r \leq 1$, we can see that $J_s \to J$ in L^2. To get convergence with probability one, we use Chebyshev's inequality to see that

$$\mathbb{P}\{|Y_{j+1} - Y_j| \geq e^{-j\xi u/4}\} \leq e^{j\xi u/2}\, \mathbb{E}[(Y_j - Y_{j-1})^2] \leq c\, e^{-j\xi u/2}.$$

By the Borel-Cantelli lemma, with probability one, for all j sufficiently large, $|Y_{j+1} - Y_j| \geq e^{-j\xi u/4}$, and hence with probability one, for each k,

$$\lim_{n\to\infty} J_{n2^{-k}} = J.$$

We can then use the monotonicity relation $J_{s+r} \leq e^{\beta r}\, J_s$, to conclude with probability one

$$\lim_{s\to\infty} J_s = J.$$

Since the convergence is in L^2, we have

$$\mathbb{E}[J] = \lim_{s\to\infty} \mathbb{E}[J_s] = \hat{c} \int_1^2 x^{-\beta}\, dx,$$

$$\begin{aligned}
\mathbb{E}[J^2] = \lim_{s\to\infty} \mathbb{E}[J_s^2] &= \lim_{s\to\infty} 2 \int_1^2 \int_x^2 \mathbb{E}[J_s(x)\, J_s(y)]\, dy\, dx \\
&= 2 \int_1^2 \int_x^2 x^{-2\beta}\, \mathbb{E}[J_s(1)\, J_s(y/x)]\, dy\, dx \\
&= 2\, \hat{c}^2 \int_1^2 \int_x^2 x^{-\beta}\, (y-x)^{-2\beta}\, h\!\left(\frac{x}{x+y}\right) dy\, dx.
\end{aligned}$$

Proof of Lemma 3.1. For notational ease, we will assume that $r = 1$ but the argument works identically for $0 \leq r \leq 1$. We write $Q_s(y) = Q_{s,1}(y) = J_{s+1}(y) - J_s(y)$. We let $\lambda = \alpha/8 \leq 1/8$. We assume that $x \geq 1 + e^{-\lambda s} \geq 1 + e^{-s/8}$. We recall that

$$|\mathbb{E}[Q_s(y)]| \leq c\, e^{-\alpha s}, \quad |Q_s(y)| \leq e^{(s+1)\beta}\, 1\{\mathrm{dist}(\gamma, y) \leq e^{-s}\}.$$

Let η_s, η_s' denote the circles of radius e^{-s} about 1 and x, respectively, and

$$\sigma_s = \inf\{t : \mathrm{dist}(\gamma_t, 1) \leq e^{-s}\}, \quad \tau_s = \inf\{t : \mathrm{dist}(\gamma_t, x) \leq e^{-s}\},$$
$$\rho_s = \inf\{t \geq \tau_{7s/8} : \mathrm{dist}(\gamma_t, 1) < e^{-s} \wedge \mathrm{dist}(\gamma_{\tau_{7s/8}}, 1)\},$$

where $\rho_s = \infty$ if $\tau_{7s/8} = \infty$. Arguing as in Lemma 2.3, we see that

$$\begin{aligned}
\mathbb{P}\{\tau_{7s/8} < \rho_s < \infty\} &\leq \mathbb{P}\{\tau_{7s/8} < \infty\}\, \mathbb{P}\{\rho_s < \infty \mid \tau_{7s/8} < \sigma_s\} \\
&\leq [c\, e^{-7s\beta/8}]\, [c\, e^{-7\beta s/8}\, e^{-\beta s}\, (x-1)^{-2\beta}] \leq c\, e^{-5\beta/2}. \qquad (53)
\end{aligned}$$

Here we are using $x \geq e^{-s/8}$.

Let $\hat\gamma = \gamma_{\tau_{7s/8}}$ where $\hat\gamma = \gamma$ if $\tau_{7s/8} = \infty$, and

$$\tilde J_s(1) = e^{\beta s}\, 1\{\mathrm{dist}(1,\hat\gamma) < e^{-s}\}, \quad \tilde J_{s+1}(1) = e^{\beta(s+1)}\, 1\{\mathrm{dist}(1,\hat\gamma) < e^{-(s+1)}\},$$
$$\tilde Q_s(1) = \tilde J_s(1) - \tilde J_{s+1}(1).$$

Since $Q_s(x) = 0$ on the event $\{\tau_{7s/8} = \infty\}$, we can write

$$\mathbb{E}[Q_s(1)\, Q_s(x)] = \mathbb{E}\big[\tilde Q_s(1)\, Q_s(x)\big] + \mathbb{E}\big[(Q_s(1) - \tilde Q_s(1))\, Q_s(x)\big].$$

Note that (53) implies that

$$\mathbb{E}\big[|Q_s(1) - \tilde Q_s(1)| Q_s(x)\big] \leq e^{2s\beta}\, \mathbb{P}\{\rho_s < \infty\} \leq c\, e^{-s/2}.$$

Also,

$$\mathbb{E}\big[\tilde Q_s(1)\, Q_s(x)\big] = \mathbb{E}\big[\tilde Q_s(1)\, \mathbb{E}(Q_s(x) \mid \hat\gamma)\big],$$

where the conditional expectation is defined to be zero if $\tau_{7s/8} = \infty$. We now use (36) to say that there exists $\alpha > 0$, $c < \infty$ such that on the event $\{\tau_{7s/8} < \infty\}$,

$$|\mathbb{E}(Q_s(x) \mid \hat\gamma)| \leq c\, e^{-\alpha s/8}\, e^{7s\beta/8}.$$

Therefore,

$$\big|\mathbb{E}[\tilde Q_s(1)\, Q_s(x)]\big| \leq c\, e^{-\alpha s/8}. \qquad \square$$

4. Asymptotically Bessel processes.

In Section 2, as is often the case in SLE, we had a simple SDE of the form

$$dX_t = F(X_t)\, dt + dB_t, \quad 0 < t < \pi.$$

Here we will discuss one way to establish the results that we needed there. The idea is to write $F(x) = [\log \Phi(x)]'$ and to consider this as a Brownian motion X_t weighted locally by $\Phi(X_t)$.

Suppose X_t is a standard Brownian motion starting at $x \in (0,\pi)$ and Φ is a (strictly) positive C^2 and L^2 function on $(0,\pi)$. Let $T = \inf\{t : X_t \in \{0,\pi\}\}$ and let

$$M_t = \Phi(X_t)\, A_t, \quad \text{where} \quad A_t = \exp\left\{ -\frac{1}{2}\int_0^t \frac{\Phi''(X_s)}{\Phi(X_s)}\, ds\right\}. \tag{54}$$

Then Itô's formula shows that M_t is a local martingale for $t < T$ satisfying

$$dM_t = [\log \Phi(X_t)]' \, M_t \, dX_t.$$

If $\epsilon > 0$ and τ is a stopping time with $\inf_{t \leq \tau} \sin X_t \geq \epsilon$, then $M_{t\wedge\tau}$ is a martingale and we can write $\mathbb{P}^x_\Phi, \mathbb{E}^x_\Phi$ for probabilities and expectations with respect to the new measure. That is, if Y is $\mathcal{F}_{t\wedge\tau}$-measurable, then

$$\mathbb{E}^x_\Phi[Y] = M_0^{-1} \, \mathbb{E}^x[Y \, M_{t\wedge\tau}] = \Phi(x)^{-1} \, \mathbb{E}^x[Y \, M_{t\wedge\tau}].$$

Girsanov's theorem states that

$$dX_t = [\log \Phi(X_t)]' \, dt + dB_t, \tag{55}$$

where B_t is a standard Brownian motion with respect to $\mathbb{P}^x_\Phi$. This holds for $t < \tau$, but since the equation does not include τ, we can write the equation for $t < T$.

For $0 < x, y < \pi$, let $p(t, x, y)$ denote the density at time t of a Brownian motion starting at x that is killed when it leaves $(0, \pi)$, that is,

$$\mathbb{P}^x\{y_1 < X_t < y_2; T > t\} = \int_{y_1}^{y_2} p(t, x, y) \, dy.$$

Let $p_\Phi(t, x, \cdot)$ be the corresponding density for X_t under the tilted measure $\mathbb{P}^x_\Phi$,

$$\mathbb{P}^x_\Phi\{y_1 < X_t < y_2; T > t\} = \int_{y_1}^{y_2} p_\Phi(t, x, y) \, dy.$$

We note that $\mathbb{P}^x_\Phi$ and $\mathbb{P}^x$, considered as measures on paths $X_s, 0 \leq s \leq t$, are mutually absolutely continuous on the event $\{T > t\}$ with Radon-Nikodym derivative

$$\frac{d\mathbb{P}^x_\Phi}{d\mathbb{P}^x} = \frac{M_t}{M_0} = \frac{\Phi(X_t)}{\Phi(X_0)} \, A_t,$$

where A_t is as in (54). If $\mathbb{P}^x_\Phi\{T < \infty\} = 0$, then $p_\Phi(t, x, y)$ is a probability density,

$$\int_0^\pi p_\Phi(t, x, y) \, dy = 1.$$

LEMMA 4.1. *If $0 < x < y < \pi$ and $t > 0$, then*

$$p_\Phi(t, x, y) \, \Phi(x)^2 = p_\Phi(t, y, x) \, \Phi(y)^2.$$

In particular, if $\mathbb{P}^x_\Phi\{T < \infty\} = 0$, then the invariant density for (55) *is $c \, \Phi^2$ where c is chosen to make this a probability measure.*

PROOF. For each path $\omega : [0, t] \to (0, \pi)$ with $\omega(0) = x$, $\omega(t) = y$, let ω^R denote the reverse path $\omega^R(s) = \omega(t - s)$ which goes from y to x. The Brownian measure on paths from y to x staying in $(0, \pi)$ can be obtained from the corresponding measure for paths from x to y by the transformation $\omega \mapsto \omega^R$. The compensating term A_t is a function of the path ω, and we can see that $A_t(\omega) = A_t(\omega^R)$. Then

$$p_\Phi(t,x,y) = p(t,x,y)\, \frac{\Phi(y)}{\Phi(x)}\, \mathbb{E}_{x,y,t}[A_t], \quad p_\Phi(t,y,x) = p(t,y,x)\, \frac{\Phi(x)}{\Phi(y)}\, \mathbb{E}_{y,x,t}[A_t],$$

where $\mathbb{E}_{x,y,t}$ denotes the probability measure associated to Brownian bridges of time duration t starting at x, ending at y, and staying in $(0,\pi)$. This quantity is not easy to compute, but the path reversal argument shows that $\mathbb{E}_{x,y,t}[A_t] = \mathbb{E}_{y,x,t}[A_t]$. This gives the first assertion, and then

$$\int_0^\pi \Phi(x)^2\, p_\Phi(t,x,y)\, dx = \int_0^\pi \Phi(y)^2\, p_\Phi(y,x,t)\, dx = \Phi(y)^2. \qquad \square$$

If Φ is L^2, but $\mathbb{P}^x_\Phi\{T < \infty\} = 1$, then an appropriate reflecting process can be defined. One way to construct it is to find a sequence $\Phi_n \uparrow \Phi$ of C^2 functions for which $\mathbb{P}^x_{\Phi_n}\{T < \infty\} = 0$ and such that $\Phi(x) = \Phi_n(x)$ for $x \in [1/n, \pi - 1/n]$. For each ϵ, if we only view the process during the excursions from $\{2\epsilon, \pi - 2\epsilon\}$ to $\{\epsilon, \pi - \epsilon\}$, then the process is the same for all Φ_n with $n > 1/\epsilon$. In particular, we can see that the invariant probability for the reflected process must also be proportional to Φ^2.

The reflected process can be a bit subtle at the boundary, but it is well understood in the case of Bessel processes. Recall that the Bessel process is obtained by tilting by the function $\Phi(x) = x^a$ where $a > -1/2$. Let $\hat{\mathbb{P}}^x$ denote the probabilities under the Bessel process reflected at 0 and killed at time T_π. Then X_t satisfies the Bessel equation

$$dX_t = \frac{a}{X_t}\, dt + dW_t,$$

where W_t is a $\hat{\mathbb{P}}^x$-Brownian motion. If $-1/2 < a < 1/2$ we must interpret this as the usual reflecting Bessel process. We will say that Φ is asymptotically a-Bessel near the origin if there exists an even, strictly positive C^2 function g such that $\Phi(x) = g(x)\, x^a, x \leq 3\pi/4$. Similarly, we say that Φ is asymptotically a-Bessel near π if $x \mapsto \Phi(\pi - x)$ is asymptotically a-Bessel near the origin. We will focus on behavior near the origin, but the same arguments work for behavior near π. We will be most interested in the example $\Phi(x) = [\sin x]^u\, [1 - \cos x]^v$, which is asymptotically $(u + 2v)$-Bessel at the origin and u-Bessel near π, and gives the equation

$$d\Theta_t = \left[\frac{v}{\sin \Theta_t} + (u+v)\, \cot \Theta_t\right] dt + dB_t. \tag{56}$$

This equation with $v = 0$ is sometimes called the *radial Bessel equation.* The invariant density ψ is proportional to Φ^2. If Θ has density ψ, then a standard calculation shows that $[1 - \cos \Theta]/2$ has a beta density

$$h(y) = \frac{\Gamma(2u + 2v + 1)}{\Gamma(u + 2v + 1/2)\, \Gamma(u + 1/2)}\, y^{u+2v-1/2}(1-y)^{u-1/2}, \quad 0 < y < 1.$$

Note that g, g', g'' are bounded on $[0, 3\pi/4]$ as is $g'(x)/[xg(x)]$ where the last function is defined to be $g''(0)/g(0)$ at $x = 0$. Itô's formula gives

$$\begin{aligned} dg(X_t) &= g'(X_t)\, dX_t + \frac{1}{2}\, g''(X_t)\, dt \\ &= g(X_t) \left[\left(\frac{a\, g'(X_t)}{g(X_t)\, X_t} + \frac{g''(X_t)}{2\, g(X_t)} \right) dt + \frac{g'(X_t)}{g(X_t)}\, dW_t \right]. \end{aligned}$$

Let $T = T_{3\pi/4} = \inf\{t : X_t = 3\pi/4\}$. If

$$M_t = g(X_t)\, A_t, \quad \text{where} \quad A_t = \exp\left\{ -\frac{1}{2} \int_0^t \left(\frac{a\, g'(X_s)}{g(X_s)\, X_s} + \frac{g''(X_s)}{2\, g(X_s)} \right) ds \right\},$$

then $M_{t\wedge T}$ is a martingale satisfying

$$dM_t = \frac{g'(X_t)}{g(X_t)}\, M_t\, dW_t, \quad t < T.$$

Also, $|\log A_{t\wedge T}|/(t \wedge T)$ is uniformly bounded. Recall that $\mathbb{P}^x_\Phi$ is obtained from $\mathbb{P}^x$ by weighting locally by $\Phi(X_t) = X_t^a\, g(X_t)$. Equivalently, we can get $\mathbb{P}^x_\Phi$ from $\hat{\mathbb{P}}^x$ by weighting locally by $g(X_t)$. We take the latter viewpoint and note that

$$dW_t = \frac{g'(X_t)}{g(X_t)}\, dt + dB_t, \quad t < T,$$

where B_t is a standard Brownian motion with respect to $\mathbb{P}^x_*$. The advantage is that the last equation can be considered for the reflected process X_t under $\hat{\mathbb{P}}^x$. We also write this as

$$dX_t = \left[\frac{a}{X_t} + \frac{g'(X_t)}{g(X_t)} \right] dt + dB_t = \frac{\Phi'(X_t)}{\Phi(X_t)}\, dt + dB_t, \tag{57}$$

where the reflection is interpreted as above. If we start the reflected process at 0 and write $\mathbb{P}_* = \mathbb{P}^0_*$, then

$$\frac{d\mathbb{P}_*}{d\mathbb{P}} = g(3\pi/4)\, A_T.$$

By analyzing the equation (57), we can see that $\mathbb{P}_*\{T < \infty\} = 1$, and hence

$$g(3\pi/4)\, \mathbb{E}[A_T] = g(0).$$

Given this representation, it becomes straightforward to establish estimates about the measure $\mathbb{P}_*$. For example, it is standard to show that if $\epsilon > 0$, then

$$\hat{\mathbb{P}}\{T \leq \epsilon\} > 0.$$

It follows immediately, that this holds for $\mathbb{P}_*$ as well. In particular, one can see that for every $\epsilon > 0$, there exists $\delta > 0$ such that if X_t^1, X_t^2 are independent processes satisfying (57) starting at different points, then with probability at least δ, $X_t^1 = X_t^2$ for some $t \leq \epsilon$. Using this and using a standard coupling procedure, we can define a process (X_t^1, X_t^2) on the same probability space satisfying (57) with $X_0^1 = x_1$, $X_0^2 = x_2$ such that

$$\mathbb{P}\{X_t^1 = X_t^2 \text{ for all } t \geq n\epsilon\} \geq 1 - (1-\delta)^n.$$

Also, by comparison with the Bessel process for which the transition density is known explicitly, there exist $0 < c_1(\epsilon) < c_2(\epsilon)$ such that for all x, y and all $t \geq \epsilon$,

$$c_1(\epsilon)\, \Phi^2(y) \leq p_\Phi(t,x,y) \leq c_2(\epsilon)\, \Phi^2(y). \tag{58}$$

Here we are writing $p_\Phi(t,x,y)$ for the probability density of the reflected process. If we start X^1 with the invariant density $c_0\, \Phi^2$ and X^2 with $X_0^2 = x$, we see that we can couple the processes so that with probability at least $1 - c\, e^{-\alpha t}$, the paths have coupled by time t and hence

$$\int_0^\pi |p_\Phi(t-1,x,y) - c_0\, \Phi(y)^2|\, dy \leq c\, e^{-\alpha t}.$$

In other words we can write

$$p_\Phi(t-1,x,y) = [1 - c\, e^{-\alpha t}]\, c_0\, \Phi(y)^2 + c\, e^{-\alpha t}\, \phi(t-1,x,y)$$

where $\phi(t,x,y) \geq 0$ with

$$\int_0^\pi \phi(t-1,x,y)\, dy = 1.$$

Using (58), we see that for each z,

$$\int_0^\pi \phi(t-1,x,z)\, \phi(1,z,y)\, dz \asymp \Phi(y)^2 = c_0\, \Phi(y)^2[1 + O(1)],$$

and hence we get

$$\left|p_\Phi(t,x,y) - c_0 \Phi^2(y)\right| \leq c\, e^{-\alpha t}\, \Phi^2(y).$$

5. SLE (κ, ρ) processes.

5.1. Definition and properties.

Two-sided chordal SLE to 1 was obtained from chordal SLE$_\kappa$ by weighting locally by X_t^{1-4a} where $X_t = g_t(1) - U_t$. By this we mean tilting in the sense of the Girsanov theorem by the local martingale $M_t = A_t\, X_t^{1-4a}$ where A_t is the C^1 process (compensator) that makes this a local martingale. More generally, we can weight locally by X_t^r, and that is what we will do in this section. Recall that SLE_κ satisfies

$$dX_t = \frac{a}{X_t}\,dt + dW_t,$$

where W_t is a standard Brownian motion. Itô's formula shows that

$$dX_t^r = X_t^r\left[\frac{r^2/2 + r(a-1/2)}{X_t^2}\,dt + \frac{r}{X_t}\,dB_t\right].$$

Differentiating the Loewner equation gives

$$\partial_t g_t'(1) = -\frac{a\,g_t'(1)}{X_t^2}, \quad g_t'(1) = \exp\left\{-\int_0^t \frac{a\,ds}{X_s^2}\right\}.$$

Therefore, if $\lambda = (r^2/2a) + r(1-1/2a)$ and

$$M_t = g_t'(1)^\lambda\, X_t^r = X_t^r \exp\left\{-\lambda\int_0^t \frac{a\,ds}{X_s^2}\right\}, \tag{59}$$

then M_t is a local martingale satisfying

$$dM_t = \frac{r}{X_t}\,M_t\,dW_t, \quad t < T.$$

If we let $\mathbb{P}^*$ denote the measure obtained by tilting by M_t, then the Girsanov theorem implies that

$$dW_t = \frac{r}{X_t}\,dt + dB_t,$$

where B_t is a $\mathbb{P}^*$-Brownian motion. In particular,

$$dX_t = \frac{r+a}{X_t}\,dt + dB_t. \tag{60}$$

If we change time $\hat{X}_t = X_{2t/a} = X_{\kappa t}$, then this equation becomes

$$d\hat{X}_t = \frac{r\kappa + 2}{\hat{X}_t}dt + \sqrt{\kappa}\,dB_t^*, \tag{61}$$

for another standard Brownian motion B_t^*. A process satisfying (61) was called an $\mathrm{SLE}(\kappa,\rho)$ process in [**13**] where $\rho = r\kappa = 2r/a$. This was perhaps a bad notation because the parameter ρ depended strongly on the parametrization of the SLE path. I find the parameter r more natural, but the important thing to remember is that the $\mathrm{SLE}(\kappa,\rho)$ process (with charge point 1) is the process obtained from SLE_κ by weighting locally as above by $X_t^{a\rho/2} = X_t^{\rho/\kappa}$. We will use the parameter $r = \rho/\kappa$ in this section, but will write SLE$(\kappa, \kappa r)$ to conform with the original notation.

We will use the radial parametrization and the notation from Section 2. Note that (22) gives

$$\begin{aligned} dJ_t &= \frac{J_t}{X_t^2}\left(1 - a - \frac{a}{1-J_t}\right)dt - \frac{J_t}{X_t}\,dW_t \\ &= \frac{J_t}{X_t^2}\left(1 - a - r - \frac{a}{1-J_t}\right)dt - \frac{J_t}{X_t}\,dB_t. \end{aligned}$$

If we change time as in that section, (25) becomes

$$d\hat{J}_t = \left[(1-2a-r) - (1-a-r)\,\hat{J}_t\right]dt + \sqrt{\hat{J}_t\,(1-\hat{J}_t)}\,d\hat{B}_t.$$

If $\hat{J}_t = [1 - \cos\hat{\Theta}_t]/2$, then

$$d\hat{\Theta}_t = \left[\frac{1-3a-r}{\sin\hat{\Theta}_t} + \left(\frac{1}{2} - a - r\right)\cot\hat{\Theta}_t\right]dt + d\hat{B}_t. \tag{62}$$

This is of the form (56) with

$$u = 2a - \frac{1}{2}, \quad v = 1 - 3a - r,$$

which is asymptotically $(2a-1/2)$-Bessel near π and asymptotically $(3/2-4a-2r)$-Bessel near 0. From this we can deduce the following known properties.

- Since $2a - 1/2 > -1/2$, the reflected process at π can be defined for all $\kappa > 0$.
- From (60), we see that if $r \geq 1/2-a$ ($\rho \geq \kappa/2-2$), then the process in the capacity parametrization exists for all times. In other words, $T_1 = \infty$.
- If $1/2 - 2a \leq r < 1/2 - a$ ($\kappa/2 - 4 \leq \rho < \kappa/2 - 2$), then $T_1 < \infty$, but $\hat{\Theta}_t$ reaches zero in finite time. This implies that $\mathrm{dist}(\gamma_{T_1}, 1) > 0$.
- If $r < 1/2 - 2a$ ($\rho < \kappa/2 - 4$), then $\hat{\Theta}_t$ exists for all times. This implies that $\mathrm{dist}(\gamma_{T_1}, 1) = 0$.

5.2. Moments of derivatives.

The Girsanov theorem gives quick proofs of some of the estimates about moments of the conformal maps g_t. This has been known for a while, but I do not believe these proofs in these cases have been written down. Anyway this gives a good example to illustrate a now standard procedure. To compute the expectation of a derivative, find an appropriate martingale or local martingale. The martingale property gives the expectation of the martingale at later times in terms of the original expectation. However, the martingale often includes extra terms. To recover the expectation of the derivative, one studies the process in the measure obtained by tilting by the local martingale.

PROPOSITION 5.1. *If γ is a chordal* SLE_κ *path with $\kappa > 0$ and*

$$\lambda > -\frac{\kappa}{4}\left(\frac{2}{\kappa} + \frac{1}{2}\right)^2,$$

then as $t \to \infty$,

$$\mathbb{E}\big[g_t'(1)^\lambda; T_1 > t\big] \sim c'\, t^{-r/2},$$

where r is the larger root of (59),

$$r = \frac{1}{2} + \frac{2}{\kappa} + \sqrt{\left(\frac{2}{\kappa} - \frac{1}{2}\right)^2 + \frac{4\lambda}{\kappa}}$$

and

$$c' = \Gamma\left(\frac{2}{\kappa} + \frac{r}{2} + \frac{1}{2}\right) \Big/ \Gamma\left(\frac{2}{\kappa} + r + \frac{1}{2}\right).$$

PROOF. We consider the local martingale $M_t = g_t'(1)^\lambda\, X_t^r$. Let $\mathbb{P}^*, \mathbb{E}^*$ denote probabilities and expectations with respect to the tilted measure under which X_t satisfies (60) with B_t a $\mathbb{P}^*$-Brownian motion. Then

$$\mathbb{E}\big[g_t'(1)^\lambda; T_1 > t\big] = \mathbb{E}\big[M_t\, X_t^{-r}; T_1 > t\big] = \mathbb{E}^*\big[X_t^{-r}; T_1 > t\big] = t^{-r/2}\, \mathbb{E}^*\big[(X_t/\sqrt{t})^{-r}\big].$$

The last equality uses $r + a > 1/2$ which implies that $\mathbb{P}^*\{T_1 > t\} = 1$. The final expectation is with respect to a Bessel process and can be given explicitly. To understand the asymptotics it is useful to let $Y_t = e^{-t}\, X_{e^{2t}}$ which satisfies

$$dY_t = \left[\frac{a+r}{Y_t} - Y_t\right] dt + dW_t,$$

for a standard Brownian motion W_t. This is the equation one gets by starting with a standard Brownian motion Y_t with $Y_0 = 1$ and weighting locally by $\Phi(Y_t)$ where

$$\Phi(x) = x^{a+r}\, e^{-x^2/2}.$$

This is a positive recurrent diffusion on $(0, \infty)$ with invariant density $\phi(x) = c\, \Phi(x)^2$. Using this, we see that

$$\lim_{t\to\infty} \mathbb{E}^*\big[(X_t/\sqrt{t})^{-r}\big] = \int_0^\infty x^{-r}\, \phi(x)\, dx = \frac{\int_0^\infty x^{2a+r}\, e^{-x^2}\, dx}{\int_0^\infty x^{2a+2r}\, e^{-x^2}\, dx} = \frac{\Gamma(a + r/2 + 1/2)}{\Gamma(a + r + 1/2)}.$$

The last equality uses

$$2\int_0^\infty x^q\, e^{-x^2}\, dx = \int_0^\infty u^{\frac{q-1}{2}}\, e^{-u}\, du = \Gamma\Big(\frac{q+1}{2}\Big). \qquad \square$$

The proof shows the stronger fact that if I is an interval then

$$\lim_{t\to\infty} t^{r/2}\, \mathbb{E}\big[g_t'(1)^\lambda; X_t/\sqrt{t} \in I, T_1 > t\big] = \frac{\int_I x^{2a+r}\, e^{-x^2}\, dx}{\Gamma(a+r+1/2)}.$$

We can get more information. Let $\hat{g}_t(z) = g_{e^{2t}}(z)$. Then,

$$-\log \hat{g}_t'(1) = -\log g_{e^{2t}}'(1) = \int_0^{e^{2t}} \frac{a\, ds}{X_s^2} = \int_0^t \frac{a[2\, e^{2u}\, du]}{X_{e^{2u}}^2} = \int_0^t \frac{2a\, du}{Y_u^2}.$$

Since Y_t is a positive recurrent diffusion with invariant probability $c\,\Phi^2$, we see that with $\mathbb{P}^*$-probability one,

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{2a\, du}{Y_u^2} &= \int_0^\infty 2a\, y^{-2}\, c\,\Phi^2(y)\\ &= \frac{\int_0^\infty x^{2a+2r-2}\, e^{-x^2}\, dx}{\int_0^\infty x^{2a+2r}\, e^{-x^2}\, dx} = \frac{2}{2a+2r-1}.\end{aligned}$$

In other words, the typical value of $g_{e^{2t}}'(1)$ for t large *with respect to the measure tilted by* M is $(e^{2t})^{-1/(2a+2r-1)}$.

5.3. $r < 1/2 - 2a$.

We will consider the case $r < 1/2 - 2a$ for which the process satisfying (62) avoids the origin for all $t > 0$ and either avoids π if $\kappa \le 4$ or can be defined by reflection at π for $\kappa < 4 < \infty$. We write $\xi = r + \lambda$ and note that

$$\xi = \frac{r^2}{2a} + r\Big(2 - \frac{1}{2a}\Big), \quad r = \frac{1}{2} - 2a - \sqrt{\Big(2a - \frac{1}{2}\Big)^2 + 2\xi}. \tag{63}$$

Alternatively, we could start with $\xi > -(2a - 1/2)^2/2$, set $r = r_\xi$ as above, and let $\lambda = \xi - r$.

Since the curve approaches 1 in the tilted measure, the radial parametrization (24) works well. We use the notation from Section 2, and let $\hat{g}_t(z) = g_{\sigma(t)}(z)$. We consider the local martingale

$$\hat{M}_t = \hat{X}_t^r\, \hat{g}_t'(1)^\lambda = \hat{\Upsilon}_t^r\, \hat{J}_t^{-r}\, \hat{g}_t'(1)^{\lambda+r} = e^{art}\, \hat{J}_t^{-r}\, \hat{g}_t'(1)^\xi.$$

The two-sided chordal SLE is the case $\xi = 0$, $r = 1 - 4a$, $\lambda = 0$. If we let $\mathbb{P}^*$ denote probabilities with respect to the tilted measure then some calculation shows that the analogue of (26) is

$$d\hat{\Theta}_t = \left[\frac{1-3a-r}{\sin\hat{\Theta}_t} + \left(\frac{1}{2} - a - r\right)\cot\Theta_t\right] + d\hat{B}_t,$$

where $\hat{B}_t$ is a $\mathbb{P}^*$-Brownian motion. This is of the form (56) with $u = 2a - 1/2$, $v = 1 - 3a - r$. The invariant density for $\hat{J}_t$ is

$$h(y) = \frac{\Gamma(2-2a-2r)}{\Gamma(2-4a-2r)\,\Gamma(2a)}\, y^{1-4a-2r}\,(1-y)^{2a-1}.$$

Similarly to before, we can find $\alpha = \alpha_{r,\kappa} > 0$ such that

$$\begin{aligned}\mathbb{E}\big[g'_{\sigma(t)}(1)^{\xi};\Upsilon_\infty < e^{-at}\big] &= e^{art}\,\mathbb{E}\big[\hat{M}_t\,\hat{J}_t^r;\sigma(t)<\infty\big]\\ &= e^{art}\,\mathbb{E}^*\big[\hat{J}_t^r\big] = e^{-art}\,c'\,[1+O(e^{-\alpha s})],\end{aligned}$$

where

$$c' = \int_0^1 y^r\,h(y)\,dy = \frac{\Gamma(2-2a-2r)\,\Gamma(2-4a-r)}{\Gamma(2-4a-2r)\,\Gamma(2-2a-r)}. \tag{64}$$

Note that we need $r < 1-2a$ for the integral to be finite.

Proposition 5.2. *If $r < 1/2 - 2a$, then there exists $\alpha = \alpha(r,\kappa) > 0$ such that if $\tau_\epsilon = \inf\{t : \Upsilon_t = \epsilon\}$, then*

$$\mathbb{E}\big[g'_{\tau_\epsilon}(1)^{\xi};\tau_\epsilon < \infty\big] = c'\,\epsilon^r[1+O(\epsilon^\alpha)],$$

where ξ, c' are as in (63) *and* (64)*.*

By looking at the tilted measure, we also see that typical value of $g'_{\tau_\epsilon}(1)$ in the expectation. Let $L_t = -\log\hat{g}'_t(z)$. Note that

$$\partial_t L_t = \frac{a}{X^2_{\sigma(t)}}\,\dot{\sigma}(t) = \frac{a\,(1-\hat{J}_t)}{\hat{J}_t}, \quad L_t = a\int_0^t \frac{1-\hat{J}_s}{\hat{J}_s}\,ds,$$

and

$$\int_0^1 \frac{1-y}{y}\,h(y)\,dy = \frac{\Gamma(1-4a-2r)\,\Gamma(2a+1)}{\Gamma(2-4a-2r)\,\Gamma(2a)} = \frac{2a}{1-4a-2r}.$$

Using positive recurrence of the diffusion under $\mathbb{P}^*$, we see that with $\mathbb{P}^*$-probability one,

$$\lim_{t\to\infty}\frac{L_t}{t} = am, \quad m = m_{r,\kappa} = \frac{2a}{1-4a-2r} = \frac{4}{(1-2r)\kappa - 8}.$$

We expect that $L_t = m\,t + O(t^{1/2})$. Indeed, one can show that for u in a neighborhood of the origin,

$$\mathbb{E}^*\left[\exp\left\{u\frac{L_t - amt}{\sqrt{t}}\right\}\right] < \infty.$$

PROPOSITION 5.3. *If $\sigma_s = \inf\{t : \mathrm{dist}(\gamma_t, 1) \le e^{-s}\}$, then*

$$\mathbb{E}\big[g'_{\sigma_s}(1)^{\xi}; \sigma_s < \infty\big] \asymp e^{-rs}.$$

Moreover, the expectation is carried on an event where $g'_{\sigma_s}(1)^{\xi} \approx e^{-ms}$. More precisely, for u in a neighborhood of the origin,

$$\mathbb{E}\left[\exp\left\{\frac{u\,[sm + \log g'_{\sigma_\epsilon}(1)]}{\sqrt{s}}\right\} g'_{\sigma_s}(1)^{\xi}; \sigma_s < \infty\right] \le c\,e^{-rs}.$$

Roughly speaking, the expectation $[g'_{\sigma_s}(1)^{\xi}; \sigma_s < \infty]$ is carried on an event on which $g'_{\sigma_s}(1) \approx e^{-m}$ and the probability of this event is on the order $e^{-s(r-\xi m)}$.

5.4. One-point estimate for SLE $(\kappa, r\kappa)$ processes.

Our definition of SLE$(\kappa, r\kappa)$ process used the point 1 as the force point. Here we assume that the force point, which we call x_0, lies in $[0+, 1)$. To be more precise, using the notation of Section 2, let $O_t = g_t(x_0) - U_t$ which satisifes

$$dO_t = \frac{a+r}{O_t}\,dt + dW_t, \quad O_0 = x_0$$

for a standard Brownian motion W_t. Here we assume that $a + r > -1/2$ so that this is well defined, perhaps with reflection at the origin. Hence, if $X_t = g_t(1) - U_t$,

$$dX_t = \left(\frac{a}{X_t} + \frac{r}{O_t}\right)dt + dW_t.$$

Let Y_t, J_t, Υ_t be as in Section 2, and note that $Y_0 = J_0 = \Upsilon_0 = 1 - x_0$.

PROPOSITION 5.4. *Suppose $0 < \kappa < 8$, $r\kappa > \max\{\kappa/2 - 4, -2\}$, and γ is an SLE$(\kappa, r\kappa)$ process with force point $x_0 \in [0+, 1)$. Then there exists $\alpha > 0$ such that if $0 < \epsilon \le 1/2$,*

$$\mathbb{P}\{\Upsilon_\infty \le \epsilon\,(1 - x_0)\} = c_*\,\epsilon^{\beta(1+r/a)}\,(1 - x_0)^{\beta}[1 + O(\epsilon^{\alpha})],$$

where

$$\beta = 4a + 2r - 1, \quad c_* = \frac{\Gamma(6a + 4r)}{\Gamma(4a + 2r)\,\Gamma(2a + 2r - 1)}.$$

Note that the assumptions imply that $r > (1/2 - 2a) \vee (-a)$, and hence $\beta > 0$, $\beta(1 + r/a) > 0$.

PROOF. The case $r = 0$ was done in Section 2 and we follow the same approach. Let $M_t = \Upsilon_t^{\beta(1+r/a)}\, J_t^{-\beta}$. Itô's formula shows that M_t is a local martingale satisfying

$$dM_t = -\frac{\beta}{X_t}\, M_t\, dW_t, \quad M_0 = (1 - x_0)^{-\beta r/a}.$$

If we tilt by $\tilde{M}_t$, then

$$dX_t = \left(\frac{1 - 3a - 2r}{X_t} + \frac{r}{O_t}\right) dt + dB_t,$$

$$dJ_t = \frac{J_t}{X_t^2}\left(3a + 2r - \frac{r + a}{1 - J_t}\right) dt - \frac{J_t}{X_t}\, dB_t,$$

where B_t is a standard Brownian motion in the new measure $\mathbb{P}^*$. If we change time setting $\hat{J}_t = J_{\sigma(t)}$ where $\sigma(t) = \inf\{t : \Upsilon_t = e^{-at}\}$, then

$$d\hat{J}_t = \big[(2a + r) - (2r + 3a)\, \hat{J}_t\big] dt + \sqrt{\hat{J}_t\, (1 - \hat{J}_t)}\, d\hat{B}_t,$$

for a standard Brownian motion $\hat{B}_t$. If we let $\hat{J}_t = (1/2)[1 - \cos\Theta_t]$, then

$$d\Theta_t = \left[\frac{a}{\sin\Theta_t} + \left(3a + 2r - \frac{1}{2}\right)\cot\Theta_t\right] dt + d\hat{B}_t.$$

This is of the form of (56) with $u = 2a + 2r - 1/2$, $v = a$. Our assumptions imply that $u > -1/2$, $u + v > 1/2$, and hence the process exists for all times, perhaps with reflection at π, but not reaching the origin in finite time. The invariant density of $\hat{J}_t$ is

$$h(x) = \frac{\Gamma(6a + 4r)}{\Gamma(4a + 2r)\, \Gamma(2a + 2r)}\, x^{4a+2r-1}\, (1 - x)^{2a+2r-1},$$

and

$$\int_0^1 x^{1-2r-4a}\, h(x)\, dx = (2a + 2r)\, \frac{\Gamma(6a + 4r)}{\Gamma(4a + 2r)\, \Gamma(2a + 2r)} = \frac{\Gamma(6a + 4r)}{\Gamma(4a + 2r)\, \Gamma(2a + 2r - 1)}.$$

Using exponential rate of convergence to the invariant distribution, we see that

$$\mathbb{E}^*\big[\hat{J}_t^{\beta}\big] = c_* + O(e^{-\alpha t}),$$

and hence, if $e^{-at} \leq (1 - x_0)$,

$$
\begin{aligned}
\mathbb{P}\{\Upsilon_\infty \le e^{-at}\} &= \mathbb{E}\big[1\{\Upsilon_\infty \le e^{-at}\}\big] \\
&= e^{-a\beta(1+r/a)t}\, \mathbb{E}\big[\tilde{M}_t\, \tilde{J}_t^{\beta}; \Upsilon_\infty \le e^{-at}\big] \\
&= e^{-a\beta(1+r/a)t}\, M_0^{-1}\, \mathbb{E}^*\big[\tilde{J}_t^{\beta}\big] \\
&= c_*\, e^{-a\beta(1+r/a)t}\, (1-x_0)^{-\beta r/a}[1+O(e^{-ut})].
\end{aligned}
$$

The third equality uses the fact that with $\mathbb{P}^*$-probability one, $\Upsilon_t \to 0$. Therefore,

$$
\mathbb{P}\{\Upsilon_\infty \le \epsilon\,(1-x_0)\} = c_*[\epsilon(1-x_0)]^{\beta(1+r/a)}\,(1-x_0)^{-\beta r/a}[1+O(\epsilon^{\alpha})]. \qquad \square
$$

When $0<\beta(1+r/a)<1$, then a "back of the envelope" calculation suggests that the Hausdorff dimension of the $\gamma(0,\infty)\cap[0,\infty)$ should be $1-\beta(1+r/a) = 1-\kappa^{-1}\,(2+r\kappa)\,(4+r\kappa-\kappa/2)$. Indeed, this can be proved with standard techniques once a two-point estimate analogous to Lemma 2.3 is established. One can do it in this case but it is a little more difficult than the proof of Lemma 2.3. We choose not to do it here. One reason to omit it is that a different proof of this result is available in [**18**]. Similarly, we expect (although have not proved) that the one-point estimate with distance replacing conformal distance and a Minkowski content result can be proved.

References

[1] T. Alberts and M. Kozdron, Intersection probabilities for a chordal SLE path and a semicircle, Electron Comm. Probab., **13** (2008), 448–460.

[2] T. Alberts and S. Sheffield, Hausdorff dimension of the SLE curve intersected with the real line, Electron J. Probab., **40** (2008), 1166–1188.

[3] T. Alberts and S. Sheffield, The covariant measure of SLE on the boundary, Probab. Th. Rel. Fields, **149** (2011), 311–371.

[4] V. Beffara, The dimension of SLE curves, Annals of Probab., **36** (2008), 1421–1452.

[5] P. Duren, Univalent Functions, Springer-Verlag, 1983.

[6] L. Field and G. Lawler, Escape probability and transience for SLE, Electron. J. Probab., **20** (2015), Article no. 10.

[7] C. Garban, S. Rohde and O. Schramm, Continuity of the SLE trace in simply connected domains, Israel J. Math., **187** (2012), 23–36.

[8] K. Itô and H. McKean, Diffusion Processes and their Sample Paths, Springer-Verlag, 1974.

[9] F. Johansson Viklund and G. Lawler, Almost sure multifractal spectrum for the tip of an SLE curve, Acta Math., **209** (2012), 265–322.

[10] G. Lawler, Schramm-Loewner evolution, In: Statistical Mechanics, (eds. S. Sheffield and T. Spencer), IAS/Park City Mathematical Series, AMS, 2009, 231–295.

[11] G. Lawler, Multifractal analysis of the reverse flow for SLE, In: Fractal Geometry and Stochastics IV, (eds. C. Brandt, P. Mörters, and M. Zähle), Birkhäuser, 2009, 73–107.

[12] G. Lawler, Continuity of radial and two-sided radial SLE_κ at the terminal point, Contemporary Mathematics, **590** (2013), 101–124.

[13] G. Lawler, O. Schramm and W. Werner, Conformal restriction: the chordal case, J. Amer. Math. Soc., **16** (2003), 917–955.

[14] G. Lawler and M. Rezaei, Minkowski content and natural parameterization for the Schramm-Loewner evolution, Annals of Probab., **43** (2015), 1082–1120.

[15] G. Lawler and S. Sheffield, A natural parametrization for the Schramm-Loewner evolution, Annals of Probab., **39** (2011), 1896–1937.

[16] G. Lawler and B. Werness, Multi-point Green's functions for SLE and an estimate of Beffara,

Annals of Probab., **41** (2013), 1513–1555.

[17] G. Lawler and W. Zhou, SLE curves and natural parametrization, Annals of Probab., **41** (2013), 1556–1584.

[18] J. Miller and H. Wu, Intersections of SLE Paths: the double and cut point dimension of SLE, arXiv:1303.4725.

[19] M. Rezaei, Hausdorff dimension of SLE curves, arXiv:1212:5847.

[20] S. Rohde and O. Schramm, Basic properties of SLE, Ann. of Math., **161** (2005), 879–920.

[21] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math., **118** (2000), 221–288.

[22] O. Schramm and W. Zhou, Boundary proximity of SLE, Probab. Th. Rel. Fields, **146** (2010), 435–450.

[23] S. J. Taylor, The exact Hausdorff measure of the sample path for planar Brownian motion, Proc. Cambridge Philos. Soc., **60** (1964), 253–258.

Gregory F. Lawler
Department of Mathematics
University of Chicago
Chicago
IL 60637-1546, U.S.A.
E-mail: lawler@math.uchicago.edu