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Sum formula for finite multiple zeta values

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Abstract. The sum formula is one of the most well-known relations among multiple zeta values. This paper proves a conjecture of Kaneko predicting that an analogous formula holds for finite multiple zeta values.

1. Introduction.

1.1. Finite multiple zeta values.

The *multiple zeta values* (MZVs) and *multiple zeta-star values* (MZSVs) are defined by

$$\zeta(k_1,\dots,k_n)=\sum_{m_1>\dots>m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}},$$
$$\zeta^\star(k_1,\dots,k_n)=\sum_{m_1\geq\dots\geq m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}$$

for $k_1,\dots,k_n\in\mathbb{Z}_{\geq 1}$ with $k_1\geq 2$. They are both generalizations of the Riemann zeta values $\zeta(k)$ at positive integers.

Among a large number of variants of the MZ(S)Vs, there has recently been growing interest in *finite multiple zeta(-star) values* (FMZ(S)Vs). Set $\mathcal{A}=(\prod_p\mathbb{Z}/p\mathbb{Z})/(\bigoplus_p\mathbb{Z}/p\mathbb{Z})$, where p runs over all primes; in other words, the elements of $\mathcal{A}$ are of the form $(a_p)_p$, where $a_p\in\mathbb{Z}/p\mathbb{Z}$, and two elements (a_p) and (b_p) are identified if and only if $a_p=b_p$ for all but finitely many primes p. We shall simply write a_p for (a_p) since no confusion is likely. The following definition is due to Zagier (see [**6**]):

Definition 1.1. For $k_1,\dots,k_n\in\mathbb{Z}_{\geq 1}$, we define

$$\zeta_{\mathcal{A}}(k_1,\dots,k_n)=\sum_{p>m_1>\dots>m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}\in\mathcal{A},$$
$$\zeta_{\mathcal{A}}^\star(k_1,\dots,k_n)=\sum_{p>m_1\geq\dots\geq m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}\in\mathcal{A}$$

and call them *finite multiple zeta*(*-star*) *values.*

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We spell out two easy properties of FMZ(S)Vs that will be used later; see Theorems 4.3 and 6.1 in [**3**] for the proofs. See also [**6**], [**8**] and the introduction of [**7**].

PROPOSITION 1.2. (1) *We have* $\zeta_{\mathcal{A}}(k)=0$ *for all* $k\in\mathbb{Z}_{\geq 1}$.
(2) *For* $k_1,k_2\in\mathbb{Z}_{\geq 1}$, *we have*

$$\zeta_{\mathcal{A}}(k_1,k_2)=\zeta^{\star}_{\mathcal{A}}(k_1,k_2)=(-1)^{k_1}\binom{k_1+k_2}{k_1}\frac{B_{p-k_1-k_2}}{k_1+k_2}.$$

Here the numbers B_m are the Bernoulli numbers given by

$$\sum_{m=0}^{\infty}B_m\frac{x^m}{m!}=\frac{x}{1-e^{-x}}\in\mathbb{Q}[[x]].$$

1.2. Sum formula.

The sum formula is a basic class of relations among MZ(S)Vs and has been generalized in various directions. For $k,n\in\mathbb{Z}$ with $1\leq n\leq k-1$, set

$$I_{k,n}=\{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geq 1}\mid k_1+\cdots+k_n=k,\ k_1\geq 2\}.$$

THEOREM 1.3 (Sum formula [**1**], [**2**]). *For* $k,n\in\mathbb{Z}$ *with* $1\leq n\leq k-1$, *we have*

$$\sum_{(k_1,\dots,k_n)\in I_{k,n}}\zeta(k_1,\dots,k_n)=\zeta(k),$$

$$\sum_{(k_1,\dots,k_n)\in I_{k,n}}\zeta^{\star}(k_1,\dots,k_n)=\binom{k-1}{n-1}\zeta(k).$$

Kaneko [**5**] conjectured the following analogous relations for FMZ(S)Vs:

$$\sum_{(k_1,\dots,k_n)\in I_{k,n}}\zeta_{\mathcal{A}}(k_1,\dots,k_n)=\left(1+(-1)^n\binom{k-1}{n-1}\right)\frac{B_{p-k}}{k},$$

$$\sum_{(k_1,\dots,k_n)\in I_{k,n}}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_n)=\left((-1)^n+\binom{k-1}{n-1}\right)\frac{B_{p-k}}{k}.$$

The aim of this paper is to prove the conjecture and its generalizations given below.

For $k,n,i\in\mathbb{Z}$ with $1\leq i\leq n\leq k-1$, set

$$I_{k,n,i}=\{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geq 1}\mid k_1+\cdots+k_n=k,\ k_i\geq 2\};$$

note that $I_{k,n,1}=I_{k,n}$.

THEOREM 1.4 (Main theorem). *For* $k,n,i\in\mathbb{Z}$ *with* $1\leq i\leq n\leq k-1$, *we have*

$$\sum_{(k_1,\dots,k_n)\in I_{k,n,i}} \zeta_{\mathcal{A}}(k_1,\dots,k_n) = (-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^n\binom{k-1}{n-i}\right)\frac{B_{p-k}}{k},$$

$$\sum_{(k_1,\dots,k_n)\in I_{k,n,i}} \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_n) = (-1)^{i-1}\left((-1)^n\binom{k-1}{i-1} + \binom{k-1}{n-i}\right)\frac{B_{p-k}}{k}.$$

Setting $i=1$ gives Kaneko's conjecture.

2. Proof of the main theorem.

For notational simplicity, we write the sums to be computed as

$$S_{k,n,i} = \sum_{(k_1,\dots,k_n)\in I_{k,n,i}} \zeta_{\mathcal{A}}(k_1,\dots,k_n), \qquad S^{\star}_{k,n,i} = \sum_{(k_1,\dots,k_n)\in I_{k,n,i}} \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_n)$$

for $k,n,i\in\mathbb{Z}$ with $1\le i\le n\le k-1$.

2.1. Recurrence relations.

We begin the proof by establishing recurrence relations for $S_{k,n,i}$ and $S^{\star}_{k,n,i}$. We will show the recurrence relations by expressing products of FMZ(S)Vs as sums of FMZ(S)Vs via the *harmonic product* (see [**4**]). Since explaining the harmonic product in its full generality is unnecessarily cumbersome, we shall only illustrate it by examples. If $k_1,k_2,l\in\mathbb{Z}_{\ge1}$, then Proposition 1.2 (1) shows that

$$\begin{aligned}
0 &= \zeta_{\mathcal{A}}(k_1,k_2)\zeta_{\mathcal{A}}(l)\\
&= \left(\sum_{m_1>m_2}\frac{1}{m_1^{k_1}m_2^{k_2}}\right)\left(\sum_{m}\frac{1}{m^l}\right)\\
&= \left(\sum_{m>m_1>m_2} + \sum_{m_1>m>m_2} + \sum_{m_1>m_2>m} + \sum_{m_1=m>m_2} + \sum_{m_1>m_2=m}\right)\frac{1}{m_1^{k_1}m_2^{k_2}m^l}\\
&= \zeta_{\mathcal{A}}(l,k_1,k_2)+\zeta_{\mathcal{A}}(k_1,l,k_2)+\zeta_{\mathcal{A}}(k_1,k_2,l)+\zeta_{\mathcal{A}}(k_1+l,k_2)+\zeta_{\mathcal{A}}(k_1,k_2+l),
\end{aligned}$$

where m_1, m_2, and m are all assumed to be positive integers less than p, and similarly that

$$0 = \zeta^{\star}_{\mathcal{A}}(l,k_1,k_2)+\zeta^{\star}_{\mathcal{A}}(k_1,l,k_2)+\zeta^{\star}_{\mathcal{A}}(k_1,k_2,l)-\zeta^{\star}_{\mathcal{A}}(k_1+l,k_2)-\zeta^{\star}_{\mathcal{A}}(k_1,k_2+l).$$

An analogous procedure leads to the following lemma:

LEMMA 2.1. *For $n\in\mathbb{Z}_{\ge2}$ and $k_1,\dots,k_{n-1},l\in\mathbb{Z}_{\ge1}$, we have*

$$\sum_{j=1}^{n}\zeta_{\mathcal{A}}(k_1,\dots,k_{j-1},l,k_j,\dots,k_{n-1}) + \sum_{j=1}^{n-1}\zeta_{\mathcal{A}}(k_1,\dots,k_{j-1},k_j+l,k_{j+1},\dots,k_{n-1}) = 0,$$

$$\sum_{j=1}^{n} \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_{j-1},l,k_j,\dots,k_{n-1}) - \sum_{j=1}^{n-1} \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_{j-1},k_j+l,k_{j+1},\dots,k_{n-1}) = 0.$$

Proof. Expand the left-hand sides of $\zeta_{\mathcal{A}}(k_1,\dots,k_{n-1})\zeta_{\mathcal{A}}(l) = 0$ and $\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_{n-1})\zeta^{\star}_{\mathcal{A}}(l) = 0$. □

Proposition 2.2 (Recurrence relations). *For $k,n,i \in \mathbb{Z}$ with $2 \le i+1 \le n \le k-1$, we have*

$$(n-i)S_{k,n,i} + iS_{k,n,i+1} + (k-n)S_{k,n-1,i} = 0,$$
$$(n-i)S^{\star}_{k,n,i} + iS^{\star}_{k,n,i+1} - (k-n)S^{\star}_{k,n-1,i} = 0.$$

Proof. Summing the equations in Lemma 2.1 over all $(k_1,\dots,k_{n-1},l) \in I_{k,n,i}$ gives the desired recurrence relations. Indeed, the map

$$(k_1,\dots,k_{n-1},l) \mapsto (k_1,\dots,k_{j-1},l,k_j,\dots,k_{n-1})$$

defined on $I_{k,n,i}$ is a bijection onto $I_{k,n,i+1}$ for $j = 1,\dots,i$ and onto $I_{k,n,i}$ for $j = i+1,\dots,n$; under the map

$$\big((k_1,\dots,k_{n-1},l),j\big) \mapsto (k_1,\dots,k_{j-1},k_j+l,k_{j+1},\dots,k_{n-1})$$

from $I_{k,n,i} \times \{1,\dots,n-1\}$ to $I_{k,n-1,i}$, the preimage of each $(k'_1,\dots,k'_{n-1}) \in I_{k,n-1,i}$ is of cardinality

$$\sum_{\substack{1\le j\le n-1\\ j\ne i}} (k'_j-1) + (k'_i-2) = k-n. \qquad \square$$

2.2. Computation of $S^{\star}_{k,n,i}$.

Lemma 2.3 (Initial values). *For $k,i \in \mathbb{Z}$ with $1 \le i \le k-1$, we have*

$$S^{\star}_{k,k-1,i} = (-1)^{i-1}\binom{k}{i}\frac{B_{p-k}}{k}.$$

Proof. By the duality theorem for FMZSVs [**3**, Theorem 4.6] and Proposition 1.2 (2), we find that

$$S^{\star}_{k,k-1,i} = \zeta^{\star}_{\mathcal{A}}(\underbrace{1,\dots,1}_{i-1},2,\underbrace{1,\dots,1}_{k-i-1}) = -\zeta^{\star}_{\mathcal{A}}(i,k-i) = (-1)^{i-1}\binom{k}{i}\frac{B_{p-k}}{k}. \qquad \square$$

Proposition 2.4. *For $k,n,i \in \mathbb{Z}$ with $1 \le i \le n \le k-1$, we have*

$$S^{\star}_{k,n,i} = (-1)^{i-1}\left((-1)^n\binom{k-1}{i-1} + \binom{k-1}{n-i}\right)\frac{B_{p-k}}{k}.$$

PROOF. The proof is by backward induction on n.

We first consider the case $n = k-1$. If k is even, then the identity trivially follows from Lemma 2.3 because $B_{p-k} = 0$ (in $\mathbb{Q}$ and so in $\mathbb{Z}/p\mathbb{Z}$ as well) whenever p is a prime at least $k+3$. If k is odd, then the identity again follows from Lemma 2.3 because

$$(-1)^n\binom{k-1}{i-1}+\binom{k-1}{n-i}=\binom{k-1}{i-1}+\binom{k-1}{k-i-1}=\binom{k}{i}.$$

Now assume that the identity holds for n. Then Proposition 2.2 shows that

$$\begin{aligned}(k-n)S^{\star}_{k,n-1,i} &= (n-i)S^{\star}_{k,n,i}+iS^{\star}_{k,n,i+1}\\ &= (n-i)(-1)^{i-1}\left((-1)^n\binom{k-1}{i-1}+\binom{k-1}{n-i}\right)\frac{B_{p-k}}{k}\\ &\quad + i(-1)^i\left((-1)^n\binom{k-1}{i}+\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k}\\ &= (-1)^{i-1}\left((n-i)(-1)^n\binom{k-1}{i-1}+(k-n+i)\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k}\\ &\quad + (-1)^i\left((k-i)(-1)^n\binom{k-1}{i-1}+i\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k}\\ &= (k-n)(-1)^{i-1}\left((-1)^{n-1}\binom{k-1}{i-1}+\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k}.\end{aligned}$$

Therefore the identity holds for $n-1$ as well and the proof is complete. □

2.3. Computation of $S_{k,n,i}$.

Observe that each (F)MZV can be written as a $\mathbb{Z}$-linear combination of (F)MZSVs and vice versa, an example being

$$\begin{aligned}\zeta_{\mathcal{A}}(k_1,k_2,k_3) &= \sum_{m_1>m_2>m_3}\frac{1}{m_1^{k_1}m_2^{k_2}m_3^{k_3}}\\ &= \left(\sum_{m_1\ge m_2\ge m_3}-\sum_{m_1=m_2\ge m_3}-\sum_{m_1\ge m_2=m_3}+\sum_{m_1=m_2=m_3}\right)\frac{1}{m_1^{k_1}m_2^{k_2}m_3^{k_3}}\\ &= \zeta^{\star}_{\mathcal{A}}(k_1,k_2,k_3)-\zeta^{\star}_{\mathcal{A}}(k_1+k_2,k_3)-\zeta^{\star}_{\mathcal{A}}(k_1,k_2+k_3)+\zeta^{\star}_{\mathcal{A}}(k_1+k_2+k_3),\end{aligned}$$

where m_1, m_2, and m_3 are all assumed to be positive integers less than p.

LEMMA 2.5. *For $k,n\in\mathbb{Z}$ with $1\le n\le k-1$, we have*

$$S_{k,n,1}=\sum_{j=0}^{n-1}(-1)^j\binom{k-n+j-1}{j}S^{\star}_{k,n-j,1}.$$

Proof. Each $\zeta_{\mathcal{A}}(k_1, \ldots, k_n)$, where $(k_1, \ldots, k_n) \in I_{k,n,1}$, can be written as a sum of the values of the form $(-1)^j \zeta^{\star}_{\mathcal{A}}(k'_1, \ldots, k'_{n-j})$ where $j = 0, \ldots, n-1$ and $(k'_1, \ldots, k'_{n-j}) \in I_{k,n-j,1}$. Moreover, each $(k'_1, \ldots, k'_{n-j}) \in I_{k,n-j,1}$ appears in this manner exactly as many times as there are ways of adding j bars to the $n-j-1$ existing bars in the gaps in the following sequence of stars, in such a way that no bar separates the leftmost two stars and no two bars are in the same gap:

$$\underbrace{\boxed{\star\star}\cdots\star}_{k'_1}|\cdots|\underbrace{\star\cdots\star}_{k'_{n-j}}$$

Since there are $(k'_1 - 2) + (k'_2 - 1) + \cdots + (k'_{n-j} - 1) = k - n + j - 1$ gaps that accept bars, the number of ways is $\binom{k-n+j-1}{j}$. $\square$

Lemma 2.6 (Initial values). *For $k, n \in \mathbb{Z}$ with $1 \le n \le k-1$, we have*

$$S_{k,n,1} = \left(1 + (-1)^n \binom{k-1}{n-1}\right) \frac{B_{p-k}}{k}.$$

Proof. By Proposition 2.4 and Lemma 2.5, we have

$$\begin{aligned} S_{k,n,1} &= \sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} \left((-1)^{n-j} + \binom{k-1}{n-j-1} \right) \frac{B_{p-k}}{k} \\ &= \left((-1)^n \sum_{j=0}^{n-1} \binom{k-n+j-1}{j} + \sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} \binom{k-1}{n-j-1} \right) \frac{B_{p-k}}{k}. \end{aligned}$$

Recall that $(1-x)^{-m} = \sum_{j=0}^{\infty} \binom{m+j-1}{j} x^j \in \mathbb{Q}[[x]]$ for $m \in \mathbb{Z}_{\ge 1}$. Looking at the coefficient of x^{n-1} in the product of $(1-x)^{-(k-n)}$ and $(1-x)^{-1}$ gives

$$\sum_{j=0}^{n-1} \binom{(k-n)+j-1}{j} = \binom{(k-n+1)+(n-1)-1}{n-1} = \binom{k-1}{n-1};$$

looking at the coefficient of x^{n-1} in the product of $(1+x)^{-(k-n)}$ and $(1+x)^{k-1}$ gives

$$\sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} \binom{k-1}{n-j-1} = 1.$$

The proof is now complete. $\square$

Proposition 2.7. *For $k, n, i \in \mathbb{Z}$ with $1 \le i \le n \le k-1$, we have*

$$S_{k,n,i} = (-1)^{i-1} \left(\binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \frac{B_{p-k}}{k}.$$

Proof. The proof is by induction on i, the case $i = 1$ being Lemma 2.6. Assume that the identity holds for i. Then Proposition 2.2 shows that

$$\begin{aligned}
iS_{k,n,i+1} &= -(n-i)S_{k,n,i} - (k-n)S_{k,n-1,i} \\
&= -(n-i)(-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^n\binom{k-1}{n-i}\right)\frac{B_{p-k}}{k} \\
&\quad - (k-n)(-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^{n-1}\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k} \\
&= -(-1)^{i-1}\left((n-i)\binom{k-1}{i-1} + (k-n+i)(-1)^n\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k} \\
&\quad - (-1)^{i-1}\left((k-n)\binom{k-1}{i-1} + (k-n)(-1)^{n-1}\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k} \\
&= (-1)^i\left((k-i)\binom{k-1}{i-1} + i(-1)^n\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k} \\
&= i(-1)^i\left(\binom{k-1}{i} + (-1)^n\binom{k-1}{n-i-1}\right)\frac{B_{p-k}}{k}.
\end{aligned}$$

Therefore the identity holds for $i+1$ as well and the proof is complete. □

Combining Propositions 2.4 and 2.7, we have completed the proof of the main theorem (Theorem 1.4).

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