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On the Siegel Eisenstein series of degree two for low weights

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Abstract. In this paper, we give the Fourier coefficients of Siegel Eisenstein series of degree 2, level p, in order to calculate the dimensions of the space of Eisenstein series for low weights. The main methods of the calculation is to compute the Siegel series of level p directly, following the similar way to that of Kaufhold.

1. Introduction.

Let Z be an element of the Siegel upper half space $\mathbb{H}_g$. For a congruence subgroup $\Gamma \subset Sp(g, \mathbb{Z})$, the Siegel Eisenstein series $E^k(Z; \Gamma)$ are defined by

$$E^k(Z; \Gamma) = \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} \det(C_\gamma Z + D_\gamma)^{-k}, \tag{1.1}$$

here P_0 is the subgroup of $Sp(g, \mathbb{Z})$ consisting all the elements whose lower-left (g, g)-block is the zero matrix. The infinite sum of the right-hand side converges uniformly on $\mathbb{H}_g$, when $k > g + 1$. For example let $\Gamma = \Gamma^g(N)$ be the principle congruence subgroup of level N. Put $M_k(\Gamma^g(N))$ the space of Siegel modular forms of weight k and $L_k(\Gamma^g(N))$ the subspace of $M_k(\Gamma^g(N))$ consisting of the functions whose constant term of the Fourier expansion vanishes at each 0-dimensional cusp. We set $\mathcal{E}_k(\Gamma^g(N)) = M_k(\Gamma^g(N))/L_k(\Gamma^g(N))$. Then it is easy to show that $\mathcal{E}_k(\Gamma^g(N))$ is spanned by $\{E^k(Z; \Gamma^g(N))|_k \gamma\}_{\gamma \in \Gamma^g}$ if $k > g + 1$.

Now we consider the low weight cases. Since the right-hand side of (1.1) does not converge, we use the "Hecke trick". For $s \in \mathbb{C}$, the non-holomorphic Siegel Eisenstein series are defined by

$$E^k(Z, s; \Gamma) = \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} \det(C_\gamma + D_\gamma)^{-k} |\det(C_\gamma + D_\gamma)|^{-2s},$$

which has an analytic continuation to whole s-plane. The famous paper of Shimura [**Sh2**] starts from the following questions.

(1) For each $Z \in \mathbb{H}_g$, $E^k(Z, s; \Gamma)$ is holomorphic at $s = 0$?
(2) If so, $E^k(Z, 0; \Gamma)$ is holomorphic in Z?
(3) If so, the Fourier coefficients of $E^k(Z, 0, \Gamma)$ are algebraic numbers?

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One of the main results of [**Sh2**] says that all of the above questions are affirmative when $k \geq g+1$, that is we can construct Eisenstein series of 1 lower weight than before.

In the classical case of elliptic modular forms, stronger results are shown by Hecke. Let $\Gamma = \Gamma^1(N)$. Then for $k = 1, 2$, all the elements of $\mathcal{E}_k(\Gamma^1(N))$ are constructed by $\{E^k(Z, 0, \Gamma^1(N))|_k\gamma \mid \gamma \in SL(2,\mathbb{Z})\}$.

In this paper we consider the following problem:

(4) Calculate the dimension spanned by Eisenstein series for low weight.

We mainly consider the case $g = 2$ and $\Gamma = \Gamma^2(p)$ or $\Gamma_0^2(p)$ for an odd prime p. It suffices to consider the case for $\Gamma_0^2(p)$, since the case of $\Gamma^2(p)$ can be induced from the results for the case of $\Gamma_0^2(p)$ using the representation theory of $Sp(2,\mathbb{F}_p)$. This natural question (4) is not considered in [**Sh2**], because Shimura considered only the Fourier expansion of $E^k(Z, s; \Gamma_0^2(N))|_k J_2$, and one has no information for other cusps. In order to get the answer of (4), we have to consider the Fourier expansions at all cusps, in particular cusp of infinity. The hardest part of the calculation is computing the Siegel series at p. In Section 4 we compute the Siegel series directly. Recently Takemori [**Ta**] gives the explicit formula of the Fourier expansion of $E^2_{N,\psi}$ (the definition is given below) for any natural number N and primitive Dirichlet character ψ modulo N by a similar method.

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Notations. Let $\Gamma^g = Sp(g,\mathbb{Z}) = \{\gamma \in GL_{2g}(\mathbb{Z}) \mid {}^t\gamma J_g \gamma = J_g\}$ with $J_g = \left(\begin{smallmatrix} 0 & 1_g \\ -1_g & 0 \end{smallmatrix}\right)$. For $\gamma \in \Gamma^g$, square matrices A_γ, B_γ, C_γ, and D_γ of size g are defined by $\gamma = \left(\begin{smallmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{smallmatrix}\right)$.

Throughout this paper p denotes an odd prime number. We put $\Gamma_0^g(p) = \{\gamma \in \Gamma^g \mid C_\gamma \equiv 0 \bmod p\}$ and $\Gamma^g(p) = \{\gamma \in \Gamma^g \mid \gamma \equiv 1_g \bmod p\}$. We define for $g \geq 2$,

$$M_k(\Gamma^g(p)) = \{\text{a holomorphic function } f \text{ on } \mathbb{H}_g \mid f|_k\gamma = f,\ \forall\gamma \in \Gamma^g(p)\},$$

with $f|_k\gamma(Z) = \det(C_\gamma Z + D_\gamma)^{-k} f(\gamma\langle Z\rangle)$, $\gamma\langle Z\rangle = (A_\gamma Z + B_\gamma)(C_\gamma Z + D_\gamma)^{-1}$. Moreover we define

$$M_k(\Gamma_0^g(p), \psi) = \{f \in M_k(\Gamma^g(p)) \mid f|_k\gamma = \psi(\det D_\gamma) f,\ \forall f \in \Gamma_0^g(p)\}$$

for a Dirichlet character ψ modulo p. If $g = 1$ we also require the holomorphic condition at each cusp.

In the following we consider the case $g = 2$. Let $P_0 = \{\gamma \in Sp(2,\mathbb{Z}) \mid C_\gamma = 0\}$. For a Dirichlet character ψ modulo p such that $\psi(-1) = (-1)^k$, we put

$$E^k_{p,\psi}(Z, s) = \sum_{\gamma \in P_0\backslash\Gamma_0^2(p)} \psi(\det D_\gamma)\det(C_\gamma Z + D_\gamma)^{-k}|\det(C_\gamma Z + D_\gamma)|^{-2s}.$$

Then the infinite sum of the right hand side converges absolutely and uniformly on $\mathbb{H}_g$ for $\mathrm{Re}(s)+k>3$. If $k\geq 4$ then $E^k_{p,\psi}(Z):=E^k_{p,\psi}(Z,0)\in M_k(\Gamma^2_0(p),\overline{\psi})$.

For a square matrix $A\in M_n(\mathbb{R})$, we put $\boldsymbol{e}(A)=\exp(2\pi i\,\mathrm{tr}(A))$.

2. Fourier expansion of the Siegel Eisenstein series.

In this section, we explain the Fourier expansion of $E^k_{p,\psi}$ following [**Ma**]. All the proofs of the facts below can be found in [**Ma**, Section 11, 12].

Lemma 2.1. *For the pair (C,D) of integral (g,g)-matrices, the following conditions are equivalent.*

(1) *There exist $X,Y\in M_g(\mathbb{Z})$ such that $CX+DY=1_g$.*
(2) *For $Q\in M_g(\mathbb{Q})$, $QC,QD\in M_g(\mathbb{Z})$ if and only if $Q\in M_g(\mathbb{Z})$.*
(3) *There exist $U\in GL_g(\mathbb{Z})$ and $V\in GL_{2g}(\mathbb{Z})$ such that $U(C,D)V=(1_g,0)$.*

Moreover theses conditions are stable under the left multiplication of the element of $GL_g(\mathbb{Z})$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious. For the proof of (2) $\Rightarrow$ (3), by the elementary divisor theorem, there exist $U\in GL_g(\mathbb{Z})$ and $V\in GL_{2g}(\mathbb{Z})$ such that $U(C,D)V=(T,0)$ with $T=\mathrm{diag}(t_1,\dots,t_r,0,\dots,0)$, $t_i\in\mathbb{Z}_{>0}$. If $T\neq 1_g$ one can find a diagonal matrix $R\in M_g(\mathbb{Q})\smallsetminus M_g(\mathbb{Z})$ such that RT is integral. Put $Q=RU$, which contradicts to (2). $\square$

Definition 2.1. The pair of the matrices $(C,D)\in M_{g,2g}(\mathbb{Z})$ is called *co-prime* if it satisfies one of, hence all, the equivalent condition in Lemma 2.1. If (C,D) satisfies $C\,{}^tD=D\,{}^tC$, then it is called *symmetric*.

We put

$$\mathcal{M}_g=\{(C,D)\in M_{g,2g}(\mathbb{Z})\mid (C,D)\text{ is symmetric and co-prime}\}$$

and $\mathcal{M}^r_g=\{(C,D)\in\mathcal{M}_g\mid \mathrm{rank}\,C=r\}$.

Lemma 2.2. *The pair $(C,D)\in\mathcal{M}_g$ if and only if $C=C_\gamma$ and $D=D_\gamma$ for some $\gamma\in Sp(g,\mathbb{Z})$. In particular the representative set $P_0\backslash Sp(g,\mathbb{Z})$ corresponds to $GL_g(\mathbb{Z})\backslash\mathcal{M}_g$ bijectively.*

Let $\Lambda_{g,r}$ be the set of (g,r)-matrices $Q\in M_{g,r}(\mathbb{Z})$ such that $(Q,R)\in GL_g(\mathbb{Z})$ with some $R\in M_{g,g-r}(\mathbb{Z})$.

Lemma 2.3. *For each $Q\in\Lambda_{g,r}$, fix $\widetilde{Q}=(Q,*)\in GL_g(\mathbb{Z})$. Then a representative set of $GL_g(\mathbb{Z})\backslash\mathcal{M}^r_g$ is given by*

$$\left\{\left(\begin{pmatrix}C' & 0\\ 0 & 0\end{pmatrix}{}^t\widetilde{Q},\ \begin{pmatrix}D' & 0\\ 0 & 1_{g-r}\end{pmatrix}\widetilde{Q}^{-1}\right)\ \middle|\ \begin{array}{l}(C',D')\in GL_r(\mathbb{Z})\backslash\mathcal{M}^r_r,\\ Q\in\Lambda_{g,r}/GL_r(\mathbb{Z})\end{array}\right\}.$$

These lemmas induce

$$\begin{aligned} E^k_{p,\psi}(Z,s) &= \sum_{\substack{(C,D)\in GL_2(\mathbb{Z})\backslash\mathcal{M}_2\\ C\equiv 0 \bmod p}} \psi(\det D)\det(CZ+D)^{-k}|\det(CZ+D)|^{-2s}\\ &= 1+\sum_{\substack{(q_1,q_2)\in\mathbb{Z}^2/\{\pm1\}\\ (q_1,q_2)=1}}\ \sum_{\substack{(c,d)\in\{\pm1\}\backslash\mathcal{M}_1^1\\ c\equiv 0 \bmod p}} \psi(d)(cZ[\left(\begin{smallmatrix} q_1\\ q_2\end{smallmatrix}\right)]+d)^{-k}|(cZ[\left(\begin{smallmatrix} q_1\\ q_2\end{smallmatrix}\right)]+d)|^{-2s}\\ &\quad + \sum_{\substack{(C,D)\in GL_2(\mathbb{Z})\backslash\mathcal{M}_2^2\\ C\equiv 0 \bmod p}} \psi(\det D)\det(CZ+D)^{-k}|\det(CZ+D)|^{-2s}. \end{aligned} \tag{2.1}$$

There exists a bijective map

$$GL_g(\mathbb{Z})\backslash\mathcal{M}_g^g \longrightarrow \mathrm{Sym}^g(\mathbb{Q}), \quad (C,D)\longmapsto C^{-1}D.$$

The inverse map is given as follows. For all $T\in\mathrm{Sym}^g(\mathbb{Q})$, there exist $U,V\in SL_g(\mathbb{Z})$ such that

$$UTV=\begin{pmatrix}\nu_1/\delta_1 & & \\ & \ddots & \\ & & \nu_g/\delta_g\end{pmatrix}, \quad \delta_i>0,\ (\nu_i,\delta_i)=1 \tag{2.2}$$

by the elementary divisor theorem. Then

$$\begin{pmatrix}\delta_1 & & \\ & \ddots & \\ & & \delta_1\end{pmatrix}U,\quad \begin{pmatrix}\nu_1 & & \\ & \ddots & \\ & & \nu_g\end{pmatrix}V^{-1}$$

gives the corresponding element in $\mathcal{M}_g^g$. We put $\delta(T)=\prod_i\delta_i$ and $\nu(T)=\prod_i\nu_i=\det(T)\delta(T)$ for $T\in\mathrm{Sym}^g(\mathbb{Q})$. Then $\delta(T)=|\det C|$ and $\nu(T)=\pm\det D$ for $T=C^{-1}D$ with $(C,D)\in\mathcal{M}_g^g$. Set $\mathrm{Sym}^g(\mathbb{Q})'\subset\mathrm{Sym}^g(\mathbb{Q})$ the image of $\{(C,D)\in\mathcal{M}_g^g \mid C\equiv 0 \bmod p\}$ under the above map.

The third line of (2.1) becomes

$$\begin{aligned} &\sum_{\substack{(C,D)\in GL_2(\mathbb{Z})\backslash\mathcal{M}_2^2\\ C\equiv 0 \bmod p}} \psi(\det D)\det C^{-k}|\det C|^{-2s}\det(Z+C^{-1}D)^{-k}|\det(Z+C^{-1}D)|^{-2s}\\ &= \sum_{T\in\mathrm{Sym}^2(\mathbb{Q})'} \psi(\nu(T))\delta(T)^{-k-2s}\det(Z+T)^{-k-s}\det(\overline{Z}+T)^{-s}\\ &= \sum_{\substack{T\in\mathrm{Sym}^2(\mathbb{Q})'\\ \bmod 1}} \psi(\nu(T))\delta(T)^{-k-2s}\sum_{S\in\mathrm{Sym}^g(\mathbb{Z})}\det(Z+T+S)^{-k-s}\det(\overline{Z}+T+S)^{-s}. \end{aligned}$$

Here we use the fact $\delta(T+S)=\delta(T)$ and $\nu(T+S)\equiv\nu(T) \bmod p$; indeed for $T=C^{-1}D$, we have $T+S=C^{-1}(D+CS)$ and $(C,D+CS)\in\mathcal{M}_2^2$. Now for $\alpha,\beta\in\mathbb{C}$ we consider the Fourier expansion of $\sum_{S\in\mathrm{Sym}^g(\mathbb{Z})}\det(Z+S)^{-\alpha}\det(\overline{Z}+S)^{-\beta}$ (the branches of the complex powers are determined suitably as in [**Sh1**, (1.11)]). Let

$$\mathrm{Sym}^g(\mathbb{Z})^*=\{h\in\mathrm{Sym}^g(\mathbb{Q})\mid \mathrm{tr}(hA)\in\mathbb{Z}\text{ for all }A\in\mathrm{Sym}^g(\mathbb{Z})\},$$

be the set of *half integral matrices of size* g, whose elements consist of integral diagonal entries, and half integral off-diagonal entries. Put $\boldsymbol{e}(X)=e^{2\pi i\,\mathrm{tr}(X)}$ for a square matrix X. Then the Fourier expansion is written by

$$\sum_{S\in\mathrm{Sym}^g(\mathbb{Z})}\det(Z+S)^{-\alpha}\det(\overline{Z}+S)^{-\beta}=\sum_{h\in\mathrm{Sym}^g(\mathbb{Z})^*}\xi_g(Y,h,\alpha,\beta)\boldsymbol{e}(hX),$$

with

$$\xi_g(Y,h,\alpha,\beta)=\int_{\mathrm{Sym}^g(\mathbb{R})}\det(X+iY)^{-\alpha}\det(X-iY)^{-\beta}\boldsymbol{e}(-hX)\,dX. \tag{2.3}$$

Thus the third line of (2.1) is given by

$$\sum_{h\in\mathrm{Sym}^g(\mathbb{Z})^*}S_2(\psi,h,k+2s)\xi_2(Y,h,\alpha,\beta),$$

where for $s\in\mathbb{C}$, we set

$$S_g(\psi,h,s)=\sum_{\substack{T\in\mathrm{Sym}^g(\mathbb{Q})'\\ \bmod 1}}\psi(\nu(T))\delta(T)^{-s}\boldsymbol{e}(hT),$$

which is called the (generalised) Siegel series. As a consequence we get

$$\begin{aligned}E^k_{p,\psi}(Z,s)=1&+\sum_{m\in\mathbb{Z}}\sum_{\substack{(q_1,q_2)\in\mathbb{Z}^2/\{\pm1\}\\(q_1,q_2)=1}}S_1(\psi,m,k+2s)\xi_1(Y[\left(\begin{smallmatrix}q_1\\q_2\end{smallmatrix}\right)],m,k+s,s)\boldsymbol{e}(mX[\left(\begin{smallmatrix}q_1\\q_2\end{smallmatrix}\right)])\\&+\sum_{h\in\mathrm{Sym}^2(\mathbb{Z})^*}S_2(\psi,h,k+2s)\xi_2(Y,h,k+s,s)\boldsymbol{e}(hX).\end{aligned} \tag{2.4}$$

3. Known results.

There are some results for the computations of ξ_g and S_g. We collect them in this section.

Theorem 3.1 (Shimura [**Sh1**, (4.34.K), Theorem 4.2]). *For* $h\in\mathrm{Sym}^g(Q)^*$ *with* $\mathrm{sgn}\,h=(p,q,r)$,

$$\xi_g(Y,h;\alpha,\beta) = i^{g(\beta-\alpha)}2^u\pi^v\Gamma_r\left(\alpha+\beta-\frac{g+1}{2}\right)\Gamma_{g-q}(\alpha)^{-1}\Gamma_{g-p}(\beta)^{-1}$$
$$\times \det(Y)^{(g+1)/2-\alpha-\beta}d_+(hY)^{\alpha-(g+1)/2+q/4}d_-(hY)^{\beta-(g+1)/2+p/4}$$
$$\times\, \omega(2\pi Y,h,\alpha,\beta),$$

with

$$u = (2p-g)\alpha+(2q-g)\beta+\frac{(g+r)(g+1)}{2}+\frac{pq}{2}-\frac{g(g-1)}{2},$$
$$v = p\alpha+q\beta+r+\frac{r(r-1)-pq}{2}.$$

Here $d_+(x)$ (*resp.* $d_-(x)$) *denotes the products of positive* (*resp. negative*) *eigenvalues of* x *and* $\Gamma_m(s) = \pi^{m(m-1)/4}\prod_{k=0}^{m-1}\Gamma(s-k/2)$*. Moreover* $\omega(2\pi Y,h,\alpha,\beta)$ *is an entire function with respect to* α *and* β*.*

The function ω_g can be written more explicitly in some special cases. We mainly use the following:

1. ([**Sh1**, (3.15), (4.7.K), (4.10)]). If $h>0$ then,

$$\omega_g(2\pi Y,h,\alpha,0) = 2^{-g(g+1)/2}\boldsymbol{e}(hY).$$

2. ([**Sh1**, (4.9)]).

$$\omega_g(2\pi Y,0,\alpha,\beta)=1.$$

3. ([**Sh1**, (4.35.K)]). If the signature of h is $(p,0,r)$ i.e. h is positive semi-definite then,

$$\omega_g(2\pi Y,h,(g+1)/2,\beta) = 2^{-p(g+1)/2}\pi^{pr/2}\boldsymbol{e}(-hY).$$

Next we investigate the Siegel series $S_g(\psi,h,s)$. Let $\mathrm{Sym}^g(\mathbb{Q})_q$ be the set of $T\in\mathrm{Sym}^g(\mathbb{Q})$ such that $\delta(T)$ is a q-power for prime numbers q, and $\mathrm{Sym}^g(\mathbb{Q})'_p = \mathrm{Sym}^g(\mathbb{Q})'\cap\mathrm{Sym}^g(Q)_p$. For all $T\in\mathrm{Sym}^g(\mathbb{Q})$ there exists a decomposition $T=\sum_{i=0}^r T_i\in\mathrm{Sym}^g(\mathbb{Q})$ with $T_i\in\mathrm{Sym}^g(\mathbb{Q})_{q_i}$, which is unique modulo $\mathrm{Sym}^g(\mathbb{Z})$; indeed if we write $1/\delta(T)=\sum_{i=0}^r x_i/q_i^{e_i}$ with prime divisors q_i of $\delta(T)$, each T_i is given by $q_i^{-e_i}x_i\delta(T)T$. If $T\in\mathrm{Sym}^g(\mathbb{Q})'$, one of the q_i, say q_0, equals to p and we have

$$\delta(T)=\prod_{i=0}^r\delta(T_i),\quad \nu(T)\equiv\nu(T_0)\prod_{i=1}^r\delta(T_i) \bmod p. \tag{3.1}$$

The first equation is obvious. For the second, write $T=C^{-1}D$ with $(C,D)\in\mathcal{M}_g^g$ so that $\det C=\delta(T)$. Then for the decomposition

$$D = CT_0+CT_1+\cdots+CT_r,$$

each $CT_i \in M_g(\mathbb{Z})$ and $CT_i \equiv 0 \bmod p$ for $i \geq 1$. Thus

$$\nu(T) = \det(D) \equiv \det(CT_0) \bmod p = \nu(T_0) \prod_{i=1}^{r} \delta(T_i).$$

Notice that for the decomposition $T = \sum_{i=0}^{r} T_i$, $T \in \mathrm{Sym}^g(Q)'$ if and only if $T_0 \in \mathrm{Sym}^g(\mathbb{Q})'_p$. As a consequence we have

$$S_g(\psi, h, s) = \prod_{q:\ \text{prime}} S_g^q(\psi, h, s),$$

with

$$S_g^q(\psi, h, s) = \begin{cases} \displaystyle\sum_{T \in \mathrm{Sym}^g(\mathbb{Q})_q \bmod 1} \psi(\delta(T))\delta(T)^{-s}\boldsymbol{e}(hT) & q \neq p; \\ \displaystyle\sum_{T \in \mathrm{Sym}^g(\mathbb{Q})'_p \bmod 1} \psi(\nu(T))\delta(T)^{-s}\boldsymbol{e}(hT) & q = p. \end{cases}$$

If $g = 1$, the Siegel series are easy to compute as follows.

Proposition 3.2. *Let ψ be a Dirichlet character modulo p and $q \neq p$ a prime.*

(1) *If $m = 0$, the local Siegel series is given by*

$$S_1^q(\psi, 0, s) = \frac{1 - \psi(q)q^{-s}}{1 - \psi(q)q^{1-s}}.$$

(2) *If $m = q^t m'$ with $(m', q) = 1$, we have*

$$S_1^q(\psi, m, s) = (1 - \psi(q)q^{-s})F_{q,m}(\psi(q)q^{-s}), \quad \text{with } F_{q,m}(X) = \sum_{k=0}^{t} (qX)^k.$$

Proposition 3.3. *Let ψ be a Dirichlet character modulo p.*

(1) *If $m = 0$, the Siegel series are given by*

$$S_1^p(\psi, 0, s) = \begin{cases} 0 & \psi \not\equiv 1; \\ \dfrac{p^{-s}(p-1)}{(1 - p^{1-s})} & \psi \equiv 1. \end{cases}$$

(2) *If $m = p^t m'$ with $(m', p) = 1$, then*

$$S_1^p(\psi, m, s) = \begin{cases} \overline{\psi}(m')G(\psi)p^{(1-s)t-s} & \psi \not\equiv 1; \\ (1 - q^{-s})F_{q,m}(q^{-s}) - 1 & \psi \equiv 1. \end{cases}$$

Here $G(\psi)$ is the Gaussian sum.

For $g \geq 2$, $S_g^q(\psi, h, s)$ $(q \neq p)$ are studied by many Mathematicians, for example Kaufhold, Siegel, Feit, Shimura, Kitaoka and finally Katsurada gave the explicit formula [**Kat**]. Our case $g = 2$ is the Kaufhold's result.

Theorem 3.4 (Kaufhold [**Kau**, (2,10), Hilfsatzs 10]).

$$\prod_{q \neq p} S_2^q(\psi, h, s) = \begin{cases} \dfrac{L(s-2,\psi)L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)} & h = 0; \\ \dfrac{L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)} \displaystyle\prod_{q \neq p} F_h^{(q)}(\psi(q)q^{-s}) & \operatorname{rank} h = 1; \\ \dfrac{L(s-1,\psi\chi_h)}{L(s,\psi)L(2s-2,\psi^2)} \displaystyle\prod_{q \neq p} G_h^{(q)}(\psi(q)q^{-s}) & \operatorname{rank} h = 2. \end{cases}$$

Here χ_h, $F_h^{(q)}$ and $G_h^{(q)}$ are defined as follows. For the discriminant D_h of quadratic extension $\mathbf{Q}(\sqrt{-\det(2h)})/\mathbf{Q}$, the quadratic character χ_h is given by $\chi_h(q) = (D_h/q)$ for prime numbers q. Let $\alpha_1 = \operatorname{ord}_p(\mathrm{g.c.d}(h_1, 2h_2, h_3))$ for $h = \left(\begin{smallmatrix} h_1 & h_2 \\ h_2 & h_3 \end{smallmatrix}\right)$ and $\alpha = (1/2)\operatorname{ord}_p(\det(2h)/D_h)$. Then the polynomials F_h and G_h are defined by

$$F_h^{(q)}(X) = \sum_{l=0}^{\alpha_1} (qX)^{2l},$$

$$G_h^{(q)}(X) = \sum_{l=0}^{\alpha_1} (qX)^{2l} \left\{ \sum_{m=0}^{\alpha-l} (q^3X^2)^m - \chi_h(q)qX \sum_{m=0}^{\alpha-l-1} (q^3X^2)^m \right\}.$$

Note that $F_h^{(q)} = G_h^{(q)} = 1$ for all but finite primes q.

4. Terms corresponding to p.

In this section we calculate the Siegel series $S^p(\psi, h, s)$. For our problem of the dimension of the space of Eisenstein series, it suffices to compute the constant term of $E_{p,\psi}^k$. However the Fourier expansion of $E_{p,\psi}^k$ itself is interesting problem, so we calculate $S^p(\psi, h, s)$ for all h.

First we consider the case $h = 0$.

Lemma 4.1. *Let ψ be a non-trivial Dirichlet character modulo p.*

$$S_2^p(\psi, 0, s) = \begin{cases} 0 & \psi^2 \not\equiv 1; \\ \psi(-1)\, \dfrac{(p-1)p^{1-2s}}{1 - p^{3-2s}} & \psi^2 \equiv 1, \psi \not\equiv 1. \end{cases}$$

Proof. Before the calculation, we rewrite the set of summation. Let

$$\mathcal{M}(p) = \{(C,D) \in \mathcal{M}_2^2 \mid C^{-1}D \in \mathrm{Sym}^2(\mathbb{Q})'_p\}$$

and

$$\widetilde{\mathcal{M}}(p) = \{(C,D) \in M_{2,4}(\mathbb{Z}) \mid \det C \text{ is a } p\text{-power},\ C \equiv 0 \bmod p,\ (C,D) \text{ is symmetric}\}.$$

Property (3) of the co-prime condition in Lemma 2.1 shows that

(∗) if (C,D) is symmetric, there exists $M \in M_g(\mathbb{Z})$ such that $C' = MC$, $D' = MD$ with $(C',D') \in \mathcal{M}_2^2$.

Then

$$\begin{aligned} S_2^p(\psi,h,s) &= \sum_{\substack{C \\ (C,D)\in SL(2,\mathbb{Z})\backslash\mathcal{M}(p)}} \sum_{D \bmod C} \psi(\det D)(\det C)^{-s}\boldsymbol{e}(C^{-1}Dh) \\ &= \sum_{\substack{C \\ (C,D)\in SL(2,\mathbb{Z})\backslash\widetilde{\mathcal{M}}(p)}} \sum_{D \bmod C} \psi(\det D)(\det C)^{-s}\boldsymbol{e}(C^{-1}Dh), \end{aligned}$$

by (∗). For each $(C,D) \in \widetilde{\mathcal{M}}(p)$, there exists $U, V \in SL(2,\mathbb{Z})$ such that

$$UCV = \begin{pmatrix} p^k & 0 \\ 0 & p^{k+l} \end{pmatrix} = T(k,l), \quad k \geq 1,\ l \geq 0.$$

Then $C^{-1}D = V^{-1}T(k,l)^{-1}U^{-1}D\,{}^tV\,{}^tV^{-1}$. Put $U^{-1}D\,{}^tV = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The pair (C,D) is symmetric if and only if $c = p^l b$. Now C runs through the representative set $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{Z})T(k,l)SL(2,\mathbb{Z})$. If $l \geq 1$, it is given by

$$\left\{T(k,l)W \;\middle|\; W = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ u \in \mathbb{Z}/p^l\mathbb{Z}\right\} \cup \left\{T(k,l)W \;\middle|\; W = \begin{pmatrix} pu & 1 \\ -1 & 0 \end{pmatrix},\ u \in \mathbb{Z}/p^{l-1}\mathbb{Z}\right\},$$

while it is given by a single element $T(k,k)$ if $l = 0$. For such $C = T(k,l)W$, D runs through the set

$$\left\{ \begin{pmatrix} a & b \\ p^l b & d \end{pmatrix} {}^tW^{-1} \;\middle|\; a, b \in \mathbb{Z}/p^k\mathbb{Z},\ d \in \mathbb{Z}/p^{k+l}\mathbb{Z}\right\}.$$

Now we shall compute $S_2^p(\psi,0,s)$. Assume that $\psi \not\equiv 1$. For a fixed $C = T(k,l)W$,

$$\sum_{D \bmod C} \psi(\det D)(\det C)^{-s} = \frac{1}{p^{(2k+l)s}} \sum_{\substack{a,b\in\mathbb{Z}/p^k\mathbb{Z} \\ d\in\mathbb{Z}/p^{k+l}\mathbb{Z}}} \psi(ad - p^l b^2). \tag{4.1}$$

Put $\Lambda(m) = \#\{(a,b,d) \in \mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^{k+l}\mathbb{Z} \mid ad - p^l b^2 \equiv m \bmod p\}$. We calculate $\Lambda(m)$ for each $m \in (\mathbb{Z}/p\mathbb{Z})^\times$.

case 1) $l \geq 1$. In this case $\Lambda(m) = \#\{a,b,d \mid ad \equiv m \bmod p\}$. Since $\mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^{k+l} \ni (a,d) \mapsto ad \in \mathbb{Z}/p\mathbb{Z}$ is a homomorphism, $\Lambda(m_1) = \Lambda(m_2)$ for all $m_1, m_2 \in \mathbb{Z}/p\mathbb{Z}$. Thus values of (4.1) is 0 in this case.

case 2) $l = 0$. Then

$$\begin{aligned}\Lambda(m) &= \#\{(a,b,d) \in (\mathbb{Z}/p^k\mathbb{Z})^{\oplus 3} \mid ad - b^2 \equiv m \bmod p\} \\ &= \begin{cases} p^{3k-2}(p+1) & \text{if } -m \in (\mathbb{F}_p^\times)^2; \\ p^{3k-2}(p-1) & \text{if } -m \notin (\mathbb{F}_p^\times)^2. \end{cases}\end{aligned}$$

Thus the values of (4.1) is

$$\psi(-1)p^{-2ks} \sum_{m \in (\mathbb{F}_p^\times)^2} \psi(m) 2p^{3k-2}.$$

As a consequence

$$S_2^p(\psi, 0, s) = 2\psi(-1) \sum_{m \in (\mathbb{F}_p^\times)^2} \psi(m) \sum_{k=1}^{\infty} p^{(3-2s)k-2},$$

which induces our lemma. $\square$

Next we consider $h \in \mathrm{Sym}^2(\mathbb{Z})^*$ with $\operatorname{rank} h = 1$. There exists $U \in SL(2,\mathbb{Z})$ such that $h[U] = \mathrm{diag}(t, 0)$, which shows that we only consider the diagonal case.

Lemma 4.2. *Assume that ψ is a non-trivial character. Then for $h = \left(\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $\mathrm{ord}_p\, t = m$,*

$$S_2(\psi, h, s) = \begin{cases} 0 & \psi^2 \not\equiv 1; \\ a(p^{-s}) + \dfrac{b(p^{-s})}{1 - p^{3-2s}} & \psi^2 \equiv 1, \end{cases}$$

with

$$\begin{aligned} a(p^{-s}) &= \psi(-1)\left(\frac{p-1}{p^2} \sum_{k=1}^{m+1} p^{(3-2s)k}\right), \\ b(p^{-s}) &= \psi(-1)(p-1)p^{(3-2s)m+4-4s}. \end{aligned}$$

Proof. We use the same notation as in the proof of Lemma 4.1. We have

$$S_2^p(\psi,h,s)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{W}p^{-(2k+l)s}\sum_{\substack{a,b\in\mathbb{Z}/p^k\mathbb{Z}\\ d\in\mathbb{Z}/p^{k+l}\mathbb{Z}}}\psi(ad-p^lb^2)\boldsymbol{e}\left(\frac{1}{p^{k-m}}\begin{pmatrix}a & b\\ b & dp^{-l}\end{pmatrix}(h'[W^{-1}])\right),$$

here $h'=\operatorname{diag}(t',0)$.

In the calculation below, we put $a=a_2p+a_1$, $b=b_2p+b_1$ and $d=d_2p+d_1$ with $a_1,b_1,d_1\in\mathbb{Z}/p\mathbb{Z}$, $a_2,b_2\in\mathbb{Z}/p^{k-1}\mathbb{Z}$ and $d_2\in\mathbb{Z}/p^{k+l-1}\mathbb{Z}$. First we consider the summation for $l=0$, which we put S_1.

$$\begin{aligned}S_1&=\sum_{k=1}^{\infty}p^{-2ks}\sum_{a,b,d\in\mathbb{Z}/p^k\mathbb{Z}}\psi(ad-b^2)\boldsymbol{e}\left(\frac{t'a}{p^{k-m}}\right)\\&=\sum_{k=1}^{\infty}p^{-2ks}\sum_{a_2,b_2,d_2\in\mathbb{Z}/p^{k-1}\mathbb{Z}}\boldsymbol{e}\left(\frac{t'a_2}{p^{k-m-1}}\right)\sum_{a_1,b_1,d_1\in\mathbb{Z}/p\mathbb{Z}}\psi(a_1d_1-b_1^2)\boldsymbol{e}\left(\frac{t'a_1}{p^{k-m}}\right).\end{aligned}$$

If $k>m+1$, the first summation vanishes. Let $k\le m+1$ and consider the second summation. If $a_1=0$ then

$$\sum_{b_1,d_1\in\mathbb{Z}/p\mathbb{Z}}\psi(-b_1^2)=\begin{cases}0 & \psi^2\not\equiv 1;\\ \psi(-1)p(p-1) & \psi^2\equiv 1.\end{cases}$$

If $a_1\neq 0$, then we exchange the variable $d_1\mapsto d_1+b_1^2a_1^{-1}$, and

$$\sum_{a_1\neq 0,b_1,d_1}\psi(a_1d_1)\boldsymbol{e}\left(\frac{t'a_1}{p^{k-m}}\right)=0.$$

As a consequence,

$$S_1=\begin{cases}0 & \psi^2\not\equiv 1;\\ \psi(-1)(p-1)\displaystyle\sum_{k=1}^{m+1}p^{(1-2s)k} & \psi^2\equiv 1.\end{cases}\tag{4.2}$$

Next consider the case $l\geq 1$ and $C=T(k,l)W$ with $W=\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)$, this summation is written by S_2. Then

$$S_2=\sum_{k,l=1}^{\infty}p^{-(2k+l)s}\sum_{u\in\mathbb{Z}/p^l\mathbb{Z}}\sum_{\substack{a,b\in\mathbb{Z}/p^k\mathbb{Z}\\ d\in\mathbb{Z}/p^{k+l}\mathbb{Z}}}\psi(ad)\boldsymbol{e}\left(\frac{t'}{p^{k-m}}\left(a-2ub+\frac{u^2d}{p^l}\right)\right).$$

By looking at the summation for a, we have

$$\sum_{a\in\mathbb{Z}/p^k\mathbb{Z}}\psi(a)\boldsymbol{e}\left(\frac{t'a_1}{p^{k-m}}\right)=\begin{cases}0 & k\neq m+1;\\ p^m\overline{\psi}(t')G(\psi) & k=m+1,\end{cases}$$

here $G(\psi)$ denotes the Gaussian sum. Thus we only consider the term for $k=m+1$. Thanks to the summation for b we can put $u=u_1p$ with $u_1\in\mathbb{Z}/p^{l-1}\mathbb{Z}$ and

$$S_2=\psi(t')G(\psi)\sum_{l=1}^{\infty}p^{-(2m+2+l)s+2m+1}\sum_{u_1\in\mathbb{Z}/p^{l-1}}\ \sum_{d\in\mathbb{Z}/p^{m+1+l}}\psi(d)\boldsymbol{e}\biggl(\frac{t'u_1^2d}{p^{l-1}}\biggr).$$

We know

$$\sum_{u_1\in\mathbb{Z}/p^{l-1}}\boldsymbol{e}\biggl(\frac{t'u_1^2d}{p^{l-1}}\biggr)=\begin{cases}\chi_p(t'd)p^{(l-2)/2}G(\chi_p) & l \text{ is even;}\\ p^{(l-1)/2} & l \text{ is odd,}\end{cases}$$

here χ_p is the unique non-trivial quadratic Dirichlet character modulo p. It suffices to consider the case for even l and

$$\begin{aligned}S_2&=\psi(t')\chi_p(t')G(\psi)G(\chi_p)\sum_{l=1}^{\infty}p^{-2(m+1+l)s+2m+l}\sum_{d\in\mathbb{Z}/p^{m+2l+1}}\chi_p\psi(d)\\&=\begin{cases}0 & \psi\neq\chi_p;\\ \psi(-1)(p-1)\sum_{l}p^{-2(m+l+1)s+3m+3l+1} & \psi=\chi_p,\end{cases}\end{aligned}$$

which coincides with $b(p^{-s})/(1-p^{3-2s})$.

Finally for the term of $C=T(k,l)W$ with $W=\left(\begin{smallmatrix}pa&1\\-1&0\end{smallmatrix}\right)$, one sees easily that it vanishes. This complete the proof. $\square$

Finally we consider the case for $\operatorname{rank}h=2$. Note that there is a bijection $\operatorname{Sym}^g(\mathbb{Q})_p \bmod 1\simeq\operatorname{Sym}^g(\mathbb{Q}_p)\bmod\mathbb{Z}_p$. Put $\operatorname{Sym}^g(\mathbb{Q}_p)'$ the image of $\operatorname{Sym}^g(\mathbb{Q})'_p$ then

$$S_2^p(\psi,h,s)=\sum_{T\in\operatorname{Sym}^2(\mathbb{Q}_p)'\bmod\mathbb{Z}_p}\psi(\nu(T))\delta(T)^{-s}\boldsymbol{e}(hT).$$

Since each $h\in\operatorname{Sym}^2(\mathbb{Q})$ can be diagonalised by the element of $\operatorname{Sym}^2(\mathbb{Z}_p)$, it suffices to consider the diagonal h.

Lemma 4.3. *Let $h=p^m\left(\begin{smallmatrix}\alpha&0\\0&\beta p^t\end{smallmatrix}\right)$ with $(\alpha,p)=(\beta,p)=1$. Then for a non-trivial Dirichlet character ψ modulo p we have*

$$S_2^p(\psi,h,s)=S_1+S_2,$$

where S_1 and S_2 are given in (4.4) *and* (4.5) *respectively.*

Proof. We use the same notation as above. First we calculate the term $l=0$. This term equals to

$$
\begin{aligned}
S_1 &= \sum_k p^{-2ks} \sum_{a,b,d\in\mathbb{Z}/p^k\mathbb{Z}} \psi(ad-b^2)\boldsymbol{e}\left(\frac{1}{p^{k-m}}(\alpha a+\beta dp^t)\right)\\
&= \sum_k p^{-2ks} \sum_{a'',b'',d''\in\mathbb{Z}/p^{k-1}} \boldsymbol{e}\left(\frac{1}{p^{k-m-1}}(\alpha a''+\beta d''p^t)\right)\\
&\quad\times \sum_{a',b',d'\in\mathbb{Z}/p\mathbb{Z}} \psi(a'd'-(b')^2)\boldsymbol{e}\left(\frac{1}{p^{k-m}}(\alpha a'+\beta dp^t)\right).
\end{aligned}
$$

By looking at the first term, this summation is 0 for $k-m-1\ge 0$, and we have

$$
S_1 = \sum_{k=1}^{m+1} p^{-2ks+3k-3} \sum_{a,b,d\in\mathbb{Z}/p\mathbb{Z}} \psi(ad-b^2)\boldsymbol{e}\left(\frac{1}{p^{k-m}}(\alpha a+\beta dp^t)\right).
$$

For the term $d=0$, $\sum_{a,b\in\mathbb{Z}/p\mathbb{Z}}\psi(-b^2)\boldsymbol{e}(\alpha ap^{m-k})$ is 0 if $\psi^2\not\equiv 1$, while if $\psi^2\equiv 1$, this term becomes

$$
p\sum_{a\in\mathbb{Z}/p\mathbb{Z}} \boldsymbol{e}(\alpha ap^{m-k}) = \begin{cases} p & k\le m;\\ 0 & k=m+1.\end{cases}
$$

Therefore, the term $d=0$ is given by

$$
\begin{cases} \displaystyle\sum_{k=1}^{m} p^{(3-2s)k-1} & \psi^2\equiv 1;\\ 0 & \psi^2\not\equiv 1.\end{cases} \tag{4.3}
$$

For the term $d\neq 0$, we may exchange the variable $a\mapsto a+d^{-1}b^2$, and

$$
\begin{aligned}
&\sum_{a,b,d} \psi(ad)\boldsymbol{e}\big((a\alpha+d^{-1}b^2\alpha+d\beta p^t)p^{m-k}\big)\\
&\quad= \sum_a \psi(a)\boldsymbol{e}\left(\frac{\alpha a}{p^{k-m}}\right)\sum_d\sum_b \boldsymbol{e}\left(\frac{\alpha d^{-1}b^2}{p^{k-m}}\right)\psi(d)\boldsymbol{e}\left(\frac{\beta d}{p^{k-m-t}}\right).
\end{aligned}
$$

If $k-m\le 0$, then the first term is 0. Thus we put $k=m+1$, and this term becomes

$$
\begin{aligned}
&\sum_a \psi(a)\boldsymbol{e}\left(\frac{\alpha a}{p}\right)\sum_d\sum_b \boldsymbol{e}\left(\frac{\alpha d^{-1}b^2}{p}\right)\psi(d)\boldsymbol{e}\left(\frac{\beta d}{p^{1-t}}\right)\\
&\quad= \overline{\psi}(\alpha)G(\psi)\chi_p(\alpha)\varepsilon_p\sqrt{p}\sum_{d\in\mathbb{Z}/p\mathbb{Z}} \psi\chi_p(d)\boldsymbol{e}\left(\frac{d\beta}{p^{1-t}}\right).
\end{aligned}
$$

Here

$$
\varepsilon_p = \begin{cases} 1 & p\equiv 1 \bmod 4;\\ i & p\equiv 3 \bmod 4.\end{cases}
$$

If $\psi = \chi_p$ then the last summation is -1 or $p-1$ according as $t=0$ or $t \geq 1$, while if $\psi \neq \chi_p$ then the last summation is $\overline{\psi}\chi_p(\beta)G(\psi\chi_p)$ or 0 according as $t=0$ or $t \geq 1$. As a consequence,

$$S_1 = \begin{cases} \displaystyle\sum_{k=1}^{m} p^{(3-2s)k-1} - \varepsilon_p^2 p & \text{if } \psi = \chi_p \text{ and } t = 0; \\ \displaystyle\sum_{k=1}^{m} p^{(3-2s)k-1} + (p-1)\varepsilon_p^2 p & \text{if } \psi = \chi_p \text{ and } t \geq 1; \\ \overline{\psi}\chi_p(\alpha\beta)G(\psi\chi_p)\varepsilon_p\sqrt{p} & \text{if } \psi \neq \chi_p \text{ and } t = 0; \\ 0 & \text{if } \psi \neq \chi_p \text{ and } t \geq 1. \end{cases} \tag{4.4}$$

Next we consider the term $l \geq 1$ and $W = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with $u \in \mathbb{Z}/p^l\mathbb{Z}$. Then

$$\begin{aligned} S_2 &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} p^{-(2k+l)s} \sum_{u \in \mathbb{Z}/p^l\mathbb{Z}} \sum_{\substack{a,b \in \mathbb{Z}/p^k\mathbb{Z} \\ d \in \mathbb{Z}/p^{k+l}\mathbb{Z}}} \psi(ad)\boldsymbol{e}\left(\frac{1}{p^{k-m}}(\alpha a - 2u\alpha b + dp^{-l}(au^2 + \beta p^t))\right) \\ &= \sum_{k,l} p^{-(2k+l)s} \sum_u \sum_a \psi(a)\boldsymbol{e}\left(\frac{\alpha a}{p^{k-m}}\right) \sum_b \boldsymbol{e}\left(-\frac{2u\alpha b}{p^{k-m}}\right) \sum_d \psi(d)\boldsymbol{e}\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{k+l-m}}\right). \end{aligned}$$

Now

$$\sum_{a \in \mathbb{Z}/p^k\mathbb{Z}} \psi(a)\boldsymbol{e}\left(\frac{\alpha a}{p^{k-m}}\right) = \begin{cases} p^m\overline{\psi}(\alpha)G(\psi) & k = m+1; \\ 0 & \text{otherwise,} \end{cases}$$

thus we put $k = m+1$. Then the summation for b is 0 if $(u,p) = 1$, and we replace u by pu with $u \in \mathbb{Z}/p^{l-1}\mathbb{Z}$. Hence

$$S_2 = \sum_{l=1}^{\infty} p^{-(2m+2+l)s}\overline{\psi}(\alpha)G(\psi)p^{2m+1} \sum_{d \in (\mathbb{Z}/p^{m+1+l}\mathbb{Z})^\times} \sum_{u \in \mathbb{Z}/p^{l-1}\mathbb{Z}} \boldsymbol{e}\left(\frac{\alpha d u^2}{p^{l-1}}\right)\psi(d)\boldsymbol{e}\left(\frac{\beta d}{p^{l+1-t}}\right).$$

Since

$$\sum_{a \in \mathbb{Z}/p^r\mathbb{Z}} \boldsymbol{e}\left(\frac{xa^2}{p^r}\right) = p \sum_{a \in \mathbb{Z}/p^{r-2}\mathbb{Z}} \boldsymbol{e}\left(\frac{xa^2}{p^{r-2}}\right), \quad r \geq 2,\ (p,x) = 1,$$

we have

$$\sum_{u \in \mathbb{Z}/p^{l-1}\mathbb{Z}} \boldsymbol{e}\left(\frac{\alpha d u^2}{p^{l-1}}\right) = \begin{cases} p^{(l-1)/2} & l \text{ is odd;} \\ \varepsilon_p\chi_p(\alpha d)p^{(l-1)/2} & l \text{ is even.} \end{cases}$$

Thus

$$S_2 = \sum_{n=0}^{\infty} p^{-(2m+2+2n+1)s+2m+n+1}\overline{\psi}(\alpha)G(\psi) \sum_{d\in\mathbb{Z}/p^{m+2n+2}\mathbb{Z}} \psi(d)\boldsymbol{e}\left(\frac{\beta d}{p^{2n+2-t}}\right)$$
$$+ \sum_{n=1}^{\infty} p^{-(2m+2+2n)s+2m+n+1/2}(\chi_p\overline{\psi})(\alpha)G(\psi)\varepsilon_p \sum_{d\in\mathbb{Z}/p^{m+2n+1}\mathbb{Z}} (\chi_p\psi)(d)\boldsymbol{e}\left(\frac{\beta d}{p^{2n+1-t}}\right).$$

The first summation remains only when $t = 2n+1$, which equals to

$$\begin{cases} p^{-(2m+2+t)s+3m+(3t+1)/2}\overline{\psi}(\alpha\beta)G(\psi)^2 & t \text{ is odd;} \\ 0 & t \text{ is even.} \end{cases}$$

For the second term, if $\psi \neq \chi_p$ then it remains only when $t = 2n$ and

$$\begin{cases} p^{-(2m+2+t)s+3m+(3t+1)/2}\varepsilon_p(\chi_p\overline{\psi})(\alpha\beta)G(\psi)G(\psi\chi_p) & t \geq 2 \text{ is even;} \\ 0 & t \text{ is odd or } t = 0. \end{cases}$$

On the other hand if $\psi = \chi_p$, then the second term is

$$\begin{cases} 0 & t = 0; \\ (\chi_p\overline{\psi})(\alpha)G(\psi)\varepsilon_p p^{-(2m+2)s+3m+1/2} \\ \quad \times\left\{(p-1)\displaystyle\sum_{n=1}^{(t-2)/2} p^{(3-2s)n} - p^{(3-2s)t/2}\right\} & t \geq 2 \text{ is even;} \\ (\chi_p\overline{\psi})(\alpha)G(\psi)\varepsilon_p(p-1)\displaystyle\sum_{n=1}^{(t-1)/2} p^{-(2m+2n+2)s+3m+3n+1/2} & t \text{ is odd.} \end{cases}$$

As a consequence,

$$S_2 = \begin{cases} 0 & \text{if } t = 0; \\ \varepsilon_p^2 p^{-(2m+2)s+3m+1} \\ \quad \times\left\{(p-1)\displaystyle\sum_{n=1}^{(t-2)/2} p^{(3-2s)n} - p^{(3-2s)t/2}\right\} & \text{if } \psi = \chi_p \text{ and } t \geq 2 \text{ is even;} \\ \varepsilon_p p^{-(2m+2)s+3m+1} \\ \quad \times\left\{p^{(3-2s)t+1/2}\overline{\psi}(\alpha\beta) + \varepsilon_p(p-1)\displaystyle\sum_{n=1}^{(t-1)/2} p^{(3-2s)n}\right\} & \text{if } \psi = \chi_p \text{ and } t \text{ is odd;} \\ p^{-(2m+2+t)s+3m+(3t+3)/2}\varepsilon_p(\chi_p\overline{\psi})(\alpha\beta)G(\psi)G(\psi\chi_p) & \text{if } \psi \neq \chi_p \text{ and } t \geq 2 \text{ is even;} \\ p^{-(2m+2+t)s+3m+(3t+1)/2}\overline{\psi}(\alpha\beta)G(\psi)^2 & \text{if } \psi \neq \chi_p \text{ and } t \text{ is odd.} \end{cases} \tag{4.5}$$

Finally we calculate the term for $C = T(k,l)W$ and $W = \left(\begin{smallmatrix} pu & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $u \in \mathbb{Z}/p^{l-1}\mathbb{Z}$.

$$S_3 = \sum_{k,l=1}^{\infty} \sum_{u \in \mathbb{Z}/p^{l-1}\mathbb{Z}} p^{-(2k+l)s} \sum_{a \in \mathbb{Z}/p^k\mathbb{Z}} \psi(a)\boldsymbol{e}\left(\frac{a\beta}{p^{k-m-t}}\right)$$
$$\times \sum_{b \in \mathbb{Z}/p^k\mathbb{Z}} \boldsymbol{e}\left(\frac{2b\beta u}{p^{k-m-t-1}}\right) \sum_{d} \psi(d)\boldsymbol{e}\left(\frac{(\alpha+\beta u^2 p^{t+2})d}{p^{k+l-m}}\right).$$

By looking at the first term, this summations remains only when $k = m+t+1$, and

$$S_3 = p^{-(2k+l)s+2m+2t}\overline{\psi}(\beta)G(\psi) \sum_{u \in \mathbb{Z}/p^{m+t}\mathbb{Z}} \sum_{d \in \mathbb{Z}/p^{m+t+l+1}\mathbb{Z}} \psi(d)\boldsymbol{e}\left(\frac{(\alpha+\beta u^2 p^{t+2})d}{p^{t+l+1}}\right).$$

Since $(\alpha+\beta u^2 p^{t+2}, p) = 1$, the last term remains only when $t+l=0$, but this does not happen. Thus $S_3 = 0$ and we conclude the proof. $\square$

Remark. One should notice that $S_2^p(\psi,h,s)$ depends only on $\det h = \alpha\beta p^{2m+t}$ and $\mathrm{ord}_p(\mathrm{g.c.d.}(h_1, 2h_2, h_3)) = m$ for $h = \left(\begin{smallmatrix} h_1 & h_2 \\ h_2 & h_3 \end{smallmatrix}\right)$ (i.e. not depend on α, β).

Remark. In [**Miz**], Mizuno gives the explicit form of the Fourier expansion of $E^k_{p,\psi}$, by using the Maass lift of the Eisenstein series of Jacobi forms. The above lemma will give an another proof of that.

5. Application –the dimension of the space of Eisenstein series–.

In this section we shall compute the dimension of the space of Eisenstein series for low weights. Let $C_0(f)$ be the constant term of the Fourier expansion of $f \in M_k(\Gamma^g(N))$ and

$$\mathcal{E}_k(\Gamma^g(N)) = M_k(\Gamma^g(N))/\{f \in M_k(\Gamma^g(N)) \mid C_0(f|_k\gamma) = 0,\ \forall \gamma \in Sp(g,\mathbb{Z})\}.$$

We denote $\mathcal{E}_k(\Gamma_0^g(N),\psi)$ the image of $M_k(\Gamma_0^g(N),\psi)$.

The aim of this section is to calculate the dimension of $\mathcal{E}_k(\Gamma^2(p))$. The classical theory says that for $k \geq 4$,

$$\dim \mathcal{E}_k(\Gamma^2(p)) = \frac{1}{2}(p^4-1).$$

Moreover it is shown that

Proposition 5.1 ([**Gu**, Theorem 3.1]).

$$\dim \mathcal{E}_1(\Gamma^2(p)) = \begin{cases} 0 & p \equiv 1 \bmod 4; \\ \dfrac{1}{2}(p^2+1) & p \equiv 3 \bmod 4. \end{cases}$$

Thus it suffices to consider the case $k = 2$ or 3. Before considering these cases, we explain how one can induce the results for $\Gamma^2(p)$ from that of $\Gamma_0^2(p)$.

Let $G = Sp(2, \mathbb{F}_p) = \Gamma^2/\Gamma^2(p)$. Then G acts on $M_k(\Gamma^2(p))$ (or $\mathcal{E}_k(\Gamma^2(p))$) from the left via $(f, g) \mapsto f|_k \tilde{g}^{-1}$, with $f \in M_k(\Gamma^2(p))$, $g \in G$ and a lift $\tilde{g}$ of g to Γ^2.

Recall that $P_0 = \{\gamma \in \Gamma^g \mid C_\gamma = 0\}$, which corresponds to the Siegel parabolic subgroups. Put $\overline{P}_0 = \{g \in G \mid C_g = 0,\ \det D_g \in \{\pm 1\}\}$, and u_0 the character of P_0 or $\overline{P}_0$ defined by $u_0(\gamma) = \det D_\gamma$. Notice that $\overline{P}_0$ is the image of P_0 under the canonical map $\Gamma^2 \to G$. Then we have

$$C_0(f|_k\gamma) = u_0(\gamma)^k C_0(f) \quad \text{for } \gamma \in P_0 \tag{5.1}$$

(cf. [**Gu**, Lemma 3.2]).

Let $H = \{h \in G \mid C_h = 0\}$ be a subgroup of G. Notice that H is the image of $\Gamma_0^2(p)$ under the canonical map $\Gamma^2 \to G$. The character $\widetilde{\psi}$ of H is defined by $\widetilde{\psi}(h) = \psi(\det D_h)$ for a Dirichlet character ψ modulo p.

Lemma 5.2 ([**Gu**, Lemma 3.3, 3.4]). *The representation of G on $\mathcal{E}_k(\Gamma^2(p))$ is isomorphic to a sub-representation of*

$$\mathrm{Ind}_{\overline{P}_0}^G(u_0^k) = \bigoplus_{\psi(-1)=(-1)^k} \mathrm{Ind}_H^G(\widetilde{\psi}).$$

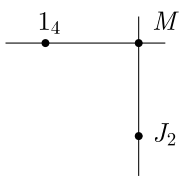
From the Frobenius reciprocity law

$$\mathrm{Hom}_G\big(\mathrm{Ind}_H^G(\widetilde{\psi}), \mathcal{E}_k(\Gamma^2(p))\big) \simeq \mathrm{Hom}_H\big(\widetilde{\psi}, \mathcal{E}_k(\Gamma^2(p))\big) \simeq \mathcal{E}_k(\Gamma_0^2(p), \overline{\psi}). \tag{5.2}$$

Recall that $\Gamma_0^2(p)\backslash\mathbb{H}_2$ has three 0-dimensional cusps, that is a representative set of $\Gamma_2^0(p)\backslash Sp(2,\mathbb{Z})/P_0$ is give by three elements

$$1_4,\ J_2,\ \text{and } M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The structure of the boundary of the Satake compactification of $\Gamma_0^2(p)\backslash\mathbb{H}_2$ is given by the following figure.



The meaning of the above figure is as follows. For $f \in M_k(\Gamma_0^2(p))$, put

$$\Phi(f)(z) = \lim_{\lambda \to \infty} f\left(\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix} \right), \quad z \in \mathbb{H}_1,$$

which is called the Siegel operator. Then

$$C_0(f|_k M) = C_0(\Phi(f)|_k J_1) = C_0(\Phi(f|_k J_2)|_k J_1). \tag{5.3}$$

We use the following Lemma.

Lemma 5.3 ([**Gu**, Lemma 3.7]). *If $\psi^2 \not\equiv 1$, then $C_0(f|_k M) = 0$ for all $f \in M_k(\Gamma_0^2(p), \psi)$.*

In particular

$$\dim \mathcal{E}_k(\Gamma^2(p), \psi) \le 2 \quad \text{if} \quad \psi^2 \not\equiv 1. \tag{5.4}$$

Finally we quote the result of Srinivasan [**Sr**], which classified all the irreducible characters of $Sp(2, \mathbb{F}_p)$. Fix a generator ξ of $\mathbb{F}_p^\times$ and define the Dirichlet character ψ_l by $\psi_l(\xi^a) = \boldsymbol{e}(al/(p-1))$. Then

$$\mathrm{Ind}_H^G(\widetilde{\psi}_l) = \begin{cases} 1_G \oplus \underbrace{\theta_9}_{p(p+1)^2/2} \oplus \underbrace{\theta_{11}}_{p(p^2+1)/2} & l = 0; \\ \underbrace{\theta_3}_{(p^2+1)/2} \oplus \underbrace{\theta_4}_{(p^2+1)/2} \oplus \underbrace{\Phi_9}_{p(p^2+1)} & l = \dfrac{p-1}{2}; \\ \underbrace{\chi_8(|l|)}_{(p+1)(p^2+1)} & -\dfrac{p-3}{2} \le l \le \dfrac{p-3}{2},\ l \ne 0. \end{cases} \tag{5.5}$$

5.1. The case of weight 3.
Let $g = 2$ and $k = 3$. We prove the following theorem.

Theorem 5.4.

$$\dim \mathcal{E}_3(\Gamma^2(p)) = \frac{1}{2}(p^4 - 1).$$

To prove this theorem, we show the following.

Theorem 5.5. *For $\psi(-1) = -1$, we have*

$$\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) = \begin{cases} 3 & \psi^2 \equiv 1; \\ 2 & \psi^2 \not\equiv 1. \end{cases}$$

Proof of Theorem 5.4 under Theorem 5.5. By Lemma 5.2 and (5.2), we know that $\dim \mathcal{E}_3(\Gamma_0^2(p),\psi)$ is the number of the irreducible component of $\mathrm{Ind}_H^G(\widetilde{\psi})$ contained in $\mathcal{E}_3(\Gamma^2(p))$. Then Theorem 5.5 and (5.5) shows that $\mathrm{Ind}_{\overline{P}_0}^G u_0 = \mathcal{E}_3(\Gamma^2(p))$, whose dimension is $(p^4-1)/2$. $\square$

Let us start to prove Theorem 5.5. Shimura proved the holomorphy of $E^3_{p,\psi}(Z) = E^3_{p,\psi}(Z,0)$ in [**Sh2**, Theorem 7.1], by considering the Fourier expansion of $E^3_{p,\psi}|_3 J_2(Z)$. Moreover one can write down the Fourier expansion of $E^3_{p,\psi}(Z,s)$ explicitly by the result of Section 3–Section 5. From (2.4), the constant term of $E^3_{p,\psi}(Z,s)$ is given by

$$1 + S_1(\psi,0,3+2s)\sum_{(q_1,q_2)=1}\xi_1(Y[\left(\begin{smallmatrix} q_1\\ q_2\end{smallmatrix}\right)],0,3+s,s) + S_2(\psi,0,3+2s)\xi_2(Y,0,3+s,s).$$

By Proposition 3.3 the second term is 0 for any odd character ψ. For the third term, by Theorem 3.1 we have

$$\xi_2(Y,0,3+s,s) = (\mathrm{const.}) \times \frac{\Gamma(2s+1/2)\Gamma(2s)}{\Gamma(s+3)\Gamma(s+1/2)\Gamma(s)\Gamma(s-1/2)},$$

which has a zero at $s=0$, while $S_2(\psi,0,3+2s)$ is 0 or at least finite at $s=0$ by Theorem 3.4 and Lemma 4.1 (note that $L(1,\psi)$ is finite since $\psi \not\equiv 1$). Hence the constant term is 1. Similarly one can check that, for h with $\operatorname{rank} h = 1$, $\xi_2(Y,h,s+3,s)S_2(\psi,h,2s+3) = 0$ at $s=0$. As a consequence we have

$$\begin{aligned} E^3(Z,s) = 1 &+ \sum_{m=1}^{\infty}\sum_{\substack{(q_1,q_2)\in\mathbb{Z}^2/\{\pm 1\}\\ (q_1,q_2)=1}} \frac{6\psi(m')\sigma_2^{\psi}(m)}{B_{3,\overline{\psi}}p^{2\,\mathrm{ord}_p m}}\boldsymbol{e}\left(m\begin{pmatrix} q_1^2 & q_1q_2\\ q_1q_2 & q_2^2\end{pmatrix}Z\right)\\ &- (2\pi)^5\sum_{\substack{h\in\mathrm{Sym}^2(Z)^*\\ h>0}}(\det h)^{3/2}S_2(\psi,h,3)\boldsymbol{e}(hZ), \end{aligned} \tag{5.6}$$

here $\sigma_k^{\psi}(m) = \sum_{d|m}\psi(d)d^k$ and $m' = m/p^{\mathrm{ord}_p m}$ for each m. Also this formula shows the holomorphy of $E^3_{p,\psi}(Z)$.

We shall compute the value of $E^3_{p,\overline{\psi}}(Z) \in \mathcal{E}_2(\Gamma_0^2(p),\psi)$ at each 0-dimensional cusp of $\Gamma_0^2(p)\backslash\mathbb{H}_2$. By (5.6) we have

$$C_0\big(E^3_{p,\overline{\psi}}(Z)\big) = 1.$$

For the value at J_2, as is calculated in [**Sh2**], one can write

$$\begin{aligned} &\big((\det Y)^s E^k_{p,\overline{\psi}}\big)|_k J_2(Z,s)\\ &\quad = \det(Y)^s \sum_{(C,D)\in\mathcal{M}_2,\ D\equiv 0 \bmod p}\psi(\det C)\det(CZ+D)^{-k}|\det(CZ+D)|^{-2s} \end{aligned}$$

$$= \frac{\det(Y)^s}{p^{2(k+2s)}} \sum_{h \in \mathrm{Sym}^2(\mathbb{Z})^*} \prod_{q \neq p} S_2^q(\psi, h, k+2s) \xi_2\left(\frac{1}{p}Y, h, k+s, s\right) \boldsymbol{e}\left(\frac{1}{p}hX\right). \tag{5.7}$$

If $\operatorname{rank} h < 2$, $\prod_{q \neq p} S_2^q$ is finite at $s = 0$, while ξ_2 has a zero at $s = 0$ thanks to the term "$\Gamma_{g-q}(\beta)^{-1}$" in Theorem 3.1. Thus we have

$$E^3_{p,\overline{\psi}}|_3 J_2(Z) = 0.$$

Finally we compute the value at M. It is hard to write down the Fourier expansion of $E^3_{p,\overline{\psi}}|_3 M(Z)$, since at the cusp M, the "Siegel series" does not have an Euler product. In order to compute $C_0(E^3_{p,\overline{\psi}}|_3 M(Z))$, we use the relation (5.3). The formula (5.6) shows that

$$\Phi\big(E^3_{p,\overline{\psi}}(Z)\big) = e^3_{p,\overline{\psi}}(z) := \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in P_0 \backslash \Gamma_0^1(p)} \overline{\psi}(d)(cz+d)^{-3},$$

whose infinite sum converges uniformly. Thus $C_0(E^3_{p,\psi}|_k M(Z)) = C_0(e^3_{p,\overline{\psi}}|_k J_1(z)) = 0$. As a consequence

$$C_0(E^3_{p,\overline{\psi}}|_3 \gamma) = \begin{cases} 1 & \gamma = 1_4; \\ 0 & \gamma = M,\ J_2. \end{cases} \tag{5.8}$$

We shall construct other functions. For $T \in \mathrm{Sym}^2(\mathbb{F}_p)$, put $\delta(T) = \left(\begin{smallmatrix} 0 & 1_2 \\ -1_2 & T \end{smallmatrix}\right)$. Then $\{\delta(T) \mid T \in \mathbb{F}_p\}$ is a representative set of $\Gamma_0^2(p) \backslash \Gamma_0^2(p) J_2 \Gamma_0^2(p)$. Fix $\gamma \in \Gamma_0^2(p)$. Since $\{\delta(T)\gamma\}_{T \in \mathrm{Sym}^2(\mathbb{F}_p)}$ is also a representative set of $\Gamma_0^2(p) \backslash \Gamma_0^2(p) J_2 \Gamma_0^2(p)$, for $T_i \in \mathrm{Sym}^2(\mathbb{F}_p)$, there exist $u \in \Gamma_0^2(p)$ and $T_j \in \mathrm{Sym}^2(\mathbb{F}_p)$ such that $\delta(T_i)\gamma = u\delta(T_j)$. By a direct computation we have $\widetilde{\psi}(u) = \widetilde{\psi}(\gamma)^{-1}$. Thus if we put

$$F^3_{p,\psi} = \sum_{T \in \mathrm{Sym}^2(\mathbb{F}_p)} E^3_{p,\psi}|_2 \delta(T), \tag{5.9}$$

then we have $F^3_{p,\psi} \in M_2(\Gamma_0^2(p), \psi)$. Using (5.1) and (5.8), an easy calculation shows

$$\Phi^0(F|_k \gamma) = \begin{cases} 1 & \gamma = J_2; \\ 0 & \gamma = 1 \text{ or } M. \end{cases}$$

Thus we have $\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) = 2$ by (5.4) for $\psi^2 \not\equiv 1$. If $\psi^2 \equiv 1$, put

$$G^3 := \sum_{c_1, d_2 \in \mathbb{Z}/p} E^3_{p,\psi}|_3 \alpha(c_1, d_2) + \sum_{d_1 \in \mathbb{Z}/p} E^3_{p,\psi}|_3 \beta(d_1) \tag{5.10}$$

with

$$\alpha(c_1,d_2)=\begin{pmatrix}0&0&0&-1\\-1&0&0&0\\c_1&1&0&d_2\\0&0&-1&c_1\end{pmatrix},\quad \beta(d_1)=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&d_1&0\\0&0&0&1\end{pmatrix}.$$

Here $\{\alpha(c_1,d_2),\beta(d_1)\mid c_1,d_2,d_1\in\mathbb{Z}/p\}$ is a representative set of $\varGamma_0^2(p)\backslash\varGamma_0^2(p)M\varGamma_0^2(p)$. Similarly one can show $G^3\in M_3(\varGamma_0^2(p),\psi)$ and

$$C_0(G^3|_3\gamma)=\begin{cases}1 & \gamma=M;\\ 0 & \gamma=1_4,\ J_2.\end{cases}$$

Thus $E^3_{p,\overline{\psi}}$, $F^3_{p,\psi}$ and G^3 form a basis of $\mathcal{E}_3(\varGamma_0^2(p),\psi)$, which completes the proof of Theorem 5.5. $\square$

5.2. The case of weight 2.

In this section we consider the case of weight 2. First assume that $\psi^2\not\equiv 1$. By using the explicit formula, one can check that $E^2_{p,\psi}(Z)=E^2_{p,\psi}(Z,0)\in M_2(\varGamma_0^2(p),\psi)$ (cf. [**Sh2**, Theorem 10.4]) and $C_0(E^2_{p,\psi}(Z))=1$. Moreover (5.7) shows $C_0(E^2_{p,\psi}|_2J_2)=0$. Thus we have $\dim\mathcal{E}_2(\varGamma_0^2(p),\psi)=2$, whose basis is given by $\{E^2_{p,\overline{\psi}},F^2_{p,\psi}\}$, where $F^2_{p,\psi}$ is defined as in (5.9).

Next we consider the case $\psi^2\equiv 1$. For such a character ψ, one can check that $E^2_{p,\psi}(Z,s)$ is finite at $s=0$, but *not* holomorphic in Z. Instead of considering $E^2_{p,\psi}(Z,0)$, we consider

$$\widetilde{E}_{p,\psi}(Z,s)=L(2+2s,\psi)L(2+4s,\psi^2)\det(Y)^sE^2_{p,\psi}(Z,s),$$

following [**BS**]. As is shown in [**BS**, Proposition 5.2. b)],

$$E^2_{p,\psi}(Z):=E^2\left(Z,-\frac{1}{2}\right)\in M_2(\varGamma_0^2(p),\psi)$$

for all even Dirichlet character ψ. This fact can also be shown by using our Fourier expansion.

We have the following two cases.

Case 1) $\psi=1_p$ is the trivial character modulo p. It is known that the space of elliptic Eisenstein series $\mathcal{E}_2(\varGamma_0^1(p))$ is one-dimensional, and a basis is given by $e^2_{p,1_p}(z,0)$, whose value at each cusp ∞ and 0 is 1 and $-1/p^2$, respectively. Thus the figure of the boundary shows that $\mathcal{E}_2(\varGamma_0^2(p))=1$.

Case 2) $\psi=\chi_p$ is the non-trivial quadratic character modulo p. Note that this case happens only when $p\equiv 1 \bmod 4$. Using E^2_{p,χ_p}, we can construct F^2_{p,χ_p} and G^2_{p,χ_p} similar to (5.9) and (5.10) respectively. Then the values of these functions at the cusps are

$$C_0\big(E^2_{p,\chi_p}|_2\gamma\big) = \begin{cases} 1 & \gamma = 1_4; \\ 0 & \gamma = M; \\ -\dfrac{1}{p^2} & \gamma = J_4, \end{cases} \quad C_0(F^2_{p,\chi_p}|_2\gamma) = \begin{cases} -p & \gamma = 1_4; \\ 0 & \gamma = M; \\ \dfrac{1}{p} & \gamma = J_4, \end{cases}$$

and

$$C_0(G^2_{p,\chi_p}|_2\gamma) = 0 \text{ for all } \gamma.$$

Thus we can get only one dimensional subspace in $\mathcal{E}^2(\Gamma_0^2(p),\chi_p)$. In order to get other elements, we use the theta series. Let $Q \in M_4(\mathbb{Z})$ be an even symmetric positive definite matrix of determinant p, and set $Q' = pQ^{-1}$. Put

$$\theta^Q(Z) = \sum_{N \in M_{4,2}(\mathbb{Z})} \boldsymbol{e}\left(\frac{1}{2}{}^tNQNZ\right), \qquad \theta^{Q'}(Z) = \sum_{N \in M_{4,2}(\mathbb{Z})} \boldsymbol{e}\left(\frac{1}{2}{}^tNQ'NZ\right),$$

then it is known that θ^Q, $\theta^{Q'} \in M_2(\Gamma_0^2(p),\chi_p)$ (cf. [**An**]). The values of θ^Q and $\theta^{Q'}$ at each cusp are given by

$$C_0(\theta^Q|_2\gamma) = \begin{cases} 1 & \gamma = 1_4; \\ -\dfrac{1}{\sqrt{p}} & \gamma = M; \\ \dfrac{1}{p} & \gamma = J_2, \end{cases} \qquad C_0(\theta^{Q'}|_2\gamma) = \begin{cases} 1 & \gamma = 1_4; \\ -\dfrac{1}{p\sqrt{p}} & \gamma = M; \\ \dfrac{1}{p^3} & \gamma = J_2. \end{cases}$$

However since

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & -\dfrac{1}{\sqrt{p}} & -\dfrac{1}{p\sqrt{p}} \\ -\dfrac{1}{p^2} & \dfrac{1}{p} & \dfrac{1}{p^3} \end{pmatrix} = 0,$$

$\widetilde{E}^2_{p,\chi_p}$, θ^Q and $\theta^{Q'}$ are linearly dependent in $\mathcal{E}^2(\Gamma_0^2(p),\chi_p)$. As a consequence we have the following.

THEOREM 5.6. *Let ψ be the even Dirichlet character modulo p.*

$$\dim \mathcal{E}^2(\Gamma_0^2(p),\psi) = \begin{cases} 2 & \psi^2 \neq 1; \\ 1 & \psi = 1_p; \\ 2 \text{ or } 3 & \psi = \chi_p. \end{cases}$$

Note that $\psi = \chi_p$ will occur only when $p \equiv 1 \bmod 4$.

Finally we consider the case of $\Gamma^2(p)$. First we study G-subspace in $\mathcal{E}^2(\Gamma^2(p))$ generated by $\{E^2_{p,1_p}\}$ or $\{E^2_{p,\chi_p}\}$.

Lemma 5.7. (1) *The subspace in $\mathcal{E}^2(\Gamma^2(p))$ spanned by $\{E^2_{p,1_p}|_2\gamma \mid \gamma \in \Gamma^2\}$ is $p(p^2+1)/2$ dimensional.*
(2) *The subspace in $\mathcal{E}^2(\Gamma^2(p))$ spanned by $\{E^2_{p,\chi_p}|_2\gamma \mid \gamma \in \Gamma^2\}$ is p^3+p dimensional.*

Proof. We only show (1), for (2) can be shown similarly. Since $E^2_{p,1_p}(Z) \in M_2(\Gamma^2_0(p))$, it suffices to consider the functions $E^2_{p,1_p}|_2\gamma$ with $\gamma \in \Gamma^2_0(p)\backslash\Gamma^2$. The representative set of $\Gamma^2_0(p)\backslash\Gamma^2$ is given by

$$\left\{\gamma(T) = \begin{pmatrix} 0 & -1_2 \\ 1_2 & T \end{pmatrix} \,\middle|\, T \in \mathrm{Sym}^2(\mathbb{F}_p)\right\} \amalg \left\{\delta(s,t) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & s & t & 0 \\ 0 & 0 & -s & 1 \end{pmatrix} \,\middle|\, s,t \in \mathbb{F}_p\right\}$$

$$\amalg \left\{\xi(u) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \end{pmatrix} \,\middle|\, u \in \mathbb{F}_p\right\} \amalg \{1_4\}.$$

For $A \in \mathrm{Sym}^2(\mathbb{F}_p)$ we put $F_A(Z) = \sum_{T\in\mathrm{Sym}^2(\mathbb{F}_p)} \boldsymbol{e}(-AT/p)\, E^2_{p,1_p}|_2\gamma(T)$ and $X = \{A \in \mathrm{Sym}^2(\mathbb{F}_p) \mid F_A(Z) \neq 0\}$. Then $\langle E^2_{p,1_p}|_2\gamma(T) \mid T \in \mathrm{Sym}^2(\mathbb{F}_p)\rangle_{\mathbb{C}} = \langle F_A \mid A \in X\rangle_{\mathbb{C}}$ and $\{F_A\}_{A\in X}$ are linearly independent since each F_A belongs to relatively distinct simultaneous eigen-space under the action of $U = \left\{\left(\begin{smallmatrix} 1_2 & T \\ 0 & 1_2 \end{smallmatrix}\right) \in \Gamma^2\right\}$. Now looking at the value at each 0-dimensional cusp of $\Gamma^2(p)\backslash\mathbb{H}_2$, we have $\sharp X = p(p^2+1)/2$.

Next we consider the set $\{\gamma(s,t)\}$. Put $G_{\alpha,s} = \sum_{t\in\mathbb{F}_p} \boldsymbol{e}(-\alpha t/p) E^2_{p,1_p}|_2\gamma(s,t)$, then $\langle E^2_{p,1_p}|_2\gamma(s,t) \mid s,t \in \mathbb{F}_p\rangle_{\mathbb{C}} = \langle G_{\alpha,s} \mid \alpha, s \in \mathbb{F}_p\rangle_{\mathbb{C}}$. Moreover by looking at the value at 0-dimensional cusps, we can check that $G_{\alpha,s}$ coincides to constant multiple of F_A with $A = \alpha\left(\begin{smallmatrix} 1 & s \\ s & s^2 \end{smallmatrix}\right)$.

For the set of $\{\xi(u)\}$, let $H_\alpha = \sum_{u\in\mathbb{F}_p} \boldsymbol{e}(-\alpha u) E^2_{p,1_p}|_2\xi(u)$. Also we can show that H_α equals to the constant multiple of F_A with $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & u \end{smallmatrix}\right)$.

Finally one can check that $F_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} = pE^2_{p,1_p}$. As a consequence we have $\dim\langle E^2_{p,1_p}|_2\gamma \mid \gamma \in \Gamma^2_0(p)\backslash\Gamma^2(p)\rangle_{\mathbb{C}} = \sharp X = p(p^2+1)/2$. This proves the lemma. $\square$

Now we decompose the space $\mathcal{E}_2(\Gamma^2(p))$ into the irreducible representation of G. If $p \equiv 1 \bmod 4$, it is the sub-representation of

$$\underbrace{1_G \oplus \theta_9 \oplus \theta_{11}}_{\mathrm{Ind}_H^G 1_G} \oplus \underbrace{\theta_3 \oplus \theta_4 \oplus \Phi_9}_{\mathrm{Ind}_H^G \chi_p} \oplus \bigoplus_{\substack{0<l\leq(p-3)/2 \\ l:\ \text{even}}} 2\chi_8(l),$$

if $p \equiv 3 \bmod 4$, it is the sub-representation of

$$\underbrace{1_G \oplus \theta_9 \oplus \theta_{11}}_{\mathrm{Ind}_H^G 1_G} \oplus \bigoplus_{\substack{0<l\le(p-3)/2 \\ l:\ \text{even}}} 2\chi_8(l).$$

The formula (5.2) says that the number of the irreducible components appearing in $\mathrm{Ind}_H^G \widetilde{\psi} \cap \mathcal{E}_2(\Gamma^2(p))$ equals to $\dim \mathcal{E}_2(\Gamma_0^2(p), \psi)$. If $\psi^2 \neq 1$, then $\mathrm{Ind}_H^G \widetilde{\psi} = \mathrm{Ind}_H^G \widetilde{\psi}^{-1} = \chi_8(l)$ for some l, thus every $\chi_8(l)$ appears 2 times in $\mathcal{E}_2(\Gamma^2(p))$. For the contribution of $\mathrm{Ind}_H^G 1_G$, Theorem 5.6 shows that only one of $\{1_G, \theta_9, \theta_{11}\}$ appears in $\mathcal{E}_2(\Gamma^2(p))$, it must be θ_{11} by the above lemma. Finally for the contribution of $\mathrm{Ind}_H^G \chi_p$ (it occurs only when $p \equiv 1 \bmod 4$), all or two of $\{\theta_3, \theta_4, \Phi_9\}$ appears in $\mathcal{E}_2(\Gamma^2(p))$. The above lemma shows that $\mathcal{E}_2(\Gamma^2(p))$ contains Φ_9, thus $\dim(\mathcal{E}_2(\Gamma^2(p)) \cap \mathrm{Ind}_H^G(\chi_p)) = (p+1)(p^2+1)$ or $(p+1/2)(p^2+1)$. As a consequence we have the following.

THEOREM 5.8. (1) *If* $p \equiv 3 \bmod 4$, *then*

$$\dim \mathcal{E}_2(\Gamma^2(p)) = \frac{1}{2}(p^2+1)(p^2-p-3).$$

(2) *If* $p \equiv 1 \bmod 4$, *then*

$$\dim \mathcal{E}_2(\Gamma^2(p)) = \frac{1}{2}(p^2+1)(p^2-p-3) \text{ or } \frac{1}{2}(p^2+1)(p^2-p-4).$$

OPEN PROBLEMS. At present, the author cannot determine whether $\dim \mathcal{E}_2(\Gamma_0^2(p), \chi_p)$ is 2 or 3. If $\dim \mathcal{E}_2(\Gamma_0^2(p), \chi_p)$ were 3, we need to construct a function by another method. On the other hand if $\dim \mathcal{E}_2(\Gamma_0^2(p), \chi_p)$ were 2, it seems difficult to show that, since we can only show $\dim \mathcal{E}_2(\Gamma_0^2(p), \chi_p) \le 3$ by the structure of the cusp.

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