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Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces

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Dedicated to the memory of Professor Kiyosi Itô

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Abstract. We give necessary and sufficient conditions for sub-Gaussian estimates of the heat kernel of a strongly local regular Dirichlet form on a metric measure space. The conditions for two-sided estimates are given in terms of the generalized capacity inequality and the Poincaré inequality. The main difficulty lies in obtaining the elliptic Harnack inequality under these assumptions. The conditions for upper bound alone are given in terms of the generalized capacity inequality and the Faber–Krahn inequality.

1. Introduction.

1.1. History and motivation.

In this paper we are concerned with heat kernel estimates in the setting of Dirichlet forms on metric measure spaces. A classical example is the heat kernel $p_t(x,y)$ in $\mathbb{R}^n$ that is the fundamental solution of the heat equation given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(- \frac{|x-y|^2}{4t} \right).$$

The notion of the heat kernel is well defined on any Riemannian manifold. Then $p_t(x,y)$ is a smooth positive function but obtaining estimates is a highly non-trivial task as the heat kernel depends significantly on the geometry of the underlying spaces. For example, on a complete Riemannian manifold of non-negative Ricci curvature the heat kernel satisfies the Li–Yau estimate

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(- c\frac{d^2(x,y)}{t} \right)$$

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where $d(x,y)$ is the geodesic distance, $V(x,r)$ is the volume of the geodesic ball of radius r centered at x, and the sign $\asymp$ means that both inequalities with $\leq$ and $\geq$ are satisfied but with different values of positive constants c, C. This and further results on heat kernel bounds on Riemannian manifolds and in $\mathbb{R}^n$ can be found in [**2**], [**11**], [**13**], [**14**], [**19**], [**20**], [**21**], [**44**], [**49**], [**50**], [**52**], [**53**], [**55**] and in many other references.

The development of Analysis on fractals in the past three decades has led to construction of diffusion processes and their associated heat kernels on wide class of fractals. For example, the diffusion on Sierpinski gasket SG in $\mathbb{R}^n$ was constructed by Barlow and Perkins [**10**] and Kusuoka [**39**]. Moreover, Barlow and Perkins [**10**] proved that the associated heat kernel $p_t(x,y)$ on SG is a continuous function of t, x, y and satisfies the estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{1/(\beta-1)}\right), \tag{1.1}$$

where $\alpha = \log(n+1)/\log 2$ is the Hausdorff dimension of SG, $\beta = \log(n+3)/\log 2$ is a so called walk dimension, and $d(x,y) = |x-y|$ (see also [**3**]). The estimate (1.1) is satisfied also on many other fractals but with different values of the parameters α, β depending on the particular fractal. For example, this is the case for Sierpinski carpets (see [**4**], [**5**]).

Kigami introduced in [**34**] the notion of p.c.f. fractals and showed the existence of the heat kernel on p.c.f. fractals with regular harmonic structure. Hambly and Kumagai [**31**] proved two sided estimates of heat kernel on p.c.f. fractals; in general such estimate look more complicated than (1.1). As a consequence of their estimates, the heat kernel satisfies the upper bound in (1.1) with the resistance metric d and the following *near-diagonal* lower bound:

$$p_t(x,y) \geq \frac{C}{t^{\alpha/\beta}} \quad \text{whenever} \quad d(x,y) \leq t^{1/\beta}. \tag{1.2}$$

The validity of the full lower bound in (1.1) depends on some additional properties of the distance function d that are not satisfied by the resistance metric (see [**30**]).

The above mentioned results motivate investigation of heat kernels associated with strongly local regular Dirichlet forms on metric measure spaces with volume doubling property. The main problem here is to provide reasonable necessary and/or sufficient conditions for the heat kernel bounds in terms of more convenient conditions. A number of results in this direction were obtained in [**1**], [**22**], [**23**], [**24**], [**25**], [**26**], [**27**], [**28**], [**29**], [**30**], [**32**], [**34**], [**35**], [**36**], [**37**] and in other papers.

In this paper we prove the necessary and sufficient conditions for the heat kernel to satisfy the upper bound in (1.1) and the lower bound (1.2) in terms of the Poincaré inequality and a generalized capacity inequality. We state the results in Subsection 1.3 after introduction of all necessary notions, and compare them with the previously known results. Our work on this subject was strongly motivated by the papers of Andres and Barlow [**1**] and Barlow, Bass and Kumagai [**7**].

Notation. The letters C, C', C_i, c, c', c_i will always refer to positive constants, whose values are unimportant and may change at each occurrence. All results of this paper are quantitative, that is, the constants in the conclusions depend only on the constants in the assumptions.

1.2. Basic setup.

Everywhere in this paper (M, d) is a locally compact separable metric space and μ is a Radon measure on M with full support (namely, $\mu(\Omega) > 0$ for any non-empty open subset Ω of M). We refer to such a triple (M, d, μ) as a *metric measure space.*

Denote by

$$B(x, r) = \{y \in M : d(x, y) < r\}$$

the open metric ball of radius $r > 0$ centered at x. If B is a ball of radius r, then λB denotes the concentric ball of radius λr.

We always assume that every ball $B(x, r)$ is precompact. In particular, the volume function

$$V(x, r) := \mu(B(x, r))$$

is finite and positive for all $x \in M$ and $r > 0$.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2 := L^2(M, \mu)$, where $\mathcal{F}$ is a dense subspace of L^2 and $\mathcal{E}$ is a bilinear form on $\mathcal{F}$ that is symmetric, non-negative definite, closed and Markovian (see [**16**]). Recall that $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$, where $C_0(M)$ is the space of all continuous functions with compact support in M, endowed with sup-norm, and the norm in $\mathcal{F}$ is $\mathcal{E}(u, u) + \|u\|_2^2$. The form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f, g) = 0$ for all functions $f, g \in \mathcal{F}$ such that their supports are compact and $f \equiv \text{const}$ in an open neighborhood of $\operatorname{supp} g$.

Let Δ be the generator of $(\mathcal{E}, \mathcal{F})$, that is, Δ is a self-adjoint and non-positive definite operator in L^2 with the domain $\mathrm{dom}(\Delta)$ that is dense in $\mathcal{F}$ and such that, for all $f \in \mathrm{dom}(\Delta)$ and $g \in \mathcal{F}$,

$$\mathcal{E}(f, g) = -(\Delta f, g),$$

where $(\cdot, \cdot)$ is the inner product in L^2. The associated *heat semigroup*

$$P_t = e^{t\Delta}, \quad t \geq 0,$$

is a family of contractive, strongly continuous, self-adjoint operators in L^2 that satisfies the Markovian property (cf. [**16**]).

It is known that P_t extends to a contractive semigroup in L^p for any $p \in [1, \infty]$. The form $(\mathcal{E}, \mathcal{F})$ is called *conservative* if $P_t 1 = 1$ for every $t > 0$.

A family $\{p_t\}_{t>0}$ of non-negative $\mu \times \mu$-measurable functions on $M \times M$ is called the *heat kernel* of the form $(\mathcal{E}, \mathcal{F})$ if p_t is the integral kernel of the operator P_t, that is, for any $t > 0$ and for any $f \in L^2$,

$$P_t f(x) = \int_M p_t(x,y) f(y) d\mu(y) \tag{1.3}$$

for μ-almost all $x \in M$.

For a non-empty open $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in the norm of $\mathcal{F}$. It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(\Omega))$ is also a regular Dirichlet form in $L^2(\Omega, \mu)$. Denote by P_t^Ω the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$ and by Δ^Ω the generator of $(\mathcal{E}, \mathcal{F}(\Omega))$.

An example of the above setting is given by a Riemannian manifold M with the geodesic distance d, the Riemannian measure μ and the classical Dirichlet form

$$\mathcal{E}(f,g) = \int_M \langle \nabla f, \nabla g \rangle d\mu \tag{1.4}$$

with the domain $\mathcal{F} = W_0^1(M, \mu)$ (cf. [**21**]). A particular case of this example is any Euclidean space $\mathbb{R}^n$.

Another class of examples that is of utmost interest for us is given by fractal spaces (cf. [**3**], [**34**], [**54**]).

In the sequel we review some properties of the energy measure associated the regular Dirichlet form (cf. [**33**], [**47**], or [**16**, Section 3.2]). Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. It is known that each $u \in \mathcal{F}$ admits a *quasi-continuous* version $\widetilde{u}$ (cf. [**16**, Theorem 2.1.3, p. 71]). In what follows we make a convention that $u \in \mathcal{F}$ is understood to be its quasi-continuous modification. For each $u \in \mathcal{F} \cap L^\infty$, there exists a unique positive Radon measure $\Gamma\langle u \rangle$ on M such that

$$\int_M f d\Gamma\langle u \rangle = \mathcal{E}(uf, u) - \frac{1}{2}\mathcal{E}(u^2, f) \text{ for any } f \in \mathcal{F} \cap C_0(M), \tag{1.5}$$

and $\Gamma\langle u \rangle(M) < \infty$. The measure $\Gamma\langle u \rangle$ is called the *energy measure* of u. Note that the energy measure $\Gamma\langle u \rangle$ can be uniquely extended to any $u \in \mathcal{F}$.

For $u, w \in \mathcal{F}$, we introduce a signed measure $\Gamma\langle u, w \rangle$ by

$$\Gamma\langle u, w \rangle = \frac{1}{2}(\Gamma\langle u + w \rangle - \Gamma\langle u \rangle - \Gamma\langle w \rangle). \tag{1.6}$$

Then $u, v \mapsto \Gamma\langle u, w \rangle$ is symmetric and bilinear. The following identity is satisfied for all $u, w \in \mathcal{F} \cap L^\infty$ and $f \in \mathcal{F} \cap C_0$

$$\int_M f d\Gamma\langle u, w \rangle = \frac{1}{2}\{\mathcal{E}(uf, w) + \mathcal{E}(u, wf) - \mathcal{E}(uw, f)\} \tag{1.7}$$

(see for example [**47**, formula (3.11)]). For all $u, w \in \mathcal{F}$ we have

$$\mathcal{E}(u, w) = \int_M d\Gamma\langle u, w \rangle. \tag{1.8}$$

Moreover, the Cauchy–Schwarz inequality holds:

$$\left|\int_M fg d\Gamma\langle u,w\rangle\right| \le \left(\int_M f^2 d\Gamma\langle u\rangle\right)^{1/2}\left(\int_M g^2 d\Gamma\langle w\rangle\right)^{1/2} \tag{1.9}$$

$$\le \frac{1}{2b}\int_M f^2 d\Gamma\langle u\rangle + \frac{b}{2}\int_M g^2 d\Gamma\langle w\rangle, \tag{1.10}$$

for all $f,g \in \mathcal{F}\cap C_0$, $u,w\in\mathcal{F}$, $b>0$.

If in addition $(\mathcal{E},\mathcal{F})$ is strongly local, then the *Leibniz* and *chain* rules hold: for all $u,w,\varphi \in \mathcal{F}\cap L^\infty$,

$$\begin{aligned} d\Gamma\langle uw,\varphi\rangle &= ud\Gamma\langle w,\varphi\rangle + wd\Gamma\langle u,\varphi\rangle \quad \text{(Leibniz rule)}\\ d\Gamma\langle \Phi(u),w\rangle &= \Phi'(u)d\Gamma\langle u,w\rangle \quad \text{(chain rule)}\end{aligned} \tag{1.11}$$

where $\Phi:\mathbb{R}\to\mathbb{R}$ is any smooth function with $\Phi(0)=0$ (see [**16**, Lemma 3.2.5, p. 127, and Theorem 3.2.2, p. 129]).

For an open subset Ω of M and for $u_1,u_2\in\mathcal{F}\cap C_0$, if $u_1|_\Omega = u_2|_\Omega$, then

$$1_\Omega d\Gamma\langle u_1\rangle = 1_\Omega d\Gamma\langle u_2\rangle \text{ on } M,$$

and if u_1 is constant in Ω and u_2 is arbitrary, then

$$1_\Omega d\Gamma\langle u_1,u_2\rangle = 0 \text{ on } M, \tag{1.12}$$

(cf. [**33**], or [**47**, formulas (3.7), (3.8), p. 387]). Finally, for all $u,w\in\mathcal{F}\cap L^\infty$, we have

$$d\Gamma\langle u_+,w\rangle = \mathbf{1}_{\{u>0\}}d\Gamma\langle u,w\rangle \text{ on } M,$$

where $u_+ = u\vee 0$.

1.3. Main results.

Before we state our main results, let us give some necessary definitions. Let (M,d,μ) be a metric measure space with precompact metric balls, and $(\mathcal{E},\mathcal{F})$ be a strongly local, regular Dirichlet form in $L^2(M,\mu)$.

Definition. We say that the *volume doubling* condition (VD) holds if there exists a constant C_D such that, for all $x\in M$ and all $r>0$,

$$V(x,2r)\le C_D V(x,r).$$

It is known that (VD) implies that, for all $x,y\in M$ and $0<r_1\le r_2$,

$$\frac{V(x,r_2)}{V(y,r_1)} \le C_D\left(\frac{r_2+d(x,y)}{r_1}\right)^\alpha, \tag{1.13}$$

for some $\alpha > 0$ (cf. [**24**]). For example, (VD) is satisfied on any Riemannian manifold of non-negative Ricci curvature and on all mentioned above fractal spaces. Moreover, on the fractal spaces one usually has a stronger *volume regularity* condition $V(x,r) \simeq r^\alpha$ for all $r > 0$.

Definition. We say that the *reverse volume doubling property* (RVD) holds if there exist positive constants α' and C such that, for all $x \in M$ and $0 < r_1 \leq r_2$,

$$\frac{V(x,r_2)}{V(x,r_1)} \geq C^{-1}\left(\frac{r_2}{r_1}\right)^{\alpha'}. \tag{1.14}$$

It is known that $(VD) \Rightarrow (RVD)$ if M is connected and unbounded (cf. [**24**]).

Throughout the paper we fix a function Ψ that is a continuous increasing bijection of $(0,\infty)$ onto itself satisfying the following condition

$$\frac{1}{C_\Psi}\left(\frac{R}{r}\right)^{\beta} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_\Psi\left(\frac{R}{r}\right)^{\beta'} \tag{1.15}$$

for all $0 < r \leq R$ and for some constants $1 < \beta \leq \beta'$ and $C_\Psi \geq 1$.

Poincaré inequality. We say that the *Poincaré inequality* $(PI)_\Psi$ holds if there exist constants $C_P > 0$ and $\sigma \in (0,1)$ such that, for any ball $B = B(x,r)$ and any function $u \in \mathcal{F}$,

$$\int_{\sigma B} |u - u_{\sigma B}|^2 d\mu \leq C_P \Psi(r) \int_B d\Gamma\langle u\rangle, \tag{1.16}$$

where u_A is the mean of the function u over A, that is,

$$u_A := \frac{1}{\mu(A)} \int_A u d\mu.$$

For example, in $\mathbb{R}^n$ and on manifolds of non-negative Ricci curvature $(PI)_\Psi$ holds with $\Psi(r) = r^2$.

Definition. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}$ is *subharmonic* (resp. *superharmonic*) in Ω if

$$\mathcal{E}(u,\varphi) \leq 0 \quad (\text{resp. } \mathcal{E}(u,\varphi) \geq 0) \tag{1.17}$$

for any $0 \leq \varphi \in \mathcal{F}(\Omega)$.

By the standard approximation argument, it suffices to have (1.17) for any non-negative $\varphi \in \mathcal{F} \cap C_0(\Omega)$.

For a local Dirichlet form $(\mathcal{E},\mathcal{F})$ the notion of subharmonicity can be extended to functions outside $\mathcal{F}$ as follows.

Definition. Let $(\mathcal{E}, \mathcal{F})$ be a local Dirichlet form and let Ω be an open subset of M. We say that a Borel function u defined in Ω is *subharmonic* (resp. *superharmonic*) in Ω if there is a function $v \in \mathcal{F}$ such that $v = u$ in Ω and v is subharmonic (resp. superharmonic) in the sense of the previous Definition.

A reason for this definition is that, by the locality, the value of $\mathcal{E}(v, \varphi)$ for $\varphi \in \mathcal{F} \cap C_0(\Omega)$ does not depend on the choice of v as long as $v = u$ in Ω.

A function is called harmonic if it is subharmonic and superharmonic.

Harnack inequality. We say that the *elliptic Harnack inequality* (H) holds on M if there exist two constants $C_H > 1$ and $\eta \in (0,1)$ such that, for any ball B in M and for any function $u \in \mathcal{F}$ that is harmonic and non-negative in B, the following inequality is satisfied:

$$\operatorname*{esup}_{\eta B} u \leq C_H \operatorname*{einf}_{\eta B} u.$$

Let us emphasize that the constants C_H and η should be independent of the ball B and function u.

The elliptic Harnack inequality is a central notion in this paper. It is known that if M is a complete Riemannian manifold then (VD) and $(PI)_\Psi$ with $\Psi(r) = r^2$ imply (H) (cf. [**17**] and [**51**]). In the present generality these two conditions are not enough to obtain (H). We need one more condition—the generalized capacity inequality, that will be described below.

Let Ω be an open subset of M and $A \Subset \Omega$ be a Borel set (where $A \Subset \Omega$ means that A is precompact and $\overline{A} \subset \Omega$).

Definition. A *cutoff function* of the pair (A, Ω) is any function $\phi \in \mathcal{F}$ such that

- $0 \leq \phi \leq 1$ in M;
- $\phi \equiv 1$ in an open neighborhood of $\overline{A}$;
- $\operatorname{supp} \phi \Subset \Omega$.

Sometimes one requires in the definition of a cutoff function also the continuity of ϕ (cf. [**23**]), but here we do not. Denote the set of all cutoff functions of (A, Ω) by $\operatorname{cutoff}(A, \Omega)$. It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then $\operatorname{cutoff}(A, \Omega)$ is non-empty (see [**16**, Lemma 1.4.2(ii), p. 29]).

Definition. For any pair (A, Ω) as above define the *capacity* $\operatorname{cap}(A, \Omega)$ by

$$\operatorname{cap}(A, \Omega) := \inf\{\mathcal{E}(\varphi) : \varphi \in \operatorname{cutoff}(A, \Omega)\}. \tag{1.18}$$

Capacity condition. We say that the *capacity condition* $(\mathrm{cap})_\Psi$ is satisfied if there exist constants $\kappa \in (0,1)$ and $C > 1$ such that, for any ball $B = B(x, r)$ of radius $r > 0$,

$$C^{-1} \frac{\mu(B)}{\Psi(r)} \leq \operatorname{cap}(\kappa B, B) \leq C \frac{\mu(B)}{\Psi(r)}. \tag{1.19}$$

DEFINITION. Let Ω be an open subset of M and $A \Subset \Omega$ be a Borel set. For any measurable function u on Ω, define the *generalized capacity* $\mathrm{cap}_u(A,\Omega)$ by

$$\mathrm{cap}_u(A,\Omega) = \inf\left\{\int_\Omega u^2 d\Gamma\langle\phi\rangle : \phi \in \mathrm{cutoff}(A,\Omega)\right\}.$$

In particular, for $u \equiv 1$ we obtain $\mathrm{cap}_u(A,\Omega) = \mathrm{cap}(A,\Omega)$.

Generalized capacity condition. We say that the *generalized capacity inequality* $(\mathrm{Gcap}_{\leq})_\Psi$ holds if there exist two positive constants c_1, c_2 such that, for any $u \in \mathcal{F} \cap L^\infty$ and for any two concentric balls $B_1 := B(x_0,R)$ and $B_2 := B(x_0,R+r)$,

$$\mathrm{cap}_u(B_1,B_2) \leq c_1 \int_{B_2\setminus B_1} d\Gamma\langle u\rangle + \frac{c_2}{\Psi(r)}\int_{B_2\setminus B_1} u^2 d\mu.$$

Using the definition of cap_u, we can restate $(\mathrm{Gcap}_{\leq})_\Psi$ as follows: for any $u \in \mathcal{F}\cap L^\infty$ there is $\phi \in \mathrm{cutoff}(B_1,B_2)$ such that

$$\int_{B_2\setminus B_1} u^2 d\Gamma\langle\phi\rangle \leq c_1 \int_{B_2\setminus B_1} d\Gamma\langle u\rangle + \frac{c_2}{\Psi(r)}\int_{B_2\setminus B_1} u^2 d\mu. \tag{1.20}$$

This condition is very close to the following condition $(CSA)_\Psi$ (*cutoff Sobolev inequality in annulus*) that was introduced by Andres and Barlow in [**1**] for the sake of proving upper bounds of heat kernel (see discussion below).

DEFINITION. We say that the condition $(CSA)_\Psi$ holds[1] if there exists a function $\phi \in \mathrm{cutoff}(B_1,B_2)$ such that (1.20) is satisfied for any $u \in \mathcal{F}\cap L^\infty$.

The condition $(\mathrm{Gcap}_{\leq})_\Psi$ is a priori weaker than $(CSA)_\Psi$, that is,

$$(CSA)_\Psi \Rightarrow (\mathrm{Gcap}_{\leq})_\Psi,$$

since in $(\mathrm{Gcap}_{\leq})_\Psi$ function ϕ may depend on u.

Note that $(CSA)_\Psi$ (and, hence, $(\mathrm{Gcap}_{\leq\Psi})$) is satisfied on any geodesically complete Riemannian manifold with $\Psi(r) = r^2$. Indeed, the standard bump function

$$\phi(x) = \frac{(R+r-d(x,x_0))_+}{r} \wedge 1$$

vanishes outside B_2, is equal to 1 on B_1 and satisfies $|\nabla\phi| \leq 1/r$, whence (1.20) follows

[1]Note that in the definition of $(CSA)_\Psi$ in [**1**] the cutoff function ϕ was assumed to be continuous and (1.20) was assumed to hold for all $u \in \mathcal{F}$. However, the analysis of the proofs of [**1**, Theorem 5.5] shows that in obtaining $(CSA)_\Psi$ from the heat kernel upper bound the boundedness of u was used and the continuity of ϕ was not established. On the other hand, the proof in the opposite direction (obtaining heat kernel upper bound from $(CSA)_\Psi$) goes through also if u in (1.20) is assumed bounded and ϕ is not necessarily continuous.

for any $u \in L^2$ with $c_1 = 0$, $c_2 = 1$.

Observe also that by [**1**, Lemma 5.1], inequality (1.20) implies (and hence, is equivalent to) the following

$$\int_{B_2 \setminus B_1} u^2 d\Gamma\langle\phi\rangle \leq \frac{1}{8}\int_{B_2 \setminus B_1} \phi^2 d\Gamma\langle u\rangle + \frac{c_2}{\Psi(r)}\int_{B_2 \setminus B_1} u^2 d\mu \tag{1.21}$$

with different value c_2.

We also remark that if two Dirichlet forms $(\mathcal{E}_1, \mathcal{F}_1)$, $(\mathcal{E}_2, \mathcal{F}_2)$ are comparable, namely, if there exist two positive constants C_1, C_2 such that

$$C_1\mathcal{E}_1(u) \leq \mathcal{E}_2(u) \leq C_2\mathcal{E}_1(u) \quad \text{for all } u \in \mathcal{F}_1 \cap \mathcal{F}_2,$$

then so are their energy measures Γ_1, Γ_2 with the same constants:

$$C_1\Gamma_1\langle u\rangle \leq \Gamma_2\langle u\rangle \leq C_2\Gamma_1\langle u\rangle,$$

see [**33**, Proposition 1.5.5(b)] or [**47**, p.389]. From this, we see that all the conditions $(PI)_\Psi$, $(CSA)_\Psi$, $(\mathrm{Gcap}_\leq)_\Psi$ are *stable* with respect to the quasi-isometry of Dirichlet forms.

The following theorem is a key result in this paper.

THEOREM 1.1. *Let (M, d, μ) be a metric measure space with precompact metric balls. Let $(\mathcal{E}, \mathcal{F})$ be a regular, strongly local Dirichlet form in $L^2(M, \mu)$ and Ψ be a function satisfying* (1.15). *If conditions (VD), (RVD) are satisfied, then the following implication takes place:*

$$(PI)_\Psi + (\mathrm{Gcap}_\leq)_\Psi \Longrightarrow (H) + (\mathrm{cap})_\Psi.$$

Choosing in $(\mathrm{Gcap}_\leq)_\Psi$ the function $u \equiv 1$, we obtain that, for any ball B of radius r,

$$\mathrm{cap}\left(\frac{1}{2}B, B\right) \leq \frac{C\mu(B)}{\Psi(r)}.$$

We refer to this condition as $(\mathrm{cap}_\leq)_\Psi$. We conjecture that, under the hypotheses of Theorem 1.1, the following stronger implication is true:

$$(PI)_\Psi + (\mathrm{cap}_\leq)_\Psi \Longrightarrow (H),$$

but we have not been able to prove this.

The proof of Theorem 1.1 will be given in Section 8. The main difficulty is to show the validity of the Harnack inequality (H), that is, the implication

$$(PI)_\Psi + (\mathrm{Gcap}_\leq)_\Psi \Longrightarrow (H).$$

Here are the main steps of that proof.

In Section 5 we prove that $(PI)_\Psi$ implies a certain Faber–Krahn inequality $(FK)_\Psi$ (an isoperimetric inequality for the first eigenvalue).

In Section 6 we use $(FK)_\Psi$ and $(\mathrm{Gcap}_\leq)_\Psi$ to prove an L^2-mean value inequality (MV) for positive subharmonic functions. This is the only place in the proof where $(\mathrm{Gcap}_\leq)_\Psi$ is used at full strength. For the proof of (MV) we adapt the method that originates from De Giorgi [**15**] (see also [**40**] and [**17**]). One uses in the proof a cutoff-function ϕ that in the setting of $\mathbb{R}^n$ and Riemannian manifolds is a standard bump function, but in the general setting is provided by the condition $(\mathrm{Gcap}_\leq)_\Psi$.

In Section 7 we prove (H). In the crucial Lemma 7.2 we use all conditions (MV), $(PI)_\Psi$, $(\mathrm{cap}_\leq)_\Psi$ to obtain some weak version of the Harnack inequality. The rest of the proof that consists of Lemmas 7.5–7.7 and Theorem 7.8 can be regarded as a long self-improvement argument leading from the weak Harnack inequality to (H). This argument is essentially due to E. M. Landis who developed it in the context of elliptic equations in divergence form in $\mathbb{R}^n$ (cf. [**43**], [**42**], [**38**], [**17**]). Note that this self-improvement argument uses only (VD) and (RVD).

Comparing our proof of the Harnack inequality with the celebrated proof of J. Moser [**48**], we mention the following. The first part of the Moser proof, that is frequently referred to as "Moser iterations", also works in our setting and also needs a cutoff function from $(\mathrm{Gcap}_\leq)_\Psi$. The second part of the Moser proof uses John–Nirenberg lemma to obtain the mean value inequality for positive superharmonic functions and can be done in our setting as well. However, a careful implementation of this method in our setting would be noticeably longer and more involved than the present approach.

Before stating the second main result, we need some more definitions. For any open set $\Omega \subset M$, set

$$\lambda_{\min}(\Omega) := \inf_{u \in \mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2}. \tag{1.22}$$

It is easy to show that $\lambda_{\min}(\Omega)$ is the bottom of the spectrum of the generator $-\Delta^\Omega$ of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\Omega))$.

DEFINITION. For an open $\Omega \subset M$, a linear operator $G^\Omega : L^2(\Omega) \to \mathcal{F}(\Omega)$ is called a *Green operator* if, for any $\varphi \in \mathcal{F}(\Omega)$ and any $f \in L^2(\Omega)$,

$$\mathcal{E}(G^\Omega f, \varphi) = (f, \varphi). \tag{1.23}$$

If G^Ω admits an integral kernel g^Ω, that is, if

$$G^\Omega f(x) = \int_\Omega g^\Omega(x,y) f(y)\, d\mu(y) \text{ for any } f \in L^2(\Omega), \tag{1.24}$$

then g^Ω is called a *Green function*.

It is known that if $\lambda_{\min}(\Omega) > 0$ then the Green operator G^Ω exists and is given by

$$G^{\Omega} f = \int_0^{\infty} P_t^{\Omega} f \, dt \tag{1.25}$$

(see [**23**, Lemma 5.1]). For an open set $\Omega \subset M$, define the *function* E^{Ω} on Ω by

$$E^{\Omega} := G^{\Omega} 1_{\Omega}. \tag{1.26}$$

The function E^{Ω} is a unique weak solution of the following Poisson-type equation

$$-\Delta^{\Omega} E^{\Omega} = 1. \tag{1.27}$$

This function has also the following probabilistic meaning: $E^{\Omega}(x)$ is the mean exit time from Ω of the process associated to $(\mathcal{E}, \mathcal{F})$ started at x.

Mean exit time bounds. We say that *condition* $(E)_{\Psi}$ holds if there exist two constants $C > 1$ and $\varepsilon \in (0,1)$ such that, for all balls B of radius $r > 0$,

$$\operatorname*{esup}_{B} E^{B} \leq C\Psi(r) \quad \text{and} \quad \operatorname*{einf}_{\varepsilon B} E^{B} \geq C^{-1}\Psi(r). \tag{1.28}$$

We will refer to the first condition in (1.28) as $(E_{\leq})_{\Psi}$ and the second one as $(E_{\geq})_{\Psi}$.

Green function bounds. We say that *condition* $(G)_{\Psi}$ holds if there exist constants $\kappa \in (0,1)$ and $\dot{C} > 0$ such that, for any ball $B := B(x,R)$, the Green kernel g^B exists, is jointly continuous off the diagonal, and satisfies

$$g^B(x,y) \leq C \int_{\kappa d(x,y)}^{R} \frac{\Psi(s) ds}{sV(x,s)} \text{ for } \mu\text{-almost all } y \in B \setminus \{x\},$$

$$g^B(x,y) \geq C^{-1} \int_{\kappa d(x,y)}^{R} \frac{\Psi(s) ds}{sV(x,s)} \text{ for } \mu\text{-almost all } y \in \kappa B \setminus \{x\}.$$

Upper bound of heat kernel. We say that *condition* $(UE)_{\Psi}$ holds if the heat kernel $p_t(x,y)$ exists and satisfies the following upper estimate

$$p_t(x,y) \leq \frac{C}{V(x, \Psi^{-1}(t))} \exp\left(-\frac{1}{2} t \Phi\left(c \frac{d(x,y)}{t} \right) \right) \tag{1.29}$$

for all $t > 0$ and μ-almost all $x, y \in M$, where c, C are positive constants, independent of x, y, t, and

$$\Phi(s) := \sup_{r>0} \left\{ \frac{s}{r} - \frac{1}{\Psi(r)} \right\}. \tag{1.30}$$

For example, if $\Psi(r) = r^{\beta}$ with some $\beta > 1$, then the supremum in (1.30) is attained at $r = (s/\beta)^{-1/(\beta-1)}$, which yields $\Phi(s) = cs^{\beta/(\beta-1)}$. In this case (1.29) takes the form

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{1/(\beta-1)}\right)$$

(cf. (1.1)).

Near-diagonal lower bound. We say that *condition* $(NLE)_\Psi$ holds if the heat kernel $p_t(x,y)$ exists and satisfies the lower estimate

$$p_t(x,y) \ge \frac{c}{V(x,\Psi^{-1}(t))}, \tag{1.31}$$

for all $t>0$ and μ-almost all $x,y \in M$ such that

$$d(x,y) \le \varepsilon \Psi^{-1}(t), \tag{1.32}$$

where $c,\varepsilon > 0$ are constants independent of x,y,t.

It is easy to show that under condition (1.32) the term $t\Phi(c(d(x,y)/t))$ in (1.29) is bounded by a constant, so that the upper bound $(UE)_\Psi$ is consistent with $(NLE)_\Psi$.

It is known (cf. [**23**] and [**9**]) that the conjunction $(UE)_\Psi + (NLE)_\Psi$ of the two estimates implies that the heat kernel $p_t(x,y)$ admits a Hölder continuous in x,y version, so that (1.29) and (1.31) are a posteriori true for all $x,y \in M$.

The following theorem is the main result of this paper about two sided estimates of the heat kernel.

Theorem 1.2. *Let (M,d,μ) be a metric measure space with precompact metric balls. Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form in $L^2(M,\mu)$ and Ψ be a function satisfying* (1.15)*. If conditions (VD), (RVD) are satisfied, then the following equivalences take place*:

$$\begin{aligned}(UE)_\Psi + (NLE)_\Psi &\Leftrightarrow (PI)_\Psi + (\mathrm{Gcap}_{\le})_\Psi \\ &\Leftrightarrow (PI)_\Psi + (CSA)_\Psi \\ &\Leftrightarrow (PI)_\Psi + (E)_\Psi \\ &\Leftrightarrow (H) + (\mathrm{cap})_\Psi \\ &\Leftrightarrow (H) + (E)_\Psi \\ &\Leftrightarrow (G)_\Psi.\end{aligned}$$

This theorem essentially follows from the implication

$$(PI)_\Psi + (\mathrm{Gcap}_{\le})_\Psi \Longrightarrow (H) + (\mathrm{cap})_\Psi$$

of Theorem 1.1 and the previously known results of [**23**], [**30**], [**1**, Lemma 5.4, Theorem 5.5] (see Section 8 for details).

We consider the equivalence

$$(UE)_\Psi + (NLE)_\Psi \Leftrightarrow (PI)_\Psi + (\mathrm{Gcap}_\leq)_\Psi$$

as the most significant part of Theorem 1.2, which provides convenient equivalent condition for the two-sided heat kernel estimate. Since condition $(\mathrm{Gcap}_\leq)_\Psi$ is quasi-isometry stable, Theorem 1.2 implies that $(UE)_\Psi + (NLE)_\Psi$ is also quasi-isometry stable.

In the setting of Riemannian manifold with $\Psi(r) = r^2$, the condition $(\mathrm{Gcap}_\leq)_\Psi$ is satisfied automatically, and we obtain

$$(UE)_\Psi + (NLE)_\Psi \Leftrightarrow (PI)_\Psi,$$

which is the result of L. Saloff-Coste [**51**] (see also [**17**]). In the setting of *geodesic* metric spaces, Barlow, Bass and Kumagai [**7**] (see also [**6**] for a setting of graphs) proved the equivalence

$$(UE)_\Psi + (NLE)_\Psi \Leftrightarrow (PI)_\Psi + (CS)_\Psi,$$

where $(CS)_\Psi$ stands for a cutoff Sobolev inequality, a rather complicated condition that, similarly to $(CSA)_\Psi$, provides the existence of a cutoff function with certain properties. The condition $(CS)_\Psi$ is also quasi-isometry stable, which was used in [**7**] to prove the stability of $(UE)_\Psi + (NLE)_\Psi$ in the setting of geodesic metric spaces. Our result implies the stability for a larger class of metric spaces, without the requirement of the metric to be geodesic.

Now let us turn to equivalent conditions for the upper bound $(UE)_\Psi$ alone.

Faber–Krahn inequality. We say that the *Faber–Krahn inequality* $(FK)_\Psi$ holds if there exist positive constants $\nu > 0$, $C_F > 0$ such that, for any ball $B \subset M$ and for any non-empty open set $\Omega \subset B$,

$$\lambda_{\min}(\Omega) \geq \frac{C_F}{\Psi(R)} \left(\frac{\mu(B)}{\mu(\Omega)} \right)^\nu, \tag{1.33}$$

where R is the radius of B.

Note that since $\mu(B) \geq \mu(\Omega)$, the value of ν in (1.33) can be chosen to be arbitrarily small.

It is known (cf. [**17**]) that $(FK)_\Psi$ with $\Psi(r) = r^2$ holds on any geodesically complete Riemannian manifold of non-negative Ricci curvature. It was proved in [**18**, Proposition 5.2] that on geodesically complete Riemannian manifolds satisfying (VD), the following equivalence holds with $\Psi(r) = r^2$:

$$(UE)_\Psi \Leftrightarrow (FK)_\Psi. \tag{1.34}$$

Andres and Barlow proved in [**1**] the following equivalent condition for $(UE)_\Psi$ in the present abstract setting:

$$(UE)_\Psi \Leftrightarrow (FK)_\Psi + (CSA)_\Psi \tag{1.35}$$

assuming that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is conservative[2].

Our main result about heat kernel upper bound is the following theorem that somewhat strengthens the result of Andres and Barlow. We denote by (C) the condition that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is conservative.

Theorem 1.3. *Let (M,d,μ) be a metric measure space with precompact metric balls. Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form in $L^2(M,\mu)$ and Ψ be a function satisfying* (1.15). *If condition (VD) is satisfied, then the following equivalences take place*

$$(UE)_\Psi + (C) \Leftrightarrow (FK)_\Psi + (\mathrm{Gcap}_\leq)_\Psi \tag{1.36}$$

$$\Leftrightarrow (FK)_\Psi + (E_\geq)_\Psi. \tag{1.37}$$

Since $(\mathrm{Gcap}_\leq)_\Psi$ with $\Psi(r)=r^2$ holds on any geodesically complete Riemannian manifold and on any such manifold (VD) implies (C), we see that the equivalence (1.36) in the case of manifolds amounts to the above mentioned result (1.34) of [**18**, Proposition 5.2].

The proof of Theorem 1.3 is given in Section 9. We conjecture that $(\mathrm{Gcap}_\leq)_\Psi$ in (1.36) can be replaced by $(\mathrm{cap}_\leq)_\Psi$.

2. Subharmonic and superharmonic functions.

In this section we present some properties of subharmonic and superharmonic functions that will be used later on.

Proposition 2.1. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and strongly local. Let u be a bounded subharmonic function in a non-empty precompact open set $\Omega\subset M$.*

(1) *If $f\in C^2(\mathbb{R})$ with $f''\geq 0$, $f'\geq 0$, $f(0)=0$ then $f(u)$ is also a subharmonic function in Ω. In particular, the function u^p is subharmonic in Ω for any $p\geq 1$.*
(2) *For any $a\geq 0$, the function $(u-a)_+$ is subharmonic in Ω.*

Proof. (1) We can assume that $u\in\mathcal{F}\cap L^\infty$, which implies that also $f(u)$ belongs to $\mathcal{F}\cap L^\infty$ (see [**16**, Theorem 1.4.2(v), p. 28]). It is enough to show that

$$\mathcal{E}(f(u),\varphi)\leq 0 \tag{2.1}$$

for any non-negative $\varphi\in\mathcal{F}(\Omega)\cap L^\infty$. Using the Leibniz and chain rules, we have

$$\begin{aligned}\mathcal{E}(f(u),\varphi) &= \int_M d\Gamma\langle f(u),\varphi\rangle = \int_M f'(u)d\Gamma\langle u,\varphi\rangle\\ &= \int_M d\Gamma\langle u, f'(u)\varphi\rangle - \int_\Omega \varphi f''(u)d\Gamma\langle u\rangle\end{aligned}$$

[2]The condition of conservativeness of $(\mathcal{E},\mathcal{F})$ was not explicitly stated in [**1**], but was implicitly used. Without the conservativeness the implication $(UE)_\Psi \Rightarrow (CSA)_\Psi$ is not true (cf. discussion in [**24**, p. 516]).

$$\leq \int_M d\Gamma\langle u, f'(u)\varphi\rangle \leq 0,$$

since u is subharmonic and $0 \leq f'(u)\varphi \in \mathcal{F}(\Omega)$. This proves (2.1), showing that $f(u)$ is subharmonic in Ω.

(2) Clearly, we have $(u-a)_+ \in \mathcal{F} \cap L^\infty$ for any $a \geq 0$. Set $g(t) = (t-a)_+$ for any $t \in \mathbb{R}$. Let $\{g\}_{k=1}^\infty$ be a sequence of functions such that each $g_k \in C^2(\mathbb{R})$, $g_k'' \geq 0$, $g_k(0) = g_k'(0) = 0$, and as $k \to \infty$, $g_k \rightrightarrows g$ uniformly while $g_k' \to g'$ everywhere except at point a. We will show that

$$\mathcal{E}(g_k(u) - g(u)) \to 0 \text{ as } k \to \infty. \tag{2.2}$$

Indeed, let $h_k := g_k - g$. Note that by (1.12), we see that $1_{\{u=a\}}\Gamma\langle u\rangle = 0$ for any $u \in \mathcal{F}$. Therefore, using the dominated convergence theorem,

$$\begin{aligned}
\mathcal{E}(g_k(u) - g(u)) = \mathcal{E}(h_k(u)) &= \int_M \left[h_k'(u)\right]^2 d\Gamma\langle u\rangle \\
&= \int_{\{u=a\}} \left[h_k'(u)\right]^2 d\Gamma\langle u\rangle + \int_{\{u\neq a\}} \left[h_k'(u)\right]^2 d\Gamma\langle u\rangle \\
&= \int_{\{u\neq a\}} \left[h_k'(u)\right]^2 d\Gamma\langle u\rangle \to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

proving (2.2).

By the above step (1), the function $g_k(u)$ is subharmonic in Ω, that is $\mathcal{E}(g_k(u), \varphi) \leq 0$. Passing to the limit as $k \to \infty$ and then using (2.2), we have that $\mathcal{E}(g(u), \varphi) \leq 0$, thus proving that $(u-a)_+$ is subharmonic in Ω. □

For non-negative subharmonic functions, we have the following general result.

PROPOSITION 2.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is strongly local and regular. Let $u \in \mathcal{F} \cap L^\infty$ be non-negative and subharmonic in a precompact open subset Ω. Then, for any $0 \leq \phi \in \mathcal{F}(\Omega) \cap L^\infty$,*

$$\int_\Omega \phi^2 d\Gamma\langle u\rangle \leq 4 \int_\Omega u^2 d\Gamma\langle \phi\rangle, \tag{2.3}$$

and

$$\mathcal{E}(u\phi) = \int_\Omega d\Gamma\langle u\phi\rangle \leq 10 \int_\Omega u^2 d\Gamma\langle \phi\rangle. \tag{2.4}$$

PROOF. Since $u\phi^2 \in \mathcal{F}(\Omega) \cap L^\infty$ and u is subharmonic in Ω, we obtain by the Leibniz rule and Cauchy–Schwarz inequality (1.10) that

$$
\begin{aligned}
0 \geq \mathcal{E}(u, u\phi^2) &= \int_M d\Gamma\langle u, u\phi^2\rangle \\
&= \int_M \phi^2 d\Gamma\langle u, u\rangle + 2\int_M \phi u d\Gamma\langle u, \phi\rangle \\
&\geq \int_M \phi^2 d\Gamma\langle u\rangle - \left(\frac{1}{2}\int_M \phi^2 d\Gamma\langle u\rangle + 2\int_M u^2 d\Gamma\langle\phi\rangle\right) \\
&= \frac{1}{2}\int_M \phi^2 d\Gamma\langle u\rangle - 2\int_M u^2 d\Gamma\langle\phi\rangle,
\end{aligned}
$$

whence (2.3) follows.

Next, using bilinearity of Γ, (1.10), and (2.3), we obtain

$$
\begin{aligned}
\mathcal{E}(u\phi) &= \int_M d\Gamma\langle u\phi\rangle \\
&= \int_M u^2 d\Gamma\langle\phi\rangle + \int_M \phi^2 d\Gamma\langle u\rangle + 2\int_M \phi u d\Gamma\langle u, \phi\rangle \\
&\leq 2\int_M u^2 d\Gamma\langle\phi\rangle + 2\int_M \phi^2 d\Gamma\langle u\rangle \\
&\leq 10\int_M u^2 d\Gamma\langle\phi\rangle,
\end{aligned}
$$

thus proving (2.4). □

Let us introduce conditions (A_1) and (A_2) to be used later to prove the L^2 mean value inequality.

CONDITION (A_1). We say that *condition* (A_1) holds if there exists a constant $C_0 > 0$ such that, for any ball $B = B(x_0, r)$ of radius r and for any bounded non-negative subharmonic function u in B, there is some $\phi \in \text{cutoff}((1/2)B, B)$ satisfying

$$
\int_B u^2 d\Gamma\langle\phi\rangle \leq \frac{C_0}{\Psi(r)}\int_B u^2 d\mu. \tag{2.5}
$$

Let us emphasize that the constant C_0 is independent of B, u, ϕ, whilst the cutoff function ϕ may depend on u.

CONDITION (A_2). We say that *condition* (A_2) holds if there exists a constant $C_1 > 0$ such that, for any ball B of radius r and for any bounded non-negative subharmonic function u in B, there is some $\phi \in \text{cutoff}((1/2)B, B)$ satisfying

$$
\mathcal{E}(u\phi) \leq \frac{C_1}{\Psi(r)}\int_B u^2 d\mu. \tag{2.6}
$$

Most likely, the both conditions (A_1) and (A_2) are *unstable* with respect to the quasi-isometry of Dirichlet forms, because the class of subharmonic functions changes uncontrollably under quasi-isometry. However, the both conditions (A_1), (A_2) are consequences of the stable condition $(\mathrm{Gcap}_{\leq})_\Psi$ as below.

PROPOSITION 2.3. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Then*

$$(\mathrm{Gcap}_{\leq})_\Psi \Rightarrow (A_1) \Rightarrow (A_2). \tag{2.7}$$

PROOF. We first show the implication

$$(\mathrm{Gcap}_{\leq})_\Psi \Rightarrow (A_1).$$

Let u be a bounded non-negative subharmonic function in a ball B of radius r. If $(\mathrm{Gcap}_{\leq})_\Psi$ holds, then it follows from (1.21) that there is some $\phi \in \mathrm{cutoff}((1/2)B, B)$ such that

$$\begin{aligned}
\int_B u^2 d\Gamma\langle\phi\rangle &\leq \frac{1}{8}\int_{B\setminus(1/2)B} \phi^2 d\Gamma\langle u\rangle + \frac{c_2}{\Psi(r)}\int_{B\setminus(1/2)B} u^2 d\mu \\
&\leq \frac{1}{8}\int_B \phi^2 d\Gamma\langle u\rangle + \frac{c_2}{\Psi(r)}\int_B u^2 d\mu \\
&\leq \frac{1}{2}\int_B u^2 d\Gamma\langle\phi\rangle + \frac{c_2}{\Psi(r)}\int_B u^2 d\mu,
\end{aligned} \tag{2.8}$$

where in the last line we have used (2.3). Clearly, (2.8) implies (2.5).

To prove the implication

$$(A_1) \Rightarrow (A_2),$$

observe that, by (2.4) and (2.5),

$$\mathcal{E}(u\phi) \leq 10\int_B u^2 d\Gamma\langle\phi\rangle \leq \frac{10C_0}{\Psi(r)}\int_B u^2 d\mu,$$

whence (2.6) follows. □

As a conclusion of this section, we state some technical results to be used later.

LEMMA 2.4. *Let $(\mathcal{E}, \mathcal{F})$ be regular and strongly local. Assume that u is a bounded superharmonic function in an open set $\Omega \subset M$ and $u \geq \varepsilon$ in Ω for some positive constant ε. Then the function $-\log u$ is subharmonic in Ω.*

PROOF. By definition of a (bounded) superharmonic function, we can assume that $u \in \mathcal{F} \cap L^\infty$. The function $\log u$ is not necessarily defined on M as u may take negative values outside Ω. To extend it to the whole of M, choose a function $l(t)$ for $t \in \mathbb{R}$ as follows: $l \in C^\infty(\mathbb{R})$, $l(t) = \log t$ for $t \geq \varepsilon$ and $l(t) = 0$ for $t \leq 0$. Then $l(u)$ is defined on M and all functions $l(u), l'(u), l''(u)$ are in $\mathcal{F} \cap L^\infty$ by a general theory of Dirichlet forms. On the other hand, in Ω we have $l(u) = \log u$, $l'(u) = 1/u$ and $l''(u) = -1/u^2$.

For any $0 \le \varphi \in \mathcal{F} \cap C_0(\Omega)$, we have by the Leibniz and chain rules,

$$\begin{aligned} d\Gamma(l(u), \varphi) &= l'(u) d\Gamma\langle u, \varphi\rangle \\ &= d\Gamma\langle u, l'(u)\varphi\rangle - \varphi l''(u) d\Gamma\langle u\rangle \\ &= d\Gamma\langle u, u^{-1}\varphi\rangle + \varphi u^{-2} d\Gamma\langle u\rangle. \end{aligned} \tag{2.9}$$

Since u is superharmonic and $u^{-1}\varphi \in \mathcal{F}(\Omega)$, we have $\mathcal{E}(u, u^{-1}\varphi) \ge 0$, and hence,

$$\int d\Gamma\langle -l(u), \varphi\rangle = -\mathcal{E}(u, u^{-1}\varphi) - \int \varphi u^{-2} d\Gamma\langle u\rangle \le 0. \tag{2.10}$$

Hence, $-l(u) = -\log u$ is subharmonic in Ω. □

Lemma 2.5. *Let u be a strictly positive bounded superharmonic function in an open set Ω. Then u^{-1} is subharmonic in Ω.*

Proof. Indeed, for any $0 \le \phi \in \mathcal{F} \cap C_0(\Omega)$, we have by the chain and product rules (similarly to the previous proof), that

$$\begin{aligned} d\Gamma\left\langle u^{-1}, \phi\right\rangle &= -u^{-2} d\Gamma\langle u, \phi\rangle \\ &= -d\Gamma\left\langle u, u^{-2}\phi\right\rangle - 2\phi u^{-3} d\Gamma\langle u, u\rangle. \end{aligned}$$

Since u is superharmonic and $u^{-2}\phi \in \mathcal{F}(\Omega)$, we have that $\mathcal{E}(u, u^{-2}\phi) \ge 0$, and hence,

$$\int_\Omega d\Gamma\left\langle u^{-1}, \phi\right\rangle = -\mathcal{E}(u, u^{-2}\phi) - 2\int_\Omega \phi u^{-3} d\Gamma\langle u\rangle \le 0,$$

so that u^{-1} is subharmonic □

3. Condition $(CSA)_\Psi$.

In this section we will prove the following implication:

$$(S)_\Psi \Rightarrow (CSA)_\Psi, \tag{3.1}$$

where condition $(S)_\Psi$ is defined as follows.

Survival estimate. We say that the condition $(S)_\Psi$ holds if there exist constants $\varepsilon, \varepsilon' \in (0,1)$ such that, for any ball B of radius $r > 0$ and for all $t \le \varepsilon' \Psi(r)$,

$$1 - P_t^B \mathbf{1}_B(x) \le \varepsilon \ \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B. \tag{3.2}$$

Condition $(S)_\Psi$ with $\Psi(r) = r^\beta$ was introduced in [**24**].

Before we can prove (3.1), we investigate the following equation

$$-\Delta^{\Omega} u_\Omega + \lambda u_\Omega = 1_\Omega \ \ = \text{weakly in } \Omega, \tag{3.3}$$

where Ω is an open subset of M, Δ^{Ω} is the infinitesimal generator of $(\mathcal{E}, \mathcal{F}(\Omega))$ as before, and $\lambda > 0$ is a constant. The function $u_\Omega \in \mathcal{F}(\Omega)$ is said to be a *weak solution* of (3.3) if

$$\mathcal{E}(u_\Omega, \varphi) + \lambda \int_\Omega u_\Omega \varphi d\mu = \int_\Omega \varphi d\mu \tag{3.4}$$

for any $\varphi \in \mathcal{F}(\Omega)$. Note that, for any $\lambda > 0$, equation (3.3) admits a unique weak solution

$$u_\Omega = \int_0^\infty e^{-\lambda t}\, P_t^{\Omega} 1_\Omega \, dt, \tag{3.5}$$

where as before $\{P_t^{\Omega}\}_{t\geq 0}$ is the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$.

Lemma 3.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. For any non-empty precompact open $\Omega \subset M$, the function* (3.5) *satisfies for all $t > 0$ the following inequalities*

$$te^{-\lambda t} P_t^{\Omega} 1_\Omega \leq u_\Omega \leq \lambda^{-1} \quad \text{in } M. \tag{3.6}$$

Proof. Since $P_t^{\Omega} 1_\Omega$ is decreasing in t, we obtain

$$u_\Omega \geq \int_0^t e^{-\lambda s}\, P_s^{\Omega} 1_\Omega \, ds \geq te^{-\lambda t}\, P_t^{\Omega} 1_\Omega,$$

which proves the left inequality in (3.6).

Since $P_t^{\Omega} 1_\Omega \leq 1$, we obtain

$$u_\Omega \leq \int_0^\infty e^{-\lambda t} dt = \lambda^{-1},$$

which finishes the proof of (3.6). □

Theorem 3.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular, strongly local Dirichlet form in L^2. Then*

$$(S)_\Psi \Rightarrow (CSA)_\Psi. \tag{3.7}$$

We use in the proof essentially the same argument as Andres and Barlow did in [**1**, Lemmas 5.3, 5.4 and Theorem 5.5] to prove $(UE)_\Psi \Rightarrow (CSA)_\Psi$, but with the necessary modifications and without reference to the stochastic process.

Proof. Set $B_1 := B(x_0, R)$ and $B_2 := B(x_0, R+r)$. Using $(S)_\Psi$, we will construct a function $\phi \in \text{cutoff}(B_1, B_2)$ such that (1.20) holds for any $u \in \mathcal{F} \cap L^\infty$. Set $A :=$

$\partial B(x_0, R + r/2)$ and

$$\Omega := B(x_0, R + r) \setminus \overline{B(x_0, R)}.$$

For any point $z \in A$ consider the ball $B_z = B(z, r/2) \subset \Omega$ (see Figure 1).

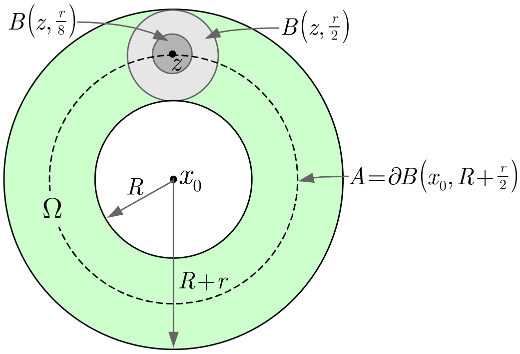

Figure 1. The ball $B(z, r/2) \subset \Omega$ and the set $A = \partial B(x_0, R + r/2)$.

Let u_Ω be as in (3.5) with $\lambda = \Psi(r)^{-1}$. Applying the survival estimate (3.2) with $t = \varepsilon' \Psi(r)$ and using (3.6) we obtain for μ-almost all $x \in (1/4)B_z$ that

$$\begin{aligned} u_\Omega(x) &\geq t e^{-\lambda t} P_t^\Omega 1_\Omega(x) \\ &\geq t e^{-\lambda t} P_t^{B_z} 1_{B_z}(x) \\ &\geq (\varepsilon' \Psi(r)) e^{-\varepsilon'} (1 - \varepsilon) = c_0 \Psi(r). \end{aligned}$$

Since z is arbitrary, the family $\{(1/4)B_z\}_{z \in A}$ covers the $r/8$-neighborhood of A, and, hence, in this neighborhood we have

$$u_\Omega \geq c_0 \Psi(r). \tag{3.8}$$

On the other hand, we have from (3.6) that in Ω (and also in M),

$$u_\Omega \leq \lambda^{-1} = \Psi(r). \tag{3.9}$$

Define the function v_Ω on M by

$$v_\Omega := \frac{u_\Omega}{c_0 \Psi(r)}, \tag{3.10}$$

where the constant c_0 is the same as in (3.8). Clearly, $v_\Omega \in \mathcal{F}(\Omega)$. Moreover, using (3.8) and (3.9), we see that $v_\Omega \geq 1$ in some neighborhood of A, and

$$v_\Omega \leq c_0^{-1} \ \text{ in } M. \tag{3.11}$$

Now we define a desired cutoff function ϕ by

$$\phi = \begin{cases} 1, & \text{in } B\left(x_0, R + \dfrac{r}{2}\right), \\ v_\Omega \wedge 1, & \text{outside } B\left(x_0, R + \dfrac{r}{2}\right). \end{cases} \tag{3.12}$$

Clearly, $\phi = 1$ in some neighborhood of the ball $B(x_0, R + r/2)$ (in particular $\phi = 1$ in B_1), and $\phi = 0$ outside B_2 because so is v_Ω. Also $\phi \in \mathcal{F}$. Thus, $\phi \in \operatorname{cutoff}(B_1, B_2)$.

It remains to show (1.20). Let us first prove that, for any $u \in \mathcal{F} \cap L^\infty$,

$$\int_M u^2 d\Gamma\langle\phi\rangle \le \int_M u^2 d\Gamma\langle v_\Omega\rangle. \tag{3.13}$$

Indeed, applying the following identity

$$\mathcal{E}(f,g) = \lim_{t\to 0+}\left[\frac{1}{2t}\int_{M\times M}(f(x)-f(y))(g(x)-g(y))P_t(x,dy)d\mu(x) + \frac{1}{t}\int_M fg(1-P_t1)d\mu\right],$$

that holds for all $f, g \in \mathcal{F}$ (cf. [**16**, (4.5.7)]), we obtain

$$\begin{aligned}\int_M u^2 d\Gamma\langle\phi\rangle &= \mathcal{E}(u^2\phi,\phi) - \frac{1}{2}\mathcal{E}(u^2,\phi^2) \\ &= \lim_{t\to 0+}\left[\frac{1}{2t}\int_{M\times M} u^2(x)(\phi(x)-\phi(y))^2 P_t(x,dy)d\mu(x) + \frac{1}{2t}\int_M \phi^2u^2(1-P_t1)d\mu\right].\end{aligned}$$

Using here inequalities

$$|\phi(x)-\phi(y)| \le |v_\Omega(x) - v_\Omega(y)|$$

and $\phi(x) \le v_\Omega(x)$, we obtain (3.13).

On the other hand, by the Cauchy–Schwarz inequality (1.10), we have

$$\begin{aligned}\int_M u^2 d\Gamma\langle v_\Omega\rangle &= \mathcal{E}(u^2v_\Omega, v_\Omega) - \frac{1}{2}\int_\Omega d\Gamma\langle u^2, v_\Omega^2\rangle \\ &= \mathcal{E}(u^2v_\Omega, v_\Omega) - 2\int_\Omega uv_\Omega d\Gamma\langle u, v_\Omega\rangle \\ &\le \mathcal{E}(u^2v_\Omega, v_\Omega) + \frac{1}{2}\int_M u^2 d\Gamma\langle v_\Omega\rangle + 2\int_\Omega v_\Omega^2 d\Gamma\langle u\rangle,\end{aligned}$$

and thus

$$\int_M u^2 d\Gamma\langle v_\Omega\rangle \leq 2\mathcal{E}(u^2 v_\Omega, v_\Omega) + 4\int_\Omega v_\Omega^2 d\Gamma\langle u\rangle. \tag{3.14}$$

By the upper bound (3.11) of v_Ω, we have

$$\int_\Omega v_\Omega^2 d\Gamma\langle u\rangle \leq \frac{1}{c_0^2}\int_\Omega d\Gamma\langle u\rangle. \tag{3.15}$$

In order to bound $\mathcal{E}(u^2 v_\Omega, v_\Omega)$, we use (3.4) with $\varphi = u^2 v_\Omega$, (3.10), (3.11) and obtain

$$\begin{aligned}
\mathcal{E}(u^2 v_\Omega, v_\Omega) &= \mathcal{E}\left(u^2 v_\Omega, \frac{u_\Omega}{c_0\Psi(r)}\right) = \frac{1}{c_0\Psi(r)}\mathcal{E}(u^2 v_\Omega, u_\Omega) \\
&= \frac{1}{c_0\Psi(r)}\left(\int_\Omega u^2 v_\Omega d\mu - \lambda\int_\Omega (u^2 v_\Omega)u_\Omega d\mu\right) \\
&\leq \frac{1}{c_0\Psi(r)}\int_\Omega u^2 v_\Omega d\mu \leq \frac{1}{c_0^2\Psi(r)}\int_\Omega u^2 d\mu.
\end{aligned}$$

Combining this with (3.14), (3.15), (3.13), we conclude that

$$\int_M u^2 d\Gamma(\phi) \leq \frac{2}{c_0^2\Psi(r)}\int_\Omega u^2 d\mu + \frac{4}{c_0^2}\int_\Omega d\Gamma(u), \tag{3.16}$$

thus proving (1.20) with $c_1 = 4/c_0^2$, $c_2 = 2/c_0^2$. □

REMARK. Note that the following implication is true:

$$(E)_\Psi \Rightarrow (S)_\Psi,$$

see [**29**, Theorem 6.13]. Combining with Theorem 3.2 and Proposition 2.3, we obtain

$$(E)_\Psi \Rightarrow (S)_\Psi \Rightarrow (CSA)_\Psi \Rightarrow (\mathrm{Gcap}_\leq)_\Psi \Rightarrow (A_1) \Rightarrow (A_2). \tag{3.17}$$

REMARK. For a large class of fractals with *effective resistance* (cf. [**34**], [**54**]), condition $(S)_\Psi$ with $\Psi(r) = r^\beta$ for some $\beta > 2$ was proved to be true, see [**29**, Theorem 6.13]. In particular, condition $(S)_\Psi$ holds on the Sierpinski gasket in $\mathbb{R}^n$ for the standard local regular conservative self-similar Dirichlet form where

$$\Psi(r) = r^{\log(n+3)/\log 2}.$$

Thus condition $(CSA)_\Psi$ is true on this class of fractals.

4. Alternative form of the Poincaré Inequality.

Here we prove some consequences from $(PI)_\psi$.

LEMMA 4.1. *The condition $(PI)_\psi$ is equivalent to the following condition: for some $c > 0$, $\sigma \in (0,1)$ and for all $f \in \mathcal{F}$ and $B = B(x_0, r)$,*

$$\int_B d\Gamma\langle f\rangle \geq \frac{c}{\Psi(r)\mu(\sigma B)} \int_{\sigma B}\int_{\sigma B} (f(x) - f(y))^2 d\mu(y) d\mu(x). \tag{4.1}$$

PROOF. Set $B' = \sigma B$ and

$$a = \frac{1}{\mu(B')} \int_{B'} f d\mu.$$

Then we have

$$\begin{aligned}
&\int_{B'}\int_{B'} (f(x) - f(y))^2 d\mu(y) d\mu(x) \\
&= \int_{B'}\int_{B'} f(x)^2 d\mu(y) d\mu(x) + \int_{B'}\int_{B'} f(y)^2 d\mu(y) d\mu(x) - 2\int_{B'}\int_{B'} f(x)f(y) d\mu(y) d\mu(x) \\
&= 2\mu(B') \int_{B'} f^2 d\mu - 2\Big(\int_{B'} f d\mu\Big)^2 \\
&= 2\mu(B')\Big(\int_{B'} f^2 d\mu - a^2\mu(B')\Big)
\end{aligned}$$

and, hence,

$$\begin{aligned}
\int_{B'} (f-a)^2 d\mu &= \int_{B'} f^2 d\mu - 2a\int_{B'} f d\mu + a^2\int_{B'} d\mu \\
&= \int_{B'} f^2 d\mu - 2a^2\mu(B') + a^2\mu(B') \\
&= \int_{B'} f^2 d\mu - a^2\mu(B') \\
&= \frac{1}{2\mu(B')} \int_{B'}\int_{B'} (f(x) - f(y))^2 d\mu(y) d\mu(x).
\end{aligned}$$

Therefore, (4.1) is equivalent to $(PI)_\psi$ with the same value of σ and with $C_P = 1/2c$. □

LEMMA 4.2. *The condition $(PI)_\psi$ implies that, for all $f \in \mathcal{F}$ and $B = B(x, r)$,*

$$\int_B d\Gamma\langle f_+\rangle \geq \frac{c}{\Psi(r)} \frac{\mu(H)}{\mu(\sigma B)} \int_{\sigma B} f_+^2 d\mu \tag{4.2}$$

where σ and c are the same as in (4.1) *and*

$$H = \{x \in \sigma B : f \le 0\}.$$

Proof. Applying (4.1) to the function f_+ and restricting integration in y in the right side to $y \in H$, we obtain

$$\begin{aligned}\int_B d\Gamma\langle f_+\rangle &\ge \frac{c}{\Psi(r)\mu(B')}\int_{B'}\left(\int_H (f_+(x)-f_+(y))^2 d\mu(y)\right)d\mu(x)\\ &= \frac{c}{\Psi(r)\mu(B')}\int_{B'} f_+(x)^2\mu(H)d\mu(x),\end{aligned}$$

which was to be proved. □

In fact, the condition (4.2) is equivalent to $(PI)_\psi$ but we do not use this.

5. The Faber–Krahn inequality.

In this section we show that the Poincaré inequality implies the Faber–Krahn inequality if both conditions (VD) and (RVD) hold. The method of proof is motivated by a similar result in [**17**] obtained in a setting of Riemannian manifolds.

Theorem 5.1. *Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form. Assume that conditions (VD), (RVD), and* (1.15) *are satisfied. Then*

$$(PI)_\Psi \Rightarrow (FK)_\Psi. \tag{5.1}$$

Proof. Let $B := B(x_0,R)$ be a ball in M and $\Omega \subset B$ be a non-empty open set. Observe first that in the definition (1.22) of $\lambda_{\min}(\Omega)$ the range of u can be restricted to $u \in \mathcal{F}\cap C_0(\Omega)$ without changing the value of inf, due to the regularity of the Dirichlet form $(\mathcal{E},\mathcal{F})$. Next, the range of u can be restricted to non-negative functions due to $\mathcal{E}(u) \ge \mathcal{E}(|u|)$. Hence, we need to show that for all non-negative functions $u \in \mathcal{F}\cap C_0(\Omega)$,

$$\mathcal{E}(u) \ge \frac{C_F}{\Psi(R)}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu \|u\|_2^2 \tag{5.2}$$

for some positive constants C_F, ν that are independent of u, Ω, B. We split the proof of (5.2) into three steps.

Step 1: *Construction of the balls $B(x,r_x)$.* Fix some $t>0$ and consider the set

$$\Omega_t = \{x \in \Omega : u(x) > t\}.$$

As u is continuous, the set Ω_t is open. Let us show that, for each $x \in \Omega_t$, there exists $r_x \in (0, C_1R)$ such that

$$\mu(B(x, r_x) \cap \Omega_t) \geq \frac{1}{2} V(x, r_x) \quad \text{and} \quad \mu\big(\overline{B}(x, r_x) \setminus \Omega_t\big) \geq \frac{1}{2} V(x, r_x) \tag{5.3}$$

where the constant $C_1 > 0$ is independent of x, t and B (see Figure 2).

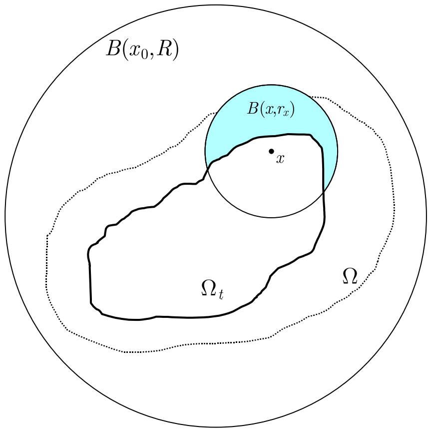

Figure 2. The sets $\Omega_t \subset \Omega \subset B(x_0, R)$ and the ball $B(x, r_x)$.

To that end consider the function

$$v(r) = \mu(B(x, r) \cap \Omega_t).$$

Since Ω_t is open and $x \in \Omega_t$, for sufficiently small $r > 0$ we have an inclusion $B(x, r) \subset \Omega_t$ and, hence,

$$v(r) = \mu(B(x, r)) = V(x, r).$$

Let us show that, for $r \geq C_1 R$,

$$v(r) \leq \frac{1}{4} V(x, r),$$

where the constant C_1 depends on the constants in the reverse volume doubling condition. Indeed, since

$$B(x, r) \cap \Omega_t \subset B(x_0, R) \subset B(x, 2R),$$

we have by (RVD)

$$v(r) \leq V(x, 2R) \leq C\left(\frac{2R}{r}\right)^{\alpha'} V(x, r) \leq \frac{1}{4} V(x, r)$$

provided $r \geq C_1 R$ with $C_1 = 2(4C)^{1/\alpha'}$. Hence, the function

$$h(r) := \frac{v(r)}{V(x,r)} = \frac{\mu(B(x,r)\cap\Omega_t)}{\mu(B(x,r))}$$

is equal to 1 for small values of r and is $\leq 1/4$ for $r \geq C_1 R$.

If h is continuous then there exists an intermediate value $0 < r_x < C_1 R$ that satisfies $h(r_x) = 1/2$, which implies the both conditions in (5.3). However, in general h does not have to be continuous, but it is always left continuous since so are the functions $V(x,r)$ and $v(r)$, which follows from the σ-additivity of measure μ. Besides, function h has right limits at any point as the ratio of two monotone functions. Furthermore, since

$$V(x,r+) := \lim_{t\to r+} V(x,t) = \mu\big(\overline{B}(x,r)\big)$$

and a similar identity holds for $v(r+)$, we obtain

$$h(r+) = \frac{\mu(\overline{B}(x,r)\cap\Omega_t)}{\mu(\overline{B}(x,r))}.$$

Setting

$$r_x = \sup\left\{r : h(r) > \frac{1}{2}\right\},$$

we obtain that $0 < r_x < C_1 R$ and

$$h(r_x) \geq \frac{1}{2} \quad\text{and}\quad h(r_x+) \leq \frac{1}{2}.$$

The first of this inequalities implies the first condition in (5.3), while the second one yields

$$\mu\big(\overline{B}(x,r_x)\cap\Omega_t\big) \leq \frac{1}{2}\mu\big(\overline{B}(x,r_x)\big)$$

and, hence,

$$\mu\big(\overline{B}(x,r_x)\setminus\Omega_t\big) \geq \frac{1}{2}\mu\big(\overline{B}(x,r_x)\big) \geq \frac{1}{2}V(x,r_x),$$

that is the second condition in (5.3).

Step 2: *Estimating of the energy of u between the level sets.* Let $\{B(x,r_x)\}_{x\in\Omega_t}$ be the family of the balls constructed as above. Set $R_x = (2/\sigma)r_x$ where $\sigma \in (0,1)$ is the constant from $(PI)_\psi$. Since the family $\{B(x,R_x)\}_{x\in\Omega_t}$ is a covering of Ω_t and condition (VD) holds, we can choose by the classical ball covering argument a countable disjoint family $\{B(x_k,R_k)\}_{k=1}^{\infty}$ of balls with $R_k = R_{x_k}$ such that

$$\Omega_t \subset \bigcup_{k=1}^{\infty} B(x_k, 5R_k).$$

Set $r_k = r_{x_k}$ and

$$U := \bigcup_{k=1}^{\infty} B(x_k, r_k)$$

and observe that by (5.3) and (VD)

$$\begin{aligned}\mu(U \cap \Omega_t) &= \sum_{k=1}^{\infty} \mu(B(x_k, r_k) \cap \Omega_t) \\ &\geq \frac{1}{2} \sum_{k=1}^{\infty} \mu(B(x_k, r_k)) \\ &\geq c_0 \sum_{k=1}^{\infty} \mu(B(x_k, 5R_k)) \\ &\geq c_0 \mu(\Omega_t), \end{aligned} \tag{5.4}$$

where $c_0 > 0$ depends on the constants C_D and σ.

Fix some pair $t' > t > 0$ and define a function f by

$$f = (u-t)_+ \wedge (t'-t) = \begin{cases} t'-t, & \text{in } \Omega_{t'}, \\ u-t, & \text{in } \Omega_t \setminus \Omega_{t'}, \\ 0, & \text{in } M \setminus \Omega_t, \end{cases}$$

see Figure 3.

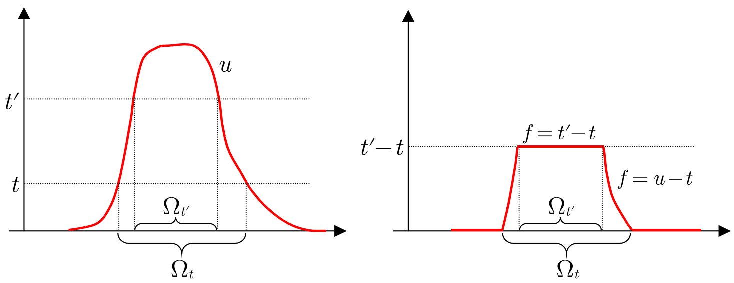

Figure 3. The function f.

By the Markov property of $(\mathcal{E}, \mathcal{F})$, we have $f \in \mathcal{F} \cap C_0(\Omega)$. Next, we would like to applying the inequality (4.2) of Lemma 4.2 to function f in the balls $B(x_k, 2r_k), B(x_k, R_k)$, whose radii have the ratio exactly σ. The set

$$H := \{x \in B(x_k, 2r_k) : f(x) = 0\}$$

contains $B(x_k, 2r_k) \setminus \Omega_t$ and, hence, $\overline{B}(x_k, r_k) \setminus \Omega_t$. Since by (5.3)

$$\mu\big(\overline{B}(x_k,r_k)\setminus\Omega_t\big)\geq\frac{1}{2}V(x_k,r_k)\geq\frac{1}{2}C_D^{-1}V(x_k,2r_k),$$

we obtain by Lemma 4.2

$$\int_{B(x_k,r_k)}f^2d\mu\leq\int_{B(x_k,2r_k)}f^2d\mu\leq C\Psi(R_k)\int_{B(x_k,R_k)}d\Gamma\langle f\rangle. \tag{5.5}$$

Let us estimate R_k from above using R and $\mu(\Omega)$. By (5.3) we have

$$\mu(\overline{B}(x_k,r_k)\cap\Omega_t)\geq\frac{1}{2}V(x_k,r_k),$$

whence

$$V(x_k,r_k)\leq 2\mu(\Omega_t)\leq 2\mu(\Omega).$$

On the other hand, by (VD) and $r_k\leq C_1R$,

$$\mu(B)=V(x_0,R)\leq V(x_k,(C_1+1)R)\leq C'\left(\frac{R}{r_k}\right)^{\alpha}V(x_k,r_k).$$

Combining this two inequalities, we obtain

$$\mu(B)\leq 2C'\left(\frac{R}{r_k}\right)^{\alpha}\mu(\Omega),$$

which implies

$$r_k\leq CR\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{1/\alpha}.$$

Further, using the monotonicity of Ψ, (1.15) and $\mu(\Omega)/\mu(B)\leq 1$, we obtain

$$\begin{aligned}\Psi(R_k)=\Psi\left(\frac{2}{\sigma}r_k\right)&\leq\Psi\left(\frac{2}{\sigma}CR\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{1/\alpha}\right)\\&\leq C_\Psi\left(\frac{2}{\sigma}C\right)^{\beta'}\Psi\left(R\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{1/\alpha}\right)\\&\leq C'\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R). \end{aligned}\tag{5.6}$$

Substituting this into (5.5), summing up in all k, and using that the balls $B(x_k,R_k)$ are disjoint, we obtain that

$$\begin{aligned}\int_U f^2 d\mu &\le C\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R)\sum_{k=1}^{\infty}\int_{B(x_k,R_k)} d\Gamma\langle f\rangle \\ &\le C\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R)\int_M d\Gamma\langle f\rangle \\ &= C\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R)\int_{\Omega_t\setminus\Omega_{t'}} d\Gamma\langle u\rangle, \end{aligned}\tag{5.7}$$

where in the last line we have also used the fact that $\Gamma\langle f\rangle = 0$ outside $\Omega_t \setminus \Omega_{t'}$ while $\Gamma\langle f\rangle = \Gamma\langle u\rangle$ inside $\Omega_t \setminus \Omega_{t'}$, due to the strong locality of $(\mathcal{E}, \mathcal{F})$.

Let us assume in addition that t' is chosen so close to t that

$$\mu\big(\Omega_t \setminus \overline{\Omega}_{t'}\big) \le \varepsilon\mu(\Omega_t), \tag{5.8}$$

where $\varepsilon = c_0/2$ and c_0 is the constant from (5.4). By (5.4), we obtain

$$\mu(U\cap\Omega_t) = \sum_{k=1}^{\infty}\mu(B(x_k, r_k)\cap\Omega_t) \ge 2\varepsilon\mu(\Omega_t),$$

which together with (5.8) implies

$$\begin{aligned}\mu\big(U\cap\overline{\Omega}_{t'}\big) &= \mu(U\cap\Omega_t) - \mu\big(U\cap(\Omega_t\setminus\overline{\Omega}_{t'})\big) \\ &\ge 2\varepsilon\mu(\Omega_t) - \mu\big((\Omega_t\setminus\overline{\Omega}_{t'})\big) \\ &\ge \varepsilon\mu(\Omega_t).\end{aligned}$$

It follows that

$$\begin{aligned}\int_U f^2 d\mu &\ge \int_{U\cap\overline{\Omega}_{t'}} f^2 d\mu \\ &= (t'-t)^2\mu\big(U\cap\overline{\Omega}_{t'}\big) \\ &\ge \varepsilon(t'-t)^2\mu(\Omega_t).\end{aligned}$$

Combining this with (5.7), we conclude that

$$\varepsilon(t'-t)^2\mu(\Omega_t) \le C'\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R)\int_{\Omega_t\setminus\Omega_{t'}} d\Gamma\langle u\rangle. \tag{5.9}$$

Step 3: *The proof of* $(FK)_\Psi$. Set for all $t \ge 0$

$$m(t) = \mu(\Omega_t) \quad\text{and}\quad \overline{m}(t) = \mu\big(\overline{\Omega_t}\big)$$

where

$$\overline{\Omega}_t = \{x \in \Omega : u(x) \geq t\}.$$

The functions $m(t)$ and $\overline{m}(t)$ are monotone decreasing, $m(t) \leq \overline{m}(t)$, and they both vanish for large enough t as u is bounded. Furthermore, $m(t)$ is right-continuous, while $\overline{m}(t)$ is left-continuous, which follows from

$$\Omega_t = \bigcup_{s>t} \Omega_s \quad \text{and} \quad \overline{\Omega}_t = \bigcap_{s<t} \overline{\Omega}_s$$

and the σ-additivity of μ.

Let us define inductively an increasing sequence $\{t_j\}_{j=0}^{\infty}$ of non-negative reals as follows: $t_0 = 0$ and

$$t_{j+1} := \sup\{s : m(s) > (1-\varepsilon)m(t_j)\}$$

for all $j = 0, 1, 2, \ldots$. By the left-continuity of $\overline{m}$ we obtain that

$$\overline{m}(t_{j+1}) = \overline{m}(t_{j+1}-) \geq m(t_{j+1}-) \geq (1-\varepsilon)m(t_j), \tag{5.10}$$

while the right-continuity of m implies that

$$m(t_{j+1}) \leq (1-\varepsilon)m(t_j). \tag{5.11}$$

It follows from (5.10) that

$$m(t_j) - \overline{m}(t_{j+1}) \leq \varepsilon m(t_j),$$

that is, the condition (5.8) is satisfied with $t' = t_{j+1}$ and $t = t_j$. Applying (5.9) with these values of t', t and then summing up over all j, we obtain that

$$\begin{aligned} \mathcal{E}(u) &= \int_{\Omega} d\Gamma\langle u\rangle = \sum_{j=0}^{\infty} \int_{\Omega_{t_j} \setminus \Omega_{t_{j+1}}} d\Gamma\langle u\rangle \\ &\geq \left\{ C' \left(\frac{\mu(\Omega)}{\mu(B)} \right)^{\beta/\alpha} \Psi(R) \right\}^{-1} \varepsilon \sum_{j=0}^{\infty} (t_{j+1} - t_j)^2 m(t_j). \end{aligned} \tag{5.12}$$

On the other hand, it is clear that

$$\|u\|_2^2 = \sum_{j=0}^{\infty} \int_{\Omega_{t_j} \setminus \Omega_{t_{j+1}}} u^2 d\mu \leq \sum_{j=0}^{\infty} t_{j+1}^2 (m(t_j) - m(t_{j+1})). \tag{5.13}$$

By [**17**, Lemma 1.2], for any increasing sequence $\{t_j\}_{j=0}^{\infty}$ with $t_0 = 0$ and for any sequence $\{m_j\}_{j=0}^{\infty}$ of non-negative reals, satisfying for some $\varepsilon \in (0, 1)$ the condition

$$m_{j+1} \leq (1-\varepsilon)m_j \quad \text{for all} \quad j = 0, 1, 2, \ldots, \tag{5.14}$$

the following inequality holds:

$$\sum_{j=0}^{\infty}(t_{j+1}-t_j)^2 m_j \geq \frac{\varepsilon}{12}\sum_{j=0}^{\infty} t_{j+1}^2(m_j - m_{j+1}) \tag{5.15}$$

(which is a consequence of the Hardy inequality). Since by (5.11) the sequence $m_j = m(t_j)$ satisfies (5.14), combining (5.12), (5.13), and (5.15) we conclude that

$$\mathcal{E}(u) \geq \frac{\varepsilon^2}{12}\left\{C'\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha}\Psi(R)\right\}^{-1}\|u\|_2^2,$$

which proves (5.2) with $\nu = \beta/\alpha$. □

6. Mean-value inequality for subharmonic functions.

In this section, we will obtain an L^2 mean value inequality for any non-negative subharmonic function assuming that conditions $(FK)_\Psi$ and $(\mathrm{Gcap}_{\leq})_\Psi$ are satisfied.

6.1. Admissible subharmonic functions.

Let us introduce a class of *admissible* functions. Fix two constants $C_A > 0$ and $\gamma \geq 0$ whose values will be determined later.

Definition. We say that a function $u \in \mathcal{F} \cap L^\infty$ is *admissible* in a ball $B(x_0, R)$ if for any $0 < r_1 \leq R$ and any $r_1/2 \leq r_2 < r_1$, there exists some $\phi \in \mathrm{cutoff}(B(x_0, r_2), B(x_0, r_1))$ satisfying

$$\mathcal{E}(\phi u) \leq \frac{C_A}{\Psi(r_1 - r_2)}\left(\frac{r_2}{r_1 - r_2}\right)^\gamma \int_{B(x_0,r_1)} u^2 d\mu. \tag{6.1}$$

Note that the cutoff function ϕ may depend on $u, B(x_0, R), r_2, r_1$.

Lemma 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and conditions (VD), (A_2) hold. Let u be a bounded, non-negative subharmonic function in a ball $B(x_0, R)$. Then u is admissible in $B(x_0, R)$.*

Proof. Let $B_2 := B(x_0, r_2)$ and $B_1 := B(x_0, r_1)$ where r_1, r_2 are as above. We will construct a function $\phi \in \mathrm{cutoff}(B_2, B_1)$ such that (6.1) holds with $\gamma = \alpha$

$$\mathcal{E}(\phi u) \leq \frac{C}{\Psi(r_1 - r_2)}\left(\frac{r_2}{r_1 - r_2}\right)^\alpha \int_{B(x_0,r_1)} u^2 d\mu \tag{6.2}$$

for some positive constant $C > 0$, where α is the same as in (1.13).

Indeed, set $s := r_1 - r_2 \in (0, r_2)$. By the doubling condition, there exists an integer N such that the ball B_2 can be covered by the union of balls $\{B(y_k, s/2)\}_{k=1}^N$ centered at B_2, whilst the balls $\{B(y_k, s/10)\}_{k=1}^N$ are disjointed (see Figure 4).

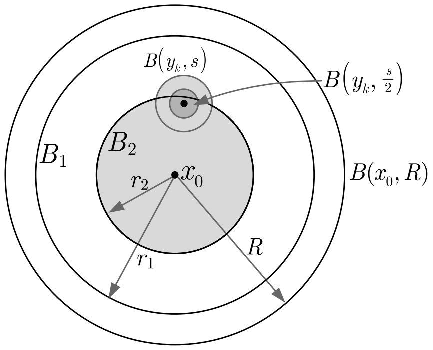

Figure 4. The balls $B(y_k, s/2)$ and B_1, B_2.

Observe that there exists a constant $C > 0$ (depending only the doubling constant C_D) such that

$$N \le C\left(\frac{r_2}{s}\right)^{\alpha}, \tag{6.3}$$

since we have from (1.13) that, for $s = r_1 - r_2 \le r_2$,

$$\begin{aligned}\frac{V(x_0, r_1)}{V(y_k, s/10)} &\le C_D\left(\frac{d(x_0, y_k) + r_1}{s/10}\right)^{\alpha}\\ &\le C_D\left(\frac{r_2 + r_1}{s/10}\right)^{\alpha} \le C\left(\frac{r_2}{s}\right)^{\alpha},\end{aligned}$$

and hence,

$$\begin{aligned}V(x_0, r_1) &\ge \sum_{k=1}^{N} V(y_k, s/10) \ge \sum_{k=1}^{N} V(x_0, r_1)C^{-1}\left(\frac{r_2}{s}\right)^{-\alpha}\\ &= V(x_0, r_1)C^{-1}\left(\frac{r_2}{s}\right)^{-\alpha} N,\end{aligned}$$

thus proving (6.3).

For simplicity, set $U_k := B(y_k, s)$. For each $1 \le k \le N$, by inequality (2.6), there is $\phi_k \in \mathrm{cutoff}((1/2)U_k, U_k)$ such that

$$\mathcal{E}(u\phi_k) \le \frac{C_1}{\Psi(s)} \int_{U_k} u^2 d\mu. \tag{6.4}$$

We set

$$\phi = \phi_1 \vee \phi_2 \vee \cdots \vee \phi_N.$$

Clearly, $\phi \in \text{cutoff}(B_2, B_1)$. We will show that (6.2) holds with this ϕ.

Indeed, as $u \geq 0$ in B_1, we see that

$$\phi u = (\phi_1 u) \vee (\phi_2 u) \vee \cdots \vee (\phi_N u).$$

Observe that for any $u, v \in \mathcal{F}$,

$$\mathcal{E}(u \vee v) \leq \mathcal{E}(u) + \mathcal{E}(v),$$

since, using the facts that

$$u \vee v = \frac{1}{2}(u + v + |u - v|) \text{ and } u \wedge v = \frac{1}{2}(u + v - |u - v|),$$

we have

$$\begin{aligned}
\mathcal{E}(u \vee v) + \mathcal{E}(u \wedge v) &= \frac{1}{4}\{\mathcal{E}(u + v + |u - v|) + \mathcal{E}(u + v - |u - v|)\} \\
&= \frac{1}{2}\{\mathcal{E}(u + v) + \mathcal{E}(|u - v|)\} \\
&\leq \frac{1}{2}\{\mathcal{E}(u + v) + \mathcal{E}(u - v)\} \\
&= \mathcal{E}(u) + \mathcal{E}(v).
\end{aligned}$$

Therefore, it follows from (6.4), $U_k \subset B_1$ and (6.3), that

$$\begin{aligned}
\mathcal{E}(\phi u) = \mathcal{E}((\phi_1 u) \vee (\phi_2 u) \vee \cdots \vee (\phi_N u)) &\leq \sum_{k=1}^{N} \mathcal{E}(\phi_k u) \\
&\leq \frac{C_1}{\Psi(s)} \sum_{k=1}^{N} \int_{U_k} u^2 d\mu \leq \frac{C_1}{\Psi(s)} \sum_{k=1}^{N} \int_{B_1} u^2 d\mu \\
&= \frac{C_1}{\Psi(s)} N \int_{B_1} u^2 d\mu \leq \frac{C'}{\Psi(s)} \left(\frac{r_2}{s}\right)^{\alpha} \int_{B_1} u^2 d\mu,
\end{aligned}$$

thus proving (6.2). Hence, u is admissible in $B(x_0, R)$. □

6.2. The L^2-mean value inequality.

We prove an L^2 mean value inequality for any non-negative (not necessarily subharmonic) function $u \in \mathcal{F} \cap L^\infty$ if $(u - k)_+$ is admissible for any $k > 0$.

THEOREM 6.2 (L^2 mean value inequality). *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form. Assume that $(FK)_\Psi$ and (VD) are satisfied. If $u \in \mathcal{F} \cap L^\infty$ is non-negative in a ball $B := B(x_0, R)$ and if $(u - k)_+$ is admissible in B for any $k > 0$, then*

$$\underset{(1/2)B}{\operatorname{esup}}\, u^2 \leq \frac{C}{\mu(B)} \int_B u^2 d\mu, \tag{6.5}$$

where the constant C is independent of B, u.

PROOF. We split the proof into two steps.

Step 1: For any $0 < r_1 \leq R$ and $r_1/2 \leq r_2 < r_1$, let $U_2 := B(x_0, r_2)$ and $U_1 := B(x_0, r_1)$. Fix some $0 < \rho_1 < \rho_2$ and set

$$a_1 := \int_{U_1} (u - \rho_1)_+^2 d\mu \quad \text{and} \quad a_2 := \int_{U_2} (u - \rho_2)_+^2 d\mu,$$

so that $a_2 \leq a_1$. We will prove the following relation between a_1 and a_2:

$$a_2 \leq \frac{C_1 \Psi(r_1)}{\Psi(r_1 - r_2)\mu(U_1)^\nu} \left(\frac{r_2}{r_1 - r_2}\right)^\gamma \frac{a_1^{1+\nu}}{(\rho_2 - \rho_1)^{2\nu}}, \tag{6.6}$$

which will be used later on to do iterations.

Choose a quasi-continuous version of u, a function $\phi \in \operatorname{cutoff}(U_2, U_1)$ (to be specified below) and consider a set

$$E = \operatorname{supp}\phi \cap \{u \geq \rho_2\} \subset U_1. \tag{6.7}$$

By the regularity of μ, for any $\varepsilon > 0$ there is an open set $\Omega \subset U_1$ that contains E and such that

$$\mu(\Omega) \leq \mu(E) + \varepsilon \tag{6.8}$$

(see Figure 5).

As $\phi(u - \rho_2)_+ = 0$ in E^c, we see that $\phi(u - \rho_2)_+ = 0$ also in $\Omega^c \subset E^c$. Since

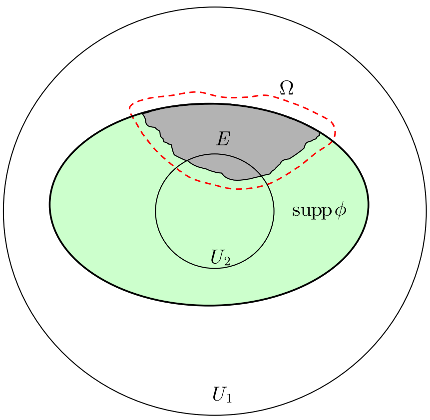

Figure 5. Sets E and Ω.

$\phi(u-\rho_2)_+$ is quasi-continuous, it follows that $\phi(u-\rho_2)_+ \in \mathcal{F}(\Omega)$ (cf. [**16**, Corollary 2.3.1, p. 98]). Since the function $(u-\rho_2)_+$ is admissible in B by assumption, the function ϕ can be chosen so that

$$\mathcal{E}(\phi(u-\rho_2)_+) \le \frac{C_A}{\Psi(r_1-r_2)}\left(\frac{r_2}{r_1-r_2}\right)^\gamma \int_{U_1}(u-\rho_2)_+^2 d\mu. \tag{6.9}$$

From this and using $(FK)_\Psi$, we obtain

$$\begin{aligned}
a_2 &= \int_{U_2}(u-\rho_2)_+^2 d\mu \le \int_{U_1}\{\phi(u-\rho_2)_+\}^2 d\mu \\
&= \int_{\Omega}\{\phi(u-\rho_2)_+\}^2 d\mu \le \frac{\mathcal{E}(\phi(u-\rho_2)_+)}{\lambda_{\min}(\Omega)} \\
&\le \frac{\Psi(r_1)}{C_F}\left(\frac{\mu(\Omega)}{\mu(U_1)}\right)^\nu \left\{\frac{C_A}{\Psi(r_1-r_2)}\left(\frac{r_2}{r_1-r_2}\right)^\gamma \int_{U_1}(u-\rho_2)_+^2 d\mu\right\} \\
&\le C_1 \frac{\Psi(r_1)}{\Psi(r_1-r_2)\mu(U_1)^\nu}\left(\frac{r_2}{r_1-r_2}\right)^\gamma \mu(\Omega)^\nu a_1. \qquad (6.10)
\end{aligned}$$

Since ε in (6.8) can be taken arbitrarily small, we obtain

$$a_2 \le C_1 \frac{\Psi(r_1)}{\Psi(r_1-r_2)\mu(U_1)^\nu}\left(\frac{r_2}{r_1-r_2}\right)^\gamma \mu(E)^\nu a_1. \tag{6.11}$$

On the other hand, by definition (6.7) of E, we have

$$a_1 = \int_{U_1}(u-\rho_1)_+^2 d\mu \ge \int_E (\rho_2-\rho_1)_+^2 d\mu = (\rho_2-\rho_1)^2\mu(E),$$

whence

$$\mu(E) \le \frac{a_1}{(\rho_2-\rho_1)^2}.$$

Substituting this into (6.11), we obtain (6.6), as desired.

Step 2: Here we prove (6.5) using (6.6) and (VD). Without loss of generality, we can assume that

$$\|u\|_{L^2(B)} = 1. \tag{6.12}$$

Fix some $\rho > 0$ to be determined later on and set

$$R_k = \left(\frac{1}{2} + 2^{-k-1}\right)R \quad \text{and } \rho_k = \rho(2-2^{-k}),\ k \ge 0. \tag{6.13}$$

The sequence $\{R_k\}_{k=0}^{\infty}$ is non-increasing, $R_0 = R$, $R_k \to (1/2)R$ as $k \to \infty$, and

$$0 < R_{k-1} - R_k = 2^{-k-1}R < R_k. \tag{6.14}$$

Similarly, the sequence $\{\rho_k\}_{k=0}^{\infty}$ is non-decreasing, $\rho_0 = \rho$, $\rho_k \to 2\rho$ as $k \to \infty$, and

$$\rho_k - \rho_{k-1} = 2^{-k}\rho.$$

For $k \geq 0$, set $U_k := B(x_0, R_k)$ and

$$a_k = \int_{U_k} (u - \rho_k)_+^2 \, d\mu. \tag{6.15}$$

Note that $U_0 = B(x_0, R)$, and $a_{k+1} \leq a_k \leq 1$ for any $k \geq 0$.

Clearly, we see from (6.13) and (6.14) that

$$\frac{R_k}{R_{k-1} - R_k} < \frac{R_{k-1}}{R_{k-1} - R_k} = \frac{(1/2 + 2^{-k})R}{2^{-k-1}R} < 2^{k+2}. \tag{6.16}$$

From this and using (1.15), we obtain

$$\frac{\Psi(R_{k-1})}{\Psi(R_{k-1} - R_k)} \leq C_\Psi \left(\frac{R_{k-1}}{R_{k-1} - R_k} \right)^{\beta'} \leq C' 2^{k\beta'}. \tag{6.17}$$

By condition (VD), we have

$$\mu(U_{k-1}) \geq \mu\left(\frac{1}{2}B\right) \geq C_D^{-1} \mu(B). \tag{6.18}$$

Applying (6.6) with $r_1 = R_{k-1}$, $r_2 = R_k$ and with ρ_1, ρ_2 being respectively replaced by ρ_{k-1}, ρ_k, and substituting (6.16), (6.17), (6.18), we obtain

$$\begin{aligned}
a_k &\leq \frac{C_1 \Psi(R_{k-1})}{\Psi(R_{k-1} - R_k)\mu(U_{k-1})^\nu} \left(\frac{R_k}{R_{k-1} - R_k} \right)^\gamma \frac{a_{k-1}^{1+\nu}}{(\rho_k - \rho_{k-1})^{2\nu}} \\
&\leq C_1 (C' 2^{k\beta'}) \big(C_D^\nu \mu(B)^{-\nu}\big) 2^{(k+2)\gamma} (2^{-k}\rho)^{-2\nu} a_{k-1}^{1+\nu} \\
&= C_2 \mu(B)^{-\nu} \rho^{-2\nu} 2^{ks} a_{k-1}^{1+\nu} := A 2^{ks} a_{k-1}^{1+\nu}
\end{aligned}$$

for any $k \geq 1$, where

$$s = \beta' + \gamma + 2\nu$$

and

$$A := C_2 \mu(B)^{-\nu} \rho^{-2\nu}. \tag{6.19}$$

Setting $q := 1+\nu$, we obtain by iteration

$$\begin{aligned} a_k &\le A2^{ks}a_{k-1}^q \le (A2^{ks})\big(A2^{(k-1)s}a_{k-2}^q\big)^q \le \cdots \\ &\le (A^{1+q+\cdots+q^{k-1}})(2^{s(k+(k-1)q+\cdots+q^{k-1})})a_0^{q^k} \\ &\le A^{(q^k-1)/(q-1)}2^{s((q^{k+1}-(k+1)q+k)/(q-1)^2)} \\ &\le A^{(q^k-1)/(q-1)}2^{s((q^k-1)(q+1)/(q-1)^2)} \\ &\le (C_3A^{1/(q-1)})^{q^k-1}, \end{aligned} \tag{6.20}$$

where we have used $a_0 \le 1$, the elementary identity

$$k+(k-1)q+\cdots+q^{k-1} = \frac{q^{k+1}-(k+1)q+k}{(q-1)^2} \le \frac{(q^k-1)(q+1)}{(q-1)^2}$$

and set $C_3 = 2^{s((q+1)/(q-1)^2)}$. Noticing that A depends on ρ, we can choose ρ so that the following equation is satisfied:

$$\frac{1}{2} = C_3A^{1/(q-1)}.$$

Indeed, by (6.19) and $q-1=\nu$ this equation is equivalent to

$$\frac{1}{2} = C_4\mu(B)^{-1}\rho^{-2},$$

which yields

$$\rho^2 = 2C_4\mu(B)^{-1}. \tag{6.21}$$

We obtain from (6.20) that, for any k,

$$\int_{B(x_0,(1/2)R)} (u-2\rho)_+^2\,d\mu \le a_k \le (C_3A^{1/(q-1)})^{q^k-1} = \left(\frac{1}{2}\right)^{q^k-1} \to 0 \quad \text{as } k\to\infty.$$

From this and (6.21), we obtain

$$\operatorname*{esup}_{B(x_0,(1/2)R)} u^2 \le 4\rho^2 = 8C_4\mu(B)^{-1},$$

which in the view of (6.12) finishes the proof. □

Let us introduce the condition (MV) (*mean value inequality*).

Mean value inequality. We say that the condition (MV) is satisfied if there exists a constant $C_{MV}>0$ such that, for any ball B and for any bounded non-negative subharmonic function u in B, the following inequality holds:

$$\operatorname*{esup}_{(1/2)B} u^2 \leq \frac{C_{MV}}{\mu(B)} \int_B u^2 d\mu. \tag{6.22}$$

The following is the main result of this section.

Theorem 6.3. *Assume that $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form and that condition (VD) holds. Then the following implication is true*:

$$(FK)_\Psi + (\mathrm{Gcap}_{\leq})_\Psi \Rightarrow (MV).$$

Proof. Let B and u be as in the above Definition of the mean value inequality. As follows from definition of subharmonic functions, there exists a function $u' \in \mathcal{F} \cap L^\infty$ such that $u = u'$ in B. By the locality of $(\mathcal{E}, \mathcal{F})$, the function u' is subharmonic in B. By Proposition 2.1, function $(u' - k)_+$ is subharmonic for any $k \geq 0$. By Proposition 2.3, we have $(\mathrm{Gcap}_{\leq})_\Psi \Rightarrow (A_2)$. Thus, by Lemma 6.1, the function u' is admissible in $B(x_0, R)$. Finally, Theorem 6.2 yields the mean value inequality for u' and, hence, for u. $\square$

7. Proof of elliptic Harnack inequality.

In this section we prove the elliptic Harnack inequality. We assume here that the following conditions are known to be true: (MV), $(PI)_\Psi$, $(\mathrm{cap}_{\leq})_\Psi$, and will prove the Harnack inequality (H). The proof follows essentially the argument of Landis [**41**], [**38**] with some simplifications (see also [**17**] for a version of this argument for parabolic Harnack inequality).

Let us introduce condition $(\mathrm{cap}_{\leq})_\Psi$.

Upper bound of capacity. We say that the *condition* $(\mathrm{cap}_{\leq})_\Psi$ holds if there exists constant $C > 0$ such that, for all balls B of radius $r > 0$,

$$\mathrm{cap}(B, 2B) \leq \frac{C\mu(B)}{\Psi(r)}. \tag{7.1}$$

Lemma 7.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that u is a strictly positive, bounded superharmonic function in a ball $2B$ where $B = B(x_0, r)$. If condition $(\mathrm{cap}_{\leq})_\Psi$ holds, then we have*

$$\int_B d\Gamma\langle \log u \rangle \leq \frac{C_1}{\Psi(r)} \mu(B), \tag{7.2}$$

where constant C_1 is independent of u, x_0, r.

Proof. Let $l(t)$ be the function from the proof of Lemma 2.4, so that $l(u), l'(u), l''(u)$ are in $\mathcal{F} \cap L^\infty$, while in $2B$ we have $l(u) = \log u$, $l'(u) = 1/u$ and $l''(u) = -1/u^2 = -l'(u)^2$.

By the chain rule, we have in $2B$

$$d\Gamma\langle l(u)\rangle = l'(u)^2 d\Gamma\langle u\rangle = -l''(u)d\Gamma(u) = -d\Gamma(u, l'(u)) = -d\Gamma\langle u, u^{-1}\rangle. \tag{7.3}$$

Let ϕ be an "almost" optimal test function for $\operatorname{cap}(B, 2B)$ so that we obtain by $(\mathrm{cap})_\Psi$

$$\mathcal{E}(\phi) \leq 2\operatorname{cap}(B, 2B) \leq \frac{2C\mu(B)}{\Psi(r)}. \tag{7.4}$$

Integrating (7.3) and using that u is superharmonic in $2B$, we obtain

$$\begin{aligned}\int \phi^2 d\Gamma\langle \log u\rangle &= -\int \phi^2 d\Gamma\langle u, u^{-1}\rangle = -\int d\Gamma\langle u, \phi^2 u^{-1}\rangle + 2\int \phi u^{-1} d\Gamma\langle u, \phi\rangle \\ &\leq 2\int \phi u^{-1} d\Gamma\langle u, \phi\rangle \leq \frac{1}{2}\int \phi^2 u^{-2} d\Gamma\langle u\rangle + 2\int d\Gamma\langle \phi\rangle \\ &= \frac{1}{2}\int \phi^2 d\Gamma\langle \log u\rangle + 2\mathcal{E}(\phi),\end{aligned}$$

where all integrals are taken over $2B$. Hence, we obtain

$$\int_{2B} \phi^2 d\Gamma\langle \log u\rangle \leq 4\mathcal{E}(\phi). \tag{7.5}$$

Combining (7.4) and (7.5), we conclude that

$$\int_B d\Gamma\langle \log u\rangle \leq \int_{2B} \phi^2 d\Gamma\langle \log u\rangle \leq \frac{8C\mu(B)}{\Psi(r)},$$

thus proving (7.2). □

For any ball B and any measurable set $A \subset M$ denote

$$\omega_B(A) = \frac{\mu(A \cap B)}{\mu(B)},$$

the *occupation measure* of the set A in B. If A has the form $\{u \geq a\}$ where u is a function and $a \in \mathbb{R}$, then we write for simplicity $\omega(A) = \omega(u \geq a)$ without additional brackets.

Lemma 7.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that (VD) holds. Let u be any non-negative bounded harmonic function in a ball $2B$ and $a > 0$ be any number. If conditions (MV), $(PI)_\Psi$, $(\mathrm{cap}_\leq)_\Psi$ hold, then (see Figure 6)*

$$\operatorname*{einf}_{(1/2)\sigma B} u \geq a \exp\left(-\frac{C}{\omega_{\sigma B}(u \geq a)}\right). \tag{7.6}$$

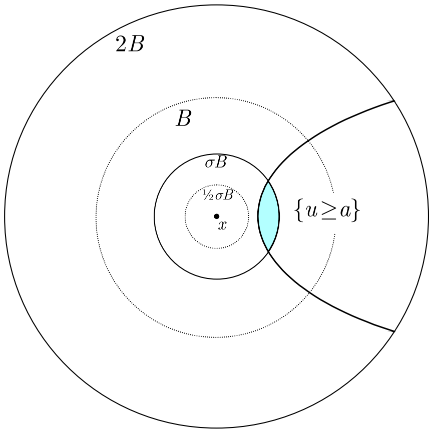

Figure 6. Set $\{u \geq a\} \cap \sigma B$.

Proof. We can assume without loss of generality, that u is strictly positive in $2B$ (otherwise apply (7.6) to function $u_\varepsilon = u + \varepsilon$ and constant $a + \varepsilon$ for any $\varepsilon > 0$ and then let $\varepsilon \to 0$).

Let $B = B(x, r)$. By Lemma 7.1 (where condition $(\mathrm{cap}_{\leq})_\Psi$ is used), we have for $f = \log(a/u)$

$$\int_B d\Gamma\langle f\rangle \leq \frac{C_1}{\Psi(r)}\mu(B).$$

By Lemma 4.2 (where condition $(PI)_\Psi$ is used), we have

$$\int_B d\Gamma\langle f_+\rangle \geq \frac{c}{\Psi(r)}\omega_{\sigma B}(f \leq 0)\int_{\sigma B} f_+^2 d\mu.$$

Since $f \leq 0 \Leftrightarrow u \geq a$ and, hence,

$$\omega_{\sigma B}(f \leq 0) = \omega_{\sigma B}(u \geq a) =: \omega,$$

we obtain

$$\frac{C_1}{\Psi(r)}\mu(B) \geq \frac{c}{\Psi(r)}\omega \int_{\sigma B} f_+^2 d\mu,$$

whence it follows that

$$⨍_{\sigma B} f_+^2 d\mu \leq \frac{C}{\omega}.$$

By Lemma 2.4, the function f is subharmonic, and so is f_+ by Proposition 2.1. Thus, we conclude by (MV) that

$$\operatorname*{esup}_{(1/2)\sigma B} f_+^2 \le C \fint_{\sigma B} f_+^2 d\mu \le \frac{C}{\omega} \le \frac{C}{\omega^2}$$

whence

$$\operatorname*{einf}_{(1/2)\sigma B} \frac{u}{a} \ge \exp\left(-\frac{C}{\omega}\right),$$

which is equivalent to (7.6). □

Lemma 7.2 plays an important part in our analysis. Once this lemma is established, we can derive the Harnack inequality (H) by using condition (VD) alone, without using the conditions (MV), $(PI)_\Psi$, $(\mathrm{cap}_\le)_\Psi$ anymore. Of course, in order to prove Lemma 7.2 we have already used all these three conditions.

In the rest of this section we assume that (VD) and the validity of Lemma 7.2 hold.

Corollary 7.3. *There exists some constant $\theta \in (0,1)$ such that for any ball B and for any bounded harmonic function u in B,*

$$\operatorname*{eosc}_{(1/4)\sigma B} u \le \theta \operatorname*{eosc}_B u, \tag{7.7}$$

where $\operatorname{eosc}_B u := \operatorname{esup}_B u - \operatorname{einf}_B u$ *is the oscillation of u over B.*

Proof. By rescaling we can assume that

$$\operatorname*{einf}_B u = 0 \quad \text{and} \quad \operatorname*{esup}_B u = 2.$$

It suffices to prove that

$$\operatorname*{eosc}_{(1/4)\sigma B} u \le 2\theta$$

for some constant $\theta \in (0,1)$ that is independent of u, B. By (7.6) with $a = 1$, we have

$$\operatorname*{einf}_{(1/4)\sigma B} u \ge \exp\left(-\frac{C}{\omega_{(1/2)\sigma B}(u \ge 1)}\right).$$

Applying (7.6) again to function $2 - u$, we obtain

$$2 - \operatorname*{esup}_{(1/4)\sigma B} u \ge \exp\left(-\frac{C}{\omega_{(1/2)\sigma B}(u \le 1)}\right).$$

Since one of the quantities $\omega_{(1/2)\sigma B}(\{u \le 1\})$, $\omega_{(1/2)\sigma B}(\{u \ge 1\})$ should be at least $1/2$, we obtain by adding up the above inequalities that

$$2 - \operatorname*{eosc}_{(1/4)\sigma B} u \ge \exp(-2C),$$

whence (7.7) follows. □

Corollary 7.4. *Any bounded harmonic function admits a Hölder continuous version.*

Proof. Let u be a bounded harmonic function in a ball B_0. Fix a ball $B := B(x, r)$ such that $B \subset B_0$. Write $B_s = B(x, s)$ for $s > 0$. It is enough to show that for any $\rho < \delta r$,

$$\operatorname*{eosc}_{B_\rho} u \leq 2\left(\frac{\rho}{r}\right)^{\gamma} \operatorname*{eosc}_{B_r} u \tag{7.8}$$

for some constant $\gamma > 0$ independent of ρ, r and u, where $\delta = (1/4)\sigma$ with the same σ as in (7.7).

Indeed, we have by (7.7) that

$$\operatorname*{eosc}_{B_{\delta r}} u \leq \theta \operatorname*{eosc}_{B_r} u. \tag{7.9}$$

For any $\rho \leq \delta r$, there exists an integer $k \geq 1$ such that

$$\delta^{k+1} r < \rho \leq \delta^k r.$$

Iterating (7.9), we obtain

$$\begin{aligned} \operatorname*{eosc}_{B_\rho} u &\leq \operatorname*{eosc}_{B_{\delta^k r}} u \leq \theta^k \operatorname*{eosc}_{B_r} u \\ &\leq \theta^{\log(r/\rho)/\log(1/\delta)-1} \operatorname*{eosc}_{B_r} u = \frac{1}{\theta}\left(\frac{\rho}{r}\right)^{\log(1/\theta)/\log(1/\delta)} \operatorname*{eosc}_{B_r} u. \end{aligned}$$

Note that the constant θ in (7.7) can be assumed to satisfy $\theta > 1/2$. Therefore, inequality (7.8) follows with $\gamma = \log(1/\theta)/\log(1/\delta)$. □

From now on we use continuous versions of harmonic functions. In particular, the inequality (7.6) of Lemma 7.2 implies

$$u(x) \geq a \exp\left(-\frac{C}{\omega_{\sigma B}(u \geq a)}\right), \tag{7.10}$$

where x is the center of B and u is non-negative and harmonic in $2B$.

Lemma 7.5. *Let u be non-negative bounded harmonic in a ball $B(x, R)$. Then, for all $y \in B(x, (\sigma/9)R)$ and $r \leq \sigma R/4$, and for any $a > 0$,*

$$u(x) \geq a\left(\frac{r}{R}\right)^{N} \exp\left(-\frac{C}{\omega_{B(y,r)}(\{u \geq a\})}\right), \tag{7.11}$$

where N *is a positive constant that depends only on the constants in the hypotheses* (*see Figure* 7)*.*

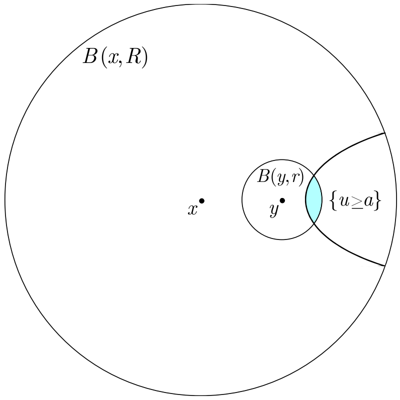

Figure 7. Set $\{u \geq a\} \cap B(y,r)$.

PROOF. Set $B = B(y,r)$ and observe that $2\sigma^{-1}B \subset B(x,R)$ since

$$d(x,y) + 2\sigma^{-1}r \leq \frac{\sigma}{9}R + \frac{R}{2} < R.$$

Applying (7.6) to the function u in the ball $2\sigma^{-1}B$, we obtain

$$\inf_{(1/2)B} u \geq a \exp\left(- \frac{C}{\omega_B(u \geq a)} \right) =: a_1. \tag{7.12}$$

If $4\sigma^{-1}B \subset B(x,R)$ then we can apply (7.6) in the ball $4\sigma^{-1}B$ and obtain

$$\inf_B u \geq a_1 \exp\left(- \frac{C}{\omega_{2B}(u \geq a_1)} \right).$$

Noticing that by (7.12) the set $\{u \geq a_1\}$ contains $(1/2)B$ and, hence,

$$\omega_{2B}(u \geq a_1) \geq \omega_{2B}\left(\frac{1}{2}B \right) \geq c,$$

we obtain

$$\inf_B u \geq a_1 \exp\left(- \frac{C}{c} \right) = \varepsilon a_1 =: a_2,$$

where $\varepsilon := \exp(-C/c)$. If $8\sigma^{-1}B \subset B(x,R)$ then in the same way

$$\inf_{2B} u \geq \varepsilon a_2 = \varepsilon^2 a_1,$$

and so on (see Figure 8).

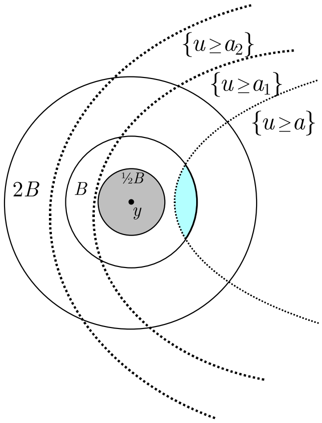

Figure 8. Level sets $\{u \geq a\}, \{u \geq a_1\}$ etc.

As long as

$$2^{k+1}\sigma^{-1}B \subset B(x,R), \tag{7.13}$$

we obtain by (7.6)

$$\inf_{2^{k-1}B} u \geq \varepsilon^k a_1 = \varepsilon^k a \exp\left(-\frac{C}{\omega_B(u \geq a)}\right).$$

Let k be the maximal integer satisfying (7.13). Then we have

$$2^{k+1}\sigma^{-1}r + d(x,y) \leq R \tag{7.14}$$

while

$$2^{k+2}\sigma^{-1}r + d(x,y) > R.$$

It follows that

$$2^{k-1}r \geq \sigma\frac{R - d(x,y)}{8} > d(x,y),$$

where the last inequality is true because

$$\left(1 + \frac{8}{\sigma}\right)d(x,y) < \left(1 + \frac{8}{\sigma}\right)\frac{\sigma}{9}R < R.$$

It follows that $x \in B(y, 2^{k-1}r)$ and, hence,

$$u(x) \geq \varepsilon^k a \exp\left(- \frac{C}{\omega_B(u \geq a)} \right).$$

Then (7.14) implies

$$2^{k+1} \leq \frac{R}{r},$$

whence

$$\varepsilon^k \geq \varepsilon^{k+1} \geq \left(\frac{r}{R} \right)^N$$

with $N = \ln(1/\varepsilon)/\ln 2$, which implies (7.11). □

LEMMA 7.6. *Let u be a bounded harmonic function in a ball $2B$ with center x. Then*

$$\sup_{2B} u \geq \left(1 + \exp\left(- \frac{C}{\omega_{\sigma B}(u \leq 0)} \right) \right) u(x). \tag{7.15}$$

PROOF. If $u(x) \leq 0$ then there is nothing to prove. So, let us assume $u(x) > 0$ and $\sup_{2B} u = 1$. Setting $v = 1 - u$ and noticing that $u \leq 0 \Leftrightarrow v \geq 1$, we obtain by (7.10)

$$v(x) \geq \exp\left(- \frac{C}{\omega_{\sigma B}(u \leq 0)} \right),$$

whence

$$u(x) \leq 1 - \exp\left(- \frac{C}{\omega_{\sigma B}(u \leq 0)} \right) \leq \frac{1}{1 + \exp(-C/\omega_{\sigma B}(u \leq 0))},$$

which is equivalent to (7.15). □

Applying (7.15) to function $u - a$ and replacing $2B$ by B, we obtain

$$\sup_{B} u \geq a + \left(1 + \exp\left(- \frac{C}{\omega_{(1/2)\sigma B}(u \leq a)} \right) \right) (u(x) - a). \tag{7.16}$$

LEMMA 7.7. *There is a constant $c > 0$ such that for any bounded harmonic function in a ball $B = B(x, r)$,*

$$\sup_{B} u \geq \exp\left(\frac{c}{\omega_B(u > 0)^{1/\alpha}} - 1 \right) u(x), \tag{7.17}$$

provided $u(x) > 0$, where α is the same as in (1.13).

Applying (7.17) to the function $u-a$ and assuming $u(x)>a$, we obtain

$$\sup_B u \geq a + \exp\left(\frac{c}{\omega_B(u>a)^{1/\alpha}} - 1\right)(u(x)-a). \tag{7.18}$$

In particular, assuming $u(x)>0$ and setting

$$a = \frac{1}{2}u(x) \quad \text{and} \quad b = \sup_B u,$$

we obtain from (7.18)

$$\omega_B(u>a) \geq \left(\frac{c}{1+\ln(b/a-1)}\right)^{\alpha}. \tag{7.19}$$

Proof. Let $\varepsilon < (1/4)\sigma$ be a positive constant to be chosen later on. We have for any $y \in B(x,(1/2)r)$

$$\frac{\mu(B(x,r))}{\mu(B(y,\varepsilon r))} \leq C\varepsilon^{-\alpha},$$

whence

$$\omega_{B(y,\varepsilon r)}(u>0) = \frac{\mu(\{u>0\}\cap B(y,\varepsilon r))}{\mu(B(y,\varepsilon r))} \leq C\varepsilon^{-\alpha}\frac{\mu(\{u>0\}\cap B)}{\mu(B(x,r))} = C\varepsilon^{-\alpha}\omega_B(u>0).$$

Now we would like to chose ε to satisfy the equality

$$C\varepsilon^{-\alpha}\omega_B(u>0) = \frac{1}{2},$$

that is, define ε by

$$\varepsilon = (2C\omega_B(u>0))^{1/\alpha}. \tag{7.20}$$

Since ε must be smaller than $(1/4)\sigma$, the choice (7.20) is possible provided

$$\omega_B(u>0) < \frac{1}{2C}\left(\frac{1}{4}\sigma\right)^{\alpha}. \tag{7.21}$$

If the opposite inequality holds, then (7.17) can be satisfied simply by choosing

$$c \leq \left(\frac{1}{2C}\right)^{1/\alpha}\frac{1}{4}\sigma.$$

Therefore, we can assume in the sequel that (7.21) is satisfied, and we choose ε from

(7.20). For this ε we have

$$\omega_{B(y,\varepsilon r)}(u > 0) \leq \frac{1}{2}$$

and, hence,

$$\omega_{B(y,\varepsilon r)}(u \leq 0) \geq \frac{1}{2}.$$

Set

$$\rho = \frac{2\varepsilon}{\sigma} r.$$

Since $\varepsilon < (1/4)\sigma$, we have $\rho < r/2$ and $B(y,\rho) \subset B(x,r)$. Applying (7.15), we obtain

$$\sup_{B(y,\rho)} u > \left(1 + \exp\left(-\frac{C}{\omega_{B(y,\varepsilon r)}(u \leq 0)}\right)\right) u(y) \geq (1+c)u(y),$$

where $c = \exp(-2C)$. It follows that there is a point y' such that

$$y' \in B(y,\rho) \quad \text{and} \quad u(y') > (1+c)u(y).$$

Applying this with $y = x$ we obtain that there is a point x_1 such that

$$x_1 \in B(x,\rho) \quad \text{and} \quad u(x_1) \geq (1+c)u(x).$$

Since $2\varepsilon/\sigma < 1/2$, the point x_1 lies in $B(x,(1/2)r)$. Applying the previous procedure with $y = x_1$, we obtain that there is a point x_2 such that

$$x_2 \in B(x_1,\rho) \quad \text{and} \quad u(x_2) \geq (1+c)u(x_1).$$

Continuing further this way, we construct a sequence $\{x_k\}_{k\geq 0}$ in B (see Figure 9) such that $x_0 = x$ and

$$x_k \in B(x_{k-1},\rho) \quad \text{and} \quad u(x_k) \geq (1+c)u(x_{k-1}).$$

It follows from the construction that

$$x_k \in B(x,k\rho).$$

As long as $k\rho < r/2$, that is, when

$$k\frac{2\varepsilon}{\sigma} < \frac{1}{2}, \tag{7.22}$$

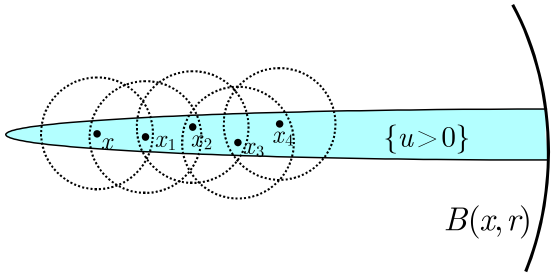

Figure 9. Sequence of balls $B(x_k, \rho)$.

we have $x_k \in B(x, (1/2)r)$, and the process of construction can be continued further to obtain x_{k+1}. Choose the maximal k with (7.22). For this k we have

$$k + 1 \geq \frac{\sigma}{4\varepsilon}$$

and

$$\begin{aligned} u(x_{k+1}) &\geq (1+c)^{k+1} u(x) \geq (1+c)^{\sigma/4\varepsilon} u(x) \\ &\geq \exp\left(\frac{c'}{\omega_B(u>0)^{1/\alpha}}\right) u(x), \end{aligned}$$

whence (7.17) follows. □

THEOREM 7.8. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that (VD) holds. Then following implication is true*:

$$(MV) + (PI)_\Psi + (\mathrm{cap}_\leq)_\Psi \Rightarrow (H).$$

PROOF. Let u be a bounded non-negative harmonic function in a ball $\eta^{-1}B$ where $B = B(x, R)$ and

$$\eta = \frac{1}{18}\sigma. \tag{7.23}$$

We will prove that if

$$\sup_B u = 2,$$

then

$$u(x) \geq c > 0, \tag{7.24}$$

which is equivalent to (H).

For that, we construct by induction a finite sequence $\{x_k\}_{k\geq 1}$ of points in $2B$ such that $u(x_k) = 2^k$. Since $\sup_B u = 2$, there exists a point $x_1 \in \overline{B}$ such that $u(x_1) = 2$ (The point $x_1 \in \partial B$ by using the maximum principle, see [**23**, Proposition 4.3]). If $x_k \in 2B$ with $u(x_k) = 2^k$ is already constructed then, for small enough $r > 0$, we have

$$\sup_{B(x_k,r)} u < 2^{k+1}.$$

Set

$$r_k = \sup\left\{r \in (0, R] : \sup_{B(x_k,r)} u \le 2^{k+1}\right\}.$$

If $r_k = R$ then the inductive process stops without constructing x_{k+1}. If $r_k < R$, then we have

$$\sup_{B(x_k,r_k)} u = 2^{k+1},$$

and we can find $x_{k+1} \in \overline{B}(x_k, r_k)$ such that $u(x_{k+1}) = 2^{k+1}$. If $x_{k+1} \in 2B$ then the inductive process goes further, while in the case $x_{k+1} \notin 2B$ the process stops.

As a result of this construction, we obtain a sequence of balls $\{B(x_k, r_k)\}_{k=1}^n$ (see Figure 10) where $x_k \in 2B$, $r_k \le R$, and

$$\sup_{B(x_k,r_k)} u \le 2u(x_k) = 2^{k+1}. \tag{7.25}$$

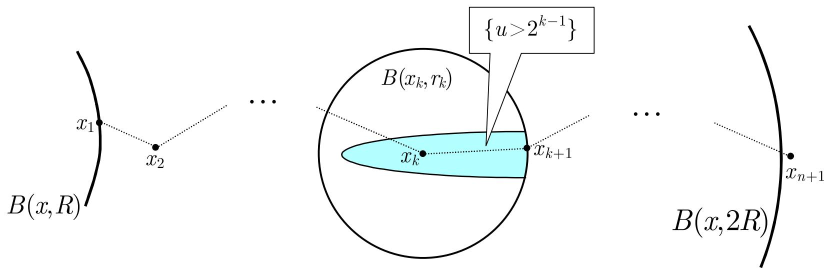

Figure 10. The sequence $\{x_k\}$.

For the largest index n in this sequence we have either $r_n = R$ or $x_{n+1} \notin 2B$. In the latter case, since $x_1 \in \overline{B}$, $d(x_k, x_{k+1}) \le r_k$ and $x_{n+1} \notin 2B$, we obtain by the triangle inequality

$$\begin{aligned} 2R \le d(x, x_{n+1}) &\le d(x, x_1) + d(x_1, x_2) + \cdots + d(x_n, x_{n+1}) \\ &\le R + r_1 + r_2 + \cdots + r_n, \end{aligned}$$

and hence,

$$r_1 + r_2 + \cdots + r_n \geq R. \tag{7.26}$$

In the former case $r_n = R$ this inequality is also trivially satisfied. It follows from (7.26) that there is $k \leq n$ such that

$$r_k \geq \frac{R}{k(k+1)}. \tag{7.27}$$

On the other hand, applying (7.19) in $B(x_k, r_k)$ with

$$a := \frac{1}{2}u(x_k) = 2^{k-1}$$

and using (7.25), that is,

$$b := \sup_{B(x_k, r_k)} u \leq 4a,$$

we obtain

$$\omega_{B(x_k,r_k)}(u > a) \geq \left(\frac{c}{1 + \ln(b/a - 1)}\right)^{\alpha} \geq \left(\frac{c}{1 + \ln 3}\right)^{\alpha} =: c'. \tag{7.28}$$

Next, we will apply Lemma 7.5 for the ball $B(x, R')$ with $R' = \eta^{-1}R$ and for $y = x_k$, $\rho = r_k$. Since $\eta = \sigma/18$, we have

$$x_k \in B(x, 2R) = B\left(x, \frac{\sigma}{9}R'\right)$$

and

$$r_k \leq R < \frac{\sigma}{4}R'.$$

Hence, the hypotheses of Lemma 7.5 are satisfied, and we obtain, using $a = 2^{k-1}$, (7.27), and (7.28),

$$\begin{aligned} u(x) &\geq a\left(\frac{r_k}{R'}\right)^N \exp\left(-\frac{C}{\omega_{B(x_k,r_k)}(\{u \geq a\})}\right) \\ &\geq \frac{2^{k-1}}{(k(k+1))^N}\eta^N \exp\left(-\frac{C}{c'}\right). \end{aligned}$$

Since

$$\inf_{k \geq 1} \frac{2^{k-1}}{(k(k+1))^N} > 0,$$

we obtain $u(x) \geq \text{const}$, which was to be proved.

It remains to consider a general (unbounded) harmonic function u. Let u be a non-negative, harmonic function in a ball $B \subset M$. Set $f_k = u \wedge k$ for any $k > 0$ and denote by u_k the solution of the Dirichlet problem

$$\begin{cases} u_k \text{ is harmonic in } B, \\ u_k = f_k \mod \mathcal{F}(B), \end{cases}$$

where $u = v \mod \mathcal{F}(B)$ means that there exists some $h \in \mathcal{F}(B)$ such that $u - v = h$ in M. Since $0 \leq f_k \leq k$ in B, we have also $0 \leq u_k \leq k$ in B (cf. [**30**, Lemma 7.2]). Since the sequence $\{f_k\}$ increases and $f_k \xrightarrow{\mathcal{F}} u$, it follows that $u_k \to u$ almost everywhere in B (cf. [**30**, Lemma 7.2]). Each function u_k is bounded and, hence, satisfies the Harnack inequality in B, that is,

$$\operatorname*{esup}_{\eta B} u_k \leq C \operatorname*{einf}_{\eta B} u_k.$$

Replacing u_k on the right-hand side by a larger function u and passing to the limit as $k \to \infty$, we obtain the same inequality for u. The proof is complete. $\square$

8. Proofs of Theorems 1.1 and 1.2.

In this section we complete the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We distinguish two steps.

Step 1: $(PI)_\Psi + (\mathrm{Gcap}_\leq)_\Psi \Rightarrow (\mathrm{cap})_\Psi$.
Indeed, we know already that

$$(\mathrm{Gcap}_\leq)_\Psi \Rightarrow (\mathrm{cap}_\leq)_\Psi.$$

To prove the lower bound

$$\mathrm{cap}(B, 2B) \geq c \frac{\mu(B)}{\Psi(r)}, \tag{8.1}$$

we use the following general fact (cf. [**23**, formula (8.19) in the proof of Lemma 8.3]):

$$\mathrm{cap}(B, 2B) \geq \mu(B) \lambda_{\min}(2B).$$

Substituting here the inequality

$$\lambda_{\min}(2B) \geq \frac{c}{\Psi(r)},$$

that is true by $(FK)_\Psi$, we obtain (8.1).

Step 2: $(PI)_\Psi + (\mathrm{Gcap}_{\leq})_\Psi \Rightarrow (H)$. By Theorem 5.1, condition $(FK)_\Psi$ holds, and then by Theorem 6.3 we obtain the mean value inequality (MV). Since $(\mathrm{cap}_{\leq})_\Psi$ holds by the previous step, we obtain the elliptic Harnack inequality (H) by Theorem 7.8.

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. The following equivalences were proved in [**23**, Theorem 3.14]:

$$\begin{aligned}(H) + (\mathrm{cap})_\Psi &\Leftrightarrow (H) + (E)_\Psi\\ &\Leftrightarrow (G)_\Psi\\ &\Leftrightarrow (UE)_\Psi + (NLE)_\Psi.\end{aligned}$$

Hence, it remains to prove that

$$\begin{aligned}(UE)_\Psi + (NLE)_\Psi &\Rightarrow (PI)_\Psi + (E)_\Psi &(8.2)\\ &\Rightarrow (PI)_\Psi + (CSA)_\Psi &(8.3)\\ &\Rightarrow (PI)_\Psi + (\mathrm{Gcap}_{\leq})_\Psi &(8.4)\\ &\Rightarrow (H) + (\mathrm{cap})_\Psi. &(8.5)\end{aligned}$$

By (3.17) we have (8.3), and (8.4) is trivial, and by Theorem 1.1 we have (8.5). The implication

$$(UE)_\Psi \Rightarrow (E)_\Psi$$

in (8.2) was proved in [**24**, Theorem 2.2]. Finally, let us prove that

$$(NLE)_\Psi \Rightarrow (PI)_\Psi,$$

which will finish the proof of Theorem 1.2. By Lemma 4.1, $(PI)_\Psi$ is equivalent to the following condition: for any ball $B = B(x_0, r)$ and for any $f \in \mathcal{F}$

$$\int_B d\Gamma\langle f\rangle \geq \frac{c}{\Psi(r)\mu(\sigma B)} \int_{\sigma B}\int_{\sigma B} (f(x) - f(y))^2 d\mu(y)d\mu(x), \tag{8.6}$$

so we will prove (8.6). Let us define a new quadratic form

$$\widetilde{\mathcal{E}}(f) = \int_B d\Gamma\langle f\rangle$$

for all $f \in \mathcal{F} \cap C_0(M)$, and define a new measure $\widetilde{\mu}$ to be the measure $\mathbf{1}_B\mu$ extended to ∂B by setting $\widetilde{\mu}(\partial B) = 0$. It can be shown that this quadratic form is closable in $L^2(\overline{B}, \widetilde{\mu})$ and its closure $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a regular Dirichlet form in $L^2(\overline{B}, \widetilde{\mu})$ (private communication of Zhen-Qing Chen, based on [**12**, Theorems 3.3.9, 6.2.13 and 6.2.14]). In fact, the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is related to the Neumann boundary value problem in B.

Let $\widetilde{P}_t$ be the heat semigroup associated with $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$. Denote by $(\cdot, \cdot)$ the scalar product in $L^2(\overline{B}, \mu)$ and by $\| \cdot \|$ its norm. It is a well known fact from the theory of Dirichlet forms, that for any $f \in \widetilde{\mathcal{F}}$, the function

$$t \to \left(\frac{f - \widetilde{P}_t f}{t}, f \right)$$

is monotone increasing as t decreases to 0, and converges to $\widetilde{\mathcal{E}}(f)$; in particular, we have

$$\widetilde{\mathcal{E}}(f) \geq \frac{1}{t} \big(f - \widetilde{P}_t f, f \big).$$

Observe that

$$\begin{aligned} 2\big(f - \widetilde{P}_t f, f\big) &= 2\|f\|^2 - 2\big(\widetilde{P}_t f, f\big) \\ &\geq 2\big(f^2, \widetilde{P}_t 1\big) - 2\big(\widetilde{P}_t f, f\big) \\ &= \big(f^2, \widetilde{P}_t 1\big) + \big(\widetilde{P}_t f^2, 1\big) - 2\big(\widetilde{P}_t f, f\big). \end{aligned}$$

Fix some point $x \in \overline{B}$ and set $a = f(x)$. Then we have

$$\widetilde{P}_t (a - f)^2 = a^2 \widetilde{P}_t 1 + \widetilde{P}_t f^2 - 2a \widetilde{P}_t f.$$

The value of this function at x is equal to

$$f(x)^2 \widetilde{P}_t 1(x) + \widetilde{P}_t f^2(x) - 2f(x) \widetilde{P}_t f(x),$$

and its inner product with 1 is equal to

$$\big(f^2, \widetilde{P}_t 1\big) + \big(\widetilde{P}_t f^2, 1\big) - 2\big(\widetilde{P}_t f, f\big).$$

Therefore,

$$2\big(f - \widetilde{P}_t f, f\big) = \big(\widetilde{P}_t (f(x)1 - f)^2(x), 1(x)\big),$$

whence we obtain

$$\widetilde{\mathcal{E}}(f) \geq \frac{1}{2t} \big(\widetilde{P}_t (f(x)1 - f)^2(x), 1(x)\big). \tag{8.7}$$

For any open set $\Omega \subset B$ consider the restriction $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}(\Omega))$ of the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$. We claim that

$$\big(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}(\Omega)\big) = (\mathcal{E}, \mathcal{F}(\Omega)),$$

which follows from the fact that $\mu = \widetilde{\mu}$ on Ω and any function $f \in \mathcal{F} \cap C_0(\Omega)$ belongs to the both spaces with

$$\widetilde{\mathcal{E}}\langle f \rangle = \int_{\overline{B}} d\Gamma\langle f \rangle = \int_{\Omega} d\Gamma\langle f \rangle = \mathcal{E}(f). \tag{8.8}$$

Consequently, the heat semigroups $\widetilde{P}_t^{\Omega}$ and P_t^{Ω} are also the same. Since for any non-negative function h

$$\widetilde{P}_t h \geq \widetilde{P}_t^{\Omega} h = P_t^{\Omega} h,$$

applying this with $h = (f(x)1 - f)^2$, we obtain that for any $f \in \mathcal{F}$

$$\big(\widetilde{P}_t(f(x)1 - f)^2(x), 1(x)\big) \geq \big(P_t^{\Omega}(f(x)1 - f)^2(x), 1(x)\big).$$

Combining with (8.7) and (8.8), we conclude that, for any $f \in \mathcal{F}$,

$$\widetilde{\mathcal{E}}(f) \geq \frac{1}{2t}\big(P_t^{\Omega}(f(x)1 - f)^2(x), 1(x)\big). \tag{8.9}$$

Now let $\Omega = B$. By a result of [**8**] (see also [**26**]) we have

$$(UE)_\Psi + (NLE)_\Psi \Rightarrow (LLE)_\Psi$$

where $(LLE)_\Psi$ is the following *local lower estimate* of the heat kernel: the heat semigroup P_t^B possesses the heat kernel $p_t^B(x, y)$ that satisfies

$$p_t^B(x, y) \geq \frac{c}{V(x_0, \Psi^{-1}(t))}, \tag{8.10}$$

for all $0 < t \leq \Psi(\varepsilon r)$ and μ-almost all $x, y \in B(x_0, \varepsilon\Psi^{-1}(t))$, where $\varepsilon \in (0, 1)$ and $c > 0$ are constant independent of B, x, y, t. Observe that the right hand side of (8.9) is equal to

$$\int_B \int_B p_t^B(x, y)(f(x) - f(y))^2 d\mu(y) d\mu(x).$$

Setting in (8.9) $t = \Psi(\varepsilon r)$ and restricting the integration to the ball $B(x_0, \varepsilon\Psi^{-1}(t)) = \varepsilon^2 B$ where (8.10) holds, we obtain

$$\begin{aligned}
\widetilde{\mathcal{E}}(f) &\geq \frac{1}{2t} \int_{\varepsilon^2 B} \int_{\varepsilon^2 B} p_t^B(x, y)(f(x) - f(y))^2 d\mu(y) d\mu(x) \\
&\geq \frac{1}{2\Psi(\varepsilon r)} \int_{\varepsilon^2 B} \int_{\varepsilon^2 B} \frac{c}{V(x_0, \varepsilon r)} (f(x) - f(y))^2 d\mu(y) d\mu(x)
\end{aligned}$$

$$\geq \frac{c_\varepsilon}{\Psi(r)\mu(\varepsilon^2 B)} \int_{\varepsilon^2 B}\int_{\varepsilon^2 B} (f(x)-f(y))^2 d\mu(y)d\mu(x),$$

which proves (8.6) with $\sigma = \varepsilon^2$. $\square$

9. Equivalent conditions for upper bound.

In this Section we prove Theorem 1.3, which will be preceded by auxiliary statements.

For any $p > 0$, we say that the L^p-mean value inequality $(MV)_p$ is satisfied if there a constant $C > 0$ such that, for any ball B and for any bounded non-negative subharmonic function u in B, the following inequality holds:

$$\operatorname*{esup}_{(1/2)B} u^p \leq \frac{C}{\mu(B)} \int_B u^p d\mu. \tag{9.1}$$

The previously used condition (MV) coincides with $(MV)_2$. In this proof we will also need $(MV)_1$.

In all the statements we assume that (VD) is satisfied.

Lemma 9.1. *Let $(MV)_2$ be satisfied. Let u be a non-negative bounded subharmonic function in an arbitrary ball $B = B(x_0, r)$. Then, for any $\delta \in (0,1)$,*

$$\operatorname*{esup}_{(1-\delta)B} u^2 \leq \frac{C\delta^{-\alpha}}{\mu(B)} \int_B u^2 d\mu, \tag{9.2}$$

where the positive constant C does not depend on B, u, δ, and where α comes from (1.13).

Proof. Indeed, for any $x \in (1-\delta)B$ let us apply $(MV)_2$ in the ball $B(x, \delta r) \subset B$ so that

$$\operatorname*{esup}_{B(x,(\delta/2)r)} u^2 \leq \frac{C}{\mu(B(x,\delta r))} \int_{B(x,\delta r)} u^2 d\mu.$$

By (1.13) we have

$$\frac{\mu(B(x_0,r))}{\mu(B(x,\delta r))} \leq C\delta^{-\alpha}$$

whence it follows that

$$\operatorname*{esup}_{B(x,(\delta/2)r)} u^2 \leq \frac{C\delta^{-\alpha}}{\mu(B)} \int_B u^2 d\mu.$$

Since $(1-\delta)B$ can be covered by a finite number of balls like $B(x, (\delta/2)r)$ with $x \in (1-\delta)B$, we obtain (9.2). $\square$

Lemma 9.2. $(MV)_2 \Rightarrow (MV)_1$.

PROOF. Choose some $0 < \tau < \tau' \le 1$. We have by (9.2) with $\delta = 1 - \tau/\tau'$, that

$$\operatorname*{esup}_{\tau B} u \le C\left(1 - \frac{\tau}{\tau'}\right)^{-\alpha/2} \mu(\tau' B)^{-1/2} \|u\|_{L^2(\tau' B)}.$$

Noting that

$$\left(\frac{\mu(B)}{\mu(\tau' B)}\right)^{1/2} \le C(\tau')^{-\alpha/2}$$

and

$$\|u\|_{L^2(\tau' B)} \le \|u\|_{L^1(\tau' B)}^{1/2} \|u\|_{L^\infty(\tau' B)}^{1/2} \le \|u\|_{L^1(B)}^{1/2} \|u\|_{L^\infty(\tau' B)}^{1/2},$$

we obtain

$$\operatorname*{esup}_{\tau B} u \le C(\tau' - \tau)^{-\alpha/2} \mu(B)^{-1/2} \|u\|_{L^1(B)}^{1/2} \|u\|_{L^\infty(\tau' B)}^{1/2}. \tag{9.3}$$

We will use this inequality to do iterations as follows.

Set

$$\tau_k = 1 - \frac{1}{2}\left(\frac{3}{4}\right)^k \quad \text{for } k = 0, 1, 2, \dots. \tag{9.4}$$

Clearly, the sequence $\{\tau_k\}_{k=0}^{\infty}$ is non-decreasing, $\tau_0 = 1/2$, $\tau_k \to 1$ as $k \to \infty$ and

$$\tau_{k+1} - \tau_k = \frac{1}{8}\left(\frac{3}{4}\right)^k.$$

Applying (9.3) with $\tau = \tau_k$ and $\tau' = \tau_{k+1}$, we obtain

$$\begin{aligned} \|u\|_{L^\infty(\tau_k B)} &\le C\left(\frac{1}{8}\left(\frac{3}{4}\right)^k\right)^{-\alpha/2} \mu(B)^{-1/2} \|u\|_{L^1(B)}^{1/2} \|u\|_{L^\infty(\tau_{k+1} B)}^{1/2} \\ &= A\left(\frac{3}{4}\right)^{-k\alpha/2} \|u\|_{L^\infty(\tau_{k+1} B)}^{1/2}, \end{aligned}$$

where

$$A := C' \mu(B)^{-1/2} \|u\|_{L^1(B)}^{1/2}.$$

Iterating this inequality, we obtain

$$
\begin{aligned}
\|u\|_{L^\infty(\tau_0 B)} &\le A\|u\|^{1/2}_{L^\infty(\tau_1 B)} \le A\left\{A\left(\frac{3}{4}\right)^{-\alpha/2}\|u\|^{1/2}_{L^\infty(\tau_2 B)}\right\}^{1/2} \le \cdots \\
&\le A^{1+(1/2)+(1/2^2)+\cdots}\left(\frac{3}{4}\right)^{-(\alpha/2)\sum_{k=0}^{\infty}(k/2^k)}\|u\|^{2^{-k}}_{L^\infty(\tau_k B)} \\
&= C''A^2\|u\|^{2^{-k}}_{L^\infty(\tau_k B)}.
\end{aligned}
$$

Setting $k \to \infty$ and noticing that $\|u\|^{2^{-k}}_{L^\infty(\tau_k B)} \to 1$, we obtain

$$
\|u\|_{L^\infty((1/2)B)} \le C''A^2 = C\mu(B)^{-1}\|u\|_{L^1(B)},
$$

which is equivalent to $(MV)_1$. $\square$

For any measurable f, we say that $u \in \mathcal{F}(\Omega)$ satisfies

$$
-\Delta u \le f \quad \text{weakly in } \Omega \tag{9.5}
$$

if, for any non-negative function $\varphi \in \mathcal{F}(\Omega)$,

$$
\mathcal{E}(u,\varphi) \le \int_\Omega f\varphi d\mu. \tag{9.6}
$$

Proposition 9.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 and let $\Omega \subset M$ be an open set with $\mu(\Omega) < \infty$. If $u \in \mathcal{F}(\Omega)$ is a non-negative solution to* (9.5) *for some $f \in L^p(\Omega)$ with $p \ge 2$, then, for any $s \ge 0$,*

$$
\|(u-s)_+\|_1 \le \frac{\mu(E_s)^{1-1/p}}{\lambda_{\min}(E'_s)}\|f\|_p, \tag{9.7}
$$

where $E_s = \{x \in \Omega : u \ge s\}$ and E'_s is any open neighborhood of E_s.

Proof. Without loss of generality, we can assume that u is quasi-continuous. Using the properties of Dirichlet forms, (9.6) with $\varphi = (u-s)_+ \in \mathcal{F}(\Omega)$, and the Hölder inequality, we obtain

$$
\begin{aligned}
\mathcal{E}((u-s)_+) &\le \mathcal{E}(u,(u-s)_+) \\
&\le \int_\Omega (u-s)_+ f d\mu \\
&\le \|(u-s)_+\|_{p'}\|f\|_p \\
&\le \mu(E_s)^{1/p'-1/2}\|(u-s)_+\|_2\|f\|_p,
\end{aligned}
$$

where $p' = p/(p-1)$ is the Hölder conjugate of p. Since $(u-s)_+ = 0$ q.e. outside E_s and, hence, outside E'_s, we obtain by definition of $\lambda_{\min}(E'_s)$ that

$$\|(u-s)_+\|_2^2 \le \frac{\mathcal{E}((u-s)_+)}{\lambda_{\min}(E_s')}.$$

By the Cauchy–Schwarz inequality we have

$$\|(u-s)_+\|_1 \le \mu(E_s)^{1/2}\|(u-s)_+\|_2.$$

Combining all the above inequalities, we obtain (9.7). □

We now derive an L^∞-estimate for any non-negative function u satisfying (9.5).

Theorem 9.4. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. For any ball B of radius $r>0$ and for any non-empty open set $\Omega \subset B$, let $u \in \mathcal{F}(\Omega)$ be non-negative and satisfy* (9.5) *for some $f \in L^p(B)$. If condition $(FK)_\Psi$ holds, then for any $p > \max\{2, \nu^{-1}\}$,*

$$\|u\|_\infty \le \left(\frac{\nu+1-1/p}{\nu-1/p}\right)^{\nu+1-1/p}\frac{\Psi(r)}{C_F}\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\nu}\left\{⨍_\Omega |f|^p d\mu\right\}^{1/p}, \tag{9.8}$$

where constants ν, C_F and function Ψ come from condition $(FK)_\Psi$. In particular, we have

$$(FK)_\Psi \Rightarrow (E_\le)_\Psi, \tag{9.9}$$

where $(E_\le)_\Psi$ refers to the first condition in (1.28).

Proof. The proof is motivated by the argument in [**40**, Lemmas 5.1 and 5.2, p. 71] and [**45**, Lemma 4].

If $\|u\|_\infty = 0$ then (9.8) is trivially satisfied. If $\|f\|_p = 0$ then it follows from (9.6) with $\phi = u$ that $u = 0$ and again (9.8) is satisfied. Hence, in the sequel we assume that $\|u\|_\infty > 0$ and $\|f\|_p > 0$. Let $E_s = \{x \in \Omega : u \ge s\}$ as before. Note that

$$\phi(s) := \|(u-s)_+\|_1 = \int_s^\infty \mu(E_t)dt$$

(see for example [**46**, p. 36]), and that

$$\phi'(s) = -\mu(E_s), \quad \phi(0) = \|u\|_1,$$

and $\phi(s) = 0$ for any $s > \|u\|_\infty$ while $\phi(s) > 0$ for $s < \|u\|_\infty$. Let E_s' be an open neighborhood of E_s. It follows from (9.7) and $(FK)_\Psi$ that

$$\phi(s) = \|(u-s)_+\|_1 \le \frac{\mu(E_s)^{1-1/p}}{\lambda_{\min}(E_s')}\|f\|_p$$

$$\le \frac{\Psi(r)}{C_F}\left(\frac{\mu(E_s')}{\mu(B)}\right)^{\nu}\mu(E_s)^{1-1/p}\|f\|_p.$$

Since $\mu(E_s')$ can be taken arbitrarily close to $\mu(E_s)$, we obtain that

$$\phi(s) \le \frac{\Psi(r)}{C_F\mu(B)^{\nu}}\mu(E_s)^{1+\nu-1/p}\|f\|_p = A\{-\phi'(s)\}^{q+1},$$

where

$$A = \frac{\Psi(r)}{C_F\mu(B)^{\nu}}\|f\|_p \quad \text{and} \quad q = \nu - 1/p > 0.$$

Assuming that $s < \|u\|_\infty$ and dividing by $\phi(s)$, we obtain

$$A^{-1/(q+1)} \le -\phi'(s)\phi(s)^{-1/(q+1)} = -\frac{q+1}{q}\frac{d}{ds}\left\{\phi(s)^{q/(q+1)}\right\}.$$

Integrating this inequality over $[0,s]$, we obtain

$$\begin{aligned} A^{-1/(q+1)}s &\le \frac{q+1}{q}\left\{\phi(0)^{q/(q+1)} - \phi(s)^{q/(q+1)}\right\} \\ &\le \frac{q+1}{q}\phi(0)^{q/(q+1)} = \frac{q+1}{q}\|u\|_1^{q/(q+1)}, \end{aligned}$$

which yields

$$s \le \frac{q+1}{q}A^{1/(q+1)}\|u\|_1^{q/(q+1)}.$$

Letting $s \uparrow \|u\|_\infty$ and using $\|u\|_1 \le \|u\|_\infty\mu(\Omega)$, we obtain

$$\|u\|_\infty \le \frac{q+1}{q}A^{1/(q+1)}\|u\|_\infty^{q/(q+1)}\mu(\Omega)^{q/(q+1)},$$

which implies

$$\begin{aligned} \|u\|_\infty &\le \left(\frac{q+1}{q}\right)^{q+1}A\mu(\Omega)^q \\ &= \left(\frac{\nu+1-1/p}{\nu-1/p}\right)^{\nu+1-1/p}\frac{\Psi(r)}{C_F}\left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\nu}\mu(\Omega)^{-1/p}\|f\|_p, \end{aligned}$$

thus proving (9.8).

Finally, as the function E^B satisfies (9.5) with $f = 1$ and $\Omega = B$, we obtain by letting $p \to \infty$ in (9.8) that

$$\|E^B\|_\infty \le \left(1+\frac{1}{\nu}\right)^{\nu+1}\frac{\Psi(r)}{C_F}, \tag{9.10}$$

thus proving condition $(E_\le)_\Psi$. □

Remark. The implication (9.9) can also be proved by combining two arguments in [**24**] as follows. By [**24**, Lemma 5.5], $(FK)_\Psi$ implies the following estimate of the heat kernel $p_t^B(x,y)$ in any ball B of radius r:

$$\operatorname*{esup}_{x,y\in B} p_t^B(x,y) \le \frac{C}{\mu(B)}\left(\frac{\Psi(r)}{t}\right)^{1/\nu}.$$

Then we use the argument from [**24**, p. 557] in the following simplified form. Integrating this inequality in y over B and then in t from 0 to ∞, we obtain, for any $T\in(0,\infty)$

$$\begin{aligned} E^B &= \int_0^\infty P_t^B 1_B dt = \int_0^T P_t^B 1_B dt + \int_T^\infty P_t^B 1_B dt \\ &\le T + C\int_T^\infty \left(\frac{\Psi(r)}{t}\right)^{1/\nu} dt \\ &= T + C'\Psi(r)^{1/\nu} T^{1-1/\nu}, \end{aligned}$$

where we have used that $\nu<1$ (note that, without loss of generality, ν can be assumed arbitrarily small). Setting $T=\Psi(r)$ we obtain

$$E^B \le C\Psi(r),$$

which finishes the proof.

Proof of Theorem 1.3. The statement will follow from the following sequence of implications:

$$\begin{aligned} (FK)_\Psi + (\mathrm{Gcap}_\le)_\Psi &\Rightarrow (FK)_\Psi + (E_\ge)_\Psi && (9.11)\\ &\Rightarrow (FK)_\Psi + (E)_\Psi && (9.12)\\ &\Rightarrow (UE)_\Psi + (C) && (9.13)\\ &\Rightarrow (FK)_\Psi + (S)_\Psi && (9.14)\\ &\Rightarrow (FK)_\Psi + (\mathrm{Gcap}_\le)_\Psi. && (9.15) \end{aligned}$$

As follows from Theorem 6.3 and Lemma 9.2, we have

$$(FK) + (\mathrm{Gcap}_\le)_\Psi \Rightarrow (MV)_1.$$

By Theorem 9.4 we have

$$(FK) \Rightarrow (E_{\leq})_{\Psi}. \tag{9.16}$$

Let us show that

$$(MV)_1 + (\mathrm{cap}_{\leq})_{\Psi} \Rightarrow (E_{\geq})_{\Psi},$$

which will give us the implication (9.11) and, by combining with (9.16), also (9.12).

Recall that $(E_{\geq})_{\Psi}$ refers to the second condition in (1.28), which we state as follows:

$$\operatorname*{einf}_{(1/8)B} E^B \geq c\Psi(r)$$

where B is any ball, $E^B := G^B 1$ and c is a positive constant. Note that the function $u = E^B = G^B 1$ is superharmonic in B. This function is also bounded by Proposition 9.4. Set $u_\varepsilon = u + \varepsilon$ for any $\varepsilon > 0$ and note that, by the strong locality, u_ε is also superharmonic in B. By (2.10) we obtain that, for any non-negative function $\varphi \in \mathrm{cutoff}((1/4)B, (1/2)B)$

$$\begin{aligned} \mathcal{E}(u_\varepsilon, u_\varepsilon^{-1}\varphi) &= \int_B d\Gamma\langle \log u_\varepsilon, \varphi\rangle - \int_B \varphi u_\varepsilon^{-2} d\Gamma\langle u_\varepsilon\rangle \\ &\leq \left(\int_{(1/2)B} d\Gamma\langle \log u_\varepsilon\rangle\right)^{1/2} \left(\int_{(1/2)B} d\Gamma\langle \varphi\rangle\right)^{1/2}. \end{aligned} \tag{9.17}$$

By Lemma 7.1 that uses only $(\mathrm{cap}_{\leq})_{\Psi}$, we have

$$\int_{(1/2)B} d\Gamma\langle \log u_\varepsilon\rangle \leq \frac{C}{\Psi(r)}\mu(B)$$

where r is the radius of B. By $(\mathrm{cap}_{\leq})_{\Psi}$, the function φ can be chosen so that

$$\mathcal{E}(\varphi) \leq \frac{C}{\Psi(r)}\mu(B).$$

Hence, we obtain from (9.17) that

$$\mathcal{E}\big(u_\varepsilon, u_\varepsilon^{-1}\varphi\big) \leq \frac{C}{\Psi(r)}\mu(B).$$

Since by the strong locality $\mathcal{E}(u+\varepsilon, u_\varepsilon^{-1}\varphi) = \mathcal{E}(u, u_\varepsilon^{-1}\varphi)$, we obtain

$$\mathcal{E}(u, u_\varepsilon^{-1}\varphi) \leq \frac{C}{\Psi(r)}\mu(B).$$

On the other hand, by $u = G^B 1$, we have

$$\mathcal{E}\big(u, u_\varepsilon^{-1}\varphi\big) = \big(1, u_\varepsilon^{-1}\varphi\big) = \int_B u_\varepsilon^{-1}\varphi d\mu \geq \int_{(1/4)B} u_\varepsilon^{-1} d\mu,$$

so that

$$\int_{(1/4)B} u_\varepsilon^{-1} d\mu \leq \frac{C}{\Psi(r)} \mu(B).$$

By Lemma 2.5, the function u_ε^{-1} is subharmonic. Applying $(MV)_1$ to this function in $(1/4)B$, we obtain

$$\operatorname*{esup}_{(1/8)B} u_\varepsilon^{-1} \leq C\mu(B)^{-1} \int_{(1/4)B} u_\varepsilon^{-1} d\mu \leq \frac{C'}{\Psi(r)},$$

whence

$$\operatorname*{einf}_{(1/8)B} u_\varepsilon \geq c\Psi(r).$$

Letting $\varepsilon \to 0$, we obtain $(E_\geq)_\Psi$, and, hence, finish the proof of (9.11) and (9.12).

By [**30**, Lemma 7.3] we have

$$(E)_\Psi \Rightarrow (C).$$

Under the standing assumption (C), the following equivalences were proved in [**24**, Theorems 2.1, 2.2][3]:

$$(FK)_\Psi + (E)_\Psi \Leftrightarrow (UE)_\Psi \Leftrightarrow (FK)_\Psi + (S)_\Psi.$$

Hence, the implications (9.13) and (9.14) follow. Finally (9.15) holds by Theorem 3.2. □

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[3]Note that the result in [**24**] was proved for $\Psi(r) = r^\beta$ but the same method goes through verbatim for any $\Psi(r)$ satisfying (1.15). Note also that the proof of $(UE)_\Psi \Rightarrow (S)_\Psi$ uses the conservativeness of $(\mathcal{E}, \mathcal{F})$.

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