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Holomorphic functions and the Itô chaos

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Dedicated to the memory of Professor Kiyosi Itô

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Abstract. This paper is concerned with the characterization of spaces of square integrable holomorphic functions on a complex manifold, G, in terms of the derivatives of the function at a fixed point $o \in G$. The reproducing kernel properties of square integrable holomorphic functions are reviewed and a number of examples are given. These examples include square integrable holomorphic functions relative to Gaussian measures on complex Euclidean spaces along with their generalizations to heat kernel measures on complex Lie groups. These results are intimately related to the Itô's chaos expansion in stochastic analysis and to the Fock space description of free quantum fields in physics.

1. Dedication.

I first met Professor Itô while still a graduate student during the Fall of 1985 while attending the annual thematic program, "Stochastic Differential Equations and their Applications," held at the Institute for Mathematics and its applications at the University of Minnesota. It was somewhat intimidating being a graduate student among the numerous well established and famous mathematicians at the program. However, I still have very fond memories of meeting Professor Itô at this venue. Not only was professor Itô an excellent expositor, he was very kind and generous to everyone including us graduate students. I feel extremely privileged to have met Professor Itô on this and other occasions throughout my professional career. More importantly, I have been deeply influenced by Professor Itô's mathematical insights and fundamental contributions to stochastic analysis and analysis in general. I am delighted to dedicate this paper to his memory.

2. Introduction.

Notation 2.1. Let (G, o) be a pointed connected complex manifold. For any open subset, $V \subset G$, let $\mathcal{H}(V)$ denote the space of holomorphic functions on V.

Definition 2.2. A measure, λ, on G (equipped with the Borel $\sigma-$ algebra) is

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smooth if for each chart z on G with domain $\mathcal{D}(z)$, there exists a smooth function $\rho_z : \mathcal{D}(z) \to [0, \infty)$ such that $d\lambda = \rho_z dm_z$ on $\mathcal{D}(z)$ where m_z is the push-forward of Lebesgue measure on $\mathcal{R}(z) := z(\mathcal{D}(z))$ by z^{-1}. We further say that λ is strictly positive if $\rho_z > 0$ on $\mathcal{D}(z)$ for all charts z. [It suffices to check this condition on any collection of charts covering G.]

As usual let, $L^2(\lambda)$ denote space of complex valued (equivalence classes) of square integrable functions on G with inner product defined by

$$(f, g)_{L^2(\lambda)} := \int_G f \cdot \bar{g}\, d\lambda \;\forall\; f, g \in \mathcal{H}(G)$$

and associated Hilbert norm defined by $\|f\|_{L^2(\lambda)} := (f, f)^{1/2}_{L^2(\lambda)}$.

In this paper we are interested in the general question of characterizing the space,

$$\mathcal{H}L^2(G, \lambda) := \left\{ f \in \mathcal{H}(G) : \|f\|_{L^2(\lambda)} < \infty \right\}, \tag{2.1}$$

of holomorphic square integrable functions on G in terms of their derivatives at some fixed base point $o \in G$. To make this problem more tractable, for most of the paper we will restrict our attention to the case where G is a complex Lie group and $o = e$ is the identity element of G.

Perhaps the most well known example of this situation is when G is a complex vector space and λ is a Gaussian measure on G. The holomorphic structure arises from a unitary isomorphism from L^2 – functions of a Gaussian measure on a real vector space to $\mathcal{H}L^2(G, \lambda)$ known as the Segal-Bargmann transform, see Segal [**36**], [**37**] and Bargmann [**1**]. The Segal-Bargmann transform was later extended to the Lie group setting Brian Hall [**23**], see Gross and Malliavin [**22**] for or more detailed background surrounding these results. The Gaussian context arises rather naturally when studying the quantum-mechanical Harmonic oscillator in both finite and infinite dimensional settings. The infinite dimensional setting relates to "free" quantum-field theories where the derivative space can be taken to be a certain completion of the symmetric tensor algebra over G referred to as Fock space, see V. A. Fock [**15**] and Notation 5.4 below.

The structure of the paper is as follows;

1. Section 3 reviews some basic properties of $\mathcal{H}L^2(G, \lambda)$ in Equation (2.1). In particular we will show $\mathcal{H}L^2(G, \lambda)$ is a reproducing kernel Hilbert space. The key observations here is the Cauchy integral formula allows one to control the derivatives of a function at a point in terms of its L^2 norm.
2. Section 4 is then mostly devoted to studying $\mathcal{H}L^2(G, \lambda)$ when $G = \mathbb{C}$ and $d\lambda(z) = \rho(|z|)dm(z)$ where ρ is a positive radial function on $\mathbb{C}$ and m is two dimensional Lebesgue measure on $\mathbb{C}$. In these examples, the space of allowed derivatives at $0 \in \mathbb{C}$ is a Hilbert subspace of sequences with a weighted ℓ^2 – norm with weights determined by the function ρ.
3. Section 5 then reviews the Gaussian setting in finite dimensions which leads to the Fock space description of the possible derivatives of a square integrable holomorphic function. The proofs and statements of the results in this section are formulated to

allow for an easier transition to the results in Section 6.

4. In Section 6, we take G to be a complex Lie group, $o = e \in G$, and λ to be a " heat kernel measure" associated to a left-invariant second order (sub)elliptic operator on $C^\infty(G)$. The results in this section have appeared in [**6**], [**10**], [**11**], [**12**]. We will however give new proofs of some these results which are more stochastic in nature than the original analytic proofs. These proofs make rather direct contact with Itô's work, especially see [**29**], [**31**].

3. Hilbert space properties of $\mathcal{H}L^2(G,\lambda)$.

In this section we will recall some of the basic properties of the square integrable holomorphic functions as introduced in Definition 2.2. After this section we will restrict our attention to the case that G is a Lie group. The next lemma is a slight extension of [**10**, Lemma 3.4].

LEMMA 3.1. *To each $o \in G$ and any open neighborhood V of o there exists a chart z on M such that $o \in \mathcal{D}(z) \subset V$ and a family of smooth probability measures, $\{\delta_g\}_{g\in\mathcal{D}(z)}$, such that*;

1. *$d\delta_g = \delta_g^z dm_z$ with δ_g^z being smooth and compactly supported in $\mathcal{D}(z)$ and the map $\mathcal{D}(z) \times G \ni (g,x) \to \delta_g^z(x) \in [0,\infty)$ being smooth.*
2. *For all $f \in \mathcal{H}(V)$ and $g \in \mathcal{D}(z)$,*

$$f(g) = \int_G f(x)\delta_g(dx) \ \forall\ f \in \mathcal{H}(G).$$

PROOF. The statement of the lemma is local, so without loss of generality we may assume that $M \doteq D_1$ where for any $R > 0$,

$$D_R \doteq \{z \in \mathbb{C}^d : |z_i| < R \ \forall\ i = 1,2,\ldots,d\}. \tag{3.1}$$

Let f be a holomorphic function on D_1 and $\varepsilon \in (0,1/2)$ be given. By the mean value theorem for holomorphic functions;

$$f(g) = (2\pi)^{-d}\int_{[0,2\pi]^d} f\big(\{g_j + r_j e^{\sqrt{-1}\theta_j}\}_{j=1}^d\big)\prod_{j=1}^d d\theta_j, \tag{3.2}$$

for any choices of $\{r_i\}_{i=1}^d \subset [0,1/2)$. Choose a smooth function $h : \mathbb{R} \to [0,\infty)$ such that h has support in $(-\varepsilon,\varepsilon)$, h is constant near 0, and

$$\int_0^1 h(r)rdr = 1.$$

Multiply (3.2) by $r_1 \cdots r_d h(r_1)\cdots h(r_d)$ and integrate each r_i over $[0,\varepsilon)$ to find:

$$f(g) = \int_{D_\varepsilon} f(g+x)\rho(x)m(dx) = \int_{D_1} f(x)\delta_g(x)m(dx),$$

where $\rho(x) = (2\pi)^{-d}h(|x_1|)\cdots h(|x_d|)$, $\delta_g(x) = \rho(x-g)$, and m is Lebesgue measure on $\mathbb{C}^d$. $\square$

THEOREM 3.2 (Pointwise Bounds). *If λ is a smooth strictly positive measure on G, then the function $C : G \to [0,\infty]$ defined by*

$$C(g) := \sup_{f\in\mathcal{H}(G)\setminus\{0\}} \frac{|f(g)|}{\|f\|_{L^2(G)}}, \tag{3.3}$$

is locally uniformly bounded on G. To be more precise for all compact sets $K \subset G$, $C(K) := \sup_{g\in K} C(g) < \infty$. More generally if $\mathcal{V} := \{V_1,\ldots,V_n\}$ is any finite collection of smooth vector fields on G, then

$$C_{\mathcal{V}}(g) := \sup_{f\in\mathcal{H}(G)\setminus\{0\}} \frac{|(V_1\ldots V_n f)(g)|}{\|f\|_{L^2(G)}} \tag{3.4}$$

is locally uniformly bounded on G.

PROOF. Let $o \in G$, z be a chart on G, and δ_g be the measures as described in Lemma 3.1. Then for $g \in \mathcal{D}(z)$,

$$f(g) = \int_{\mathcal{D}(z)} f(x)\delta_g(x)dm_z(x) = \int_{\mathcal{D}(z)} f(x)\frac{\delta_g(x)}{\lambda_z(x)}d\lambda(x). \tag{3.5}$$

From this identity and the Cauchy-Schwarz inequality,

$$|f(g)| \le \|f\|_{L^2(\mathcal{D}(z),\lambda)}\|\delta_g/\lambda_z\|_{L^2(\mathcal{D}(z);\lambda)}$$

from which it follows that $C(g) \le \|\delta_g/\lambda_z\|_{L^2(\mathcal{D}(z);\lambda)}$. The latter expression is bounded for g is a neighborhood of o and since $o \in G$ was arbitrary the proof is completed by a simple covering argument.

Differentiating Equation (3.5) implies

$$(V_1\ldots V_n f)(g) = \int_{\mathcal{D}(z)} f(x)\frac{\delta_g^{\mathcal{V}}(x)}{\lambda_z(x)}d\lambda(x)$$

where $\delta_g^{\mathcal{V}}(x) := V_1(g)V_2\ldots V_n[y \to \delta_y(x)]$. The assertion about $C^{\mathcal{V}}$ now follows by the same argument as above with δ_g replaced by $\delta_g^{\mathcal{V}}$. $\square$

The reader may also wish to consult [**5**] where very precise bounds similar to Theorem 3.2 may be found in the $d = 1$ case when $M = \mathbb{C}$, and $d\lambda(z) = e^{-\varphi(z)}dm(z)$ with $0 \le \Delta\varphi \le C < \infty$.

Corollary 3.3. *The subspace, $\mathcal{H}L^2(G,\lambda)$, is a closed subspace of $L^2(\lambda)$ and in particular $\mathcal{H}L^2(G,\lambda)$ is a Hilbert space.*

Proof. Suppose that $f \in L^2(\lambda)$ and $f_n \in \mathcal{H}L^2(G,\lambda)$ is a sequence converging in $L^2(\lambda)$ to f. Given a compact subset, K, of G we conclude from Theorem 3.2 that there exists $C(K) < \infty$ such that

$$\sup_{g\in K} |f_n(g) - f_m(g)| \le C(K)\|f_n - f_m\|_{L^2(\lambda)} \to 0 \text{ as } m, n \to \infty.$$

Thus $\{f_n\}_{n=1}^{\infty}$ is locally uniformly convergent and hence the pointwise limiting function, F, is still holomorphic. By passing to a subsequence if necessary, we may conclude that $f = \lim_{n\to\infty} f_n = F(\lambda - \text{a.e.})$ and hence the $L^2(\lambda)$ - equivalence class is represented by the unique holomorphic representative, F. □

Theorem 3.4 (Reproducing Kernel). *To each positive smooth measure, λ, on G, there exists a unique function (called the* reproducing kernel*),*

$$k = k_\lambda : G \times G \to \mathbb{C}$$

such that for all $w \in G$, $k(\cdot, w)$ is the unique element in $\mathcal{H}L^2(\lambda)$ such that

$$f(w) = (f, k(\cdot, w))_{L^2(\lambda)} \ \forall \ f \in \mathcal{H}L^2(\lambda). \tag{3.6}$$

1. *$k(w,z) = (k(\cdot,z), k(\cdot,w))$ and hence $\overline{k(w,z)} = k(z,w)$.*
2. *$k(z,w)$ is jointly C^∞ with $w \to \overline{k(z,w)}$ being holomorphic for each $z \in G$.*
3. *If $\{\varphi_n\}_{n=0}^{\infty} \subset \mathcal{H}L^2(\lambda)$ is any orthonormal basis, then*

$$k(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}, \tag{3.7}$$

   where the sum is absolutely convergent.
4. *The function, $C : G \to [0,\infty)$ in Theorem 3.2 is given by*

$$C(g) = \|k(\cdot, g)\|_{L^2(\lambda)} = \sqrt{k(g,g)}, \tag{3.8}$$

   i.e. for all $f \in \mathcal{H}L^2(G)$ the following optimal pointwise bounds *hold,*

$$|f(g)|^2 \le \|f\|^2_{L^2(\lambda)} k(g,g) \text{ for all } g \in G. \tag{3.9}$$

5. *We also have,*

$$|k(z,w)| \le \sqrt{k(z,z)\cdot k(w,w)} = \sqrt{C(z)C(w)} \ \forall \ z, w \in G. \tag{3.10}$$

Proof. By Theorem 3.2, the pointwise evaluations maps, $ev_g(f) := f(g)$, defines a continuous linear functional on the Hilbert space $\mathcal{H}L^2(G,\lambda)$. Therefore the existence and uniqueness of k satisfying Equation (3.6) is a consequence of the Riesz theorem for continuous linear functionals on a Hilbert space. We now prove the remaining items of the theorem in turn.

1. Apply Equation (3.6) with $f(\cdot) = k(\cdot, z)$.
2. Let w be a point in G and choose a complex chart, $\psi : U \to \psi(U) \subset_o \mathbb{C}^d$ such that $w \in U$ and $\psi(U)$ is closed under complex conjugation. Then for $w \in U$ let $\bar{w} := \psi^{-1}(\overline{\psi(w)})$. [Note this notion of conjugation on U is chart dependent and is not well defined in general.] As $\overline{k(z,w)} = k(w,z)$ is holomorphic for $w \in U$ we may conclude that $k(z,\bar{w})$ is also holomorphic for $w \in U$. An application of (an easy version of) Hartog's Theorem, see [**40**] now shows that $G \times U \ni (z,w) \to k(z,\bar{w})$ is jointly holomorphic and in particular smooth. (We will give a another proof of this statement as after the proof of item 5. which avoids the need for using Hartog's theorem.)
3. Equation (3.7) follows by the reproducing property of k, Item 1. and Parseval's identity;
$$\begin{aligned}\sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)} &= \sum_{n=0}^{\infty}(\varphi_n, k(\cdot,z))\overline{(\varphi_n, k(\cdot,w))}\\ &= \sum_{n=0}^{\infty}(k(\cdot,w),\varphi_n)(\varphi_n,k(\cdot,z))\\ &= (k(\cdot,w),k(\cdot,z)) = k(z,w).\end{aligned}$$
4. From Equation (3.6) and the Cauchy-Schwarz inequality,
$$|f(w)| = \left|(f,k(\cdot,w))_{L^2(\lambda)}\right| \le \|f\|_{L^2(\lambda)} \cdot \|k(\cdot,w)\|_{L^2(\lambda)} \tag{3.11}$$
where
$$\|k(\cdot,w)\|^2_{L^2(\lambda)} = \langle k(\cdot,w), k(\cdot,w)\rangle = k(w,w).$$
From this equation it follows that equality holds in Equation (3.11) when $f = k(\cdot,w)$ and therefore the optimal constant $C(z)$ as defined in Equation (3.3) is given by Equation (3.8).
5. Equation (3.10) follows by taking $f = k(\cdot,z)$ in Equation (3.11) and then using Equation (3.8).

Second proof of item 2. If P_N is orthogonal projection onto $\mathrm{span}\{\varphi_n : 0 \le n \le N\}$, then
$$k_N(z,w) := (k(\cdot,w), P_N k(\cdot,z)) = \sum_{n=0}^{N}\varphi_n(z)\overline{\varphi_n(w)}$$
satisfies; 1) $k_N(z,w) \to k(z,w)$ pointwise, 2)

$$|k_N(z,w)| \leq \|k(\cdot,w)\|\|k(\cdot,z)\| \leq C(w)C(z)$$

and so $\{k_N(z,w)\}_{N=1}^{\infty}$ is locally uniformly bounded, 3) and hence $k_N \to k$ in the L^2 -norm over compact sets. Give $w \in G$ let $U \subset G$ be an open neighborhood of w and $\bar{w}$ be defined as in the proof of item 2 above. With this notation we see that

$$k_N(z,\bar{w}) = \sum_{n=0}^{N} \varphi_n(z)\overline{\varphi_n(\bar{w})} \text{ for } (z,w) \in G \times U$$

is jointly holomorphic on $G \times U$. From Corollary 3.3 applied to any precompact neighborhood $\tilde{G} \subset G \times U$, we may now conclude that $k_N \to k$ locally uniformly and hence $G \times U \ni (z,w) \to k(z,\bar{w}) = \lim_{N\to\infty} k_N(z,\bar{w})$ is holomorphic on $(z,w) \in G \times U$. In particular, $k(z,w)$ is jointly smooth. □

3.1. Crude pointwise bounds for Lie groups.

For the remainder of this paper, we are now going to further assume that G is a complex Lie group and our base point, o, is the identity element $e \in G$. Let us further suppose that $d\lambda(g) = \rho(g)dg$ where ρ is a positive continuous function on G and dg is a fixed *right invariant Haar measure* on G. For $A \in \mathfrak{g} = \mathrm{Lie}(G) := T_eG$ let $\tilde{A}$ denote the left invariant vector field on G agreeing with A at $e \in G$.

Notation 3.5. If V is a precompact open neighborhood of $e \in G$ and $\rho : G \to (0,\infty)$ is a given function, let

$$\rho_V(g) := \inf_{x\in Vg} \rho(x) = \inf_{y\in V} \rho(yg).$$

Theorem 3.6. *If V is a precompact open neighborhood of $e \in G$ and $\rho \in C(G,(0,\infty))$, then there exists $C(V) < \infty$ such that for all $f \in \mathcal{H}(G)$,*

$$|f(g)| \leq C(V)\|f\|_{L^2(V,\lambda)} \frac{1}{\sqrt{\rho_V(g)}} \ \forall\ g \in G.$$

Proof. Applying Lemma 3.1 with $o = e \in G$, there exists a smooth probability measure of the form $\delta_e(x)dx$ with $\delta_e \in C_c^\infty(V,[0,\infty))$ such that

$$f(e) = \int_V f(x)\delta_e(x)dx \ \forall\ f \in \mathcal{H}(G).$$

Then for any g we may use this equation with f replace by $f \circ R_g$ to find;

$$\begin{aligned} f(g) = f \circ R_g(e) &= \int_V f(xg)\delta_e(x)dx \\ &= \int_{Vg} f(x)\delta_e(xg^{-1})dx = \int_{Vg} f(x)\frac{\delta_e(xg^{-1})}{\rho(x)}d\lambda(x). \end{aligned}$$

Therefore by an application of the Cauchy–Schwarz inequality,

$$|f(g)| \leq \|f\|_{L^2(Vg,\lambda)} \cdot \left\| \frac{\delta_e((\cdot)g^{-1})}{\rho(\cdot)} \right\|_{L^2(Vg,\lambda)}$$

where

$$\left\| \frac{\delta_e((\cdot)g^{-1})}{\rho(\cdot)} \right\|^2_{L^2(Vg,\lambda)} = \int_{Vg} \frac{\delta_e^2(xg^{-1})}{\rho^2(x)} \rho(x)dx \leq \frac{1}{\rho_V(g)} \int_{Vg} \delta_e^2(xg^{-1})dx$$

from which the theorem holds with

$$C(V) := \sqrt{\int_V \delta_e^2(x)dx} = \|\delta_e\|_{L^2(m)}. \qquad \square$$

4. Examples with $G = \mathbb{C}$.

Before continuing with the general theory in the complex Lie group case, let us pause to give some examples in the special case where $G = \mathbb{C}$ and $o = 0$, In this case Haar measure is Lebesgue measure, $dm(z) = dxdy$, where as usual $z = x + iy$. As above let $d\lambda(z) = \rho(z)dm(z)$ where $\rho \in C(\mathbb{C}, (0,\infty))$ is a positive continuous function on $\mathbb{C}$. We will further use the standard first order differential operators on $\mathbb{C}$,

$$\partial := \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \text{and} \quad \bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

where $\partial_x := \partial/\partial x = \partial + \bar{\partial}$, $\partial_y := \partial/\partial y = i(\partial - \bar{\partial})$. With this notation the Laplacian, Δ, on $\mathbb{C} \cong \mathbb{R}^2$ may be expressed as

$$\Delta := \partial_x^2 + \partial_y^2 = 4\partial\bar{\partial}.$$

NOTATION 4.1 (Taylor map). Given a function f on $\mathbb{C}$ which is holomorphic near 0, let

$$Tf = \{f^{(n)}(0)\}_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}_0}.$$

NOTATION 4.2 (Derivative Space). Let

$$J^0 := \left\{ \alpha := \{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C} : \limsup_{n\to\infty} \left| \frac{\alpha_n}{n!} \right|^{1/n} = 0 \right\}.$$

By the basic properties of holomorphic functions we have that $T : \mathcal{H}(\mathbb{C}) \to J^0$ is a linear isomorphism with inverse,

$$T^{-1}(\alpha)(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n \quad \text{for all} \quad z \in \mathbb{C}.$$

Our primary aims are to; 1) develop some basic properties of $\mathcal{H}L^2(\lambda)$, 2) identify the norm on J^0 which makes

$$T|_{\mathcal{H}L^2(\lambda)} : \mathcal{H}L^2(\lambda) \to J^0 \quad \textit{isometric,}$$

and 3) characterize the image, $T(\mathcal{H}L^2(\lambda)) \subset J^0$, of the Taylor map.

DEFINITION 4.3. Let $\mathcal{HP}(\mathbb{C})$ denote the space of holomorphic polynomials. Further let,

$$\mathcal{HP}_k = \{p \in \mathcal{HP}(\mathbb{C}) : \deg(p) \leq k\} = \left\{ p(z) = \sum_{n=0}^{k} a_n z^n : a_n \in \mathbb{C} \right\}.$$

NOTATION 4.4. For every $\varepsilon > 0$, let

$$\rho_\varepsilon(z) := \min\{\rho(w) : w \in D(z, \varepsilon)\}.$$

COROLLARY 4.5 (Louiville's Theorem). *Suppose there exists $c < \infty$ and $n \in \mathbb{N}_0$ such that*

$$\rho(z) \geq \frac{c}{|z|^{2n} + 1} \textit{ for all } z \in \mathbb{C}. \tag{4.1}$$

Then $\mathcal{H}L^2(\rho) = \mathcal{HP}_k$ for some $k < n$ where $\mathcal{HP}_k := \{0\}$ if $k \leq 0$.

PROOF. First off from Equation (4.1) we conclude that

$$\sup_{|z|=r} \frac{1}{\rho_\varepsilon(z)} \leq \sup_{|z| \leq r+\varepsilon} \frac{|z|^{2n} + 1}{c} = \frac{(r+\varepsilon)^{2n} + 1}{c}. \tag{4.2}$$

Hence if $m > n$ and $f \in \mathcal{H}L^2(\lambda)$, by the Cauchy estimates and the pointwise bounds in Theorem 3.6 along with Equation (4.2), we find,

$$\begin{aligned}
|f^{(m)}(0)| &\leq \frac{m!}{r^m} \sup_{|z|=r} |f(z)| \\
&\leq \frac{m!}{r^m} \frac{1}{\sqrt{\pi}\varepsilon} \|f\|_{L^2(\lambda)} \sqrt{\sup_{|z|=r} \frac{1}{\rho_\varepsilon(z)}} \\
&\leq \frac{m!}{r^m} \frac{1}{\sqrt{\pi}\varepsilon} \|f\|_{L^2(\lambda)} \sqrt{\frac{(r+\varepsilon)^{2n} + 1}{c}} \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Hence it follows by Taylor's theorem that $f \in \mathcal{HP}_n$ by Taylor's theorem. Since

$$\int_{\mathbb{C}} |z^n|^2 d\lambda(z) \geq c \int_{\mathbb{C}} \frac{|z|^{2n}}{1+|z|^{2n}} dm(z) = \infty,$$

it follows that in fact, $\mathcal{H}L^2(\lambda) \subset \mathcal{HP}_{n-1}$. Lastly if $f(z) = \sum_{j=0}^{k} a_j z^j \in \mathcal{H}L^2(\lambda)$ with $a_k \neq 0$, then there exists $M < \infty$ such that $|f(z)| \geq (1/2)|a_k||z|^k$ for $|z| \geq M$. Therefore we conclude that

$$\int_{|z|\geq M} |z|^{2k} \rho(z) dm(z) \leq \frac{|a_k|^2}{4} \int |f(z)|^2 \rho(z) dm(z) < \infty$$

from which is easily follows that $\mathcal{HP}_k \subset \mathcal{H}L^2(\lambda)$. □

EXAMPLE 4.6 (A Non-Uniform Decay Example). If $\rho(z) := (1/\pi)\exp(-|z|^2)$, then $\mathcal{HP} \subset \mathcal{H}L^2(\rho dm)$ and in particular $\dim \mathcal{H}L^2(\rho dm) = \infty$. On the other hand if φ is any nowhere vanishing holomorphic function (i.e. $\varphi(z) = e^{g(z)}$ for some $g \in \mathcal{H}$) then the map,

$$\mathcal{H}L^2(\rho dm) \ni f \to \varphi f \in \mathcal{H}L^2\left(\frac{\rho}{|\varphi|^2} dm\right)$$

is unitary and so again $\dim \mathcal{H}L^2((\rho/|\varphi|^2)dm) = \infty$. However, even though $\mathcal{H}L^2((\rho/|\varphi|^2)dm)$ is infinite dimensional, it is possible that $\mathcal{H}L^2((\rho/|\varphi|^2)dm)$ does not contain any non-zero polynomials. For example, let $\varphi(z) := \exp(-(c/2)z^2)$. Then $\mathcal{H}L^2((\rho/|\varphi|^2)dm) \cap \mathcal{HP} = \{0\}$ when $c \geq 1$ as is easy to verify since

$$\frac{\rho(z)}{|\varphi(z)|^2} = \frac{1}{\pi} \exp(-((1-c)x^2 + (1+c)y^2))$$

is now growing in the $y-$direction for fixed x.

4.1. Radially symmetric densities.

For the rest of this section let us now enforce the standing assumption that $\rho(z) = \rho(|z|)$ is a continuous positive radial function on $\mathbb{C}$ such that $\mathcal{HP} \subset \mathcal{H}L^2(\lambda)$, i.e.

$$\int_{\mathbb{C}} |z|^k \rho(z) dm(z) < \infty \text{ for all } k \in \mathbb{N}_0.$$

In this case we will use the following notation.

NOTATION 4.7. For $\alpha \in \mathbb{C}^{\mathbb{N}_0}$, let

$$\|\alpha\|_\rho^2 := \sum_{n=0}^{\infty} |\alpha_n|^2 \left(\frac{a_n}{n!}\right)^2 \text{ where } a_n^2 := \int_{\mathbb{C}} |z|^{2n} d\lambda(z).$$

We further let $J(\rho)$ be the Hilbert subspace of $\mathbb{C}^{\mathbb{N}_0}$ defined by

$$J(\rho) := \left\{\alpha = (\alpha_0, \alpha_1, \dots) \in \mathbb{C}^{\mathbb{N}_0} : \|\alpha\|_\rho^2 < \infty\right\}$$

equipped with the inner product,

$$\langle \alpha, \beta\rangle_\rho := \sum_{n=0}^{\infty} \alpha_n \bar{\beta}_n \left(\frac{a_n}{n!}\right)^2 \text{ for all } \alpha, \beta \in J(\rho).$$

PROPOSITION 4.8. *If $\rho(z) = \rho(|z|)$ and $\mathcal{HP} \subset \mathcal{H}L^2(\lambda)$, then for all $n \in \mathbb{N}_0$,*

$$(f, z^n)_{L^2(\rho)} := \int_{\mathbb{C}} f(z)\bar{z}^n \rho(|z|) dm(z) = a_n^2 \frac{f^{(n)}(0)}{n!}. \tag{4.3}$$

PROOF. Using polar coordinates we find,

$$(f, z^n) = \int_0^\infty \left(\int_{-\pi}^{\pi} f(re^{i\theta}) r^n e^{-in\theta} d\theta\right) \rho(r) r dr, \tag{4.4}$$

where, using Taylor's theorem, we have

$$\int_{-\pi}^{\pi} f(re^{i\theta}) r^n e^{-in\theta} d\theta = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \int_{-\pi}^{\pi} r^k e^{ik\theta} r^n e^{-in\theta} d\theta = 2\pi r^{2n} \frac{f^{(n)}(0)}{n!}.$$

Thus it follows that

$$(f, z^n) = \frac{f^{(n)}(0)}{n!} 2\pi \int_0^\infty r^{2n} \rho(r) r dr = \frac{f^{(n)}(0)}{n!} \int_{\mathbb{C}} |z|^{2n} d\lambda(z)$$

which proves Equation (4.3). □

THEOREM 4.9. *If $\rho(z) = \rho(|z|)$ and $\mathcal{HP} \subset \mathcal{H}L^2(\lambda)$, then;*

1. *$\{z^n/a_n\}_{n=0}^\infty$ forms an orthonormal basis for $\mathcal{H}L^2(\lambda)$.*
2. *For any $f \in \mathcal{H}L^2(\lambda)$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{4.5}$$

   converges pointwise and $L^2(\lambda)$.
3. *The Taylor map in T in Notation 4.1 is a unitary map from $\mathcal{H}L^2(\lambda) \to J(\lambda)$.*
4. *The reproducing kernel (see Theorem 3.4) for λ is given by*

$$k(z, w) = k_\lambda(z, w) = \sum_{n=0}^{\infty} \frac{1}{a_n^2} (z\bar{w})^n. \tag{4.6}$$

5. *Every $f \in \mathcal{H}(\mathbb{C})$ satisfies the (optimal) pointwise bounds,*

$$|f(z)|^2 \le \|f\|^2_{L^2(\lambda)} \left(\sum_{n=0}^{\infty} \frac{1}{a_n^2} |z|^{2n} \right). \tag{4.7}$$

6. *If $f \in \mathcal{H}(\mathbb{C})$, then*

$$\int_{\mathbb{C}} |f(z)|^2 \rho(z) dm(z) = \sum_{n=0}^{\infty} |f^{(n)}(0)|^2 \left(\frac{a_n}{n!} \right)^2.$$

PROOF. We prove each item in turn.

1. Taking $f(z) = z^m$ in Equation (4.3) shows $(z^m, z^n) = a_n^2 \delta_{m,n}$ and therefore, $\{z^n/a_n\}_{n=0}^{\infty}$ is orthonormal subset of $\mathcal{H}L^2(\lambda)$. If $f \in \mathcal{H}L^2(\lambda)$ is orthogonal to $\{z^n/a_n\}_{n=0}^{\infty}$, then according to Equation (4.3),
$$0 = \left(f, \frac{z^n}{a_n} \right) = a_n \frac{f^{(n)}(0)}{n!} \text{ for all } n \in \mathbb{N}_0.$$
As f is entire, we may conclude that f is identically zero and hence $\{z^n/a_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}L^2(\lambda)$.
2. The pointwise convergence of the sum in Equation (4.5) to $f(z)$ is a consequence of Taylor's theorem for holomorphic functions and the $L^2(\lambda)$ – convergence follows from item 1. and the observation that
$$\frac{f^{(n)}(0)}{n!} z^n = \left(f, \frac{z^n}{a_n} \right)_{L^2(\rho)} \frac{z^n}{a_n}.$$
3. The fact that $T : \mathcal{H}L^2(\lambda) \to J(\lambda)$ is unitary follows directly from item 1.
4. Item 4. is a direct consequence of item 3. of Theorem 3.4 and item 1. of this theorem.
5. The bounds in Equation (4.7) follows from item 4. of Theorem 3.4 and Equation (4.6).
6. To see the isometry property is valid for all $f \in \mathcal{H}(\mathbb{C})$, use $T : \mathcal{H}L^2(\lambda) \to J(\lambda)$ is unitary, Taylor's theorem, and Fatou's lemma, to show;

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^2 \rho(z) dm(z) &= \int_{\mathbb{C}} \liminf_{N\to\infty} \left| \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n \right|^2 \rho(z) dm(z) \\
&\le \liminf_{N\to\infty} \int_{\mathbb{C}} \left| \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n \right|^2 \rho(z) dm(z) \\
&= \liminf_{N\to\infty} \sum_{n=0}^{N} a_n^2 \left| \frac{f^{(n)}(0)}{n!} \right|^2 = \sum_{n=0}^{\infty} a_n^2 \left| \frac{f^{(n)}(0)}{n!} \right|^2.
\end{aligned}$$ □

COROLLARY 4.10 (Density of Polynomials). *If $\rho(z) = \rho(|z|) > 0$ is such that $\mathcal{HP} \subset \mathcal{H}L^2(\lambda)$, then $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$.*

PROOF. This is an immediate consequence of item 1. of Theorem 4.9. □

QUESTION. under what conditions on ρ is $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$? It is certainly not necessary for ρ to be radially symmetric. For example, by a change of variables arguments one easily shows that $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$ if $\rho(z) = \tilde{\rho}(|az+b|)$ for some $a \neq 0$ and $\tilde{\rho}$ decays sufficiently fast at infinity. Moreover, it is shown in [**13**, Theorem 3.6] (or see Theorem 5.1 below) that $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$ whenever

$$\rho(z) = C\exp(-(ax^2 + 2bxy + cy^2))$$

for some $a, c > 0$ and $b \in \mathbb{R}$ such that $b^2 - ac < 0$.

4.2. Exponential examples.

THEOREM 4.11. *If $\kappa > 0$, $\Gamma(z) := \int_0^\infty t^z e^{-t}(dt/t)$ is the gamma function, and*

$$\rho(z) = \rho_\kappa(z) := \frac{\kappa}{2\pi}\exp(-|z|^\kappa),$$

then

$$\|\alpha\|_\rho^2 = \sum_{n=0}^\infty |\alpha_n|^2 \frac{\Gamma((2n+2)/\kappa)}{(n!)^2},$$

$$k(z,w) = \sum_{n=0}^\infty \frac{1}{\Gamma((2n+2)/\kappa)}(z\bar{w})^n,$$

and for all $f \in \mathcal{H}(\mathbb{C})$,

$$|f(z)|^2 \leq \|f\|^2_{L^2(\rho_\kappa dm)}\left(\sum_{n=0}^\infty \frac{|z|^{2n}}{\Gamma((2n+2)/\kappa)}\right) \quad \textit{and}$$

$$\int_{\mathbb{C}} |f(z)|^2 \frac{\kappa}{2\pi}\exp(-|z|^\kappa)dm(z) = \sum_{n=0}^\infty |f^{(n)}(0)|^2 \frac{\Gamma((2n+2)/\kappa)}{(n!)^2}.$$

PROOF. These results all follow directly from Theorem 4.9 upon noting (after making a change of variables) that

$$a_n^2 = \kappa\int_0^\infty r^{2n+1}e^{-r^\kappa}dr = \Gamma\left(\frac{2n+2}{\kappa}\right). \qquad \square$$

EXAMPLE 4.12 ($\kappa = 1$). If $\kappa = 1$, then

$$k(z,w) = \sum_{n=0}^\infty \frac{1}{(2n+1)!}(z\bar{w})^n = \frac{\sinh(\sqrt{z\bar{w}})}{\sqrt{z\bar{w}}}$$

and for all $f \in \mathcal{H}(\mathbb{C})$,

$$\frac{1}{2\pi}\int_{\mathbb{C}}|f(z)|^2\exp(-|z|)dm(z)=\sum_{n=0}^{\infty}|f^{(n)}(0)|^2\frac{(2n+1)!}{(n!)^2},$$

and

$$|f(z)|^2\le\|f\|^2_{L^2(\lambda)}\frac{\sinh(|z|)}{|z|}\le\|f\|^2_{L^2(\lambda)}\frac{1}{2|z|}e^{|z|}.$$

EXAMPLE 4.13 ($\kappa=2$). If $\kappa=2$, then

$$k(z,w)=\sum_{n=0}^{\infty}\frac{1}{n!}(z\bar{w})^n=e^{z\bar{w}}$$

and for all $f\in\mathcal{H}(\mathbb{C})$,

$$\frac{1}{\pi}\int_{\mathbb{C}}|f(z)|^2\exp(-|z|^2)dm(z)=\sum_{n=0}^{\infty}\frac{1}{n!}|f^{(n)}(0)|^2,\ \text{and}$$

$$|f(z)|^2\le\|f\|^2_{L^2(\lambda)}e^{|z|^2}.\quad(\textit{Bargmann's}\text{ Pointwise Bounds})$$

We will revisit this last example in the next section.

5. The "classical" Gaussian example.

In this section we will take $G=\mathbb{C}^d$ and $o=0\in\mathbb{C}^d$. If $u\in\mathbb{C}^d$ we let $\partial_u f(z):=d/dt|_0 f(z+tu)$ so that

$$\frac{\partial}{\partial x_l}=\partial_{e_l}\text{ and }\frac{\partial}{\partial y_l}=\partial_{ie_l}\text{ for }1\le l\le d,$$

where $\{e_l\}_{l=1}^d$ is the standard basis for $\mathbb{C}^d$. As in the $d=1$, we will also use the standard complex differential operators,

$$\partial_l=\frac{\partial}{\partial z_l}=\frac{1}{2}\left(\frac{\partial}{\partial x_l}-i\frac{\partial}{\partial y_l}\right),\quad\bar{\partial}_l=\frac{\partial}{\partial\bar{z}_l}=\frac{1}{2}\left(\frac{\partial}{\partial x_l}+i\frac{\partial}{\partial y_l}\right).$$

THEOREM 5.1 (Density). *Let $q:\mathbb{C}^d\to\mathbb{R}$ be a positive quadratic form on $\mathbb{C}^d$ when considering $\mathbb{C}^d$ as a real vector space. Define the Gaussian measure on $\mathbb{C}^d$ by $d\lambda(z)=\rho(z)dm(z)$, where m is Lebesgue measure on $\mathbb{C}^d$, $\rho(z)=(1/Z)e^{-q(z)}$, and Z is the normalization constant which makes λ a probability measure. Then the holomorphic polynomials are dense in $\mathcal{H}L^2(\mathbb{C}^d,\lambda)$.*

This theorem was proved in [**13**, Theorem 3.6] where the authors made use of the finite dimensional version of the Itô-chaos expansion in order to reduce the problem to the density of all polynomials in the L^2 -space of a Gaussian measure. We will give an alternate proof technique here which may prove to be useful for other measures on $\mathbb{C}^d$. Let us first recall the standard integration by parts lemma.

LEMMA 5.2 (Integration by Parts). *Suppose that $f, g \in C^1(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$ such that $|fg| + |f\partial_v g| + |g\partial_v f|$ is integrable relative to Lebesgue measure m on $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} f\partial_v g\, dm = -\int_{\mathbb{R}^d} g\partial_v f\, dm. \tag{5.1}$$

PROOF. The proof is standard and is based on introducing a cutoff function, $\Psi_M(x) := \Psi(x/M)$, where $\Psi \in C_c^\infty(\mathbb{R}^d)$ such that $\Psi \geq 0$ and $\Psi(x) = 1$ if $|x| \leq 2$. One then uses the dominated convergence theorem to justify passing to the limit as $M \to \infty$ in the following identity;

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \partial_v(\Psi_M fg)\, dm \\
&= \int_{\mathbb{R}^d} \frac{1}{M}(\partial_v \Psi)\left(\frac{x}{M}\right) f(x)g(x)\, dm(x) + \int_{\mathbb{R}^d} \Psi_M \partial_v f \cdot g\, dm + \int_{\mathbb{R}^d} \Psi_M f \partial_v g\, dm. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 5.1. In order to demonstrate the method of proof of Theorem 5.1 let us first consider the special case where $d = 1$ and $q(z) = |z|^2 = z\bar{z}$ so that $\rho(z) = (1/\pi)e^{-z\bar{z}}$. We wish to show: if $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$ is orthogonal to all the holomorphic polynomials then $f \equiv 0$. Let $p(z) = \sum_{k=0}^N a_k z^k$ be a holomorphic polynomial and $p(\bar{\partial}) = \sum_{k=0}^N a_k \bar{\partial}^k$. Notice that $p(\bar{\partial})\rho(z) = p(-z)\rho(z)$. By repeated applications of the integration by parts Lemma 5.2 we find

$$\begin{aligned}
\int_{\mathbb{C}} \bar{z}^m f(z) p(-z)\rho(z) dm(z) &= \int_{\mathbb{C}} \bar{z}^m f(z) p(\bar{\partial})\rho(z) dm(z) \\
&= \int_{\mathbb{C}} p(-\bar{\partial})(\bar{z}^m f(z))\rho(z) dm(z) \\
&= \int_{\mathbb{C}} (p(-\bar{\partial})\bar{z}^m) f(z)\rho(z) dm(z) = 0
\end{aligned}$$

wherein the last equality we have used that fact that $\bar{\partial} f = 0$ (f is holomorphic) and the assumption that f is orthogonal to the holomorphic polynomials. Therefore we have shown $\int_{\mathbb{C}} f(z)\bar{z}^k z^m \rho(z) dm(z) = 0$ for all integers k and m. In particular, this shows that f is orthogonal to all polynomials and hence is zero.

The general proof will follow this same strategy with some minor complications which arise when $Q \neq I$. Let $q : \mathbb{C}^d \to \mathbb{R}$ be a positive quadratic form where $\mathbb{C}^d$ is viewed as the real vector space $\mathbb{R}^{2d}$. We may write $q(z) = z \cdot Az + 2Bz \cdot \bar{z} + \bar{z} \cdot C\bar{z}$ where A, B, and C are $d \times d$ complex matrices and $v \cdot w = \sum_{i=1}^d v_i w_i$ is the complex bilinear dot product on $\mathbb{C}^d$. Notice that $q(iz) + q(z) = 4Bz \cdot \bar{z}$ for all $z \in \mathbb{C}^d$. Therefore, $B = B^*$ and B is positive definite. In particular B is invertible. It can also be shown that $A = \bar{C}$, but this will not be needed in the proof of Theorem 5.1.

Let us now proceed to the proof of the density Theorem 5.1 in the general case. Let $p(z,\bar{z})$ be a polynomial in z and $\bar{z}$. We will call the highest power of z's appearing in $p(z,\bar{z})$ the z–degree of p. By assumption, we are given that $f:\mathbb{C}^d\to\mathbb{C}$ is holomorphic and

$$\int_{\mathbb{C}^d} f(z)p(z,\bar{z})d\lambda(z)=0 \tag{5.2}$$

for all polynomials p which have *zero* z–degree. We wish to show that Equation (5.2) holds for all polynomials $p(z,\bar{z})$. This will be proved by induction on the the z–degree of p.

So suppose $k\geq 0$ and that Equation (5.2) holds for all polynomials of z–degree less than or equal to k. Let $p(z,\bar{z})$ be any polynomial of z–degree k. Integration by parts along with $\bar{\partial}_i f=0$ and the fact that $\bar{\partial}_i p(z,\bar{z})$ has z–degree less than or equal to k leads to

$$\int_{\mathbb{C}^d} f(z)p(z,\bar{z})\bar{\partial}_i\rho(z)dm(z)=-\int_{\mathbb{C}^d} f(z)(\bar{\partial}_i p(z,\bar{z}))\rho(z)dm(z)=0. \tag{5.3}$$

Since

$$\bar{\partial}_i\rho(z)=-2(B^t z+C\bar{z})_i\rho(z)$$

and $p(z,\bar{z})(C\bar{z})_i$ is a polynomial of z–degree k, it follows from Equation (5.3) and the induction hypothesis that

$$0=\int_{\mathbb{C}^d} f(z)p(z,\bar{z})(B^t z+C\bar{z})_i\rho(z)dm(z)=\int_{\mathbb{C}^d} f(z)p(z,\bar{z})(B^t z)_i\rho(z)dm(z).$$

Since B is invertible and i is arbitrary, we may take linear combinations of the above identity in order to conclude

$$0=\int_{\mathbb{C}^d} f(z)p(z,\bar{z})z_l\rho(z)dm(z)$$

for all l and polynomials p of z–degree equal to k. Hence Equation (5.2) holds for all polynomials p of z–degree less than or equal to $k+1$. This completes the induction argument and establishes Equation (5.2) for all polynomials. This then proves the theorem since space of all polynomials is well known to be dense in $L^2(\mathbb{C}^d,\lambda)$. $\square$

For the rest of this section we will now restrict our attention to the case where λ_t (for $t>0$) is the Gaussian measure defined by

$$d\lambda_t(z):=\left(\frac{1}{\pi t}\right)^d e^{-(1/t)q(z)}dm(z) \tag{5.4}$$

with

$$q(z) = z \cdot \bar{z} = \sum_{i=1}^{d} |z_i|^2. \tag{5.5}$$

NOTATION 5.3. Let J^0 denote the elements $\alpha = (\alpha_n)_{n=0}^{\infty}$ where for each n, $\alpha_n : (\mathbb{C}^d)^n \to \mathbb{C}$ are complex multi-linear symmetric forms on $\mathbb{C}^d$. Given a holomorphic function, f, on $\mathbb{C}^d$ (or on a open neighborhood of $0 \in \mathbb{C}^d$), let $\hat{f}(0) = \alpha \in J^0$ be defined by $\alpha_0 = f(0)$ and for $n \geq 1$,

$$\alpha_n(u_1, \ldots, u_n) := (\partial_{u_1} \ldots \partial_{u_n} f)(0) \text{ for all } u_i \in \mathbb{C}^d.$$

It should be observe that $\hat{f}(0)$ is indeed in J^0. This is because; 1) f being holomorphic guarantees that α_n is complex multi-linear (an not just real multi-linear) and 2) α_n is symmetric in its arguments since mixed partial directional derivatives commute.

NOTATION 5.4 (Symmetric Fock Space). For $t > 0$ and $\alpha \in J^0$, let

$$\|\alpha\|_t^2 := \sum_{n=0}^{\infty} \frac{t^n}{n!} \|\alpha_n\|^2 \text{ where}$$
$$\|\alpha_n\|^2 := \sum_{u_1,\ldots,u_n \in S} |\alpha_n(u_1, \ldots, u_n)|^2$$

and S is a complex orthonormal basis for $\mathbb{C}^d$. We then let $J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t < \infty\}$ which is a Hilbert space when equipped with the inner product associated to the Hilbertian norm, $\|\cdot\|_t$. [The Hilbert space, J_t^0, is an example of a symmetric (or Bosonic) Fock space.]

With this notation we have the following multi-dimensional version of Example 4.13.

THEOREM 5.5. *Let $t > 0$ be fixed and $T : \mathcal{H}(\mathbb{C}^d) \to J^0$ be the Taylor map, $f \to Tf = \hat{f}(0)$. If $f \in \mathcal{H}(\mathbb{C}^d)$, then*

$$\|f\|_{L^2(\lambda_t)} = \left\|\hat{f}(0)\right\|_t = \|Tf\|_t \tag{5.6}$$

and $T : \mathcal{H}L^2(\lambda_t) \to J_t^0$ is unitary. Moreover we also have the following results.

1. *If $f \in \mathcal{H}L^2(\lambda_t)$ and $\alpha = Tf = \hat{f}(0)$, then*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n \Big(\overbrace{z, z, \ldots, z}^{n \text{ times}} \Big) \tag{5.7}$$

converges pointwise and $L^2(\lambda_t)$.

2. *$\{p_\beta(z) := z^\beta / \sqrt{\beta!} t^{|\beta|/2} : \beta \in \mathbb{N}_0^d\}$ forms an orthonormal basis for $\mathcal{H}L^2(\lambda_t)$ where $\beta! = \prod_{l=1}^d \beta_l!$ and $|\beta| = \beta_1 + \cdots + \beta_d$.*

3. *The reproducing kernel* (*see Theorem* 3.4) *for* λ_t *is given by*

$$k(z,w) = k_{\lambda_t}(z,w) = e^{(1/t)z\cdot\bar{w}}. \tag{5.8}$$

4. *Every* $f \in \mathcal{H}(\mathbb{C})$ *satisfies the* (*optimal*) *pointwise bounds,*

$$|f(z)|^2 \le \|f\|^2_{L^2(\lambda_t)} e^{1/t|z|^2} \text{ for all } z \in \mathbb{C}^d. \tag{5.9}$$

PROOF. Although this theorem could be proved by similar methods used to prove Theorem 4.9, it will be more instructive to give a different proof here which is more easily generalizes to the setting of complex Lie groups in Section 6 below. The key observation is to make use of the fact that λ_t is the convolution kernel associated to the heat flow, $e^{t\Delta/4}$ where

$$\Delta = \sum_{l=1}^{d}\left[\left(\frac{\partial}{\partial x_l}\right)^2 + \left(\frac{\partial}{\partial y_l}\right)^2\right] = 4\partial\cdot\bar{\partial} := 4\sum_{l=1}^{d}\partial_l\bar{\partial}_l.$$

In particular, for a large class of functions we have

$$\left(e^{t\Delta/4}f\right)(0) = \int_{\mathbb{C}^d} f(z)d\lambda_t(z).$$

Assuming that $f \in \mathcal{H}L^2(\lambda)$ and working formally for the moment we have,

$$\int_{\mathbb{C}^d} |f(z)|^2 d\lambda_t(z) = \left(e^{t\Delta/4}|f|^2\right)(0) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left((\partial\cdot\bar{\partial})^n|f|^2\right)(0). \tag{5.10}$$

As f is holomorphic, $\partial\bar{f} = 0 = \bar{\partial}f$ and $\bar{\partial}\bar{f} = \overline{\partial f}$, we conclude that

$$(\partial\cdot\bar{\partial})|f|^2 = \sum_{l=1}^{d}\partial_l f\cdot\overline{\partial_l f} = \sum_{l=1}^{d}|\partial_l f|^2$$

and similarly by induction that

$$(\partial\cdot\bar{\partial})^n|f|^2 = \sum_{l_1,\dots,l_n=1}^{d}|\partial_{l_1}\dots\partial_{l_n}f|^2.$$

Evaluating this expression at $z = 0$ shows

$$\|\alpha_n\|^2 = \sum_{l_1,\dots,l_n=1}^{d}|\partial_{l_1}\dots\partial_{l_n}f|^2(0) = \left((\partial\cdot\bar{\partial})^n|f|^2\right)(0)$$

which combined with Equation (5.10) formally proves Equation (5.6).

Although the above computation was formal it is easy to justify when $f \in \mathcal{HP}(\mathbb{C}^d)$ in which case all sums are finite sums. Thus Equation (5.6) holds for all $f \in \mathcal{HP}(\mathbb{C}^d)$ and then by a simple limiting argument along with the density result in Theorem 5.1 we may conclude that Equation (5.6) is valid for $f \in \mathcal{H}L^2(\lambda_t)$. [The key points to carrying out this limiting arguments are: 1) if $\mathcal{HP}(\mathbb{C}^d) \ni f_k \to f \in \mathcal{H}L^2(\lambda_t)$ in the $L^2(\lambda_t)$ – norm, then $\{Tf_k\}_{k=1}^\infty$ is Cauchy in J_t^0 and 2) the Cauchy estimates show f_k and all of its derivatives converge to the analogous derivatives of f as $k \to \infty$.] Since $T[\mathcal{HP}(\mathbb{C}^d)]$ is the dense subspace of finite rank tensors on J_t^0 (i.e. those $\alpha \in J_t^0$ with $\alpha_n \equiv 0$ for all sufficiently large n), we may easily conclude that T maps $\mathcal{H}L^2(\lambda_t)$ onto J_t^0, i.e. T is unitary.

If $f \in \mathcal{H}(\mathbb{C})$ with $\|f\|_{L^2(\lambda_t)} = \infty$, then let $\alpha = Tf$ and set

$$f_k(z) := \sum_{n=0}^{k} \frac{1}{n!} \alpha_n(z, z, \ldots, z)$$

be the k^{th} – order Taylor approximation to f. Then by Taylor's theorem for entire functions on $\mathbb{C}^d$ we know that $f_k(z) \to f(z)$ as $k \to \infty$ and therefore by Fatou's lemma along with the Equation (5.6) for holomorphic polynomials we find,

$$\infty = \|f\|^2_{L^2(\lambda_t)} \leq \liminf_{k\to\infty} \|f_k\|^2_{L^2(\lambda_t)} = \liminf_{k\to\infty} \sum_{n=0}^{k} \frac{t^n}{n!} \|\alpha_n\|^2 = \|\alpha\|_t^2.$$

Thus we have shown that Equation (5.6) holds for all $f \in \mathcal{H}(\mathbb{C})$. We now prove the remaining results in items 1–4 in turn.

1. The pointwise convergence in Equation (5.7) is guaranteed by Taylor's theorem for holomorphic functions. The $L^2(\lambda_t)$ – convergence follows from the isometry property already proved which allows us to conclude that $\{z \to (1/n!)\alpha_n(z, z, \ldots, z)\}_{n=0}^\infty$ is a collection of orthogonal functions in $L^2(\lambda_t)$ with

$$\sum_{n=0}^{\infty} \left\| z \to \frac{1}{n!} \alpha_n(z, z, \ldots, z) \right\|^2_{L^2(\lambda_t)} = \sum_{n=0}^{\infty} \|\alpha_n\|^2_{J_t^0} = \|\alpha\|^2_{J_t^0} < \infty.$$

2. Let $\beta \in \mathbb{N}_0^d$ and $n = |\beta|$. A little thought shows

$$\partial_{l_1} \ldots \partial_{l_n} z^\beta|_{z=0} = \begin{cases} \beta! & \text{if } \partial_{l_1} \ldots \partial_{l_n} = \partial^\beta \\ 0 & \text{otherwise} \end{cases}$$

where $\partial^\beta = \partial_1^{\beta_1} \ldots \partial_d^{\beta_d}$. From this observation and the isometry property already proved, it follows that $\{z^\beta : \beta \in \mathbb{N}_0^d\}$ is an orthogonal subset of $\mathcal{H}L^2(\lambda_t)$. Moreover, since

$$\frac{n!}{\beta!} = \#\{(l_1, \ldots, l_n) \in \{1, \ldots, d\}^n : \partial_{l_1} \ldots \partial_{l_n} = \partial^\beta\},$$

we may conclude that

$$\|z \to z^\beta\|^2_{L^2(\lambda_t)} = \frac{t^n}{n!}(\beta!)^2 \cdot \frac{n!}{\beta!} = t^n \beta! = t^{|\beta|}\beta!.$$

These computations along with the fact that holomorphic polynomials are dense in $\mathcal{H}L^2(\lambda_t)$ completes the proof of item 2.
3. & 4. The proof of items 3. and 4. now follow as in the proof of Theorem 4.9. □

For infinite dimensional versions of this theory see Shigekawa [**38**], Sugita [**39**], and Gross and Malliavin [**22**] where the reader will numerous relevant references. The interested reader may consult [**13**] and [**24**] for a number of related results involving more general Gaussian measure in both finite and infinite dimensional settings and their relations to the Itô-chaos expansion and Brian Hall's isometry theorem first introduced in [**23**].

6. Heat kernel complex Lie group theory.

The goal of this section is to describe the results in [**10**], [**11**], [**12**] which generalize the results of the previous section to the context of complex Lie groups. In this setting we will be replacing the Gaussian measure λ_t in Equation (5.4) by more general " heat kernel" measures (see Theorem 6.16) associated to both elliptic and hypo-elliptic second order left invariant differential operators on a complex Lie group. Along the way, I will also indicate alternative proofs to some of the results in [**10**], [**11**], [**12**] which are likely to be more palatable to stochastic analysts. These new proofs also show more clearly the connection of the results in [**10**], [**11**], [**12**] to Itô's chaos expansions, see [**29**].

6.1. Algebraic setup.

For the rest of this paper we are going to assume that G is a complex simply connected Lie group. Let $\mathfrak{g} := \mathrm{Lie}(G) = T_eG$ be the complex Lie algebra of G and $\mathfrak{g}^*$ be the (complex) dual space of $\mathfrak{g}$. For example, as above we might take $G = \mathbb{C}^d$ or some other complex simply connected nilpotent Lie group, see [**12**]. Another example to have in mind is $G = SL(n, \mathbb{C})$ (with $n \geq 2$) in which case $\mathfrak{g} = \mathrm{Lie}(SL(n, \mathbb{C})) = sl(n, \mathbb{C})$ – the space of $n \times n$ trace free matrices.

Definition 6.1. for $A \in \mathfrak{g}$ and $g \in G$ let $\tilde{A}(g) = L_{g*}A$ so that $\tilde{A}$ is the unique left invariant vector field on G which agrees with A at the identity.

Notation 6.2. Given a complex Lie algebra $\mathfrak{g}$, let;

1. $T(\mathfrak{g}) = \bigoplus_{n=0}^\infty \mathfrak{g}^{\otimes n}$ be the complex algebraic tensor algebra over $\mathfrak{g}$,
2. $T(\mathfrak{g})' = \prod_{k=0}^\infty (\mathfrak{g}^{\otimes k})^*$ denote the algebraic dual to $T(\mathfrak{g})$.
3. $J \subset T(\mathfrak{g})$ be the two sided ideal in $T(\mathfrak{g})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}$.

Let $\mathcal{D}$ denote the complex linear differential operators on $\mathcal{H}(G)$ which preserve $\mathcal{H}(G)$. By the universal property of the tensor algebra, $T(\mathfrak{g})$, there exists an algebra homomorphism, $T(\mathfrak{g}) \ni \beta \to \tilde{\beta} \in \mathcal{D}$, determined uniquely by setting $\tilde{1} = id$ and $\tilde{A}$ to be as in Definition 6.1 for all $A \in \mathfrak{g}$. [Notice that; if $z \in \mathbb{C}$, $A \in \mathfrak{g}$, and $f \in \mathcal{H}(G)$, then $(zA)^{\sim} f = z \cdot \tilde{A} f$ by the Cauchy-Riemann equations.] It will also be useful to let, for $x \in G$, and $k \in \mathbb{N}_0$, $(D^k f)(x) \in (\mathfrak{g}^{\otimes k})^*$ be uniquely determined by

$$\big\langle (D^k f)(x), A_1 \otimes \cdots \otimes A_k \big\rangle = \big(\tilde{A}_1 \ldots \tilde{A}_k f\big)(x). \tag{6.1}$$

Definition 6.3 (Taylor map). For $f \in \mathcal{H} = \mathcal{H}(G)$ and $x \in G$, let $\hat{f}(x) \in T(\mathfrak{g})'$ be defined by

$$\langle \hat{f}(x), \beta \rangle := \hat{f}(x)(\beta) = (\tilde{\beta} f)(x) \ \forall \ \beta \in T(\mathfrak{g}).$$

We call $\hat{f}(x) \in J^0$ the *Taylor coefficients of* f at x and refer to the linear map,

$$\mathcal{H}(G) \ni f \xrightarrow{T} Tf := \hat{f}(e) \in J^0, \tag{6.2}$$

as the *Taylor map.* We also will denote $\hat{f}(e)$ by α_f.

To be more explicit $\hat{f}(x)$ is determined by $\langle \hat{f}(x), 1 \rangle = f(x)$ and

$$\langle \hat{f}(x), A_1 \otimes \cdots \otimes A_n \rangle = \langle (D^n f)(x), A_1 \otimes \cdots \otimes A_n \rangle = \big(\tilde{A}_1 \ldots \tilde{A}_n f\big)(x)$$

for any $n \in \mathbb{N}$ and $A_i \in \mathfrak{g}$ for $1 \leq i \leq n$.

Lemma 6.4. *If $f \in \mathcal{H}(G)$, then $\alpha_f(\beta) = 0$ for all $\beta \in J$.*

Proof. As is well known, if $\gamma_1, \gamma_2 \in T(\mathfrak{g})$ and $\xi, \eta \in \mathfrak{g}$, then

$$[\gamma_1 \otimes (\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]) \otimes \gamma_2]^{\sim} = \tilde{\gamma}_1 \big(\tilde{\xi}\tilde{\eta} - \tilde{\eta}\tilde{\xi} - \widetilde{[\xi, \eta]}\big) \tilde{\gamma}_2 = 0$$

since by the definition of the Lie bracket, $\widetilde{[\xi, \eta]} = [\tilde{\xi}, \tilde{\eta}]$. In other words, $\tilde{\beta} = 0$ for all $\beta \in J$ and in particular $(\tilde{\beta} f)(e)$ for all $\beta \in J$. □

Definition 6.5. Let $J^0 = \{\alpha \in T(\mathfrak{g})' : \alpha|_J \equiv 0\}$ be the (raw) *derivative space.* [The derivative space may be identified with the dual of the universal enveloping algebra, $U(\mathfrak{g}) = T(\mathfrak{g})/J$.]

We now suppose that q is a non-negative Hermitian form on $\mathfrak{g}^*$. The proof of the following simple linear algebra lemma may be found in [**11**, Lemma 2.2]

Lemma 6.6. *If q is a non-negative Hermitian form on $\mathfrak{g}^*$, then there exists $m \leq \dim_{\mathbb{C}}(\mathfrak{g})$ and a linearly independent subset, $\{X_l\}_{l=1}^m \subset \mathfrak{g}$, such that*

$$q(\alpha, \beta) = \sum_{l=1}^{m} \alpha(X_l)\overline{\beta(X_l)} \text{ for all } \alpha, \beta \in \mathfrak{g}^*.$$

DEFINITION 6.7 (Horizontal subspace). The horizontal subspace associated to q is $H = H(q) := \text{span}_{\mathbb{C}}(X_l : 1 \le l \le m)$ with the inner product: $(X_l, X_k)_H := \delta_{lk}$. [Let me emphasize that m may be strictly less than $\dim_{\mathbb{C}} \mathfrak{g}$.]

EXAMPLE 6.8. If $\mathfrak{g} = sl(n,\mathbb{C})$ we might take q to be the dual norm to the Hilbert–Schmidt inner product on $\mathfrak{g}$ defined by $\langle A, B\rangle_{HS} := \operatorname{tr}(B^*A)$.

Next we extend q to $(\mathfrak{g}^{\otimes k})^*$ by defining

$$q^{\otimes k}(\alpha,\beta) = \sum_{l_1,\dots,l_k=1}^{m} \alpha(X_{l_1}\otimes\cdots\otimes X_{l_k})\overline{\beta(X_{l_1}\otimes\cdots\otimes X_{l_k})}$$

for all $\alpha, \beta \in (\mathfrak{g}^{\otimes k})^*$.

DEFINITION 6.9. For each $t > 0$, and $\alpha \in T(\mathfrak{g})'$ let

$$q_t(\alpha) := \sum_{k=0}^{\infty} \frac{t^k}{k!} q^{\otimes k}(\alpha,\alpha) \tag{6.3}$$

and then set

$$J_t^0 := \{\alpha \in J^0 : q_t(\alpha) < \infty\}. \tag{6.4}$$

For $\alpha, \beta \in J_t^0$ we also let

$$q_t(\alpha,\beta) := \sum_{k=0}^{\infty} \frac{t^k}{k!} q^{\otimes k}(\alpha,\beta) \tag{6.5}$$

which is the polarization of Equation (6.3).

DEFINITION 6.10 (Hörmander's condition). We say q satisfies *Hörmander's condition* if $\text{Lie}(H(q)) = \mathfrak{g}$ where $\text{Lie}(H(q))$ is the smallest Lie-subalgebra of $\mathfrak{g}$ which contains the horizontal subspace, $H(q)$.

The next theorem gives simple necessary and sufficient condition on q in order that q_t is an inner product on J_t^0 for one and hence for all $t > 0$.

THEOREM 6.11 ([**11**, Theorem 2.7]). *The following are equivalent*:

1. *Hörmander's condition holds, i.e.* $\text{Lie}(H) = \mathfrak{g}$,
2. $T(\mathfrak{g}) = T(H) + J$,
3. *for any* $t > 0$, $q_t|_{J_t^0}$ *is an inner product on* J_t^0.

6.2. The reconstruction series.

If z is a point in $\mathbb{C}^n$ and $z^{\otimes k}$ is its k^{th} tensor power in $(\mathbb{C}^n)^{\otimes k}$ then the conventional power series representation of a holomorphic function f in a neighborhood of $0 \in \mathbb{C}^n$ may be written $f(z) = \langle \alpha, \Phi(z)\rangle$, where $\Phi(z) := \sum_{k=0}^{\infty}(k!)^{-1}z^{\otimes k}$ is an element of the (suitably completed) tensor algebra over $\mathbb{C}^n$ and α is in the dual space. In order to recover a holomorphic function f on a complex Lie group G from a knowledge of its Taylor coefficient $\alpha = \hat{f}(e)$, we will need to represent f globally on G by an analogous kind of power series. Of course we do not, in general, have a global coordinate system as on $\mathbb{C}^n$. Proposition 6.13 below is a special case of [**11**, Proposition 5.13] which in turn relies on the machinery in [**6**] and [**10**]. To state the proposition we need the following notation.

NOTATION 6.12. If $g : [0, 1] \to G$ is a piecewise C^1 path such that $g(0) = e$ then for each $n \in \mathbb{N}$ we let

$$\Psi_t^n(g) := \int_{\Delta_n(t)} b'(s_1) \otimes \cdots \otimes b'(s_n)\; d\mathbf{s} \in \mathfrak{g}^{\otimes n}$$

where

$$\Delta_n(t) := \{(s_1, \ldots, s_n) : 0 \le s_1 \le s_2 \le \cdots \le s_n \le t\},$$

$d\mathbf{s} = ds_1 \ldots ds_n$, $b(s) := \int_0^s \theta(g'(r))dr$ and θ is the Maurer-Cartan form defined by $\theta(g'(r)) := L_{g(r)^{-1}*}g'(r) \in \mathfrak{g}$. Here $L_x : G \to G$ denotes the holomorphic function given by left multiplication by $x \in G$. We further let $\Psi_t^0(g) = 1$ and

$$\Psi_t(g) := \sum_{n=0}^{\infty} \Psi_t^n(g) \in \prod_{n=0}^{\infty} \mathfrak{g}^{\otimes n}.$$

PROPOSITION 6.13 ([**11**, Proposition 5.13]). *If $f \in \mathcal{H}(G)$, $\alpha = Tf = \hat{f}(e)$, and $g : [0, 1] \to G$ is a piecewise C^1 path, such that $g(0) = e$, then*

$$f(g(1)) = \langle \alpha, \Psi_1(g)\rangle := \sum_{k=0}^{\infty} \langle \hat{f}(e), \Psi_1^k(g)\rangle \tag{6.6}$$

where the sum in Equation (6.6) *is absolutely convergent.*

PROOF. I will give a sketch of the proof here and refer the interested reader to [**6**, Proposition 5.1] for the details. There are three basic steps to the proof.

1. For each $z = x + iy \in \mathbb{C}$, let

$$V_z(t, g) \equiv L_{g*}[zb'(t)] = L_{g*}(xb'(t) + yib'(t)).$$

   One then shows that the unique solution, $g_z(t) \in G$, to the ordinary differential equation:

$$\dot{g}_z(t) = V_z(t, g_z(t)) = L_{g*}[zb'(t)] \text{ with } g_z(0) = e \tag{6.7}$$

   is holomorphic in z. Consequently $z \to f(g_z(1))$ is a holomorphic function of z and hence has an expansion of the form,

$$f(g_z(1)) = \sum_{n=0}^{\infty} a_n z^n \text{ for all } z \in \mathbb{C} \tag{6.8}$$

which is absolutely convergent.

2. By repeated use of the fundamental theorem of calculus one shows (see [**6**, Lemma 5.2]),

$$f(g_z(1)) = \sum_{n=0}^{N-1} z^n \int_{\Delta_n(1)} \left\langle D^n f(e), b'(s_1) \otimes \cdots \otimes b'(s_n) \right\rangle \boldsymbol{ds} + z^N R_N(z) \tag{6.9}$$

$$= \sum_{n=0}^{N-1} z^n \int_{\Delta_n(1)} \langle \alpha_f, \Psi_1^n(g) \rangle \boldsymbol{ds} + z^N R_N(z) \tag{6.10}$$

where

$$R_N(z) \equiv \int_{\Delta_N(1)} \left\langle D^N u(g_z(s_1)), b'(s_1) \otimes \cdots \otimes b'(s_N) \right\rangle \boldsymbol{ds}. \tag{6.11}$$

3. By comparing the series in Equations (6.8) and (6.10) we may conclude that in fact $a_n = \int_{\Delta_n(1)} \langle \alpha_f, \Psi_1^n(g) \rangle \boldsymbol{ds}$ and therefore,

$$f(g_z(1)) = \sum_{n=0}^{\infty} z^n \cdot \int_{\Delta_n(1)} \langle \alpha_f, \Psi_1^n(g) \rangle \boldsymbol{ds} \text{ for all } z \in \mathbb{C}.$$

The result now follows by taking $z = 1$ in the previous equation. □

REMARK 6.14. The absolute convergence of the sum in Equation (6.6) is solely a consequence of the fact that f is holomorphic on G.

6.3. Heat Kernel Measures.

DEFINITION 6.15. Given a non-negative Hermitian form, q, on $\mathfrak{g}^*$ we associate a second order differential operator on $C^\infty(G)$ defined by

$$\Delta = \Delta_q := \sum_{l=1}^{m} \left[\widetilde{X_l}^2 + \widetilde{Y_l}^2 \right]$$

where $\{X_l\}_{l=1}^m$ form a basis for $H(q)$ as described in Lemma 6.6 and $Y_l := iX_l = \sqrt{-1}X_l$ for $1 \le l \le m$. We refer to Δ_q as the *Laplacian* associated to q.

The following theorem is by now standard and the reader is referred to [**11**, Section 3] for references to the literature where the proofs and more details may be found.

THEOREM 6.16. *Assuming* $\mathrm{Lie}(H) = \mathfrak{g}$, *there exists a convolution* (heat kernel) *semi-group,* $\{\rho_t\}_{t>0} \subset C^\infty(G, (0, \infty))$ *such that*

$$\left(e^{t\Delta/4}f\right)(e) = \int_G f(g)d\lambda_t(g)$$

where $d\lambda_t(g) = \rho_t(g)dg$ *and* dg *denotes a* right Haar *measure on* G. [*The fact that* ρ_t *is smooth is a consequence Hörmander's theorem* [**25**].]

The following Theorem 6.17 is the main theorem in [**11**]. The remaining goal of this paper is to explain some aspects of the proof of this theorem. Along the way, we will give some alternate Itô calculus explanations of the some the steps in the proof which were previously done by more pure analytic techniques.

Theorem 6.17 (Taylor Isomorphism Theorem, [**11**, Theorem 6.1]). *Let G be a connected, simply connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds,* (*cf. Definition* 6.10). *Let ρ_t denote the heat kernel as in Theorem* 6.16. *Then the Taylor map,*

$$\mathcal{H}L^2(G,\lambda_t) \ni f \to Tf = \hat{f}(e) = \alpha_f \in J_t^0$$

is a unitary map from $\mathcal{H}L^2(G,\lambda_t)$ onto J_t^0.

Proof. Before getting started let us first note that the same type of formal argument used in the proof of Theorem 5.5 carry over to this setting to formally show for $f \in \mathcal{H}L^2(\lambda_t)$ that

$$\begin{aligned} \|f\|^2_{L^2(\lambda_t)} &= e^{t\Delta/4}|f|^2(e) = \sum_{n=0}^{\infty} \frac{1}{n!}\left((Z\cdot\bar{Z})^n|f|^2\right)(e) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{l_1,\dots,l_k=1}^{m}\left|\tilde{X}_{l_1}\dots\tilde{X}_{l_k}f(e)\right|^2 = \left\|\hat{f}\right\|_t^2, \end{aligned} \tag{6.12}$$

where

$$Z_l = \frac{1}{2}\left(\tilde{X}_l - i\tilde{Y}_l\right) \text{ and } \bar{Z}_l = \frac{1}{2}\left(\tilde{X}_l + i\tilde{Y}_l\right).$$

However, it is much more difficult to make this argument rigorous in this case as there are, in general, no known dense subspaces in $\mathcal{H}L^2(\lambda_t)$ and J_t^0 respectively on which these computations are easily justified. For example it is shown in [**21**] that for general $\mathfrak{g}$ there are typically *no* finite rank tensors in J^0. On the other hand, if $\mathfrak{g}$ is "stratified," then the finite rank tensors in J^0 are dense in J_t^0 and the above proof goes through, see [**12**, Lemma 3.5].

The rigorous proof for general $\mathfrak{g}$ involves three steps. The proofs given here of the first two steps will be different than those given in [**11**].

1. First we wish to show $\|\hat{f}(e)\|^2_{J_t^0} \le \|f\|^2_{L^2(\rho_t)}$ for all $f \in \mathcal{H}L^2(G,\lambda_t)$ and hence for all $f \in \mathcal{H}(G)$. This is the content of Corollary 6.23 below.

2. Secondly we show $\|f\|_{L^2(\rho_t)}^2 \leq \|\hat{f}(e)\|_{J_t^0}^2$ for all $f \in \mathcal{H}(G)$, see Corollary 6.28 below. These two steps together shows the isometry property in Equation (6.12) holds for all $f \in \mathcal{H}(G)$.
3. The last step is to show the Taylor map, $\mathcal{H}L^2(G, \lambda_t) \ni f \to \hat{f}(e) \in J_t^0$ is surjective. This is where the fact that G is simply connected comes into play. The construction of $f \in \mathcal{H}(G)$ from $\alpha \in J_t^0$ is rather subtle and I have nothing new to add here over the proof given in [**11**, Section 6]. □

The remainder of this section is devoted to the proofs of Corollaries 6.23 and 6.28 below. The proofs we give here will make heavy use of Brownian motion, Itô calculus, and multiple Itô integrals in particular. This is in contrast to the more analytic proofs given in [**11**].

Remark 6.18. The Taylor Isomorphism Theorem 6.17 was proved for *non-degenerate* q in [**6**] for complexifications of compact type Lie groups and then for general complex Lie groups in [**10**]. The most general form given here is found in [**11**] as already mentioned. There are by now numerous infinite dimensional version of this theorem, see Gordina [**17**], [**18**], [**19**], Cecil [**4**], Driver and Gordina [**7**], [**8**], [**9**], and Gordina and Melcher [**20**]. Moreover, the methods describe above for the informal proof of Equation (6.12) have proved useful for understanding the large-N limit of the Segal-Bargmann-Hall transform in the context of $SL(N, \mathbb{C})$, see [**14**].

6.4. Itô expansion results.

Let $(\Omega, \{\mathcal{B}_s\}_{s\geq 0}, P)$ be a filtered probability space satisfying the usual assumptions. We further assume that $\{b(s) \in H(q)\}_{s\geq 0}$ is a "*horizontal*" Brownian motion by which we mean

$$b(s) = \frac{1}{\sqrt{2}} \sum_{j=1}^{m} (x_j(s) X_j + y_j(s) Y_j) \in H(q) \tag{6.13}$$

where $\{(x_1(s), \ldots, x_m(s))\}_{s\geq 0}$ and $\{(y_1(s), \ldots, y_m(s))\}_{s\geq 0}$ are two independent $\mathbb{R}^m$ – valued Brownian motions. Let $g(s) \in G$ denote the solution to the stochastic differential equation;

$$\delta g \doteq L_{g*} \delta b \text{ with } g(0) = e, \tag{6.14}$$

or equivalently g solves

$$\delta g \doteq \sum_{j=1}^{d} \left\{ \tilde{X}_j(g) \delta x_j + \tilde{Y}_j(g) \delta y_j \right\} \text{ with } g(0) = e. \tag{6.15}$$

Here and in the rest of the paper we use δb to denote the Stratonovich differential while db denotes the Itô differential.

The interpretation of such equations again goes back to Itô [**27**], [**28**], [**30**], [**32**]. By definition of (6.15), if $u \in C^\infty(G)$, then

$$\begin{aligned} du(g) &\doteq \sum_{j=1}^{d} \left\{ (\tilde{X}_j u)(g) \delta x_j + (\tilde{Y}_j u)(g) \delta y_j \right\} \\ &= \sum_{j=1}^{d} \left\{ (\tilde{X}_j u)(g) dx_j + (\tilde{Y}_j u)(g) dy_j \right\} + \frac{1}{2} \sum_{j=1}^{d} \left\{ (\tilde{X}_j^2 u)(g) + (\tilde{Y}_j^2 u)(g) \right\} \frac{1}{2} ds \\ &= \langle Du(g), db \rangle + \frac{1}{4} (\Delta u)(g) ds, \end{aligned} \tag{6.16}$$

where $Du(g)$ is viewed as an element of $\mathfrak{g}_{\mathbb{R}}^*$ and $\mathfrak{g}_{\mathbb{R}}$ denotes $\mathfrak{g}$ thought of as a real vector space. Thus $g(s)$ is a Brownian motion on G with generator Δ and hence

$$\mathrm{Law}(g(t)) = \lambda_t \text{ for all } t > 0. \tag{6.17}$$

If $u = f \in \mathcal{H}(G)$ then $\Delta f = 0$ (by the Cauchy-Riemann equations) so that (6.16) simplifies to

$$df(g) = \langle Df(g), db \rangle. \tag{6.18}$$

We will need a few auxiliary results in order to exploit this stochastic description of λ_t. Our first goal is to get control of size of $D^n f$ for $f \in \mathcal{H}L^2(G, \lambda_t)$ so that we can conclude that the local martingale, $f(g_s)$, is in fact a square integrable martingale for $0 \leq s \leq t$.

THEOREM 6.19. *If $f \in \mathcal{H}L^2(G, \lambda_t)$, then for all $n \in \mathbb{N}$ and $0 < s < t$, there exists $C_s = C(n, s) < \infty$ such that*

$$\|D^n f\|_{L^2(\lambda_\tau)} \leq C_s \|f\|_{L^2(\lambda_t)} \textit{ for all } 0 < \tau \leq s.$$

PROOF. To this end we start with Lemma 3.1 in order to see that there exists a smooth function $\delta : G \times G \to \mathbb{C}$ supported near $(e, e) \in G \times G$ such that

$$f(x) = \int_G f(y) \delta(x, y) dy$$

which then implies,

$$\begin{aligned} \big(\tilde{A}_1 \dots \tilde{A}_n f\big)(e) &= \int_G f(y) \big(\tilde{A}_1 \dots \tilde{A}_n \delta\big)(e, y) dy \\ &= \int_G f(y) \frac{(\tilde{A}_1 \dots \tilde{A}_n \delta)(e, y)}{\rho_\tau(y)} \rho_\tau(y) dy, \end{aligned} \tag{6.19}$$

where

$$\big(\tilde{A}_1 \dots \tilde{A}_n \delta\big)(e, y) := \tilde{A}_1 \dots \tilde{A}_n [x \to \delta(x, y)]|_{x=e}.$$

If we replace f by $f \circ L_x$ in Equation (6.19), it follows that

$$\begin{aligned}\big(\tilde{A}_1 \dots \tilde{A}_n f\big)(x) &= \big(\tilde{A}_1 \dots \tilde{A}_n f\big) \circ L_x(e) = \big(\tilde{A}_1 \dots \tilde{A}_n [f \circ L_x]\big)(e) \\ &= \int_G f(xy)\big(\tilde{A}_1 \dots \tilde{A}_n \delta\big)(e,y)dy \\ &= \int_G f(xy)\frac{(\tilde{A}_1 \dots \tilde{A}_n \delta)(e,y)}{\rho_\tau(y)}\rho_\tau(y)dy.\end{aligned}$$

Now let $0 < \varepsilon = t - s$ and then defining $C_\varepsilon < \infty$ by

$$C_\varepsilon := \sup\left\{\left|\frac{(\tilde{A}_1 \dots \tilde{A}_n \delta)(e,y)}{\rho_\tau(y)}\right|^2 : |A_i| = 1, y \in G, \varepsilon \le \tau \le t\right\}.$$

Then for $0 \le \tau \le s < t$ we have $\varepsilon \le t - \tau \le t$ and therefore,

$$\begin{aligned}\big|(\tilde{A}_1 \dots \tilde{A}_n f)(x)\big|^2 &\le \int_G |f(xy)|^2 \left|\frac{(\tilde{A}_1 \dots \tilde{A}_n \delta)(e,y)}{\rho_\tau(y)}\right|^2 \rho_\tau(y)dy \\ &\le |A_1|^2 \dots |A_n|^2 C_\varepsilon \int_G |f(xy)|^2 \rho_\tau(y)dy.\end{aligned}$$

Now multiply this equation by $\rho_{t-\tau}(x)$ and then integrating the result while making us of the fact that $\{\rho_\tau\}_{\tau>0}$ is a convolution semi-group we learn that

$$\begin{aligned}\int \big|(\tilde{A}_1 \dots \tilde{A}_n f)(x)\big|^2 \rho_{t-\tau}(x)dx &\le |A_1|^2 \cdots |A_n|^2 C_\varepsilon \int_G |f(xy)|^2 \rho_\tau(y)\rho_{t-\tau}(x)dydx \\ &= |A_1|^2 \cdots |A_n|^2 C_\varepsilon \int_G |f(y)|^2 \rho_t(y)dy\end{aligned}$$

from which the result easily follows. □

COROLLARY 6.20. *If $f \in \mathcal{H}L^2(G, \lambda_t)$, then $e^{s(\Delta/4)} f = f$ a.e. for $0 \le s < t$, where by definition,*

$$\big(e^{s(\Delta/4)} f\big)(x) := \int_G f(y)\rho_s(x^{-1}y)m(x)dy$$

and m is the modular function defined by $\int_G f(xy)dy = m(x)\int_G f(y)dy$.

PROOF. Let $M_s := f(g_s)$. Then by Itô's formula along with the fact that $\Delta f = 0$ allows us to conclude that

$$M_s = M_0 + \int_0^s \langle Df(g_r), db_r\rangle$$

is a local martingale. Moreover, by Theorem 6.19 we may conclude that for any $s < t$ that

$$\int_0^s \mathbb{E}|Df(g_r)|^2 dr \leq C_s \int_0^s \|f\|^2_{L^2(\rho_t)} dr < \infty$$

and hence $\{M_s\}_{0 \leq s < t}$ is a square integrable martingale. This fact and the Markov property for $\{g_\tau\}_{\tau \geq 0}$ shows for any $0 \leq r < s < t$ that

$$f(g_r) = M_r = \mathbb{E}[M_s|\mathcal{B}_r] = \mathbb{E}[f(g_s)|\mathcal{B}_r] = (e^{((s-r)/4)\Delta} f)(g_r) \text{ a.s.}$$

From this we conclude (and the positivity of the ρ_r) that $e^{((s-r)/4)\Delta} f = f$ a.e. which suffices to conclude the proof. □

Corollary 6.21. *If $f \in \mathcal{H}L^2(G, \lambda_t)$, then $M_s \doteq f(g(s))$ is a square integrable martingale for $s \in [0, t]$.*

Proof. From Corollary 6.20 and the Markov property, we conclude that

$$f(g_s) = (e^{(t-s)(\Delta/4)} f)(g_s) = \mathbb{E}[f(g_t)|\mathcal{B}_s].$$

This completes the proof since $f(g_t)$ is a square integrable random variable and therefore $M_s := \mathbb{E}[f(g_t)|\mathcal{B}_s]$ is a square integrable martingale for $0 \leq s \leq t$. □

Corollary 6.22. *If $f \in \mathcal{H}L^2(G, \lambda_t)$, then*

$$\|f\|_{L^2(\rho_s)} \uparrow \|f\|_{L^2(\rho_t)} \text{ as } s \uparrow t. \tag{6.20}$$

Proof. From Corollary 6.21 and the convergence theorem for L^2 – martingales, $\|f\|^2_{L^2(\rho_s)} = \mathbb{E}|M_s|^2$ increases to $\mathbb{E}|M_t|^2 = \|f\|^2_{L^2(\rho_t)}$. □

Corollary 6.23. *If $f \in \mathcal{H}L^2(G, \lambda_t)$, then $\|\hat{f}(e)\|^2_{J_t^0} \leq \|f\|^2_{L^2(\rho_t)}$.*

Proof. By the stochastic analogue of the iteration scheme used to prove Equation (6.9), we find

$$f(g_s) = \sum_{k=0}^{n} \left\langle D^k f(e), \int_{\Delta_k(s)} db_{t_1} \otimes \cdots \otimes db_{t_k} \right\rangle + R_s(n) \tag{6.21}$$

where

$$R_s(n) := \left\langle D^{n+1} f(g_{t_0}), \int_{\Delta_{n+1}(s)} db_{t_0} \otimes db_{t_1} \otimes \cdots \otimes db_{t_n} \right\rangle.$$

Because of Theorem 6.19, we know that

$$\int_{\Delta_{n+1}(s)} \mathbb{E}\|D^{n+1}f(g_{t_0})\|^2 dt_0 \ldots dt_n < \infty$$

for all $0 \le s < t$. Therefore it follows by the basic properties of iterated Itô integrals that

$$\begin{aligned}
\mathbb{E}|f(g_s)|^2 &= \sum_{k=0}^{n} \mathbb{E}\left\langle D^k f(e), \int_{\Delta_k(s)} db_{t_1} \otimes \cdots \otimes db_{t_k} \right\rangle^2 + \mathbb{E}|R_s(n)|^2 \\
&= \sum_{k=0}^{n} \frac{s^k}{k!} \|D^k f(e)\|^2 + \int_{\Delta_{n+1}(s)} \mathbb{E}\|D^{n+1}f(g_{t_0})\|^2 dt_0 \ldots dt_n \\
&\ge \sum_{k=0}^{n} \frac{s^k}{k!} \|D^k f(e)\|^2. \qquad (6.22)
\end{aligned}$$

We may now let $n \uparrow \infty$ to learn,

$$\left\|\hat{f}(e)\right\|_{J_s^0}^2 \le \mathbb{E}|f(g_s)|^2 = \|f\|_{L^2(\rho_s)}^2.$$

We may now pass to the limit as $s \uparrow t$ to conclude that $\|\hat{f}(e)\|_{J_t^0}^2 \le \|f\|_{L^2(\rho_t)}^2$ wherein we have used Equation (6.20) for the right side and the monotone convergence theorem to conclude,

$$\left\|\hat{f}(e)\right\|_{J_t^0}^2 = \uparrow \lim_{s \uparrow t} \left\|\hat{f}(e)\right\|_{J_s^0}^2. \qquad \square$$

Remark 6.24. It is tempting to try to prove Equation (6.22) without the aid of Theorem 6.19 using the following simpler (false) " argument." The false argument (which appears in various places in the literature) states; by the basic properties of multiple Itô integrals, all terms in Equation (6.21) are orthogonal so of course,

$$\begin{aligned}
\mathbb{E}|f(g_s)|^2 &= \sum_{k=0}^{n} \mathbb{E}\left\langle D^k f(e), \int_{\Delta_k(s)} db_{t_1} \otimes \cdots \otimes db_{t_k} \right\rangle^2 + \mathbb{E}|R_s(n)|^2 \\
&\ge \sum_{k=0}^{n} \mathbb{E}\left\langle D^k f(e), \int_{\Delta_k(s)} db_{t_1} \otimes \cdots \otimes db_{t_k} \right\rangle^2. \qquad (6.23)
\end{aligned}$$

To see this type of argument is in general false consider the following example that the author learned from Remi Leandre many years ago.

Example 6.25. Let $\{B_t\}_{t \ge 0}$ be a Brownian motion in $\mathbb{R}^3$ and let $f(y) = 1/|x-y|$ for some fixed point $x \in \mathbb{R}^3 \setminus \{0\}$. Then it is well known that $\Delta f(y) = 0$ for $y \neq x$ and $\Delta f = -4\pi\delta_x$ in the sense of Schwarz-distributions. As $\{x\}$ is a polar set for B, we may apply Itô's lemma to conclude,

$$f(B_t) = f(0) + \int_0^t \nabla f(B_s) \cdot dB_s.$$

Hence, if the argument prior to Equation (6.23) were true in this setting we would conclude that $\int_0^t \nabla f(B_s) \cdot dB_s$ is orthogonal to the constant functions, i.e.

$$0 = \mathbb{E}\left[\int_0^t \nabla f(B_s) \cdot dB_s\right] = \mathbb{E}[f(B_t) - f(0)] = \mathbb{E}f(B_t) - \frac{1}{|x|}. \tag{6.24}$$

To see this is false, let

$$p_t(y) := \frac{1}{(2\pi t)^{3/2}} \exp\left(-\frac{1}{2t}|y|^2\right) \text{ for } y \in \mathbb{R}^3.$$

Then

$$\begin{aligned}\frac{d}{dt}\mathbb{E}f(B_t) &= \frac{d}{dt}\int_{\mathbb{R}^3} f(y)p_t(y)dy = \frac{1}{2}\int_{\mathbb{R}^3} f(y)\Delta p_t(y)dy \\ &= \frac{1}{2}(-4\pi)\int_{\mathbb{R}^3} \delta_x(y)p_t(y)dy := -2\pi p_t(x) < 0\end{aligned}$$

and therefore using the easily verified fact that $[0,\infty) \ni t \to \mathbb{E}f(B_t)$ is continuous in t, we learn

$$\mathbb{E}f(B_t) < \mathbb{E}f(B_0) = \frac{1}{|x|} \text{ for all } t > 0$$

which violates Equation (6.24).

The next theorem is a close relative of the Veretennikov-Krylov formula in [**41**].

THEOREM 6.26. *Suppose that $f \in \mathcal{H}(G)$ and $\alpha := Tf = \hat{f}(e) \in J^0$. Then for all $t \in [0,\infty)$,*

$$f(g(t)) = \sum_{k=0}^{\infty} \left\langle \alpha_k, \int_{\Delta_k(t)} db(s_1) \otimes \cdots \otimes db(s_k) \right\rangle \tag{6.25}$$

where the sum is almost surely absolutely convergent.

PROOF. This theorem is the stochastic analogue of Proposition 6.13. We will sketch two proofs of this result.

1. The first method is to repeat the proof in of Proposition 6.13 in the stochastic context making use of basic facts about multiple Itô integrals and stochastic flows. Results on stochastic flows may be found (for example) in Kunita [**34**], see Theorem 4.8.4. In truth it may be difficult to find the exact results which are needed here.
2. The second method is to make use of Terry Lyons' rough path analysis to give a proof

of Proposition 6.13 which is valid in the context of Brownian motion. Recall that we may choose a version of

$$\mathbb{B}_{st} := \int_s^t [b(\tau) - b(s)] \otimes_{\mathbb{R}} \delta b(\tau) \text{ for } 0 \le s \le t < \infty$$

such that $(b_{s,t} = b(t) - b(s), \mathbb{B}_{s,t})$ is a geometric rough path. [The subscript $\mathbb{R}$ on the tensor symbol indicates we are, for the moment, taking tensor products of $\mathfrak{g}$ over $\mathbb{R}$ rather than $\mathbb{C}$.] In fact, we may choose piecewise linear approximations, $b^n(t)$ to $b(t)$ so that

$$\left(b^n_{s,t}, \mathbb{B}^n_{s,t}\right) = \left(b^n(t) - b^n(s), \int_s^t [b^n(\tau) - b^n(s)] \otimes_{\mathbb{R}} db^n(\tau)\right)$$

converges to $(b_{s,t}, \mathbb{B}_{s,t})$ in $(p, p/2)$ -variation norm for any $2 < p < 3$, see for example [**35**, Theorem 4.1.1] or [**16**, Proposition 3.6]. The exceptional null set, E, is now fixed once and for all. Moreover for any $1/3 < \alpha < 1/2$, there exists a finite non-negative random variable $C(\alpha, T)$ such that off of E (expanded a bit if necessary) the following Hölder-estimates hold;

$$\|b_{st}\|_{\mathfrak{g}} \le C(\alpha, T)|t-s|^{\alpha} \text{ and } \|\mathbb{B}_{s,t}\|_{\mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g}} \le C(\alpha, T)|t-s|^{2\alpha} \ \forall\ s, t \in [0, T],$$

where $\|\cdot\|_{\mathfrak{g}}$ and $\|\cdot\|_{\mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g}}$ are any two fixed norms on $\mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}$ respectively.

3. From Theorem 4.20 of [**3**], we may now construct a solution $g_z(t)$ to the rough path analogue of Equation (6.7). This rough path solution is uniquely determined in this setting by requiring $g_z(t)$ satisfies; for all smooth functions, $u : G \to \mathbb{C}$, there exists a constant $C = C(u, z) < \infty$ such that

$$\left|\begin{matrix} u(g_z(t)) - u(g_z(s)) - (Du)(g_z(s))\langle L_{g_z(s)*}[zb_{st}]\rangle \\ -\left(D^2_{\mathbb{R}}u\right)(g_z(s))\langle [L_{g_z(s)*}M_z \otimes_{\mathbb{R}} L_{g_z(s)*}M_z]\mathbb{B}_{st}\rangle \end{matrix}\right| \le C|t-s|^{3\alpha} \tag{6.26}$$

for all $0 \le s \le t \le T$. Here $M_z A := zA$ for all $z \in \mathbb{C}$ and $A \in \mathfrak{g}$ and $(D^2_{\mathbb{R}}u)(g)$ is the real linear functional on $\mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}$ determined by,

$$\left(D^2_{\mathbb{R}}u\right)(g)\langle A \otimes_{\mathbb{R}} B\rangle := \left(\tilde{A}\tilde{B}u\right)(g) \ \forall\ g \in G \ \&\ A, B \in \mathfrak{g}.$$

It is worth observing that if $u = f \in \mathcal{H}(G)$, then

$$\left(D^2_{\mathbb{R}}f\right)(g)\langle A \otimes_{\mathbb{R}} B\rangle = (D^2 f)(g)\langle A \otimes B\rangle.$$

4. Now let $g^n_z(t)$ denote the solution to Equation (6.7) with $b(t)$ replaced by $b^n(t)$ as in step 2. Then by Terry Lyons' universal limit theorem, we know that $g^n_z(t) \to g_z(t)$ locally uniformly in (z, t) and so by Morera's theorem, $z \to g_z(t)$ is still holomorphic in z. Moreover, the universal limit theorem [**35**, Theorem 4.1.1] (or see [**16**, Theorem 8.5 and Chapter 9]) also implies

$$\int_{\Delta_n(t)} db^n(s_1) \otimes \cdots \otimes db^n(s_n)$$

converges to a version of the Stratonovich iterated integral,

$$\int_{\Delta_n(t)} \delta b(s_1) \otimes \cdots \otimes \delta b(s_n).$$

So following the same arguments as in the proof of Proposition 6.13, we may conclude off the exceptional set, E, that

$$f(g(1)) = \sum_{k=0}^{\infty} \left\langle \hat{f}(e), \int_{\Delta_k(t)} \delta b(s_1) \otimes \cdots \otimes \delta b(s_k) \right\rangle.$$

5. Explicit formulas relating multiple Stratonovich integrals to multiple Itô integrals are well known, see for example [**26**], [**2**, Proposition 1], [**33**], and [**13**, Definition 4.10]. From these formula one shows

$$\int_{\Delta_k(t)} \delta b(s_1) \otimes \cdots \otimes \delta b(s_k) = \int_{\Delta_k(t)} db(s_1) \otimes \cdots \otimes db(s_k) \text{ a.s.} \tag{6.27}$$

and so, at the expense of increasing the size of exceptional null set E, we also have

$$f(g(t)) = \sum_{k=0}^{\infty} \left\langle \hat{f}(e), \int_{\Delta_k(t)} db(s_1) \otimes \cdots \otimes db(s_k) \right\rangle.$$

A direct proof of Equation (6.27) may also be given as follows.
The key point is that differential of the *complex* quadratic variation tensor (denoted by $db \otimes db$) is zero because,

$$\begin{aligned} db(t) \otimes db(t) &= \frac{1}{2} \sum_{j=1}^{m} [X_j \otimes X_j + Y_j \otimes Y_j] dt \\ &= \frac{1}{2} \sum_{j=1}^{m} \left[X_j \otimes X_j + (iX_j) \otimes (iX_j)\right] dt \\ &= \frac{1}{2} \sum_{j=1}^{m} [X_j \otimes X_j - X_j \otimes X_j] dt = 0, \end{aligned}$$

wherein we have used $Y_j := iX_j$ as in the definition of $b(t)$ in Equation (6.13). Therefore if we let

$$S_k(t) := \int_{\Delta_k(t)} \delta b(s_1) \otimes \cdots \otimes \delta b(s_k),$$

then

$$\begin{aligned}
S_{k+1}(t) &= \int_0^t S_k(\tau)\otimes\delta b(\tau) = \int_0^t S_k(\tau)\otimes db(\tau) + \frac{1}{2}\int_0^t dS_k(\tau)\otimes db(\tau)\\
&= \int_0^t S_k(\tau)\otimes db(\tau) + \frac{1}{2}\int_0^t S_{k-1}(\tau)\otimes db(\tau)\otimes db(\tau)\\
&= \int_0^t S_k(\tau)\otimes db(\tau)
\end{aligned}$$

and the result now follows by induction on k. □

REMARK 6.27. The expansion in Equation (6.25) converges not because of some size restriction imposed on $\alpha = \hat{f}(e)$ but because f is assumed to be holomorphic on G. For example when $G = \mathbb{C}$, Equation (6.25) reduces to

$$f(b(t)) = \sum_{k=0}^{\infty} \alpha_k \cdot \int_{\Delta_k(t)} db(s_1)\dots db(s_k) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} b(t)^k \tag{6.28}$$

and the only information we can infer about the coefficients, $\alpha_k = f^{(k)}(0)$, is that $\limsup_{k\to\infty} \|\alpha_k/k!\|^{1/k} = 0$, i.e. the radius of convergence of the power series in Equation (6.28) is infinite.

COROLLARY 6.28. *If $f \in \mathcal{H}(G)$, then $\|f\|_{L^2(\lambda_t)} \le \|\hat{f}(e)\|_{J_t^0}$ and in particular, if $\hat{f} \in J_t^0$, then $f \in L^2(\lambda_t)$.*

PROOF. Let $\alpha = \hat{f}(e)$. By Fatou's lemma along with Theorem 6.26,

$$\begin{aligned}
\|f\|^2_{L^2(\lambda_t)} &= \mathbb{E}|f(g_t)|^2 = \mathbb{E}\left[\lim_{N\to\infty}\left|\sum_{k=0}^{N}\int_{\Delta_k(t)}\langle\alpha_k, db^{\otimes k}\rangle\right|^2\right]\\
&\le \liminf_{N\to\infty}\mathbb{E}\left[\left|\sum_{k=0}^{N}\int_{\Delta_k(t)}\langle\alpha_k, db^{\otimes k}\rangle\right|^2\right]\\
&= \liminf_{N\to\infty}\mathbb{E}\left[\sum_{k=0}^{N}\frac{t^k}{k!}\|\alpha_k\|^2\right] = \sum_{k=0}^{\infty}\frac{t^k}{k!}\|\alpha_k\|^2 = \|\alpha\|^2_{J_t^0}.
\end{aligned}$$ □

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