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The equivariant cohomology rings of Peterson varieties

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Abstract. The main result of this note gives an efficient presentation of the S^1-equivariant cohomology ring of Peterson varieties (in type A) as a quotient of a polynomial ring by an ideal J, in the spirit of the well-known Borel presentation of the cohomology of the flag variety. Our result simplifies previous presentations given by Harada-Tymoczko and Bayegan-Harada. In particular, our result gives an affirmative answer to a conjecture of Bayegan and Harada that the defining ideal J is generated by quadratics.

1. Introduction.

The main result of this paper is an explicit and efficient presentation of the S^1-equivariant cohomology ring[1] of type A Peterson varieties in terms of generators and relations, in the spirit of the well-known Borel presentation of the cohomology of the flag variety. Our presentation is significantly simpler than the computations given in [**6**] (respectively [**1**]) which uses the Monk formula (respectively Giambelli formula) for type A Peterson varieties. In particular, our result gives an affirmative answer to the conjecture formulated in [**1**, Remark 3.12] by showing that the defining ideal for the S^1-equivariant cohomology ring of type A Peterson varieties can be generated by quadratic polynomials.

We briefly recall the setting of our results. *Peterson varieties* in type A can be defined as the following subvariety $\mathcal{Y}$ of $\mathcal{F}\ell ags(\mathbb{C}^n)$:

$$\mathcal{Y} := \{V_\bullet \mid NV_i \subseteq V_{i+1} \text{ for all } i = 1, \dots, n-1\} \tag{1.1}$$

where $V_\bullet$ denotes a nested sequence $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n$ of subspaces of $\mathbb{C}^n$ and $\dim_{\mathbb{C}} V_i = i$ for all i and $N : \mathbb{C}^n \to \mathbb{C}^n$ denotes a principal nilpotent operator. These varieties have been much studied due to its relation to the quantum cohomology of the flag variety [**7**], [**8**]. Thus it is natural to study their topology, e.g. the structure of their (equivariant) cohomology rings.

There is a natural circle subgroup of $U(n, \mathbb{C})$ which acts on $\mathcal{Y}$ (recalled in Section 2). The inclusion of $\mathcal{Y}$ into $\mathcal{F}\ell ags(\mathbb{C}^n)$ induces a natural ring homomorphism

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[1]In this note, all cohomology rings are with coefficients in $\mathbb{C}$.

$$H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathcal{Y}) \tag{1.2}$$

where T is the subgroup of diagonal matrices of $U(n,\mathbb{C})$ acting in the usual way on $\mathcal{F}\ell ags(\mathbb{C}^n)$. The content of this manuscript is to give an efficient presentation of the equivariant cohomology ring $H^*_{S^1}(\mathcal{Y})$. Our proof uses Hilbert series and regular sequences, in a similar spirit to previous work of Fukukawa, Ishida, and Masuda [**2**], [**3**] which computes the graph cohomology of the GKM graphs of the flag varieties of classical type and of G_2.

This paper is organized as follows. We briefly recall the necessary background in Section 2. The main theorem, Theorem 3.3, is formulated in Section 3. Hilbert series and regular sequences are introduced in Section 4 to prove the main result. The proof of one key lemma used in the proof of the main theorem occupies Section 5.

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2. Peterson varieties and S^1-fixed points.

In this section we briefly recall the definitions of our main objects of study. We also record some key facts. For details and proofs, we refer the reader to [**6**]. Since we work exclusively in Lie type A, we henceforth omit it from our terminology.

Let n be a fixed positive integer which we assume throughout is ≥ 2. The flag variety $\mathcal{F}lags(\mathbb{C}^n)$ is the space of nested subspaces in $\mathbb{C}^n$, i.e.,

$$\mathcal{F}lags(\mathbb{C}^n) = \{V_\bullet = (V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i\}.$$

Let N be the $n \times n$ principal nilpotent operator which, written with respect to the standard basis of $\mathbb{C}^n$, is associated to the matrix with a single $n \times n$ Jordan block of eigenvalue 0:

$$N := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then the *Peterson variety* $\mathcal{Y}$ is the subvariety of $\mathcal{F}lags(\mathbb{C}^n)$ defined as

$$\mathcal{Y} := \{V_\bullet \in \mathcal{F}lags(\mathbb{C}^n) \mid NV_i \subset V_{i+1} \text{ for all } i = 1, 2, \dots, n-1\}. \tag{2.1}$$

The Peterson variety is a (singular) projective variety of complex dimension $n-1$.

The n-dimensional compact torus T consisting of diagonal $n \times n$ unitary matrices acts on $\mathcal{F}lags(\mathbb{C}^n)$ in a natural way. This torus action does not preserve $\mathcal{Y}$. However, we may consider the following circle subgroup of T:

$$S := \{\mathrm{diag}(g^n, g^{n-1}, \dots, g) \mid g \in \mathbb{C}, \|g\| = 1\}. \tag{2.2}$$

It is not hard to show that there is a 2-dimensional subtorus T^2 of T which acts on N by scalars (i.e. N is a T^2-weight vector) under conjugation. The above S is a subgroup of this T^2, acting by weight 1 on N. From this it immediately follows from (2.1) that S preserves Y. The S-fixed points in $\mathcal{Y}$ are the T-fixed points in $\mathcal{F}lags(\mathbb{C}^n)$ that lie in $\mathcal{Y}$, i.e.,

$$\mathcal{Y}^S = \mathcal{F}lags(\mathbb{C}^n)^T \cap \mathcal{Y}. \tag{2.3}$$

As is standard, we identify $\mathcal{F}lags(\mathbb{C}^n)^T$ with the set of permutations on n letters S_n. More specifically, it is straightforward to see that $V_\bullet$ is in $\mathcal{F}lags(\mathbb{C}^n)^T$ precisely when there exists $w \in S_n$ such that

$$V_i = \langle e_{w(1)}, e_{w(2)}, \dots, e_{w(i)} \rangle \quad \text{for } i = 1, 2, \dots, n \tag{2.4}$$

where e_k denotes the k-th element in the standard basis of $\mathbb{C}^n$. It is shown in [**6**] that a permutation $w \in S_n \cong \mathcal{F}lags(\mathbb{C}^n)^T$ is in $\mathcal{Y}^S$ precisely when the one-line notation of w^{-1} (equivalently w, since in this case $w = w^{-1}$) is of the form

$$w = \underbrace{j_1\ j_1-1 \cdots 1}_{j_1 \text{ entries}}\ \underbrace{j_2\ j_2-1 \cdots j_1+1}_{j_2 - j_1 \text{ entries}} \cdots \underbrace{n\ n-1 \cdots j_m+1}_{n - j_m \text{ entries}}, \tag{2.5}$$

where $1 \le j_1 < j_2 < \cdots < j_m < n$ is any sequence of strictly increasing integers.

3. A ring presentation of $H_S^*(\mathcal{Y})$.

In this section we formulate our main result, Theorem 3.3 below, which gives a ring presentation via generators and relations of the S-equivariant cohomology ring of $\mathcal{Y}$. Our presentation is more efficient than previous computations of this ring (cf. Remark 3.5 below).

Consider the commutative diagram

$$\begin{tikzcd} H_T^*(\mathcal{F}lags(\mathbb{C}^n)) \arrow[r, "\iota_1"] \arrow[d, "\pi_1"'] & \bigoplus_{w \in \mathcal{F}lags(\mathbb{C}^n)^T = S_n} H_T^*(w) \arrow[d, "\pi_2"] \\ H_S^*(\mathcal{Y}) \arrow[r, "\iota_2"] & \bigoplus_{w \in \mathcal{Y}^S \subset S_n} H_S^*(w) \end{tikzcd} \tag{3.1}$$

where the maps are induced from the inclusions $\mathcal{Y} \hookrightarrow \mathcal{F}lags(\mathbb{C}^n)$, $\mathcal{Y}^S \hookrightarrow \mathcal{F}lags(\mathbb{C}^n)^T$ and $S \hookrightarrow T$. Since $H^{odd}(\mathcal{F}lags(\mathbb{C}^n))$ and $H^{odd}(\mathcal{Y})$ vanish, the maps ι_1 and ι_2 above are both injective. Moreover, it is known that the map π_1 above is surjective [**6**, Theorem 4.12]. Therefore, $H_S^*(\mathcal{Y})$ is isomorphic to the image of $H_T^*(\mathcal{F}lags(\mathbb{C}^n))$ by $\pi_2 \circ \iota_1$.

Through the map $\mathrm{diag}(g^n, g^{n-1}, \dots, g) \to g$, we may identify S with the unit circle S^1 of $\mathbb{C}$ so that we have an identification

$$H_S^*(\mathrm{pt}) = H^*(BS) \cong H^*(BS^1) \cong \mathbb{C}[t].$$

The torus $T \subset U(n)$ of diagonal unitary matrices has a natural product decomposition $T \cong (S^1)^n$. This decomposition identifies BT with $(BS^1)^n$ and induces an identification

$$H_T^*(\mathrm{pt}) = H^*(BT) \cong \bigotimes_{i=1}^n H^*(BS^1) \cong \mathbb{C}[t_1, \ldots, t_n],$$

where t_i $(i = 1, 2, \ldots, n)$ denotes the element corresponding to the fixed generator t of $H^2(BS^1)$. Then from the explicit description of S as the subgroup $\mathrm{diag}(g^n, g^{n-1}, \ldots, g)$ of T it readily follows that

$$\pi_2(t_i) = (n + 1 - i)t. \tag{3.2}$$

We now briefly recall a well-known ring presentation of the equivariant cohomology ring $H_T^*(\mathcal{F}lags(\mathbb{C}^n))$. There is a tautological filtration of the trivial rank n vector bundle over $\mathcal{F}lags(\mathbb{C}^n)$

$$\mathcal{F}lags(\mathbb{C}^n) \times \{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U_n = \mathcal{F}lags(\mathbb{C}^n) \times \mathbb{C}^n$$

where the fiber of U_i over a point in $\mathcal{F}lags(\mathbb{C}^n)$ corresponding to a flag $V_\bullet$ is precisely the vector space V_i of $V_\bullet$. The bundles U_i are all T-equivariant and the quotient bundles U_i/U_{i-1} $(i = 1, 2, \ldots, n)$ are T-equivariant line bundles over $\mathcal{F}lags(\mathbb{C}^n)$. We define

$$\tau_i := c_1^T(U_i/U_{i-1}) \in H_T^2(\mathcal{F}lags(\mathbb{C}^n)) \quad \text{for } i = 1, 2, \ldots, n, \tag{3.3}$$

where c_1^T denotes the equivariant first Chern class. Then it is known (see e.g. [**4**, Equation (2.1)] or [**2**]) that

$$H_T^*(\mathcal{F}lags(\mathbb{C}^n)) = \mathbb{C}[\tau_1, \ldots, \tau_n, t_1, \ldots, t_n]/I \tag{3.4}$$

where I is the ideal generated by

$$e_i(\tau) - e_i(t) \quad \text{for } i = 1, 2, \ldots, n$$

and $e_i(\tau)$ (resp. $e_i(t)$) denotes the i-th elementary symmetric polynomial in the variables $\tau_1, \ldots, \tau_n$ (resp. $t_1, \ldots, t_n$). By slight abuse of notation, in the discussion below we denote by τ_i, t_i the corresponding cohomology classes in $H_T^*(\mathcal{F}lags(\mathbb{C}^n))$ (i.e. the equivalence classes of τ_i, t_i in the quotient ring $\mathbb{C}[\tau_1, \ldots, \tau_n,\ t_1, \ldots, t_n]/I$).

It follows from (2.4) and (3.3) that for $w \in S_n$ we have

$$\iota_1(\tau_i)|_w = t_{w(i)} \tag{3.5}$$

and clearly

$$\iota_1(t_i)|_w = t_i, \tag{3.6}$$

where $*|_w$ denotes the w-th component of $*$ in the direct sum $\bigoplus_{w\in S_n} H_T^*(w)$.

We define

$$p_k := \pi_1\left(\sum_{i=1}^{k}(t_i - \tau_i)\right) \quad \text{for } k = 1, \ldots, n. \tag{3.7}$$

The following lemma computes the images of the p_k in $H_S^*(\mathcal{Y}^S) \cong \bigoplus_{w\in\mathcal{Y}^S} H_S^*(w)$ under the map ι_2 in (3.1).

Lemma 3.1. *Let $p_k \in H_S^*(\mathcal{Y})$ for $1 \leq k \leq n$ be defined as above. Then*

$$\iota_2(p_k)|_w = \sum_{i=1}^{k}(w(i) - i)t.$$

Proof. For $w \in \mathcal{Y}^S$ and $1 \leq k \leq n$ we have

$$\begin{aligned}
\iota_2(p_k)|_w &= \iota_2\left(\pi_1\left(\sum_{i=1}^{k}(t_i-\tau_i)\right)\right)\Bigg|_w && \text{by definition of } p_k \\
&= \pi_2\left(\iota_1\left(\sum_{i=1}^{k}(t_i-\tau_i)\right)\right)\Bigg|_w && \text{by commutativity of (3.1)} \\
&= \pi_2\left(\sum_{i=1}^{k}(t_i - t_{w(i)})\right) && \text{by (3.5) and (3.6)} \\
&= \sum_{i=1}^{k}(w(i)-i)t && \text{by (3.2)}
\end{aligned} \tag{3.8}$$

as desired. □

Since $\sum_{i=1}^{n} i = \sum_{i=1}^{n} w(i)$, Lemma 3.1 immediately implies that $\iota_2(p_n)|_w = 0$ for any $w \in \mathcal{Y}^S$. In particular we may conclude

$$p_n = 0 \tag{3.9}$$

since ι_2 is injective.

The next lemma derives some relations which are satisfied among the elements $p_k \in H_S^*(\mathcal{Y})$. By slight abuse of notation we denote also by t the element in $H_S^*(\mathcal{Y})$ which is the image of $t \in H_S^*(\mathrm{pt})$ under the canonical map $H_S^*(\mathrm{pt}) \to H_S^*(\mathcal{Y})$. Note that $\iota_2(t)|_w = t$ for all $w \in \mathcal{Y}^S$.

Lemma 3.2. *Let $p_k \in H_S^*(\mathcal{Y})$ for $1 \leq k \leq n$ be defined as above. Then $p_k(p_k - (1/2)p_{k-1} - (1/2)p_{k+1} - t) = 0$ for $k = 1, 2, \ldots, n-1$.*

Proof. Let $w \in \mathcal{Y}^S \subset S_n$. It follows from Lemma 3.1 that

$$\begin{aligned}
&\iota_2\left(p_k - \frac{1}{2}p_{k-1} - \frac{1}{2}p_{k+1} - t\right)\Big|_w \\
&= \sum_{i=1}^{k}(w(i)-i)t - \frac{1}{2}\sum_{i=1}^{k-1}(w(i)-i)t - \frac{1}{2}\sum_{i=1}^{k+1}(w(i)-i)t - t \\
&= \frac{1}{2}(w(k) - w(k+1) - 1)t. \qquad (3.10)
\end{aligned}$$

Since w is in $\mathcal{Y}^S$, we know it must be of the form given in (2.5). If $k = j_q$ for some $1 \leq q \leq m$, then $\sum_{i=1}^k i = \sum_{i=1}^k w(i)$. Otherwise, $w(k+1) = w(k) - 1$. Therefore, for any $w \in \mathcal{Y}^S$ and for any k, either (3.8) or (3.10) vanishes. This implies the lemma because ι_2 is injective. □

Our main result states that the relations given in Lemma 3.2 are enough to determine the ring structure.

Theorem 3.3. *Let n be a positive integer, $n \geq 2$. Let $\mathcal{Y} \subseteq \mathcal{F}lags(\mathbb{C}^n)$ be the Peterson variety defined in* (2.1). *Let the circle group S act on $\mathcal{Y}$ as described in Section* 2. *Then the S-equivariant cohomology ring of $\mathcal{Y}$ can be presented by generators and relations as follows*:

$$H_S^*(\mathcal{Y}) \cong \mathbb{C}[t, p_1, \ldots, p_{n-1}]/J,$$

where J is the ideal generated by the quadratic polynomials

$$p_k\left(p_k - \frac{1}{2}p_{k-1} - \frac{1}{2}p_{k+1} - t\right) \quad \textit{for } k = 1, 2, \ldots, n-1 \qquad (3.11)$$

where we take $p_0 = p_n = 0$.

Since $H^{odd}(\mathcal{Y}) = 0$ and $H_S^*(\mathcal{Y}) = H^*(BS) \otimes H^*(\mathcal{Y})$ as $H^*(BS)$-modules, we also obtain the following corollary.

Corollary 3.4. *Let $\check{p}_k$ be the restriction of p_k to $H^*(\mathcal{Y})$. Then*

$$H^*(\mathcal{Y}) = \mathbb{C}[\check{p}_1, \ldots, \check{p}_{n-1}]/\check{J},$$

where $\check{J}$ is the ideal generated by

$$\check{p}_k\left(\check{p}_k - \frac{1}{2}\check{p}_{k-1} - \frac{1}{2}\check{p}_{k+1}\right) \quad \textit{for } k = 1, 2, \ldots, n-1$$

with $\check{p}_0 = \check{p}_n = 0$.

REMARK 3.5. In [**1**] it is also shown that the equivariant cohomology ring $H^*_S(\mathcal{Y})$ is a quotient of a polynomial ring $\mathbb{C}[t, p_1, \ldots, p_{n-1}]$ by an ideal generated by polynomials denoted as $q_{i,\mathcal{A}}$ [**1**, Theorem 3.8]. The ring presentation in [**1**] is a simplification of the presentation given in [**6**, Theorem 6.12 and Corollary 6.14] by decreasing both the number of variables in the polynomial ring and the number of generators of the ideal of relations. In fact, it was conjectured in [**1**, Remark 3.12] that things could be made even simpler, namely, that the ideal of relations for the presentation in [**1**, Theorem 3.8] is in fact generated by just the quadratics. Our Theorem 3.3 proves that this is in fact the case. Indeed, it is straightforward to see from the definitions in [**1**], [**6**] that the polynomials (3.11) in Theorem 3.3 correspond to the $q_{i,\mathcal{A}}$ for the special case $\mathcal{A} = \{i\}$. These are precisely the quadratic polynomials among all the $q_{i,\mathcal{A}}$.

4. Hilbert series and regular sequences.

In this section we prove our main result, Theorem 3.3, modulo one key lemma whose proof we postpone to Section 5.

Since the map π_1 in the diagram (3.1) is known to be surjective, it follows from (3.4), (3.7), (3.9) and Lemma 3.2 that the natural homomorphism of graded rings

$$\varphi \colon \mathbb{C}[t, p_1, \ldots, p_{n-1}]/J \twoheadrightarrow H^*_S(\mathcal{Y}) \tag{4.1}$$

is surjective. Forgetting the S-action, this induces a surjective homomorphism of graded rings

$$\check{\varphi} \colon \mathbb{C}[\check{p}_1, \ldots, \check{p}_{n-1}]/\check{J} \twoheadrightarrow H^*(\mathcal{Y}). \tag{4.2}$$

We next recall the definition of Hilbert series. Suppose $A^* = \bigoplus_{i=0}^{\infty} A^i$ is a graded module over $\mathbb{C}$. Then its associated Hilbert series $F(A^*, s)$ is defined to be the formal power series

$$F(A^*, s) := \sum_{i=0}^{\infty} (\dim_{\mathbb{C}} A^i) s^i.$$

When comparing Hilbert series of different rings, we use the notation $\sum a_i s^i \geq \sum b_i s^i$ to mean that $a_i \geq b_i$ for all i.

In our setting, taking the Hilbert series of both rings appearing in (4.1) and (4.2) yields

$$F(\mathbb{C}[t, p_1, \ldots, p_{n-1}]/J, s) \geq F(H^*_S(\mathcal{Y}), s) \tag{4.3}$$

$$F(\mathbb{C}[\check{p}_1, \ldots, \check{p}_{n-1}]/\check{J}, s) \geq F(H^*(\mathcal{Y}), s) \tag{4.4}$$

since both φ and $\check{\varphi}$ are surjective. Note that φ (resp. $\check{\varphi}$) is an isomorphism if and only if the inequality in (4.3) (resp. (4.4)) is in fact an equality.

The Hilbert series of the right hand sides of (4.3) and (4.4) are known to be as follows. It is shown in [**10**] that

$$F(H^*(\mathcal{Y}), s) = (1+s^2)^{n-1}. \tag{4.5}$$

Moreover, since $H^*_S(\mathcal{Y}) = H^*(BS) \otimes H^*(\mathcal{Y})$ as $H^*(BS)$-modules, (4.5) implies

$$F(H^*_S(\mathcal{Y}), s) = \frac{(1+s^2)^{n-1}}{1-s^2}. \tag{4.6}$$

The following lemma computes the left hand side of (4.4). Its proof will be given in Section 5 in a more general setting.

Lemma 4.1. $F(\mathbb{C}[\check{p}_1, \dots, \check{p}_{n-1}]/\check{J}, s) = (1+s^2)^{n-1}$.

Assuming Lemma 4.1, we now complete the proof of Theorem 3.3. For this we use the following notion from commutative algebra (see e.g. [**9**]).

Definition. Let R be a graded commutative algebra over $\mathbb{C}$ and let R_+ denote the positive-degree elements in R. Then a homogeneous sequence $\theta_1, \dots, \theta_r \in R_+$ is a *regular sequence* if θ_k is a non-zero-divisor in the quotient ring $R/(\theta_1, \dots, \theta_{k-1})$ for every $1 \le k \le r$. This is equivalent to saying that $\theta_1, \dots, \theta_r$ is algebraically independent over $\mathbb{C}$ and R is a free $\mathbb{C}[\theta_1, \dots, \theta_r]$-module.

It is a well-known fact (see for instance [**9**, p. 35]) that a homogeneous sequence $\theta_1, \dots, \theta_r \in R_+$ is a regular sequence if and only if

$$F(R/(\theta_1, \dots, \theta_r), s) = F(R, s) \prod_{k=1}^{r} (1 - s^{\deg \theta_k}). \tag{4.7}$$

A sketch of the proof of this fact is as follows. Let $\theta_1, \dots, \theta_r$ be a homogeneous sequence of R and set $R_k := R/(\theta_1, \dots, \theta_k)$ for $1 \le k \le r$. Consider the exact sequence

$$R_{k-1} \xrightarrow{\times \theta_k} R_{k-1} \to R_k \to 0 \quad \text{for } 1 \le k \le r,$$

where $\times\theta_k$ denotes multiplication by θ_k, the map $R_{k-1} \to R_k$ is the quotient map and $R_0 := R$. The regularity of the sequence $\theta_1, \dots, \theta_r$ implies that the map $\times\theta_k$ is injective for every $1 \le k \le r$, which in turn implies

$$F(R_k, s) = F(R_{k-1}, s)(1 - s^{\deg \theta_k}) \quad \text{for any } 1 \le k \le r.$$

The desired fact then follows.

Returning to our setting, we have the following lemma.

Lemma 4.2. *In the polynomial ring* $\mathbb{C}[t, p_1, \dots, p_{n-1}]$*, the sequence*

$$\theta_k := p_k \left(p_k - \frac{1}{2} p_{k-1} - \frac{1}{2} p_{k+1} - t \right) \quad \textit{for } 1 \le k \le n-1,$$

$$\theta_n := t.$$

is regular.

Proof. Since $\theta_n = t$, from the definitions of θ_k and the ideals J and $\check{J}$ given in the statements of Theorem 3.3 and Corollary 3.4 it follows that

$$\begin{aligned}
&F(\mathbb{C}[t, p_1, \ldots, p_{n-1}]/(\theta_1, \ldots, \theta_{n-1}, \theta_n), s) \\
&\quad = F(\mathbb{C}[\check{p}_1, \ldots, \check{p}_{n-1}]/\check{J}, s) \\
&\quad = (1+s^2)^{n-1}
\end{aligned}$$

where the last equality follows from Lemma 4.1. This implies that (4.7) is satisfied in our setting because $\deg \theta_i = 4$ for $1 \le i \le n-1$, $\deg \theta_n = 2$ and

$$F(\mathbb{C}[t, p_1, \ldots, p_{n-1}], s) = \frac{1}{(1-s^2)^n}. \tag{4.8}$$

The result follows. □

We can now prove the main theorem.

Proof of Theorem 3.3. From the definition of a regular sequence it is clear that the subsequence $\theta_1, \ldots, \theta_{n-1}$ of a regular sequence $\theta_1, \ldots, \theta_n$ is again a regular sequence. Hence it follows from (4.7) and (4.8) that

$$\begin{aligned}
F(\mathbb{C}[t, p_1, \ldots, p_{n-1}]/J, s) &= F(\mathbb{C}[t, p_1, \ldots, p_{n-1}]/(\theta_1, \ldots, \theta_{n-1}), s) \\
&= \frac{1}{(1-s^2)^n} \prod_{k=1}^{n-1} (1 - s^{\deg \theta_k}) \\
&= \frac{(1+s^2)^{n-1}}{1-s^2}.
\end{aligned}$$

This together with (4.6) shows that the equality holds in (4.3). Hence the map φ in (4.1) is an isomorphism, as desired. □

5. Proof of Lemma 4.1.

This section is devoted to the proof of Lemma 4.1. Note first that Lemma 4.1 is equivalent to the statement that the sequence of homogeneous elements

$$\check{p}_k \left(\check{p}_k - \frac{1}{2}\check{p}_{k-1} - \frac{1}{2}\check{p}_{k+1} \right) \quad (k = 1, 2, \ldots, n-1),$$

(where $\check{p}_0 = \check{p}_n$ are both defined to be 0) is a regular sequence in the polynomial ring $\mathbb{C}[\check{p}_1, \ldots, \check{p}_{n-1}]$. We now recall a criterion which characterizes when such a homogenous sequence in a polynomial ring is regular. We learned this criterion from S. Murai.

Proposition 5.1. *A sequence of positive-degree homogeneous elements $\theta_1, \dots, \theta_r$ in the polynomial ring $\mathbb{C}[z_1, \dots, z_r]$ is a regular sequence if and only if the solution set in $\mathbb{C}^r$ of the equations $\theta_1 = 0, \dots, \theta_r = 0$ consists only of the origin $\{0\}$.*

Proof. First we claim that the homogeneous sequence $\theta_1, \dots, \theta_r$ is regular if and only if the Krull dimension of $\mathbb{C}[z_1, \dots, z_r]/(\theta_1, \dots, \theta_r)$ is zero. To see this, observe that by definition, if $\theta_1, \dots, \theta_r$ is a regular sequence then the $\theta_1, \dots, \theta_r$ are algebraically independent. This implies that the Krull dimension of $\mathbb{C}[z_1, \dots, z_r]/(\theta_1, \dots, \theta_r)$ is zero (note that the number of generators of the polynomial ring $\mathbb{C}[z_1, \dots, z_r]$ is equal to the length of the regular sequence). In the other direction, if $\mathbb{C}[z_1, \dots, z_r]/(\theta_1, \dots, \theta_r)$ has Krull dimension 0, then the $\theta_1, \dots, \theta_r$ are a homogeneous system of parameters for $\mathbb{C}[z_1, \dots, z_r]$ [**9**, Definition 5.1]. Moreover, since the polynomial ring $\mathbb{C}[z_1, \dots, z_r]$ is Cohen-Macaulay, by [**9**, Theorem 5.9] we may conclude that the homogeneous system of parameters $\theta_1, \dots, \theta_r$ is a regular sequence.

Next we observe that by Hilbert's Nullstellensatz the quotient ring

$$\mathbb{C}[z_1, \dots, z_r]/(\theta_1, \dots, \theta_r)$$

has Krull dimension 0 if and only if the algebraic set in $\mathbb{C}^r$ defined by the equations $\theta_1 = 0, \dots, \theta_r = 0$ is zero-dimensional. Since the polynomials $\theta_1, \dots, \theta_r$ are assumed to be homogeneous, the corresponding zero-dimensional algebraic set in $\mathbb{C}^r$ must consist of only the origin. This proves the proposition. □

By Proposition 5.1, in order to prove Lemma 4.1 it suffices to check that the solution set in $\mathbb{C}^r$ of the equations

$$z_i^2 = \frac{1}{2} z_i (z_{i-1} + z_{i+1}) \quad (i = 1, 2, \dots, r) \tag{5.1}$$

(where $z_0 = z_{r+1} = 0$) consists of only the origin. To prove this, we consider a more general set of equations in $\mathbb{C}^r$ $(r \geq 2)$, namely:

$$\begin{aligned} z_1^2 &= b_1 z_1 z_2 \\ z_i^2 &= z_i (a_{i-1} z_{i-1} + b_i z_{i+1}) \quad (i = 2, \dots, r-1) \\ z_r^2 &= a_{r-1} z_{r-1} z_r \end{aligned} \tag{5.2}$$

where a_i, b_i for $i = 1, 2, \dots, r-1$ are fixed complex numbers.

Lemma 5.2. *In the setting above, set $c_i := a_i b_i$ for $i = 1, 2, \dots, r-1$. If*

$$1 - \cfrac{c_i}{1 - \cfrac{c_{i+1}}{\ddots \over 1 - \cfrac{c_{j-1}}{1 - c_j}}} \neq 0 \tag{5.3}$$

for all $1 \leq i \leq j \leq r-1$*, then the solution set of the equations* (5.2) *consists of only the origin in* $\mathbb{C}^r$*.*

Proof. We prove the lemma by induction on r, the number of variables. It is easy to check the lemma directly for the base case $r = 2$. Now suppose that $r \geq 3$ and the result of the lemma holds for $r-1$. Note that the equations in (5.2) which involve the variable z_r are the two equations

$$\begin{aligned} z_{r-1}^2 &= z_{r-1}(a_{r-2}z_{r-2} + b_{r-1}z_r) \\ z_r^2 &= a_{r-1}z_{r-1}z_r. \end{aligned}$$

From the latter equation we can conclude that either $z_r = 0$ or $z_r = a_{r-1}z_{r-1}$.

Now we take cases. Suppose $z_r = 0$. Then the equations (5.2) become

$$\begin{aligned} z_1^2 &= b_1 z_1 z_2 \\ z_i^2 &= z_i(a_{i-1}z_{i-1} + b_i z_{i+1}) \quad (i = 2, \ldots, r-2) \\ z_{r-1}^2 &= a_{r-2}z_{r-2}z_{r-1}. \end{aligned}$$

By the induction assumption, the solution set of these equations consists of only the origin since (5.3) is satisfied for all $1 \leq i \leq j \leq r-2$.

Next suppose $z_r = a_{r-1}z_{r-1}$. In this case the equations (5.2) turn into

$$\begin{aligned} z_1^2 &= b_1 z_1 z_2 \\ z_i^2 &= z_i(a_{i-1}z_{i-1} + b_i z_{i+1}) \quad (i = 2, \ldots, r-2) \\ z_{r-1}^2 &= \frac{a_{r-2}}{1 - a_{r-1}b_{r-1}} z_{r-2}z_{r-1}. \end{aligned} \tag{5.4}$$

Here we know that $1 - a_{r-1}b_{r-1} \neq 0$ from the condition (5.3) with $i = j = r-1$. Again by the induction assumption, the solution set of the equations (5.4) consists of only the origin if

$$1 - \cfrac{c'_i}{1 - \cfrac{c'_{i+1}}{\cfrac{\ddots}{1 - \cfrac{c'_{j-1}}{1 - c'_j}}}} \neq 0 \quad \text{for all } 1 \leq i \leq j \leq r-2, \tag{5.5}$$

where

$$c'_k = c_k \ (1 \leq k \leq r-3), \qquad c'_{r-2} = \frac{c_{r-2}}{1 - c_{r-1}}.$$

From the definition of the c_k' it is clear that (5.5) is equivalent to (5.3) for i and j with $1 \le i \le j \le r-3$. Further, the case $i = j = r-2$ of (5.5) follows from the $i = r-2$, $j = r-1$ case of (5.3), and the case $i < j = r-2$ of (5.5) follows from the $i \le j = r-1$ case of (5.3). Hence (5.5) holds for all choices of i and j and by the induction assumption the solution set consists of only the origin, as desired. □

Remark 5.3. It is not difficult to see that the "only if" part of Lemma 5.2 also holds, but we do not need this implication in what follows.

We now return to our special case, for which $a_i = b_i = 1/2$ and hence $c_i = 1/4$ for all $1 \le i \le r-1$. Below, we give a sufficient condition for (5.3) to be satisfied when $c_i = a_i b_i$ $(i = 1, 2, \dots, r-1)$ is a constant c independent of i. This will suffice to prove Lemma 4.1. For this purpose, consider the numerical sequence $\{x_m\}_{m=0}^{\infty}$ defined by the following recurrence relation and with $x_0 = 1$:

$$x_m = 1 - \frac{c}{x_{m-1}} \quad \text{for } m \ge 1. \tag{5.6}$$

In the situation when the c_i are all equal, it is straightforward to see that the condition (5.3) is equivalent to the statement that $x_m \neq 0$ for $m = 1, 2, \dots, r-1$. We have the following.

Lemma 5.4. *Let $\{x_m\}$ be the sequence defined in* (5.6). *Then*:

1. *if $0 \le c \le 1/4$, then $x_m \ge (1+\sqrt{1-4c})/2$ for any $m \ge 1$, and*
2. *if $c < 0$, then $x_m \ge 1$ for any $m \ge 1$.*

In particular, if c is any real number $\le 1/4$, then $x_m > 0$ for all $m \ge 1$.

Proof. Let $0 \le c \le 1/4$ and suppose that

$$x_{m-1} \ge \frac{1+\sqrt{1-4c}}{2} > 0 \quad \text{for some } m \ge 1. \tag{5.7}$$

Then it follows from (5.6) and (5.7) that

$$x_m = 1 - \frac{c}{x_{m-1}} \ge 1 - \frac{2c}{1+\sqrt{1-4c}} = \frac{1+\sqrt{1-4c}}{2}.$$

This proves (1) in the lemma since the inequality (5.7) is satisfied for $m = 1$. A similar argument proves (2). □

The proof of Lemma 4.1 is now straightforward.

Proof of Lemma 4.1. The statement of Lemma 4.1 follows from Proposition 5.1, Lemma 5.2 and Lemma 5.4. □

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