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Edges, orbifolds, and Seiberg–Witten theory

By Claude LeBrun

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Abstract. Seiberg–Witten theory is used to obtain new obstructions to the existence of Einstein metrics on 4-manifolds with conical singularities along an embedded surface. In the present article, the cone angle is required to be of the form $2\pi/p$, p a positive integer, but we conjecture that similar results will also hold in greater generality.

Recent work on Kähler–Einstein metrics by Chen, Donaldson, Sun, and others [**5**], [**12**], [**13**], [**17**], [**26**] has elicited wide interest in the existence and uniqueness problems for Einstein metrics with conical singularities along a submanifold of real codimension 2. This article will show that Seiberg–Witten theory gives rise to interesting obstructions to the existence of 4-dimensional Einstein metrics with conical singularities along a surface. These results are intimately tied to known phenomena in Kähler geometry, and reinforce the overarching principle that Kähler metrics play a uniquely privileged role in 4-dimensional Riemannian geometry, to a degree that is simply unparalleled in other dimensions.

We now recall the definition [**2**] of an edge-cone metric on a 4-manifold. Let M be a smooth compact 4-manifold, let $\Sigma \subset M$ be a smoothly embedded compact surface. Near any point $p \in \Sigma$, we can thus find local coordinates (x^1, x^2, y^1, y^2) in which Σ is given by $y^1 = y^2 = 0$. Given any such adapted coordinate system, we then introduce an associated *transversal polar coordinate* system (ρ, θ, x^1, x^2) by setting $y^1 = \rho\cos\theta$ and $y^2 = \rho\sin\theta$. Now fix some positive constant $\beta > 0$. An edge-cone metric g of cone angle $2\pi\beta$ on (M, Σ) is a smooth Riemannian metric on $M - \Sigma$ which takes the form

$$g = d\rho^2 + \beta^2\rho^2(d\theta + u_j dx^j)^2 + w_{jk}dx^j dx^k + \rho^{1+\varepsilon}h \tag{1}$$

in a suitable transversal polar coordinate system near each point of Σ, where the symmetric tensor field h on M is required to have *infinite conormal regularity* along Σ. This last assumption means that the components of h in (x^1, x^2, y^1, y^2) coordinates have continuous derivatives of all orders with respect to collections of smooth vector fields (e.g. $\rho\,\partial/\partial\rho$, $\partial/\partial\theta$, $\partial/\partial x^1$, $\partial/\partial x^2$) which have vanishing normal component along Σ.

Thus, an edge-cone metric g behaves like a smooth metric in directions parallel to Σ, but is modelled on a 2-dimensional cone in the transverse directions.

An edge-cone metric g is said to be *Einstein* if its Ricci tensor r satisfies

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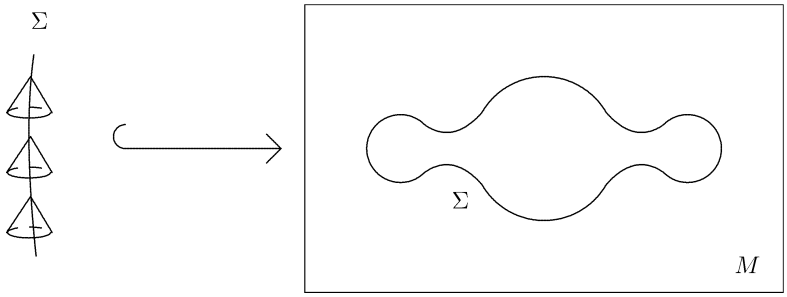

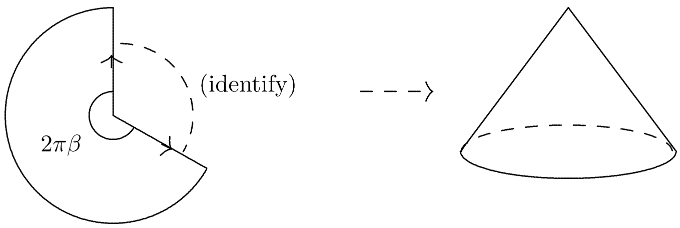

$$r = \lambda g \tag{2}$$

on $M - \Sigma$, where λ is an undetermined real constant, called the Einstein constant of g. In other words, an edge-cone metric is Einstein if and only if it has constant Ricci curvature on the complement of Σ.

Many interesting edge-cone metrics arise as orbifold metrics. Suppose that M is a smooth oriented 4-manifold and that $\Sigma \subset M$ is a smooth oriented surface. By the tubular neighborhood theorem, Σ therefore has a neighborhood which is diffeomorphic to the unit disk bundle in a complex line bundle $L \to \Sigma$. Choosing some conformal structure on Σ, we can then introduce local complex coordinates $(w, z) \in \mathbb{C}^2$ near any point of Σ, so that $z = 0$ is a local defining function for Σ, and such that $|z| < 1$ corresponds to the given tubular neighborhood. Now choose a natural number $p \geq 2$, and set $\beta = 1/p$. If we may then introduce an auxiliary complex coordinate ζ by declaring[1] that $z = \zeta^p/|\zeta|^{p-1}$. This allows us to think of $\mathbb{C}^2$ as $\mathbb{C}^2/\mathbb{Z}_p$, where the action of $\mathbb{Z}_p$ on $\mathbb{C}^2$ is generated by $(w, \zeta) \mapsto (w, e^{2\pi i/p}\zeta)$, and we can thus equip M with an orbifold structure by supplementing our original smooth atlas on $M - \Sigma$ with multi-valued coordinates (x^1, x^2, x^3, x^4) near points of Σ, where $(w, \zeta) = (x^1 + ix^2, x^3 + ix^4)$. Following standard practice in discussing orbifolds [**11**], [**44**], we will use the term *local uniformizing chart* to refer to either a smooth chart around a point of $M - \Sigma$, or to one of these multi-valued coordinate systems on a neighborhood of a point of Σ; the *local uniformizing group* is then the trivial group $\{1\}$ for a standard smooth chart, or $\mathbb{Z}_p$ for a multi-valued chart near a point of Σ. Modulo self-diffeomorphisms of the pair (M, Σ), the orbifold constructed in this way is then independent of choices, and will be denoted by (M, Σ, β), where $\beta = 1/p$.

[1] Here we have divided by $|\zeta|^{p-1}$ in order to ensure that $|z| = |\zeta|$. Later, however, it will sometimes be useful to ignore this factor, instead viewing it as representing a fixed self-homeomorphism of M that stretches our tubular neighborhood radially, away from Σ.

An orbifold metric g on (M, Σ, β) is then an object which is locally represented near Σ as a $\mathbb{Z}_p$-invariant Riemannian metric in local uniformizing coordinates, and which, on $M - \Sigma$, is a smooth Riemannian metric in the usual sense. An orbifold metric is said to be Einstein if and only if equation (2) holds in a locally uniformizing chart near each point.

If $\beta = 1/p$ for some integer $p \geq 2$, every orbifold metric on (M, Σ, β) may be viewed as an edge-cone metric on (M, Σ) of cone angle $2\pi\beta$. Of course, the converse is generally false; our definition allows edge-cone metrics to have so little regularity that most of them do not even have bounded curvature as one approaches Σ. However, the converse *does* hold in the Einstein case: Proposition 1.1, proved in Section 1 below, shows that, when $\beta = 1/p$, every edge-cone Einstein metric of cone angle $2\pi\beta$ is simply an orbifold Einstein metric described from an alternate perspective. This fact, which holds in arbitrary dimension, will allow us to obtain results concerning Einstein-edge cone metrics with $\beta = 1/p$ by applying ideas from the theory of orbifolds.

In the author's joint paper with Atiyah [**2**], topological obstructions to the existence of Einstein edge-cone metrics were obtained. These obstructions, which generalize the Hitchin–Thorpe inequality [**6**], [**23**], [**47**] for non-singular Einstein metrics, are stated purely in terms of homeomorphism invariants of the pair (M, Σ). While these obstructions do reflect several peculiar features of 4-dimensional Riemannian geometry, there is every reason to believe that they barely scratch the surface of what must actually be true. For example, in the non-singular case, there are also obstructions [**32**], [**35**], [**36**] to the existence of Einstein metrics which depend on *diffeomorphism* invariants rather than on homeomorphism invariants. The purpose of this paper is to explore the manner in which these results, which are proved using Seiberg–Witten theory, generalize to the edge-cone setting. Our main result is the following:

Theorem A. *Let X be a smooth compact oriented 4-manifold, and let $\Sigma \subset X$ be a smooth compact oriented embedded surface. Fix some integer $p \geq 2$, and set $\beta = 1/p$. Suppose that X admits a symplectic form ω_0 for which Σ is a symplectic submanifold, and such that*

$$(c_1(X) + (\beta - 1)[\Sigma]) \cdot [\omega_0] < 0 \tag{3}$$

where $[\omega_0]$, $c_1(X)$, $[\Sigma] \in H^2(X)$ are respectively the de Rham class of ω_0, the first Chern class of (X, ω_0), and the Poincaré dual of Σ. Choose a non-negative integer ℓ such that

$$\ell \geq \frac{1}{3}(c_1(X) + (\beta - 1)[\Sigma])^2, \tag{4}$$

let $M \approx X \# \ell \overline{\mathbb{CP}}_2$ be the manifold obtained by blowing X up at ℓ points of $X - \Sigma$, and notice that Σ can also be viewed as a submanifold of M. Then (M, Σ) does not carry any Einstein edge-cone metrics of cone angle $2\pi\beta$.

It should be emphasized that the symplectic form ω_0 plays a purely auxiliary role here, and merely guarantees that a certain differential-topological invariant is non-trivial; the putative Einstein metrics under discussion here are emphatically *not* assumed to satisfy any local condition involving ω_0. The constraint $\beta = 1/p$ on the cone angle is essential for our proof, but is presumably just an artifact of our method. In any case, we will see in Section 5 below that Theorem A obstructs the existence of orbifold Einstein metrics in concrete circumstances where the results of [**2**] do not lead to such a conclusion.

Notice that, since $\beta = 1/p < 1$ and $\int_\Sigma \omega_0 > 0$, condition (3) automatically holds whenever $c_1(X) \cdot [\omega_0] \leq 0$. Results of Taubes [**46**] and Liu [**41**] therefore imply that (3) automatically holds whenever X is not a blow-up of $\mathbb{CP}_2$ or some ruled surface. But even in these exceptional cases, (3) will still hold whenever $\Sigma \subset X$ has sufficiently high degree with respect to $[\omega_0]$.

Our proof of Theorem A uses Proposition 1.1 to first reduce the problem to a question regarding orbifold metrics. An orbifold version of Seiberg–Witten theory is then used to obtain curvature estimates without imposing the Einstein condition, but assuming orbifold regularity. Our main result in this direction is the following:

Theorem B. *Let (X, ω_0) be a compact symplectic 4-manifold, and let $\Sigma \subset X$ be a compact embedded symplectic surface. Let $M \approx X \# \ell \overline{\mathbb{CP}}_2$ be the manifold obtained by blowing up X at $\ell \geq 0$ points that do not belong to Σ. For a positive integer p, set $\beta = 1/p$, and let (M, Σ, β) be the orbifold version of M for which the total angle around Σ is $2\pi\beta$. If (3) holds, then the curvature of any orbifold Riemannian metric g on (M, Σ, β) satisfies*

$$\int_M s^2 d\mu \geq 32\pi^2 (c_1(X) + (\beta - 1)[\Sigma])^2 \tag{5}$$

$$\int_M (s - \sqrt{6}|W_+|)^2 d\mu \geq 72\pi^2 (c_1(X) + (\beta - 1)[\Sigma])^2 \tag{6}$$

where s, W_+ and $d\mu$ respectively denote the scalar curvature, self-dual Weyl curvature, and volume form of g. Moreover, both inequalities are strict unless $\ell = 0$ and the orbifold metric g on (X, Σ, p) is Kähler–Einstein.

Trailblazing work by Kronheimer and Mrowka [**29**] gives one reason to hope that such results might eventually be proved to hold for general edge-cone metrics, and not just for metrics with orbifold regularities. At the moment, however, this remains unsettled, even when $\beta = 1/p$. Nonetheless, it seems reasonable to hope that results like Theorem A will in fact still hold for more general cone angles. For the present, however, we believe that it is useful to explain what currently can be proved for these special cone angles, in the hope that these results will provide a reliable indicator of what one might expect to hold in greater generality.

1. Edges and orbifolds.

Let us begin our discussion by proving a regularity result, inspired by [**14**], which will play a key role in what follows. The gist is that, when $\beta = 1/p$, every Einstein edge-cone metric is actually an orbifold metric.

Proposition 1.1. *Let g be an Einstein edge-cone metric on (M, Σ) of cone angle*

$2\pi\beta$, where $\beta = 1/p$ for some positive integer p. Then g naturally extends to (M, Σ, β) as an orbifold Einstein metric.

PROOF. After passing to the ramified cover obtained by introducing the new angular coordinate $\tilde{\theta} = \beta\theta$ and then setting $x^3 = \rho\cos\tilde{\theta}$ and $x^4 = \rho\sin\tilde{\theta}$, our edge-cone metric (1) takes the form

$$g = g_0 + \rho^{1+\varepsilon} h$$

where g_0 is a smooth metric in (x^1, x^2, x^3, x^4) coordinates, $\varepsilon > 0$ is a small positive constant, and the tensor field h has infinite conormal regularity along the x^1x^2 coordinate plane. Now notice that

$$\begin{aligned}\frac{\partial}{\partial x^3}\rho^{1+\varepsilon}h_{jk} &= (\cos\tilde{\theta})\frac{\partial}{\partial\rho}\rho^{1+\varepsilon}h_{jk} - \frac{\sin\tilde{\theta}}{\rho}\frac{\partial}{\partial\tilde{\theta}}\rho^{1+\varepsilon}h_{jk} \\ &= \rho^{\varepsilon}\left[(\cos\tilde{\theta})\left(\rho\frac{\partial}{\partial\rho} + (1+\varepsilon)\right)h_{jk} - (\sin\tilde{\theta})\frac{\partial}{\partial\tilde{\theta}}h_{jk}\right]\end{aligned}$$

is of class $C^{0,\varepsilon}$, and similar computations for the other first partial derivatives of $\rho^{1+\varepsilon}h_{jk}$ show that they, too, belong to $C^{0,\varepsilon}$. It follows that the components g_{jk} of g in (x^1, x^2, x^3, x^4) coordinates are of class $C^{1,\varepsilon}$. Similarly, on a neighborhood of, say, the origin, the second derivatives of g_{jk} are bounded by a constant times $\rho^{\varepsilon-1}$. Since $\rho^{2(\varepsilon-1)}$ has finite integral on a disk about the origin in $\mathbb{R}^2$, this means that, on a neighborhood of the origin in $\mathbb{R}^4$, the metric components g_{jk} actually belong to $C^{1,\varepsilon} \cap L^2_2$.

Since g is of class $C^{1,\varepsilon}$, there exist harmonic coordinates $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4)$, which depend on (x^1, x^2, x^3, x^4) via a $C^{2,\varepsilon}$ diffeomorphism [**16**, Lemma 1.2], and rewriting the metric in these coordinates does not alter the fact that the metric components g_{jk} belong to $C^{1,\varepsilon} \cap L^2_2$. In these harmonic coordinates, the Einstein equation (2) now takes the form

$$\Delta g_{jk} = 2\lambda g_{jk} + Q_{jk}(g, \partial g) \tag{7}$$

and our hypotheses tell us that this equation is satisfied *almost everywhere* in the classical sense. Here the metric Laplacian takes the simplified form

$$\Delta = -g^{\ell m}\partial_\ell\partial_m$$

in our harmonic coordinates [**16**], while the term $Q_{jk}(g, \partial g)$ is quadratic in first derivatives of g, with coefficients expressed in terms of g and its inverse, and so, in our case, is of class $C^{0,\varepsilon}$. Thus, for any choice of j and k, the function $u = g_{jk} \in L^2_2 \cap L^\infty$ solves an elliptic equation

$$a^{\ell m}\partial_\ell\partial_m u = f$$

almost everywhere, where the coefficients $a^{\ell m} = -g^{\ell m}$ and the inhomogeneous term

$f = -2\lambda g_{jk} - Q_{jk}(g, \partial g)$ both belong to $C^{0,\varepsilon}$. By a classic result of Ladyzhenskaya–Ural′tseva [**30**, Chapater 3, Theorem 12.1], it follows that the function $u = g_{jk}$ is actually a function of class $C^{2,\varepsilon}$. In particular, by continuity, g_{jk} must solve (7) everywhere in the classical sense, so that g is a bona fide Einstein metric in our entire coordinate domain. Bootstrapping then allows one to conclude that g_{jk} is smooth. Indeed, a stronger result of DeTurck–Kazdan [**16**, Theorem 5.2] actually tells us that g is real analytic in these harmonic coordinates; moreover, it is real analytic in geodesic normal coordinates as well.

Since the given $\mathbb{Z}_p$ action preserved our C^1 metric g in the original (x^1, x^2, x^3, x^4) coordinate system, it sends geodesics of our real-analytic metric to other geodesics. Hence this finite group of isometries becomes a group of linear maps of $\mathbb{R}^4$ in geodesic normal coordinates centered at a point of Σ. Modding out by this action makes this chart into a local uniformizing chart for (M, Σ, β). Moreover, the transition functions between two such charts preserves the real-analytic metric g, and are therefore real-analytic. In particular, these charts are compatible with our smooth atlas on $M - \Sigma$. □

While the above proof has, for notational consistency, only been presented here in dimension 4, one can define an edge-cone metric in arbitrary dimension [**2**], and it is therefore worth pointing out that the same argument works without essential changes in dimension n. In this setting, one thus obtains the same regularity result for Einstein edge-cone metrics g with cone angle $2\pi/p$ along an arbitrary submanifold $\Sigma^{n-2} \subset M^n$ of codimension 2.

2. Orbifolds, indices, and all that.

In this section, we explain various simple facts about orbifolds that will turn out to be vital for our purposes.

Recall [**11**], [**44**] that a smooth n-dimensional orbifold X is a Hausdorff second-countable space, equipped with an open covering $\{U_{\mathfrak{J}}\}$ and a collection of homeomorphisms $\phi_{\mathfrak{J}} : U_{\mathfrak{J}} \to V_{\mathfrak{J}}/\Gamma_{\mathfrak{J}}$, where $V_{\mathfrak{J}} \subset \mathbb{R}^n$ is an open set and $\Gamma_{\mathfrak{J}} < GL(n, \mathbb{R})$ is a finite matrix group, such that the transition functions $\phi_{\mathfrak{J}\mathfrak{K}} := \phi_{\mathfrak{J}} \circ \phi_{\mathfrak{K}}^{-1}$ lift as diffeomorphisms $\tilde{\phi}_{\mathfrak{J}\mathfrak{K}}$ between appropriate regions of $\mathbb{R}^n$. The multi-valued maps $U_{\mathfrak{J}} \multimap V_{\mathfrak{J}} \subset \mathbb{R}^n$ induced by the $\phi_{\mathfrak{J}}$ are called *local uniformizing charts* for X, while the groups $\Gamma_{\mathfrak{J}}$ are called the associated *local uniformizing groups*. Any orbifold can be written as the disjoint union of its *regular set* and its *singular set*. The singular set consists of points which correspond to fixed points of $\Gamma_{\mathfrak{J}}$ in some local uniformizing chart; its complement, the regular set, is open and dense, and our definition canonically endows the regular set with the structure of a smooth n-manifold.

If M is a smooth compact oriented 4-manifold, if $\Sigma \subset M$ is a compact oriented surface, and if $p \geq 2$ is any integer, we have already observed that we may endow M with an orbifold structure by using the usual smooth charts on $M - \Sigma$, while modeling neighborhoods of point of Σ on $\mathbb{R}^4/\mathbb{Z}_p$, where the $\mathbb{Z}_p$ action on $\mathbb{R}^4 = \mathbb{C}^2$ is generated by $(z_1, z_2) \mapsto (z_1, e^{2\pi i/p} z_2)$. Throughout this article, the resulting orbifold is denoted by (M, Σ, β), where $\beta = 1/p$. For these examples, the singular set is of course Σ, whereas $M - \Sigma$ is the regular set.

2.1. De Rham cohomology and Hodge theory.

A tensor field ψ on a smooth orbifold X is an object which is represented by a smooth, $\Gamma_{\mathfrak{J}}$-invariant tensor field on the co-domain $V_{\mathfrak{J}}$ of each local uniformizing chart, such that these local representatives transform under the transition functions $\tilde{\phi}_{\mathfrak{J}\mathfrak{K}}$ via the appropriate representation of the Jacobian matrix. In other words, a tensor field on X is a smooth tensor field on the regular set which extends across each singular point as a $\Gamma_{\mathfrak{J}}$-invariant tensor field in a local uniformizing chart. For example, as we have already seen, an orbifold metric is an object which, in locally uniformizing charts, is simply a Riemannian metric which is invariant under the action of the locally uniformizing group.

Another important class of tensor fields consists of the skew-symmetric covariant tensors, better known as differential forms. If X is a smooth orbifold, we thus have, for each non-negative integer k, a sheaf $\mathcal{E}^k$ of real-valued differential k-forms on X. This sheaf is fine, in the sense that it admits partitions of unity, and hence is acyclic. The usual Poincaré lemma, combined with averaging over the action of $\Gamma_{\mathfrak{J}}$, shows that the Poincaré lemma also holds for orbifolds. The orbifold de Rham complex therefore provides an acyclic resolution

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^k \xrightarrow{d} \cdots$$

of the constant sheaf. The abstract de Rham theorem [**49**] thus shows that the de Rham cohomology of any orbifold computes the Čech cohomology with real coefficients; and since any orbifold is locally contractible, this Čech cohomology is in turn isomorphic to the singular cohomology of the underlying topological space, with real coefficients:

$$H^k(X,\mathbb{R}) \cong \check{H}^k(X,\mathbb{R}) \cong H^k_{DR}(X,\mathbb{R}) := \frac{\ker d : \mathcal{E}^k(X) \to \mathcal{E}^{k+1}(X)}{\mathrm{im}\ d : \mathcal{E}^{k-1}(X) \to \mathcal{E}^k(X)}.$$

Cohomology with complex coefficients can similarly be computed using complex-valued orbifold forms $\mathcal{E}^\bullet_{\mathbb{C}}$:

$$H^k(X,\mathbb{C}) \cong \check{H}^k(X,\mathbb{C}) \cong H^k_{DR}(X,\mathbb{C}) := \frac{\ker d : \mathcal{E}^k_{\mathbb{C}}(X) \to \mathcal{E}^{k+1}_{\mathbb{C}}(X)}{\mathrm{im}\ d : \mathcal{E}^{k-1}_{\mathbb{C}}(X) \to \mathcal{E}^k_{\mathbb{C}}(X)}.$$

Moreover, the cup product of orbifold de Rham classes is induced by the wedge product:

$$[\varphi] \smile [\psi] = [\varphi \wedge \psi].$$

This can be proved by exactly the same argument one uses in the non-singular case [**8**, Theorem 14.28].

It makes perfectly good sense to integrate an n-form on a smooth compact oriented connected orbifold X, and integration gives rise to an isomorphism $H^n_{DR}(X) \cong \mathbb{R}$, exactly as in the manifold case. Of course, the fact that integration is defined as an operation on cohomology depends on the observation that Stokes theorem is also valid for orbifolds.

If X is a smooth compact orbifold, equipped with an orbifold Riemannian metric g, then every de Rham class is represented by a unique harmonic representative. To make this precise, let us assume, for simplicity, that the n-dimensional compact orbifold X is oriented, which is to say that there is a fixed global choice $d\mu$ of volume n-form which is compatible with the given metric g. We can then define the Hodge star operator

$$\star : \mathcal{E}^k(X) \to \mathcal{E}^{n-k}(X)$$

by requiring that

$$\varphi \wedge \star\psi = \langle \varphi, \psi \rangle \; d\mu$$

for any two k-forms φ and ψ, where $d\mu$ is the metric orbifold n-volume form and where the point-wise inner product of forms is the one induced by g. The space of (orbifold) harmonic k-forms on X is then defined by

$$\mathcal{H}^k(X, g) := \big\{\varphi \in \mathcal{E}^k(X) \mid d\varphi = 0, \;\; d(\star\varphi) = 0\big\}.$$

The *Hodge theorem* for orbifolds asserts that the tautological map

$$\begin{array}{ccc} \mathcal{H}^k(X, g) & \longrightarrow & H^k_{DR}(X, \mathbb{R}) \\ \varphi & \longmapsto & [\varphi] \end{array}$$

is an isomorphism. The injectivity of this map is elementary; if φ is harmonic, it is L^2-orthogonal to any exact form, so that

$$\int_X \|\varphi + d\psi\|^2 d\mu = \int_X \|\varphi\|^2 d\mu + \int_X \|d\psi\|^2 d\mu,$$

making φ the unique minimizer of the L^2-norm in its de Rham class.

2.2. Vector V-bundles.

Just as tensor fields on a manifold are sections of appropriate vector bundles, tensor fields on orbifolds are sections of vector V-bundles [**11**], [**43**], [**44**]. A vector V-bundle over an orbifold X has a total space E which is an orbifold, and a projection $\varpi : E \to X$ which is a smooth submersion of orbifolds. However, in contrast to the situation for ordinary vector bundles, we do not require this projection to be locally trivial. Instead, we merely require that, for some fixed real or complex vector space $\boldsymbol{V}$, there is a system of local uniformizing charts $U_{\mathfrak{J}} \to V_{\mathfrak{J}}/\Gamma_{\mathfrak{J}}$ for X which is compatible with a a system of locally uniformizing charts $\varpi^{-1}(U_{\mathfrak{J}}) \to (V_{\mathfrak{J}} \times \boldsymbol{V})/\Gamma_{\mathfrak{J}}$ for E, where the action of $\Gamma_{\mathfrak{J}}$ on $V_{\mathfrak{J}} \times \boldsymbol{V}$ is the given action on $V_{\mathfrak{J}}$ times some representation $\varrho : \Gamma_{\mathfrak{J}} \to \mathrm{End}(\boldsymbol{V})$; as usual, the transition functions for E are moreover required to lift to fiber-wise linear maps. Thus, the fiber $\varpi^{-1}(x)$ over any regular point $x \in X$ will be a copy of $\boldsymbol{V}$. However, if $x \in X$ is a singular point, $\varpi^{-1}(x)$ is in principle merely the quotient of $\boldsymbol{V}$ by a finite group, and so, in particular, may not even be a vector space. For example, the tangent bundle $TX \to X$ is a V-bundle, as are the tensor bundles $(\bigotimes^k TX) \otimes (\bigotimes^\ell T^*X) \to X$ and their complexifications.

A *section* of a V-bundle $\varpi : E \to X$ is a right inverse $f : X \to E$ for ϖ which is locally represented by an equivariant smooth function $V_{\mathfrak{J}} \to \boldsymbol{V}$. For example, a real tensor field is exactly a section of one of the V-bundles $(\bigotimes^k TX) \otimes (\bigotimes^\ell T^*X) \to X$; a complex tensor field is a section of the complexification of one of these bundles. The space of sections of a vector V-bundle $E \to X$ will be denoted by $\mathcal{E}(X, E)$; considering sections over arbitrary open subsets of $U \subset X$ gives rise to a sheaf of X which will simply be denoted by $\mathcal{E}(E)$. In the same vein, we will use $\mathcal{E}^k(E)$ to denote the sheaf of sections of the V-bundle $(\Lambda^k T^*X) \otimes_{\mathbb{R}} E$.

It also makes sense to talk about connections on vector V-bundles. By definition, a connection ∇ on a vector V-bundle $E \to M$ is a linear operator

$$\nabla : \mathcal{E}(E) \to \mathcal{E}^1(E)$$

which in local uniformizing charts is represented by a $\Gamma_{\mathfrak{J}}$-equivariant connection on the trivial vector bundle $V_{\mathfrak{J}} \times \boldsymbol{V}$.

We will need to pay especially close attention to the special case of V-line bundles, which are by definition vector V-bundles with generic fiber $\mathbb{C}$. For such a V-bundle $\varpi : L \to X$, there is thus a system of local uniformizing charts $U_{\mathfrak{J}} \to V_{\mathfrak{J}}/\Gamma_{\mathfrak{J}}$ for X which is compatible with a a system of locally uniformizing charts $\varpi^{-1}(U_{\mathfrak{J}}) \to (V_{\mathfrak{J}} \times \mathbb{C})/\Gamma_{\mathfrak{J}}$ for L, where the action of $\Gamma_{\mathfrak{J}}$ on $V_{\mathfrak{J}} \times \mathbb{C}$ is the given action on $V_{\mathfrak{J}}$ times some representation $\Gamma_{\mathfrak{J}} \to U(1)$. Since $\varpi : L \to X$ may not be locally trivial, so we cannot immediately invoke standard machinery to define its Chern class. However, tensor products of V-line bundles are again V-line bundles; in particular, if $L \to X$ is any V-line bundle, its tensor powers $L^{\otimes q}$ are also V-line bundles. Now assume the orders $|\Gamma_{\mathfrak{J}}|$ of the local uniformizing groups are bounded—as is automatically true if X is compact. If q is the least common multiple of $\{|\Gamma_{\mathfrak{J}}|\}$, then $L^{\otimes q}$ is locally trivial, and so is a line bundle in the conventional sense. We may then define the *orbifold Chern class* of L by

$$c_1^{\mathrm{orb}}(L) = \frac{1}{q} c_1(L^{\otimes q}) \in H^2(X, \mathbb{Q}).$$

If L happens to be a complex line bundle in the conventional sense, this reproduces its usual rational Chern class. Of course, this construction ignores the torsion part of the usual Chern class, but this loss of information will not be an issue for present purposes.

The orbifold Chern class of a V-line bundle can also be obtained directly, via Chern–Weil theory. If ∇ is any connection on the V-line bundle $L \to X$, its curvature F_∇ is a closed 2-form on X. Moreover, any other connection on L can be expressed as

$$\tilde{\nabla} = \nabla + \mathcal{A}$$

for a unique complex-valued 1-form $\mathcal{A}$, and the new connection's curvature is expressible in terms of the old one by

$$F_{\tilde{\nabla}} = F_\nabla + d\mathcal{A}.$$

Thus the de Rham class of F_∇ is independent of the choice of connection, and is therefore an invariant of L. In fact, one can show that

$$c_1^{\mathrm{orb}}(L) = \left[\frac{i}{2\pi}F_\nabla\right] \in H^2_{DR}(M,\mathbb{C})$$

where we have identified $H^2(X,\mathbb{Q})$ with a subset of $H^2(X,\mathbb{C}) \cong H^2_{DR}(X,\mathbb{C})$ in the usual way. Indeed, curvature forms are additive under tensor products, so it suffices to prove the assertion for the case when L is locally trivial; and for locally trivial line bundles, the usual Čech-de Rham proof [**49**, Section III.4] immediately generalizes from manifolds to orbifolds with only cosmetic changes.

Since the space of connections on a V-line bundle is an affine space modeled on $\mathcal{E}^1_{\mathbb{C}}$, we shall engage in a standard abuse of notation by referring to a connection on L as $\mathcal{A}$ rather than as ∇; in this context, $\mathcal{A}$ is actually to be understood as a connection 1-form up on $L^\times \subset L$ rather than as an ordinary 1-form down on X. We can (and will) also restrict our choice of connection ∇ by requiring that it be compatible with some fixed Hermitian structure $\langle\ ,\ \rangle$ on L. The curvature of such a connection has vanishing real part, and the space of such a connections is an affine space modeled on $\mathcal{E}^1$. This in particular makes it obvious that $c_1^{\mathrm{orb}}(L)$ belongs to real cohomology, although the fact that it is actually belongs to rational cohomology can of course only be explained by invoking other ideas.

2.3. Self-duality.

If (X,g) is an oriented Riemannian orbifold of dimension n, the Hodge star operator $\star$ may be regarded as a homomorphism

$$\star : \Lambda^p \to \Lambda^{n-p}$$

between the V-bundles of differential forms of complementary degrees, and

$$\star^2 : \Lambda^p \to \Lambda^p$$

equals $(-1)^{p(n-p)}$ times the identity. In particular, if $n = 4$, $\star$ is an involution of the V-bundle Λ^2 of 2-forms, which can therefore be expressed as the direct sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{8}$$

where $\Lambda^\pm$ is the (± 1)-eigenspace of $\star$. Sections of Λ^+ (respectively, Λ^-) are called self-dual (respectively, anti-self-dual) 2-forms. The decomposition (8) is, moreover, *conformally invariant*, in the sense that it is left unchanged if the orbifold metric g is multiplied by an arbitrary orbifold-smooth positive function. Any 2-form can thus be uniquely expressed as

$$\varphi = \varphi^+ + \varphi^-,$$

where $\varphi^\pm \in \Lambda^\pm$, and we then have

$$\varphi \wedge \varphi = \big(|\varphi^+|^2 - |\varphi^-|^2\big) d\mu_g,$$

where $d\mu_g$ denotes the metric volume form associated with our orientation.

The decomposition (8) also leads to a decomposition of the Riemann curvature tensor for any metric on an oriented 4-dimensional orbifold. Indeed, viewing the curvature tensor of g as a self-adjoint linear map

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$$

we obtain a decomposition

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right). \tag{9}$$

Here $W_+ \in \mathrm{End}(\Lambda^+)$ is the trace-free piece of its block, and is the called the *self-dual Weyl curvature* of (M, g); the anti-self-dual Weyl curvature W_- is defined analogously. Both of the objects are conformally invariant, with appropriate conformal weights. Note that the scalar curvature s is understood to act in (9) by scalar multiplication, while the trace-free Ricci curvature $\mathring{r}$ acts on 2-forms by contraction and projection to the alternating piece.

Now suppose that our oriented 4-dimensional Riemannian orbifold is also *compact*. Then Hodge theory tells us that $H^2(M, \mathbb{R}) \cong \mathcal{H}^2_g$, where

$$\mathcal{H}^2_g := \mathcal{H}^2(X, g) = \{\varphi \in \mathcal{E}^2(X) \mid d\varphi = 0,\ d(\star\varphi) = 0\}$$

is the space of harmonic 2-forms on X with respect to the given orbifold metric g. However, $\star$ is an involution of $\mathcal{H}^2_g$, so we obtain a decomposition

$$H^2(X, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g \tag{10}$$

where

$$\mathcal{H}^{\pm}_g = \{\varphi \in \mathcal{E}(X, \Lambda^{\pm}) \mid d\varphi = 0\}$$

is the space of self-dual (respectively, anti-self-dual) harmonic 2-forms. Since

$$\int_X \varphi \wedge \varphi = \int_X \big(|\varphi^+|^2 - |\varphi^-|^2\big) d\mu,$$

the intersection pairing

$$\begin{array}{ccc} H^2_{DR}(X, \mathbb{R}) \times H^2_{DR}(X, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) & \longmapsto & \int_X \varphi \wedge \psi \end{array}$$

is therefore positive-definite on $\mathcal{H}^+$ and negative-definite on $\mathcal{H}^-$, whereas $\mathcal{H}^+$ and $\mathcal{H}^-$ are also orthogonal with respect to the intersection pairing. Since the intersection pairing can be identified with the topologically-defined pairing

$$\begin{aligned} H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\ \boldsymbol{a}\ ,\ \boldsymbol{b}\ ) \qquad &\longmapsto \langle \boldsymbol{a} \smile \boldsymbol{b}, [X]\rangle \end{aligned}$$

on singular cohomology, it follows that the integers

$$b_\pm(X) = \dim \mathcal{H}^\pm$$

are oriented topological invariants of the Poincaré space X, and one may then define the *signature* of X to be

$$\tau(X) = b_+(X) - b_-(X).$$

In the special case in which $X = (M, \Sigma, \beta)$, so that X and M are homeomorphic as topological spaces, $b_\pm(X) = b_\pm(M)$ and $\tau(X) = \tau(M)$ therefore coincide with familiar topological invariants of M.

If (X, g) is a compact oriented 4-dimensional Riemannian orbifold, we can then use the decomposition (10) to define a projection

$$\begin{aligned} H^2(X,\mathbb{R}) \longrightarrow \mathcal{H}^+_g &\subset H^2(X,\mathbb{R}) \\ \boldsymbol{a} \longmapsto \boldsymbol{a}^+ & \end{aligned}$$

and thus associate a non-negative number $(\boldsymbol{a}^+)^2 = \langle \boldsymbol{a}^+ \smile \boldsymbol{a}^+, [X]\rangle$ with any cohomology class $\boldsymbol{a} \in H^2_{DR}(X,\mathbb{R})$. Of course, this number ostensibly still depends on g via the decomposition (10).

Lemma 2.1. *Let (X, g) be a compact oriented 4-dimensional Riemannian orbifold, and $\boldsymbol{a} \in H^2(X,\mathbb{R})$. If ψ is any closed orbifold 2-form such that $[\psi] = \boldsymbol{a}$ in de Rham cohomology, then*

$$\int_X |\psi^+|^2 d\mu \geq (\boldsymbol{a}^+)^2$$

with equality if and only if ψ is harmonic.

Proof. Let φ be the unique harmonic 2-form with $[\varphi] = \boldsymbol{a}$, so that

$$(\boldsymbol{a}^+)^2 = \int_X |\varphi^+|^2 d\mu.$$

Since φ is the unique minimizer of the L^2-norm in its cohomology class,

$$\int_X (|\psi^+|^2 + |\psi^-|^2) d\mu \geq \int_X (|\varphi^+|^2 + |\varphi^-|^2) d\mu$$

with equality if and only if $\psi = \varphi$. On the other hand,

$$\int_X (|\psi^+|^2 - |\psi^-|^2) d\mu = \int_X (|\varphi^+|^2 - |\varphi^-|^2) d\mu$$

because both sides compute the self-intersection number $[\psi]^2 = \boldsymbol{a}^2 = [\varphi]^2$. Averaging these two formulas then yields the desired inequality. □

The harmonic 2-forms on a compact Riemannian orbifold are exactly those orbifold 2-forms which are killed by the Hodge Laplacian

$$\Delta_d = (d + d^*)^2 = - \star d \star d - d \star d \star .$$

If ψ is a self-dual 2-form, then $\Delta_d \psi$ is also self-dual, and can, moreover, be re-expressed by means of the Weitzenböck formula [**10**]

$$\Delta_d \psi = \nabla^* \nabla \psi - 2W_+(\psi, \cdot) + \frac{s}{3}\psi. \tag{11}$$

This formula leads to various interesting interplays between curvature and topology, and even supplies interesting information about self-dual 2-forms which are not assumed to satisfy any equation at all. Indeed, taking the L^2 inner product of (11) with ψ tells us

$$\int_M \left(|\nabla \psi|^2 - 2W_+(\psi, \psi) + \frac{s}{3}|\psi|^2 \right) d\mu \geq 0,$$

since Δ_d is a non-negative operator. On the other hand, since $W_+ : \Lambda^+ \to \Lambda^+$ is self-adjoint and trace-free,

$$|W_+(\psi, \psi)| \leq \sqrt{\frac{2}{3}} |W_+||\psi|^2,$$

so it follows that any self-dual 2-form ψ satisfies

$$\int_M |\nabla \psi|^2 d\mu \geq \int_M \left(-2\sqrt{\frac{2}{3}}|W_+| - \frac{s}{3} \right) |\psi|^2 d\mu. \tag{12}$$

Moreover, assuming that $\psi \not\equiv 0$, equality holds if and only if ψ is closed, belongs the lowest eigenspace of W_+ at each point, and the two largest eigenvalues of W_+ are everywhere equal. Of course, this last assertion crucially depends on the fact [**1**], [**3**] that if $\Delta_d \psi = 0$ and $\psi \not\equiv 0$, then $\psi \neq 0$ on a dense subset of M.

2.4. Almost-complex structures.

An *almost-complex structure* J on an orbifold X is by definition a section of the V-bundle $\mathrm{End}(TX) = T^*X \otimes TX$ such that $J^2 = -\mathbf{1}$. An almost-complex structure is said to be *integrable*, or to be a *complex structure*, it X can be covered by local uniformizing charts in which it becomes the standard (constant coefficient) structure on $\mathbb{C}^m$ for some m. The latter happens if and only if there is some torsion-free orbifold connection ∇ on TX such that $\nabla J = 0$.

Suppose that M is a smooth compact 4-manifold which is equipped with an almost-complex structure J_0, and let $\Sigma \subset M$ be a smooth compact embedded surface. Suppose that Σ is a pseudo-holomorphic curve with respect to J_0, by which we mean that $J_0(T\Sigma) = T\Sigma$ at every point of Σ. Then $TM|_\Sigma$ can be made into a complex vector bundle of rank 2 by equipping it with J_0, and this can then be split as a sum $T\Sigma \oplus N$ of complex line bundles. Now choose any $U(1)$ connection on N, and use it to endow N with the structure of a holomorphic line bundle by endowing it with a $\bar{\partial}$ operator. This makes N into a complex manifold. Now use the tubular neighborhood theorem to endow a neighborhood $U \supset \Sigma$ with an integrable complex structure J_2 which exactly agrees with J_0 along Σ. In particular, the $(-i)$-eigenspace $T^{0,1}_2$ of J_2 is in general position to the $(+i)$-eigenspace $T^{1,0}_1$ of J_0 at Σ; hence this also holds in a neighborhood U' of Σ. It follows that, on U', $T^{0,1}_2$ may be expressed as the graph of a unique section of ϕ of $\Lambda^{0,1}_0 \otimes T^{1,0}_0 = \mathrm{Hom}(T^{0,1}_0, T^{1,0}_0)$, where ϕ vanishes identically along Σ. Moreover, by shrinking U' if necessary, we may also assume that $\mathrm{tr}[\bar{\phi} \circ \phi] < 1/2$ everywhere. Now let $f : M \to [0,1]$ be a cut-off function which is identically 1 on a neighborhood of Σ and which is supported on a compact subset of U'. Then the graph of $f\phi$ is then in general position to its conjugate, and so is the $(-i)$-eigenspace $T^{0,1}_1$ of a unique almost-complex structure J_1 on M which coincides with J_0 outside of U', and coincides with J_2 in a neighborhood of Σ. Moreover, the family of tensor fields $tf\phi$, $t \in [0,1]$, gives rise to a homotopy of almost-complex structure J_t which interpolates between the given almost-complex structure J_0 and the constructed almost-complex structure J_1. All of these almost-complex structures J_t exactly coincide along Σ and outside a tubular neighborhood of Σ. Insofar as homotopy classes of almost-complex structures are actually the important objects for our purposes, the point is that, by merely replacing J_0 with a homotopic almost-complex structure J_1 if necessary, we may always assume the given almost-complex structure J_0 is integrable in a neighborhood of Σ. In the same way, we may also assume, if necessary, that it is integrable in a neighborhood of any given finite collection of points of $M - \Sigma$.

Choose some integer $p \geq 2$, set $\beta = 1/p$, and consider the orbifold (M, Σ, β) with total angle $2\pi\beta$ around Σ. Of course, this is only defined modulo diffeomorphisms of M, so we are free to choose the orbifold charts for (M, Σ, β) to be adapted to the integrable complex structure we have just chosen on a neighborhood of Σ. Our objective now is to construct a homotopy class of almost-complex structures J on (M, Σ, β) that is determined by the homotopy class of almost-complex structures determined by J_0, where the homotopies on both M and (M, Σ, β) are both subject to the constraint that Σ is to remain a pseudo-holomorphic curve for all values of the time parameter t.

We will describe two different useful ways of understanding the construction. The first of these, which we will call the *holomorphic model*, is especially useful when J_0 is a complex structure on M, but can be carried out even when J_0 is merely integrable in a neighborhood of Σ. If $p \in \Sigma$ is any point, let $(w, z) \in \mathbb{C}^2$ be local holomorphic coordinates on (M, J_0) such that $z = 0$ is a local defining function for Σ. We then introduce local uniformizing complex coordinates on (M, Σ, β) by declaring that $z = \zeta^p$. This convention is often used in complex geometry [**13**], [**43**], because, when (M, J_0) is a complex manifold, the sheaf of holomorphic orbifold functions on (M, Σ, β) then coincides with the sheaf $\mathcal{O}$ of holomorphic functions on (M, J_0). We may then equip (M, Σ, β) with the unique complex structure J which agrees with J_0 on $M - \Sigma$, and coincides with the usual integrable complex structure tensor on $\mathbb{C}^2$ in the local uniformizing charts we have just introduced. However, one caveat must be borne in mind: this convention is not consistent with standard conventions regarding the definition of orbifolds! However, this is actually not a serious problem. We can hew to the standard definition by instead using what we'll call the *origami model* of (M, Σ, β), where we instead introduce uniformizing complex coordinates $(w, \tilde{\zeta})$ such that $z = \tilde{\zeta}^p / |\tilde{\zeta}|^{p-1}$. Then, if we equip M with a self-homeomorphism that simply rescales the radius function ρ within the tubular neighborhood, equals the identity outside the tubular neighborhood, is smooth on $M - \Sigma$, and behaves like $\rho \mapsto \text{const } \rho^{1/p}$ for small ρ, we then induce a diffeomorphism between the two different models for (M, Σ, β). Notice that this self-homeomorphism of M is moreover homotopic to the identity.

LEMMA 2.2. *Let M be equipped with an almost-complex structure J_0 for which Σ is a pseudo-holomorphic curve, and equip (M, Σ, β) with the associated homotopy class of almost-complex structures J. Then the Chern classes of these two spaces are related by*

$$c_1^{\text{orb}}(M, \Sigma, \beta) = c_1(M) + (\beta - 1)[\Sigma]$$

where $\beta = 1/p$.

PROOF. Let $\mathcal{I} \subset \mathcal{E}_{M,\mathbb{C}}$ be the ideal sheaf on M consisting of smooth complex-valued functions on M which vanish along Σ and whose first derivatives at Σ are J_0-linear; this is a rank-1 free $\mathcal{E}_{M,\mathbb{C}}$-module, and in fact is exactly the sheaf of sections of the smooth line bundle $L \to M$ with $c_1(L) = -[\Sigma]$. Similarly, let $\hat{\mathcal{I}}$ be the sheaf on (X, Σ, β) whose sections on $V_{\mathfrak{J}} = U_{\mathfrak{J}}/\Gamma_{\mathfrak{J}}$ are smooth complex-valued functions f on $U_{\mathfrak{J}}$ which vanish along Σ, have J-linear first derivatives there, and, when $\Gamma_{\mathfrak{J}} = \mathbb{Z}_p$, transform under the action of $e^{2\pi i/p} \in \mathbb{Z}_p$ by $f \mapsto e^{2\pi i/p} f$; this is a locally free rank-1 sheaf of $\mathcal{E}_{(M,\Sigma,\beta),\mathbb{C}}$-modules, locally generated by the complex coordinate ζ, and is actually the sheaf of orbifold-smooth sections of a V-line bundle $\hat{L}$ over (M, Σ, β). Using the holomorphic model, we have a pull-back morphism $\mathcal{I} \to \hat{\mathcal{I}}^{\otimes p}$ induced by $(w, z) = (w, \zeta^p)$, which gives rise to an isomorphism $L \cong \hat{L}^{\otimes p}$. Hence

$$c_1^{\text{orb}}(\hat{L}) = \frac{1}{p} c_1(L) = -\frac{1}{p}[\Sigma]$$

in rational cohomology. On the other hand, since $dz \wedge dw$ pulls back to become $p\zeta^{p-1} d\zeta \wedge dw$, we have an induced isomorphism

$$K = K^{\text{orb}} \otimes \hat{L}^{p-1}$$

where $K=\Lambda^{2,0}_{J_0}$ and $K^{\rm orb}=\Lambda^{2,0}_{J}$. Thus

$$\begin{aligned} c_1(K) &= c_1^{\rm orb}(K^{\rm orb})+(p-1)c_1^{\rm orb}(\hat{L})\\ &= c_1^{\rm orb}(K^{\rm orb})+\left(1-\frac{1}{p}\right)c_1(L)\\ &= c_1^{\rm orb}(K^{\rm orb})-(1-\beta)[\Sigma] \end{aligned}$$

in rational cohomology, and since

$$c_1^{\rm orb}(M,\Sigma,\beta)=-c_1^{\rm orb}(K^{\rm orb}),\qquad c_1(M)=-c_1(K),$$

we therefore have

$$c_1^{\rm orb}(M,\Sigma,\beta)=c_1(M)+(\beta-1)[\Sigma],$$

thus proving the claim. □

We will also need to explicitly understand the almost-complex structure J in the origami model of (M,Σ,β). Let us take transverse a polar coordinate system (ρ,θ,x^1,x^2) about some point of Σ. If $d\theta+\alpha$ is the imaginary part of a $U(1)$ connection on the normal bundle N of $\Sigma\subset M$, then we may take J_0 to be integrable near Σ, and given by

$$\begin{aligned} J_0=\frac{\partial}{\partial\theta}\otimes\frac{d\rho}{\rho}-\rho\frac{\partial}{\partial\rho}\otimes d\theta+\left(\frac{\partial}{\partial x^2}-\alpha_2\frac{\partial}{\partial\theta}-\alpha_1\rho\frac{\partial}{\partial\rho}\right)\otimes dx^1\\ -\left(\frac{\partial}{\partial x^1}-\alpha_1\frac{\partial}{\partial\theta}+\alpha_2\rho\frac{\partial}{\partial\rho}\right)\otimes dx^2 \end{aligned}$$

in our transverse polar coordinates. Passing to orbifold coordinates near Σ just involves replacing the polar angle θ with a new polar angle $\tilde{\theta}=\theta/p$, so that the pull-back of the above becomes

$$\begin{aligned} J_0=\frac{\partial}{\partial\tilde\theta}\otimes p\frac{d\rho}{\rho}-\frac{1}{p}\rho\frac{\partial}{\partial\rho}\otimes d\tilde\theta+\left(\frac{\partial}{\partial x^2}-p\alpha_2\frac{\partial}{\partial\tilde\theta}-\alpha_1\rho\frac{\partial}{\partial\rho}\right)\otimes dx^1\\ -\left(\frac{\partial}{\partial x^1}-p\alpha_1\frac{\partial}{\partial\tilde\theta}+\alpha_2\rho\frac{\partial}{\partial\rho}\right)\otimes dx^2. \end{aligned}$$

By contrast, the branched-cover complex structure is given by

$$\begin{aligned} \frac{\partial}{\partial\tilde\theta}\otimes\frac{d\rho}{\rho}-\rho\frac{\partial}{\partial\rho}\otimes d\tilde\theta+\left(\frac{\partial}{\partial x^2}-p\alpha_2\frac{\partial}{\partial\tilde\theta}-p\alpha_1\rho\frac{\partial}{\partial\rho}\right)\otimes dx^1\\ -\left(\frac{\partial}{\partial x^1}-p\alpha_1\frac{\partial}{\partial\tilde\theta}+p\alpha_2\rho\frac{\partial}{\partial\rho}\right)\otimes dx^2 \end{aligned}$$

and one can interpolate between these two by taking our orbifold complex structure to be the integrable complex structure

$$J = \frac{\partial}{\partial\tilde{\theta}} \otimes f(\rho)\frac{d\rho}{\rho} - \frac{1}{f(\rho)}\rho\frac{\partial}{\partial\rho} \otimes d\tilde{\theta} + \left(\frac{\partial}{\partial x^2} - p\alpha_2\frac{\partial}{\partial\tilde{\theta}} - \frac{p}{f(\rho)}\alpha_1\rho\frac{\partial}{\partial\rho}\right) \otimes dx^1$$
$$- \left(\frac{\partial}{\partial x^1} - p\alpha_1\frac{\partial}{\partial\tilde{\theta}} + \frac{p}{f(\rho)}\alpha_2\rho\frac{\partial}{\partial\rho}\right) \otimes dx^2$$

where $f(\rho)$ is a smooth positive function with $f \equiv 1$ for, say, $\rho < \epsilon$ and $f \equiv p$ for, say, $\rho > 10\epsilon$. In fact, this J is simply the pull-back of J_0 via a suitable homeomorphism which is smooth away from Σ; indeed, if we set $r = \exp(1/p)\int f(\rho)d\rho/\rho$, then J becomes J_0, with ρ replaced by r. Since, for an appropriate choice of constant of integration, $r = \text{const}\ \rho^{1/p}$ for small ρ and $r = \rho$ for large ρ, we see that this harmonizes the holomorphic and origami pictures in exactly the manner previously promised.

2.5. Symplectic structures.

PROPOSITION 2.3. *Let (M, ω_0) be a symplectic manifold, and suppose that $\Sigma \subset M$ is an embedded surface to which ω_0 restricts as an area form. Choose any integer $p \geq 2$, and set $\beta = 1/p$. Then the orbifold (M, Σ, β) also admits a symplectic form ω with $[\omega] = [\omega_0]$ in $H^2(M, \mathbb{R})$.*

PROOF. By the Weinstein tubular neighborhood theorem [**48**], a tubular neighborhood of Σ is determined up to symplectomorphism by the induced area form and the symplectic normal bundle of Σ. Thus, choosing a complex line bundle $E \to \Sigma$ of degree $[\Sigma]^2$, the symplectic form is expressible in transverse polar coordinates (ρ, θ, x^1, x^2) about some point of Σ as

$$\omega = d\left(\frac{\rho^2}{2}(d\theta + \alpha)\right) + \varpi^*\omega_\Sigma$$
$$= d\rho \wedge \rho(d\theta + \alpha) + \frac{1}{2}\rho^2\varpi^*\Omega + \varpi^*\omega_\Sigma$$

where ω_Σ is the induced area form on Σ, $\varpi : E \to \Sigma$ is the bundle projection, and $d\theta + \alpha$ is the imaginary part of a $U(1)$ connection form on E, with curvature $i\Omega = id\alpha$, expressed for concreteness in terms of a local connection form α on Σ. If we now set $\tilde{\theta} = \beta\theta$ and $r = \rho/\sqrt{\beta}$, where $\beta = 1/p$, we then have

$$\omega = d\left(\frac{r^2}{2}(d\tilde{\theta} + p\alpha)\right) + \varpi^*\omega_\Sigma$$
$$= dr \wedge r(d\tilde{\theta} + p\alpha) + \frac{1}{2}r^2\varpi^*(p\Omega) + \varpi^*\omega_\Sigma$$

which may be viewed, in the origami model, as an orbifold symplectic structure on the tubular neighborhood. □

The key point is that there is no *symplectic* difference between a 2-dimensional cone and a 2-dimensional disk. The fact that they are *metrically* different reflects different choices of almost-complex structure. Now notice that ω is invariant under the action of the almost-complex structure J explicitly described in the origami model at the end of Section 2.4, and that $\omega(\cdot, J\cdot)$ is moreover positive-definite. In the symplectic case, this gives a self-contained characterization of the homotopy class of J on (M, Σ, β).

2.6. The Todd genus.

If (M, J_0) is a complex surface, and if $\Sigma \subset M$ is a holomorphic curve, then, for any integer $p \geq 2$, the so-called holomorphic model of (M, Σ, β), $\beta = 1/p$, has some remarkable advantages. In particular, the structure sheaf $\mathcal{O}_{(M,\Sigma,p)}$ is actually just equal to the usual structure sheaf $\mathcal{O}_M$ of holomorphic functions on the complex manifold M. Indeed, a holomorphic function $f(w, \zeta)$ is invariant under the action of $(w, \zeta) \mapsto (w, e^{2\pi i/p}\zeta)$ if and only if it can be expressed as $F(w, z)$, where $z = \zeta^p$. The interesting point is that the orbifold point of view then leads to a non-standard acyclic resolution

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0}_{(M,\Sigma,\beta)} \stackrel{\bar\partial}{\to} \mathcal{E}^{0,1}_{(M,\Sigma,\beta)} \stackrel{\bar\partial}{\to} \mathcal{E}^{0,2}_{(M,\Sigma,\beta)} \to 0$$

of the structure sheaf, and so the abstract de Rham theorem tells us that

$$H^{0,k}(M, \Sigma, \beta) = H^k(M, \mathcal{O}) = H^{0,k}(M)$$

for every integer k. In particular, the index of the orbifold elliptic operator

$$\bar\partial + \bar\partial^* : \mathcal{E}((M, \Sigma, \beta), \Lambda^{0,0} \oplus \Lambda^{0,2}) \to \mathcal{E}((M, \Sigma, \beta), \Lambda^{0,1})$$

is exactly the Todd genus of the original manifold M:

$$\chi((M, \Sigma, p), \mathcal{O}) = \chi(M, \mathcal{O}) = \frac{(\chi + \tau)(M)}{4}.$$

We can put this in a broader context by thinking of the V-bundles

$$\mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$
$$\mathbb{V}_- = \Lambda^{0,1}$$

as actually being twisted versions

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2}$$

of the spin bundles for some orbifold metric g adapted to J; here $L = K^{-1}$ is the anti-canonical V-bundle. If g is Kähler, then $\sqrt{2}(\bar\partial + \bar\partial^*)$ is then just [**24**] the spin^c Dirac operator $\not{D}$ associated to an appropriate connection on L. However, even for more general metrics, the $\not{D}$ and $\sqrt{2}(\bar\partial + \bar\partial^*)$ will have the same symbol, and hence the same index. This idea naturally leads to the following key observation:

PROPOSITION 2.4. *Let (M, J_0) be a 4-manifold with almost-complex structure, and let $\Sigma \subset M$ be any compact embedded pseudo-holomorphic curve. Let $p \geq 2$ be an integer, let $\beta = 1/p$, and let (M, Σ, β) be the orbifold obtained from M by declaring the total angle around Σ to be $2\pi\beta$. Let J be an orbifold almost-complex structure on (M, Σ, β) in the homotopy class discussed in Section 2.4. Let $\not{D}_0$ be a spin^c Dirac operator on M for the spin^c structure induced by J_0, and let $\not{D}$ be a spin^c Dirac operator on (M, Σ, β) for the spin^c structure induced by J. Then $\not{D}$ and $\not{D}_0$ have the same index*:

$$\mathrm{Ind}(\not{D}) = \mathrm{Ind}(\not{D}_0) = \frac{(\chi + \tau)(M)}{4}.$$

PROOF. The index theorem for elliptic operators on orbifolds [**28**] implies that the difference $\mathrm{Ind}(\not{D}) - \mathrm{Ind}(\not{D}_0)$ is expressible in terms of the Euler characteristic and self-intersection of Σ, since these numbers also determine the restriction of the symbol to the singular set. However, we have already seen that $\mathrm{Ind}(\not{D}) - \mathrm{Ind}(\not{D}_0) = 0$ if $\Sigma \subset M$ is a holomorphic curve in a compact complex surface. Since $\chi(\Sigma)$ and $[\Sigma]^2$ take all possible values in such examples, it follows that $\mathrm{Ind}(\not{D}) - \mathrm{Ind}(\not{D}_0)$ must vanish in full generality. □

2.7. The generalized Hitchin–Thorpe inequality.

The Euler characteristic χ and signature τ of a smooth compact 4-manifold M may both be calculated by choosing any smooth Riemannian metric g on M, and then integrating appropriate universal quadratic polynomials in the curvature of g. When g has an edge-cone singularity, however, correction terms must be introduced in order to compensate for the singularity of the metric along the given surface $\Sigma \subset M$. In [**2**], the following formulas were proved for any edge-cone metric g of cone angle $2\pi\beta$ on a pair (M, Σ), where M is a smooth compact oriented 4-manifold, and $\Sigma \subset M$ is a smoothly embedded compact oriented surface:

$$\chi(M) - (1-\beta)\chi(\Sigma) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \tag{13}$$

$$\tau(M) - \frac{1}{3}(1-\beta^2)[\Sigma]^2 = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu. \tag{14}$$

In particular, it follows that these formulas are valid when $\beta = 1/p$ and g is an orbifold metric on (M, Σ, β). Indeed, the orbifold versions of these formulas are implicit in the earlier work of other authors [**25**], [**27**], [**44**], and one of the two proofs of (13)–(14) given in [**2**] shows that the validity of these formulas in general is actually a logical consequence of their validity for orbifolds.

The Hitchin–Thorpe inequality [**6**], [**23**], [**47**] provides an important obstruction to the existence of Einstein metrics on 4-manifolds, and this fact has a natural generalization [**2**] to the setting of edge-cone metrics. Indeed, if g is an edge-cone metric of cone angle $2\pi\beta$ on (M, Σ), equations (13)–(14) tell us that

$$2\big[\chi(M) - (1-\beta)\chi(\Sigma)\big] \pm 3\left[\tau(M) - \frac{1}{3}(1-\beta^2)[\Sigma]^2\right]$$

$$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

so that the topological expression on the left-hand-side is necessarily non-negative if the given metric g is Einstein.

In this article, we are primarily interested in the special case of $\beta = 1/p$, where Proposition 1.1 tells us that any Einstein edge-cone metric is actually an orbifold metric. Our focus will be on the case when M admits an almost-complex structure J_0 for which Σ is a pseudo-holomorphic curve; as we saw in Section 2.4, the orbifold (M, Σ, β) then carries an almost-complex structure J which is induced by J_0. Now notice that the orbifold Chern class of this line bundle satisfies

$$\begin{aligned}(c_1^{\rm orb})^2(M,\Sigma,\beta) &= (c_1 + (\beta-1)[\Sigma])^2\\ &= c_1^2 + 2(\beta-1)c_1\cdot[\Sigma] + (\beta-1)^2[\Sigma]^2\\ &= c_1^2 + 2(\beta-1)\chi(\Sigma) + 2(\beta-1)[\Sigma]^2 + (1-2\beta+\beta^2)[\Sigma]^2\\ &= c_1^2 - 2(1-\beta)\chi(\Sigma) - (1-\beta^2)[\Sigma]^2\\ &= 2[\chi(M) - (1-\beta)\chi(\Sigma)] + 3\left[\tau(M) - \frac{1}{3}(1-\beta^2)[\Sigma]^2\right]\end{aligned}$$

so that our previous computation tells us that this quantity can be expressed in terms of the curvature of an arbitrary metric g (M, Σ, β) by

$$(c_1^{\rm orb})^2(M,\Sigma,\beta) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \tag{15}$$

and that $(c_1^{\rm orb})^2$ is therefore necessarily non-negative if (M, Σ, β) admits an Einstein metric. This gives us a useful special case of [**2**, Theorem A]:

PROPOSITION 2.5. *Let (M, J_0) be a smooth compact 4-manifold with almost-complex structure, let $\Sigma \subset M$ be a compact embedded pseudo-holomorphic curve. Choose an integer $p \geq 2$, and equip the orbifold (M, Σ, β), $\beta = 1/p$, with the almost-complex structure J which is determined, up to homotopy, by J_0. If (M, Σ, β) admits an orbifold Einstein metric g, then $(c_1^{\rm orb})^2(M, \Sigma, \beta) \geq 0$, with equality if and only if g is Ricci-flat and anti-self-dual.*

Requiring that g be Ricci-flat ($r \equiv 0$) and anti-self-dual ($W_+ \equiv 0$) is equivalent to asking that Λ^+ be flat. This happens precisely when g has restricted holonomy $\subset SU(2)$, and amounts to saying that g is *locally hyper-Kähler*. The next two results provide all the global information about such orbifolds that will be needed to prove Theorems A and B.

PROPOSITION 2.6. *Let M be a smooth compact oriented 4-manifold such that $b_+(M) \neq 0$, and let $\Sigma \subset M$ be a non-empty smooth compact oriented surface, possibly with several connected components. For some integer $p \geq 2$, set $\beta = 1/p$, and let*

(M, Σ, β) be the orbifold version of M with total angle $2\pi\beta$ around Σ. Suppose that (M, Σ, β) does not admit orbifold metrics of positive scalar curvature, but does admit a scalar-flat anti-self-dual orbifold metric g. Then $b_+(M) = 1$, $p \in \{2, 3, 4\}$, and g is Kähler and Ricci-flat. Moreover, M carries an integrable complex structure J_0 such that $\Sigma \subset M$ is a holomorphic curve, and (M, J_0) is either a rational complex surface or a finite quotient of $\mathbb{CP}_1$ times an elliptic curve. Finally, the orbifold (M, Σ, β) is a global quotient: if g is flat, it is the product of two elliptic curves, divided by a finite group; otherwise, it is a Calabi–Yau $K3$ divided by an isometric action of $\mathbb{Z}_p$.

Proof. Since $b_+(M) = b_+(M, \Sigma, \beta)$ is non-zero, the orbifold (M, Σ, β) carries some non-trivial self-dual harmonic 2-form ψ. Because we have assumed that s and W_+ vanish identically, the Weitzenböck formula (11) simplifies to read

$$0 = (d + d^*)^2\psi = \nabla^*\nabla\psi$$

so that $\int_M |\nabla\psi|^2 d\mu$ vanishes, and every such harmonic form ψ must therefore be parallel. However, any point $p \in \Sigma \subset M$ is covered by a local uniformizing chart with local uniformizing group $\mathbb{Z}_p$; moreover, we can take this chart to consist of geodesic normal coordinates about p, so that $\mathbb{Z}_p < SO(2)$ acts on $\mathbb{R}^4$ by rotation about $\mathbb{R}^2$. But $SO(2) < SO(4)$, and hence $\mathbb{Z}_p < SO(4)$, acts faithfully on Λ^+ via rotation about an axis; in particular, this action preserves only a 1-dimensional subspace of Λ^+. It follows that there is at most a 1-dimensional space of parallel self-dual 2-forms ψ on (M, Σ, β), and since this space coincides with $\mathcal{H}^+_g \neq 0$, it follows it is exactly 1-dimensional. In particular, $b_+(M) = 1$. Moreover, since the point-wise norm of a parallel form ψ is constant, there is, up to sign, a unique parallel self-dual 2-form ω on M with $|\omega| \equiv \sqrt{2}$. This form is the Kähler form associated with a unique complex structure J on (M, Σ, β), and in a locally uniformizing chart near any $p \in \Sigma$, this become a complex structure on $\mathbb{R}^4$ which is preserved the local $\mathbb{Z}_p$ action. In particular, the local $\mathbb{Z}_p$-action is by holomorphic maps with fixed point set Σ, so that Σ is a holomorphic curve in local complex coordinates, and M can locally be thought of as a p-fold cyclic branched quotient. Choosing two generators for the $\mathbb{Z}_p$-invariant local holomorphic functions now gives us complex coordinates on M. This equips M with an integrable almost-complex structure J_0, and turns $\Sigma \subset M$ into a complex curve in the compact complex surface (M, J_0). Moreover, since $b_+(M) = 1$ is odd, (M, J_0) is necessarily [**4**] of Kähler type.

Since we have also assumed that (M, Σ, β) does not admit any metrics of positive scalar curvature, an argument due to Bourguignon [**6**], [**9**] shows that our scalar-flat metric must be Ricci-flat; indeed, one could otherwise produce a metric with $s > 0$ by following the Ricci flow for a short time, and then conformally rescaling by the lowest eigenfunction of the Yamabe Laplacian. Thus $((M, \Sigma, \beta), g, J)$ is actually a Ricci-flat Kähler orbifold, and in particular has $c_1^{\mathrm{orb}} = 0$. Lemma 2.2 therefore now tells us that

$$0 = c_1^{\mathrm{orb}}(M, \Sigma, \beta) = c_1(M, J_0) + \left(\frac{1}{p} - 1\right)[\Sigma]$$

in $H^2(M, \mathbb{Q})$. In particular, the canonical line bundle K of the compact complex surface

(M, J_0) has negative degree respect to any Kähler form ω_0, so no positive power of K can have a holomorphic section. Thus, the Kodaira dimension of the Kähler surface (M, J_0) is $-\infty$, and surface classification [**4**], [**21**] tells us that (M, J_0) is either rational or ruled.

In particular, $H_1(M, \mathbb{Z})$ is torsion-free, and the same therefore goes for $H^2(M, \mathbb{Z})$. The equation

$$[\Sigma] = p\big([\Sigma] - c_1(M, J_0)\big)$$

is therefore valid in integer cohomology, rather than just rationally. It follows that the divisor line bundle D of Σ has a p^{th} root $E \to M$, so that $D = E^{\otimes p}$ as holomorphic line bundles. We can therefore construct a p-fold cyclic branched covering $\varpi : Y \to M$ branched along Σ by setting

$$Y := \{\zeta \in E \mid \zeta^{\otimes p} = f\}$$

where $f \in H^0(M, \mathcal{O}(D))$ vanishes exactly at Σ. The pull-back $\hat{g} = \varpi^* g$ then makes $(Y, \hat{g})$ into a compact Ricci-flat Kähler surface, and displays $((M, \Sigma, \beta), g, J)$ as a global quotient $(Y, \hat{g})/\mathbb{Z}_p$ of some locally hyper-Kähler manifold. In particular, surface classification tells us that Y is finitely covered by either $K3$ or T^4. Thus $\chi(Y) \leq 24$, with equality if and only if Y is a $K3$ surface.

The possible values of p are severely constrained. Indeed, since

$$c_1(M, J_0) = (p-1)\big([\Sigma] - c_1(M, J_0)\big)$$

in integer cohomology, $c_1(M, J_0)$ must be divisible by $(p-1)$. If (M, J_0) is ruled, and if $\mathcal{F}$ is the fiber class, we have $c_1 \cdot \mathcal{F} = 2$, so $p-1$ divides 2, and p is either 2 or 3. If (M, J_0) is non-minimal, it contains an exceptional curve $\mathcal{E}$ on which $c_1 \cdot \mathcal{E} = 1$, so $p-1$ divides 1, and $p = 2$. Finally, if $M = \mathbb{CP}_2$, it contains a line $\mathcal{L}$ on which $c_1 \cdot \mathcal{L} = 3$, so $p-1$ divides 3, and $p = 2$ or $p = 4$.

Now consider what happens if $b_1(M) = 0$. In this case, $h^1(M, \mathcal{O}) = h^2(M, \mathcal{O}) = 0$, so $\chi(M, \mathcal{O}) = 1$. Now, since Y is a p-fold branched cyclic cover,

$$\chi(Y) = p\chi(M) + (1-p)\chi(\Sigma).$$

On the other hand, $c_1^{\text{orb}} = c_1(M) + (\beta - 1)\Sigma = 0$, so $\chi(\Sigma) + \beta[\Sigma]^2 = 0$ by adjunction, and hence $\chi(\Sigma) = -[p/(p-1)^2]c_1^2$. We therefore conclude that

$$\begin{aligned}
\chi(Y) &= \frac{p}{p-1}\big[(p-1)\chi(M) + c_1^2(M)\big] \\
&= \frac{p}{p-1}\big[(p-2)\chi(M) + 12\chi(M, \mathcal{O})\big] \\
&= \frac{p}{p-1}\big[(p-2)\chi(M) + 12\big].
\end{aligned}$$

If $p = 2$, it follows that $\chi(Y) = 24$. If $p = 3$, M must be a Hirzebruch surface, with

$\chi(M) = 4$, so again $\chi(Y) = 24$. If $p = 4$, M must be $\mathbb{CP}_2$, with $\chi(M) = 3$, and once again $\chi(Y) = 24$. Thus, the hyper-Kähler manifold Y must be a $K3$ surface whenever $b_1(M) = 0$.

Now suppose that $b_1(M) \neq 0$. Then there is a holomorphic 1-form $\varphi \not\equiv 0$ on $((M, \Sigma, \beta), J)$, and this pulls back to a a holomorphic 1-form $\hat{\varphi}$ on Y. However, Y is Ricci-flat, so $\hat{\varphi} \not\equiv 0$ is both parallel and $\mathbb{Z}_p$-invariant. It follows that $(Y, \hat{g})$ is flat, and that φ restricts to Σ as a non-zero holomorphic 1-form. Each component of Σ is therefore a 2-torus, and submerses holomorphically onto the base of our ruled surface via the Albanese map. Thus (M, J_0) must be a ruled surface over an elliptic curve T^2. Moreover, since φ is non-zero everywhere, every fiber of $M \to T^2$ is non-singular; and since φ is parallel, every fiber of the ruling is totally geodesic, and so is itself a flat orbifold. This means that the fibers of $Y \to T^2$ are elliptic curves which are p-to-1 cyclic branched covers of the $\mathbb{CP}_1$ fibers of $M \to T^2$, with 4 branched points if $p = 2$, or 3 branched points if $p = 3$. The parallel $(1, 0)$-vector field $\xi = \bar{\varphi}^\sharp$ on Y is necessarily holomorphic and $\mathbb{Z}_p$-invariant, so Y also carries a $\mathbb{Z}_p$-invariant holomorphic foliation transverse to the fibers of $Y \to T^2$, and this foliation induces a flat projective connection on $M \to T^2$ with monodromy consisting of permutations of the branch points in which Σ meets the fibers. Pulling this back to a finite cover of T^2 then gives us a finite cover of M biholomorphic to $\mathbb{CP}_1 \times T^2$, and the corresponding cover of Y is then biholomorphic to the product of two elliptic curves. □

PROPOSITION 2.7. *Let M be a smooth compact oriented 4-manifold such that $b_+(M) \neq 0$, and let $\Sigma \subset M$ be a (non-empty) smooth compact oriented surface. For some integer $p \geq 2$, set $\beta = 1/p$, let (M, Σ, β) be the orbifold version of M with total angle $2\pi\beta$ around Σ, and suppose that (M, Σ, β) admits a scalar-flat anti-self-dual orbifold metric g. Then g is Kähler, and $b_+(M) = 1$. Moreover, g is Ricci-flat if and only if (M, Σ, β) does not admit orbifold metrics of positive scalar curvature.*

PROOF. Given that g is anti-self-dual and scalar-flat, the Weitzenböck formula (11) again shows that every self-dual harmonic 2-form is parallel. Since $b_+(M) \neq 0$, there must be at least one non-zero parallel self-dual form, so g is globally Kähler. If it is not Ricci-flat, then Λ^+ is not flat, and so there cannot be a self-dual harmonic form which is linearly independent from the first one, so $b_+(M) = 1$; moreover, Bourguignon's argument [**6**], [**9**] shows that (M, Σ, β) also admits positive scalar curvature metrics.

On the other hand, if g is Ricci-flat Kähler, the proof of Proposition 2.6 shows that we still have $b_+(M) = 1$, but that (M, Σ, β) is in this case a global quotient of $K3$ or T^4 by a finite group. However, neither $K3$ nor T^4 admits Riemannian metrics of positive scalar curvature [**6**], [**22**], [**33**], [**40**]. Since orbifold metrics on (M, Σ, β) are really just Riemannian metrics on $K3$ or T^4 which are invariant under the action of the appropriate finite group, it follows that (M, Σ, β) then does not admit orbifold metrics of positive scalar curvature. □

2.8. Almost-Kähler geometry.

A rather special set of techniques can be applied if, for a given the orbifold metric g on X, there happens to be a harmonic self-dual 2-form $\omega \in \mathcal{H}^+_g$ with constant point-wise norm $|\omega|_g \equiv \sqrt{2}$. In this case, there is an associated orbifold almost-complex structure

$J : TX \to TX$, $J^2 = 1$, defined by

$$g(J\cdot, \cdot) = \omega(\cdot, \cdot),$$

and this almost-complex structure then acts on TX in a g-preserving fashion. The triple (X, g, ω) is then said to be an *almost-Kähler* 4*-orbifold*. Because J allows one to think of TX as a complex vector bundle, it is only natural to look for a connection on its anti-canonical V-line bundle $L = \wedge^2 T^{1,0}_J \cong \Lambda^{0,2}_J$ in order to use the Chern–Weil theorem in order to express $c_1^{\rm orb}(J)$ as

$$c_1^{\rm orb}(J) = \left[\frac{i}{2\pi}F\right] \in H^2_{DR}(X, \mathbb{R}),$$

where F is the curvature of the relevant connection on L. A particular choice of Hermitian connection on L was first introduced in the manifold context by Blair [**7**], and later rediscovered by Taubes [**45**] for entirely different reasons. Of course, all the local calculations this entails are also valid for orbifolds. In particular, the curvature $F_{\mathscr{B}} = F^+_{\mathscr{B}} + F^-_{\mathscr{B}}$ of this *Blair connection* is given [**18**], [**38**] by

$$iF^+_{\mathscr{B}} = \frac{s + s^*}{8}\omega + W^+(\omega)^\perp \tag{16}$$

$$iF^-_{\mathscr{B}} = \frac{s - s^*}{8}\hat{\omega} + \mathring{\varrho} \tag{17}$$

where the so-called *star-scalar curvature* is given by

$$s^* = s + |\nabla\omega|^2 = 2W_+(\omega, \omega) + \frac{s}{3},$$

while $W^+(\omega)^\perp$ is the component of $W^+(\omega)$ orthogonal to ω,

$$\mathring{\varrho}(\cdot, J\cdot) = \frac{\mathring{r} + J^*\mathring{r}}{2},$$

and where the anti-self-dual 2-form $\hat{\omega} \in \Lambda^-$ is defined only on the open set where $s^* - s \neq 0$, and satisfies $|\hat{\omega}| \equiv \sqrt{2}$.

An important special case occurs when $\nabla\omega = 0$. This happens precisely when J is integrable, and g is a Kähler metric compatible with J. In this case, $s = s^*$, ω is an eigenvector of the W_+, r is J-invariant, and $iF_{\mathscr{B}}$ is the Ricci form of (g, J). In fact, ω is an eigenvector of W_+ with eigenvalue $s/6$, whereas the elements of $\omega^\perp = \Re e\Lambda^{2,0}_J$ are eigenvectors of eigenvalue $-s/12$.

Kähler metrics of constant negative scalar curvature s play a privileged role in 4-dimensional geometry. For our purposes, though, it will sometimes be useful to regard them as belonging to the following broader class of almost-Kähler metrics:

Definition 2.1. An almost-Kähler metric g on a 4-dimension orbifold will be said to be *saturated* if

- $s + s^*$ is a negative constant;
- the associated symplectic form ω belongs to the lowest eigenspace of $W_+ : \Lambda^+ \to \Lambda^+$ at each point; and
- the two largest eigenvalues of $W_+ : \Lambda^+ \to \Lambda^+$ are everywhere equal.

3. Seiberg–Witten theory.

Suppose that M is a smooth compact oriented 4-manifold, let J_0 be an almost-complex structure on M, and let $\Sigma \subset M$ be a pseudo-holomorphic curve; that is, let Σ be a compact, smoothly embedded surface such that $J_0(T\Sigma) = T\Sigma$. We now choose some integer $p \geq 2$, set $\beta = 1/p$, and let (M, Σ, β) be the smooth compact oriented orbifold with underlying topological space M, regular set $M - \Sigma$, and with total angle $2\pi\beta$ around Σ. As we have saw in Section 2.4, we can then endow (M, Σ, β) with an almost-complex structure J which agrees with J_0 outside a tubular neighborhood of Σ, and such that Σ is also a pseudo-holomorphic curve with respect to J. This allows us to define vector V-bundles $\Lambda^{0,k}$, and we can then set

$$\mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$
$$\mathbb{V}_- = \Lambda^{0,1}$$

so that $\det \mathbb{V}_+ = \det \mathbb{V}_- = \Lambda^{0,2}$. If we choose an orbifold metric g which is J-invariant, these bundles then become the twisted spinor bundles of an orbifold spin^c structure on (M, Σ, β), and we formally have

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2}$$

where $L = K^{-1} \cong \Lambda^{0,2}$ is the anti-canonical line bundle of J. Notice that, by Lemma 2.2, we have

$$c_1^{\mathrm{orb}}(L) = c_1(M, J_0) + (\beta - 1)[\Sigma].$$

Since any other orbifold metric on our orbifold is obtained from g by a self-adjoint automorphism of the tangent bundle, this construction also induces unique choices of twisted spinor bundles $\mathbb{V}_\pm$ for any other metric on (M, Σ, β).

We now endow L with a fixed Hermitian inner product $\langle\ ,\ \rangle$. Every unitary connection $\mathcal{A}$ on then L induces a unitary connection

$$\nabla_{\mathcal{A}} : \mathcal{E}(\mathbb{V}_+) \to \mathcal{E}(\Lambda^1 \otimes \mathbb{V}_+),$$

and composition of this with the natural *Clifford multiplication* homomorphism

$$\Lambda^1 \otimes \mathbb{V}_+ \to \mathbb{V}_-$$

gives one [**24**], [**31**] a spin^c Dirac operator

$$\not{D}_{\mathcal{A}} : \mathcal{E}(\mathbb{V}_+) \to \mathcal{E}(\mathbb{V}_-).$$

Because of our special choice of spin^c structure, this is an elliptic operator whose index, in complex-linear terms, was shown in Proposition 2.4 to equal the Todd genus

$$\mathrm{Ind}_{\mathbb{C}}(\not{D}_{\mathcal{A}}) = \dim_{\mathbb{C}} \ker(\not{D}_{\mathcal{A}}) - \dim_{\mathbb{C}} \ker(\not{D}^*_{\mathcal{A}}) = \frac{(\chi+\tau)(M)}{4}$$

despite the fact that we are working on the orbifold (M, Σ, β) rather than on the original 4-manifold M.

We will obtain our main results by studying the Seiberg–Witten equations

$$\not{D}_{\mathcal{A}}\Phi = 0 \tag{18}$$

$$F_A^+ = i\sigma(\Phi), \tag{19}$$

where both the twisted spinor Φ and the unitary connection $\mathcal{A}$ are treated as unknowns. Here $F_{\mathcal{A}}^+$ is the self-dual part of the curvature of $\mathcal{A}$, while the natural real-quadratic map $\sigma : \mathbb{V}_+ \to \Lambda^+$ defined by $\sigma(\Phi) = -(1/2)\Phi \odot \bar{\Phi}$ satisfies

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

These equations are non-linear, but they become an elliptic first-order system once one imposes the 'gauge-fixing' condition

$$d^*(\mathcal{A} - \mathcal{A}_0) = 0 \tag{20}$$

relative to some arbitrary background connection $\mathcal{A}_0$; this largely eliminates the natural action of the 'gauge group' of automorphisms of the Hermitian line bundle $L \to M$, reducing it to the action of the 1-dimensional group $U(1) \rtimes H^1(M, \mathbb{Z})$ of harmonic maps $M \to S^1$.

In order to obtain an invariant that may in principle force the system (18)–(19) to have a solution, one first considers the *perturbed* Seiberg–Witten equations

$$\not{D}_{\mathcal{A}}\Phi = 0 \tag{21}$$

$$iF_A^+ + \sigma(\Phi) = \eta, \tag{22}$$

where η is a real-valued self-dual 2-form. We will say that the perturbation η is *good* if

$$\eta^H \neq 2\pi[c_1^{\mathrm{orb}}(L)]^+$$

where $\eta^H \in \mathcal{H}^+ \subset \mathcal{E}^2(M, \Sigma, \beta)$ is the L^2-orthogonal projection of η onto the harmonic 2-forms and where $[c_1^{\mathrm{orb}}(L)]^+ \in \mathcal{H}^+ \subset H^2(M, \mathbb{R})$ is the cup-orthogonal projection into the de Rham classes with self-dual representatives relative to g. Whenever η is a good perturbation, any solution $(\Phi, \mathcal{A})$ of (21)–(22) is necessarily *irreducible*, in the sense that $\Phi \not\equiv 0$. Even in the presence of the gauge-fixing condition (20), this implies that

$U(1) \rtimes H^1(M,\mathbb{Z})$ acts freely on the space of solutions.

If (g,η) is a pair consisting of a orbifold-smooth metric and a self-dual 2-form which is a good perturbation with respect to g, we will say that (g,η) is a good pair. The set of good pairs is connected if $b_+(M) > 1$. If $b_+(M) = 1$, it instead consists of two connected components, called *chambers*. Indeed, when $b_+(M) = 1$, the intersection form on $H^2(M,\mathbb{R})$ is of Lorentz type, so that $H^2(M,\mathbb{R})$ can be thought of as a copy of $b_2(M)$-dimensional Minkowski space. The set of vectors $\boldsymbol{a} \in H^2(M,\mathbb{R})$ with $\boldsymbol{a}^2 > 0$ then becomes the set of time-like vectors, and has two connected components, and we may give this Minkowski space a time-orientation by labeling one the *future-pointing*, and the other *past-pointing*, time-like vectors. For any good pair (g,η), the vector $2\pi[c_1(L)]^+ - \eta$ is a time-like vector. One chamber, which we will label $\uparrow$, consists of those pairs (g,η) for which $2\pi[c_1(L)]^+ - \eta$ is future-pointing; the other chamber, which we will call $\downarrow$, consists of those pairs (g,η) for which $2\pi[c_1(L)]^+ - \eta$ is past-pointing.

Equations (21)–(22) imply the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_{\mathcal{A}}\Phi|^2 + s|\Phi|^2 + |\Phi|^4 - 8\langle \eta, \sigma(\Phi)\rangle \tag{23}$$

and from this one obtains the important C^0 estimate that any solution must satisfy

$$|\Phi|^2 \leq \max(2\sqrt{2}|\eta| - s, 0).$$

By bootstrapping and the Rellich–Kondrachov theorem, it follows that the space of solutions of (20)–(22) of regularity L^q_{k+1}, modulo the restricted gauge action of $U(1) \rtimes H^1(M,\mathbb{Z})$, is compact in the L^q_k topology. This argument also applies to the space of all solutions as (g,η) varies over some compact family of pairs (g,η).

LEMMA 3.1. *Let (M, J_0) be a orbifold-smooth compact 4-manifold with almost-complex structure, let $\Sigma \subset M$ be a compact embedded pseudo-holomorphic curve. Choose an integer $p \geq 2$, and equip the orbifold (M,Σ,β), $\beta = 1/p$, with the spin^c-structure obtained by lifting J_0 to an almost-complex structure J. For some orbifold-smooth metric g, let $(\Phi,\mathcal{A})$ be an L^q_{k+1} solution of* (20)–(22) *for some $q > 4$, $k \geq 0$, and let*

$$Л : L^q_{k+1}(\mathbb{V}_+ \oplus \Lambda^1) \to L^q_k(\mathbb{V}_- \oplus \Lambda^+ \oplus \Lambda^0)$$

be the linearization of (20)–(22) *at $(\Phi,\mathcal{A})$. Then Л is a Fredholm operator of index* 0.

PROOF. The linearization Л differs from the elliptic operator $\not{D}_{\mathcal{A}} \oplus d^+ \oplus d^*$ by lower order terms, and so has the same index. However, the kernel of

$$d^+ \oplus d^* : L^q_{k+1}(\Lambda^1) \to L^q_k(\Lambda^+ \oplus \Lambda^0)$$

is the space of harmonic 1-forms $\mathcal{H}^1$, while its cokernel is $\mathcal{H}^+ \oplus \mathcal{H}^0$, the space of self-dual harmonic 2-forms direct sum the constants. The index of Л is therefore

$$\begin{aligned}\mathrm{Ind}_{\mathbb{R}}(\text{Л}) &= 2\,\mathrm{Ind}_{\mathbb{C}}(\not{D}_{\mathcal{A}}) + (b_1 - b_+ - b_0)(M)\\ &= 2\frac{(\chi+\tau)(M)}{4} - \frac{(\chi+\tau)(M)}{2}\\ &= 0,\end{aligned}$$

using the value for the index of $\not{D}_{\mathcal{A}}$ found in Proposition 2.4. □

This index calculation gives rise to an immediate generalization of the familiar definition of the Seiberg–Witten invariant of 4-manifolds to the present orbifold context. The space of solutions of (20)–(21) with L^q_{k+1} regularity and $\Phi \not\equiv 0$ is a Banach manifold $\mathcal{B}^q_{k+1}$, because, letting $L^q_k(0) \subset L^q_k$ be the codimension-1 subspace of functions of integral 0, because (20)–(21) may be interpreted as defining this $\mathcal{B}^q_{k+1}$ to be the set of $(\Phi, \mathcal{A})$ which are sent to $(\mathbf{0}, \mathbf{0})$ by a smooth map $[L^q_{k+1}(\mathbb{V}_+) - \mathbf{0}] \times L^q_{k+1}(\Lambda^1) \to L^q_k(\mathbb{V}_-) \times L^q_k(0)$ for which $(\mathbf{0}, \mathbf{0})$ is a regular value. Equation (22) then defines a smooth Fredholm map $\mathcal{B}^q_{k+1} \to L^q_k(\Lambda^+)$ which, by Lemma 3.1, has index 1. For regular values $\eta \in L^q_k(\Lambda^+)$ of this map which are also good, we therefore conclude that the solutions of (20)–(22) form a 1-manifold. Moreover, $U(1) \rtimes H^1(M, \mathbb{Z})$ acts freely on this 1-manifold, and the *moduli space* $\mathfrak{M}(g, \eta)$, obtained by dividing the space of solutions by the free action of $U(1) \rtimes H^1(M, \mathbb{Z})$, is a 0-manifold. Compactness therefore implies that the moduli space $\mathfrak{M}(g, \eta)$ is finite. Moreover, one obtains the following:

Proposition 3.2. *Let (M, J_0) be a smooth compact 4-manifold with almost-complex structure, let $\Sigma \subset M$ be a compact embedded pseudo-holomorphic curve. Choose some positive integer $p \geq 2$, and equip the orbifold (M, Σ, β), $\beta = 1/p$, with the spin^c-structure obtained by lifting J_0 to an almost-complex structure J. Then, for any metric g, and for a residual set of good $\eta \in L^q_k(\Lambda^+)$ the Seiberg–Witten moduli space $\mathfrak{M}(g, \eta)$ is finite. If $b_+(M) > 1$, then, for the fixed spin^c structure, any two such moduli spaces $\mathfrak{M}(g, \eta)$ and $\mathfrak{M}(\tilde{g}, \tilde{\eta})$ are cobordant. If $b_+(M) = 1$, two such moduli spaces $\mathfrak{M}(g, \eta)$ and $\mathfrak{M}(\tilde{g}, \tilde{\eta})$ are cobordant if, in addition, the good pairs (g, η) and $(\tilde{g}, \tilde{\eta})$ belong to the same chamber.*

If $\mathfrak{c}$ denotes the spin^c structure on the oriented orbifold (M, Σ, β) induced by J, and if $b_+(M) > 1$, the cobordism-invariance of the moduli space allows us to define the mod-2 Seiberg–Witten invariant of (M, Σ, β) by setting

$$SW_{\mathfrak{c}}(M, \Sigma, \beta) = \mathrm{Cardinality}(\mathfrak{M}(g, \eta)) \bmod 2$$

for any good pair (g, η) for which η is a regular value of the relevant map. When $b_+(M) = 1$, this definition instead gives rise to two $\mathbb{Z}_2$-valued invariants, $SW^{\uparrow}_{\mathfrak{c}}(M, \Sigma, \beta)$ and $SW^{\downarrow}_{\mathfrak{c}}(M, \Sigma, \beta)$, where the arrow indicates whether the generic good pair (g, η) at which we carry out our mod-2 count belongs to the $\uparrow$ or the $\downarrow$ component of the space of good pairs.

If $b_+(M) > 1$ and $SW_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, then the Seiberg–Witten equations (18)–(19) must have a solution for any orbifold metric g on (M, Σ, β). Indeed, if $[c_1^{\mathrm{orb}}(L)]^+ \neq 0$ for the given metric g, then $\eta = 0$ is a good perturbation; if there were no solution, it would also be a regular value, and we would obtain a contradiction by performing a mod-2 count of solutions and concluding that $SW_{\mathfrak{c}} = 0$. If $[c_1(L)]^+ = 0$, we still have a solution, albeit a reducible one, obtained by setting $\Phi \equiv 0$ and arranging for $F_{\mathcal{A}}$ to be harmonic. Note that elliptic regularity actually guarantees that any solution $(\Phi, \mathcal{A})$ of (18)–(20) is actually orbifold-smooth, assuming that the metric g is orbifold-smooth, too.

If $b_+(M) > 1$ and $SW^{\downarrow}_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, the same argument shows that the Seiberg–Witten equations (18)–(19) must have a solution for any orbifold metric g on (M, Σ, β) such that $[c_1^{\mathrm{orb}}(L)]^+$ is past-pointing. Similarly, if $SW^{\uparrow}_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, the Seiberg–Witten equations (18)–(19) must have a solution for any orbifold metric g on (M, Σ, β) such that $[c_1^{\mathrm{orb}}(L)]^+$ is future-pointing. Here, it is worth recalling once again that, in the case that concerns us here,

$$c_1^{\mathrm{orb}}(L) = c_1(M, J) + (\beta - 1)[\Sigma]$$

where $\beta = 1/p$.

Of course, all this would be completely useless without some geometric criterion to sometimes guarantee that our orbifold version of the Seiberg–Witten invariant is actually non-zero. Fortunately, such a criterion is implicit in the work of Taubes [**45**]:

THEOREM 3.3. *Suppose the 4-manifold M admits a symplectic form ω_0 which restricts to $\Sigma \subset M$ as an area form. Choose an almost-complex structure J_0 which is compatible with ω_0 and which makes Σ a pseudo-holomorphic curve. Let J be the lift of J_0 to (M, Σ, β), and let $\mathfrak{c}$ be the spin^c structure on (M, Σ, β) induced by J as above. The following then hold*:

- *If $b_+(M) > 1$, then $SW_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$.*
- *If $b_+(M) = 1$, then $SW^{\downarrow}_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, where $H^2(M, \mathbb{R})$ has been time-oriented so that $[\omega_0]$ is future-pointing.*

PROOF. Choose an orbifold metric g on (M, Σ, β) which is adapted to the lift ω of ω_0 to (M, Σ, β). Let $t \gg 0$ be a large positive real constant, and let $\mathcal{A}_0$ be the Blair connection [**7**] on $L = \Lambda^{2,0}_J$. Then $\Phi_0 = (\sqrt{t}, 0) \in \mathcal{E}^{0,0} \oplus \mathcal{E}^{0,2}$ solves [**38**], [**45**]

$$D\!\!\!/_{\mathcal{A}_0} \Phi = 0.$$

Hence $(\Phi_0, \mathcal{A}_0)$ can be viewed as a solution of the perturbed Seiberg–Witten equations (21)–(22) for an appropriate choice of perturbation η, namely the self-dual form given, in the notation of (16), by

$$\eta = \frac{t}{4}\omega + iF^+_{\mathcal{A}_0} = (2t + s + s^*)\frac{\omega}{8} + W_+(\omega)^\perp.$$

If t is sufficiently large, Taubes [**45**] then shows that, for this choice of η, *any* solution $(\Phi, \mathcal{A})$ of (21)–(22) of the system is then gauge equivalent to $(\Phi_0, \mathcal{A}_0)$, and that η is moreover a regular value of the Seiberg–Witten map. (While Taubes assumes that he is working on a manifold rather than an orbifold, all of his arguments go through in the present context without change.) Since $\mathfrak{M}(g, \eta)$ consists of a single point, our mod-2 count therefore gives a non-zero answer. When $b_+(M) > 1$, this shows that $SW_{\mathfrak{c}}(M,\Sigma,\beta) \neq 0$. When $b_+(M) = 1$, it instead shows that the mod-2 Seiberg–Witten invariant is non-zero for some chamber, and since $[\omega]\cdot(2\pi[c_1^{\rm orb}(L)]^+ - \eta) = -t[\omega]^2/4 < 0$, the relevant chamber is the past-pointing one relative to $[\omega] = [\omega_0]$. □

4. Curvature estimates.

The Seiberg–Witten equations (18)–(19) imply the Weitzenböck formula

$$0 = 2\Delta|\Phi|^2 + 4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4, \tag{24}$$

where $\nabla = \nabla_{\mathscr{A}}$. This leads to some rather surprising curvature estimates; cf. [**50**], [**34**], [**35**], [**36**], [**38**]. We begin by proving some estimates in terms of the self-dual part $c_1^{\rm orb}(L)^+ \in \mathcal{H}_g^+$ of $c_1^{\rm orb}(L)$.

Lemma 4.1. *Let (M, J_0) be an almost-complex 4-manifold, and let $\Sigma \subset M$ be a compact embedded pseudo-holomorphic curve. Choose some integer $p \geq 2$, and let (M,Σ,β), $\beta = 1/p$, be the orbifold obtained by declaring the total angle around Σ to be $2\pi\beta$. Let $\mathfrak{c}$ be the spin^c structure induced by the lifting J_0 to an almost-complex structure J on (M,Σ,β).*

- *If $b_+(M) > 1$, and if $SW_{\mathfrak{c}}(M,\Sigma,\beta) \neq 0$, then the scalar curvature s_g of any orbifold metric g on (M,Σ,β) satisfies*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2[c_1^{\rm orb}(L)^+]^2, \tag{25}$$

  with equality if and only if g is an orbifold Kähler metric of constant negative scalar curvature which is compatible with a complex structure on (M,Σ,β) with $c_1^{\rm orb} = c_1^{\rm orb}(L)$ and with $(c_1^{\rm orb})^+$ a negative multiple of the Kähler class $[\omega]$.
- *If $b_+(M) = 1$, and if $SW_{\mathfrak{c}}^{\downarrow}(M,\Sigma,\beta) \neq 0$ for a fixed time orientation for $H^2(M,\mathbb{R})$, then the same conclusion holds for any g such that $c_1^{\rm orb}(L)^+$ is past-pointing.*

Proof. Our hypotheses guarantee that there is a solution $(\Phi,\mathscr{A})$ of (18)–(19) for the given metric and spin^c structure. Integrating (24), we have

$$0 = \int [4|\nabla\Phi|^2 + s|\Phi|^2 + |\Phi|^4] d\mu,$$

and it follows that

$$\left(\int s^2 d\mu\right)^{1/2}\left(\int |\Phi|^4 d\mu\right)^{1/2} \geq \int (-s)|\Phi|^2 d\mu \geq \int |\Phi|^4 d\mu,$$

so that

$$\int s^2 d\mu \geq \int |\Phi|^4 d\mu = 8\int |F_{\mathscr{A}}^+|^2 d\mu.$$

Since $iF_{\mathcal{A}}$ belongs to the de Rham class $2\pi c_1^{\rm orb}(L)$, we moreover have

$$\int |F_{\mathcal{A}}^+|^2 d\mu \geq 4\pi^2 [c_1^{\rm orb}(L)^+]^2$$

by Lemma 2.1. Together, these two inequalities imply (25).

If equality holds in (25), the above argument shows that $\nabla\Phi \equiv 0$, and that $|\Phi|^2$ and $(-s)$ are both non-negative constants. If $c_1^{\rm orb}(L)^+ \neq 0$, the constants $(-s)$ and $|\Phi|^2$ must both be positive, and the parallel section Φ of $\mathbb{V}_+$ is non-zero. This implies that the self-dual 2-form $\sigma(\Phi)$ is parallel and non-zero, so that g must Kähler. This shows that g is a constant-scalar-curvature Kähler metric with $s < 0$. Moreover, $\Phi \otimes \Phi$ is a non-zero section of $\Lambda^{2,0} \otimes L$, so this Kähler metric has $c_1^{\rm orb} = c_1^{\rm orb}(L)$.

It only remains to see what happens when $b_+(M) > 1$, $SW_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, and both sides of (25) vanish. In this case, the non-vanishing of the Seiberg–Witten invariant implies that there cannot be a positive-scalar-curvature orbifold metric on (M, Σ, β), because otherwise there would exist solutions of the perturbed Seiberg–Witten equations (21)–(22) for a good perturbation η with $s - 2\sqrt{2}|\eta| > 0$, leading to a contradiction via the perturbed Weitzenböck formula (23). On the other hand, we have assumed that both sides of (25) vanish, so g is scalar flat; and if it were not Ricci-flat, there would be a nearby metric of positive scalar curvature [**6**], [**9**]. Thus g must be Ricci-flat, and so, in particular, Einstein. By (15), this implies that $[c_1^{\rm orb}(L)]^2 \geq 0$, with equality if and only if g is scalar-flat and anti-self-dual. But since we have assumed that both sides of (25) vanish, we have $[c_1^{\rm orb}(L)^+]^2 = 0$, so that

$$[c_1^{\rm orb}(L)]^2 = [c_1^{\rm orb}(L)^+]^2 - |[c_1^{\rm orb}(L)^-]^2| \leq 0.$$

Hence $[c_1^{\rm orb}(L)]^2 = 0$, and our scalar-flat metric g is also anti-self-dual. But this implies $b_+(M) = 1$ by Proposition 2.6. Thus the case under discussion never occurs, and we are done. □

We will also need an analogous curvature estimate for a mixture of the scalar and Weyl curvatures. To this end, let $f > 0$ be a positive orbifold-smooth function, and consider the *rescaled Seiberg–Witten equations* [**37**]

$$D\!\!\!/_{\mathcal{A}} \Phi = 0 \tag{26}$$

$$F_{\mathcal{A}}^+ = if\sigma(\Phi), \tag{27}$$

which, for reasons related to the conformal invariance of the Dirac operator [**24**], [**31**], [**42**], are simply a disguised form of the Seiberg–Witten equations for the conformally related metric $f^{-2}g$. When $b_+(M) > 1$, the non-vanishing of $SW_{\mathfrak{c}}(M)$ is therefore enough to guarantee that (26)–(27) have a solution for any g and any f. Similarly, when $b_+(M) = 1$, the non-vanishing of $SW_{\mathfrak{c}}^{\downarrow}(M)$ suffices to guarantee that, for any $f > 0$, (26)–(27) must have a solution for any metric g such that $c_1^{\rm orb}(L)^+$ is past-pointing. Will now exploit the fact that (26)–(27) imply the Weitzenböck formula

$$0=2\Delta|\Phi|^2+4|\nabla\Phi|^2+s|\Phi|^2+f|\Phi|^4, \tag{28}$$

to obtain the desired curvature estimate.

Lemma 4.2. *Let (M,Σ,β) and $\mathfrak{c}$ be as in Lemma 4.1.*

- *If $b_+(M)>1$, and if $SW_{\mathfrak{c}}(M,\Sigma,\beta)\neq 0$, then the scalar curvature s_g and self-dual Weyl curvature $(W_+)_g$ of any orbifold metric g on (M,Σ,β) satisfy*

$$\int_M (s-\sqrt{6}|W_+|)^2 d\mu_g \geq 72\pi^2[c_1^{\mathrm{orb}}(L)^+]^2,$$

with equality if and only if g is a saturated almost-Kähler metric, in the sense of Definition 2.1, with $c_1^{\mathrm{orb}}=c_1^{\mathrm{orb}}(L)$, and with $(c_1^{\mathrm{orb}})^+$ a negative multiple of the symplectic class $[\omega]$.
- *If $b_+(M)=1$, and if $SW_{\mathfrak{c}}^{\downarrow}(M,\Sigma,\beta)\neq 0$ for a fixed time orientation for $H^2(M,\mathbb{R})$, then the same conclusion holds for any g for which $c_1^{\mathrm{orb}}(L)^+$ is past-pointing.*

Proof. For any smooth function $f>0$ on M, our hypotheses guarantee that (26)–(27) must admits a solution $(\Phi,\mathscr{A})$. We now proceed, *mutatis mutandis*, as in [**39**]. Multiplying (28) by $|\Phi|^2$ and integrating, this solution must then satisfy

$$0\geq \int_M \big(4|\Phi|^2|\nabla_{\mathscr{A}}\Phi|^2+s|\Phi|^4+f|\Phi|^6\big)d\mu.$$

On the other hand, the self-dual 2-form $\psi=2\sqrt{2}\sigma(\Phi)$ automatically satisfies

$$|\Phi|^4=|\psi|^2, \qquad 4|\Phi|^2|\nabla_{\mathscr{A}}\Phi|^2\geq|\nabla\psi|^2,$$

so it follows that

$$0\geq\int_M \big(|\nabla\psi|^2+s|\psi|^2+f|\psi|^3\big)d\mu.$$

However, inequality (12) also tells us that

$$\int_M|\nabla\psi|^2d\mu\geq\int_M\bigg(-2\sqrt{\frac{2}{3}}|W_+|-\frac{s}{3}\bigg)|\psi|^2d\mu,$$

and combining these facts yields

$$0\geq\int_M\bigg[\big(s-\sqrt{6}|W_+|\big)|\psi|^2+\frac{3}{2}f|\psi|^3\bigg]d\mu.$$

Repeated use of Hölder inequalities gives us

$$\left(\int_M f^4 d\mu\right)^{1/3}\left(\int_M \left|s-\sqrt{6}|W_+|\right|^3 f^{-2}d\mu\right)^{2/3} \geq \int_M \left(\frac{3}{2}f|\psi|\right)^2 d\mu,$$

and application of Lemma 2.1 to the right-hand-side then yields

$$\left(\int_M f^4 d\mu\right)^{1/3}\left(\int_M \left|s-\sqrt{6}|W_+|\right|^3 f^{-2}d\mu\right)^{2/3} \geq 72\pi^2[c_1^{\rm orb}(L)^+]^2 \tag{29}$$

for any smooth positive function f. Choosing a sequence f_j of smooth positive functions such that

$$f_j \searrow \sqrt{\left|s-\sqrt{6}|W_+|\right|}$$

we then have

$$\int_M f_j^4 d\mu \geq 72\pi^2[c_1^{\rm orb}(L)^+]^2,$$

and taking the limit as $j\to\infty$ therefore gives us

$$\int_M \left(s-\sqrt{6}|W_+|\right)^2 d\mu \geq 72\pi^2[c_1^{\rm orb}(L)^+]^2 \tag{30}$$

as desired.

If equality holds, we must have

$$s-\sqrt{6}|W_+| = \text{constant}$$

because g minimizes the left-hand-side in its conformal class. If this constant were zero, we would then have $s=\sqrt{6}|W_+|\equiv 0$, since g would otherwise be conformal to a metric of positive scalar curvature. However, we would also have $[c_1^{\rm orb}(L)^+]^2 = 0$, and our hypotheses exclude this because Proposition 2.6 forces $b_+(M)=1$. Thus

$$f=\sqrt{\left|s-\sqrt{6}|W_+|\right|}$$

is positive, and we would necessarily obtain equality in (29) for this choice of positive function; and our use of Lemma 2.1 in the previous argument would force ψ to be self-dual harmonic. Moreover, our use of the Hölder inequality guarantees that $|\psi|$ must be a non-zero constant. In particular, Φ is everywhere non-zero, and $\Phi\otimes\Phi$ now defines a non-zero section of $\Lambda_J^{2,0}\otimes L$, so that L is actually the anti-canonical line bundle of J. Finally, the fact that (12) must be saturated guarantees that ω belongs to the lowest eigenspace of W_+, and the fact that $s-\sqrt{6}|W_+|$ is a negative constant now translates into the statement that $s+s^*$ is a negative constant. Thus g is a saturated almost-Kähler metric with respect to the symplectic form ω. □

While these Lemmas do provide us with interesting lower bounds for certain curvature quantities, the right-hand-sides of the inequalities still depend on the decomposition $H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g$, and so are still somewhat metric-dependent. Fortunately, however, they do imply other estimates which do not suffer from this short-coming:

Theorem 4.3. *Let (X, J_0) be a compact almost-complex 4-manifold, and let $\Sigma \subset X$ be a compact embedded pseudo-holomorphic curve. For some integer $p \geq 2$, set $\beta = 1/p$, let (X, Σ, β) be the orbifold version of X with total angle $2\pi\beta$ around Σ, and let (M, Σ, β) be the orbifold obtained by blowing up (X, Σ, β) at $\ell \geq 0$ points away from Σ. Give (M, Σ, β) an almost-complex structure J by first lifting J_0 to the orbifold (X, Σ, β), and then to its blow-up (M, Σ, β); and let $\mathfrak{c}$ be the spin^c structure on (M, Σ, β) induced by J. If $b_+(M) > 1$, suppose that $SW_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$; if $b_+(M) = 1$, instead suppose that $SW^{\downarrow}_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$, and that $c_1^{\mathrm{orb}}(X)$ is time-like or null past-pointing. Then the curvature of any orbifold Riemannian metric g on (M, Σ, β) satisfies*

$$\int_M s^2 d\mu \geq 32\pi^2 (c_1^{\mathrm{orb}})^2(X, \Sigma, \beta)$$

$$\int_M (s - \sqrt{6}|W_+|)^2 d\mu \geq 72\pi^2 (c_1^{\mathrm{orb}})^2(X, \Sigma, \beta)$$

and both inequalities are strict unless $\ell = 0$ and g is an orbifold Kähler–Einstein metric on (X, Σ, β) with negative scalar curvature.

Proof. If $(c_1^{\mathrm{orb}})^2(X, \Sigma, \beta) < 0$, there is nothing to prove, so we may assume from the outset that $(c_1^{\mathrm{orb}})^2(X, \Sigma, \beta) \geq 0$. When $b^+(X) = 1$, this means that $c_1^{\mathrm{orb}}(X, \Sigma, \beta)$ is either time-like or null. Our hypotheses moreover imply that it must be past-pointing, and s its pull-back to M is therefore also time-like or null and past-pointing with respect to the given time-orientation. Here we are using the fact that

$$H^2(M,\mathbb{R}) = H^2(X,\mathbb{R}) \oplus \underbrace{H^2(\overline{\mathbb{CP}}_2,\mathbb{R}) \oplus \cdots \oplus H^2(\overline{\mathbb{CP}}_2,\mathbb{R})}_{\ell}$$

by applying Mayer–Vietoris to $M \approx X \# \ell \overline{\mathbb{CP}}_2$; the summands are mutually orthogonal with respect to the intersection form, and the intersection form restricts to each of these subspaces as the intersection form of the corresponding topological building block.

If $E_1, \ldots, E_\ell \in H^2(M)$ are standard generators for the ℓ copies of $H^2(\overline{\mathbb{CP}}_2, \mathbb{R})$ obtained by taking Poincaré duals of the ℓ projective lines $S_1, \ldots, S_\ell$ introduced by blow-up, then the given spin^c structure $\mathfrak{c}$ has Chern class

$$c_1^{\mathrm{orb}}(M, \Sigma, \beta) = c_1^{\mathrm{orb}}(X, \Sigma, \beta) - E_1 - \cdots - E_\ell.$$

However, there are self-diffeomorphisms [**19**], [**35**] of M which are the identity outside a tubular neighborhood of a given S_j, but which send E_j to $-E_j$. Applying these to $\mathfrak{c}$ give us 2^ℓ different spin^c structures, with

$$\widetilde{c_1^{\rm orb}}(M,\Sigma,\beta) = c_1^{\rm orb}(X,\Sigma,\beta) \pm E_1 \pm \cdots \pm E_\ell$$

for any desired choice of signs; and when $b_+(M) = b_+(X) = 1$, these diffeomorphisms moreover preserve the given time orientation of M. Given a metric g on M, let us now choose $\epsilon_j = \pm E_j$ so that

$$[c_1^{\rm orb}(X,\Sigma,\beta)]^+ \cdot \epsilon_j^+ = [c_1^{\rm orb}(X,\Sigma,\beta)]^+ \cdot \epsilon_j \geq 0$$

and then observe that, when $b_+(M) = 1$, $[\widetilde{c_1^{\rm orb}}]^+$ is past-pointing for this spin^c structure and this metric. Moreover, we then have

$$\begin{aligned}\left([\widetilde{c_1^{\rm orb}}(M,\Sigma,\beta)]^+\right)^2 &= \left([c_1^{\rm orb}(X,\Sigma,\beta)]^+\right)^2 + 2[c_1^{\rm orb}(X,\Sigma,\beta)]^+ \cdot \sum_j \epsilon_j^+ + \Big(\sum_j \epsilon_j^+\Big)^2 \\ &\geq \left([c_1^{\rm orb}(X,\Sigma,\beta)]^+\right)^2 \\ &\geq \left(c_1^{\rm orb}(X,\Sigma,\beta)\right)^2\end{aligned}$$

so Lemmas 4.1 and 4.2 immediately give us the two promised curvature inequalities.

If equality were to hold in either of these inequalities, we would necessarily have $c_1^{\rm orb}(X,\Sigma,\beta)^- = 0$, and $[c_1^{\rm orb}(X,\Sigma,\beta)]^+ \cdot \epsilon_j = 0$ for $j = 1,\ldots,\ell$. In particular, if we replace ϵ_j with $-\epsilon_j$ for each $j = 1,\ldots,\ell$, we would obtain a second spin^c structure which also saturated the relevant inequality. But these two spin^c structures have identical $(c_1^{\rm orb})^+$ with respect to the given metric. Lemma 4.1 or 4.2 then tell us that g is almost-Kähler with respect to two symplectic forms ω and $\tilde\omega$ which are harmonic representatives of the same negative multiple of $(c_1^{\rm orb})^+$ and whose Chern classes differ by $2\sum_j \epsilon_j$. The latter of course implies that $\omega = \tilde\omega$, so that we must have $\sum_{j=1}^{\ell} \epsilon_j = 0$, and this can only happen if $\ell = 0$.

Thus equality implies that $M = X$, and that $(c_1^{\rm orb})^- = 0$. Moreover, we know that g is a saturated almost-Kähler metric, and that the curvature $F_{\mathscr{B}}$ of the Blair connection is the harmonic representative of $-2\pi i c_1^{\rm orb}$. In particular, in our present situation, $F_{\mathscr{B}}$ is a self-dual form. However, by specializing (16)–(17) to the saturated case, the curvature of the Blair connection of such a metric is given by

$$iF_{\mathscr{B}}^+ = \frac{s+s^*}{8}\omega, \qquad iF_{\mathscr{B}}^- = \frac{s-s^*}{8}\hat\omega + \mathring{\varrho},$$

where $s+s^*$ is a non-positive constant, $\mathring{\varrho}$ encodes the J-invariant piece of the trace-free Ricci curvature $\mathring{r}$, and where the bounded anti-self-dual 2-form $\hat\omega \in \Lambda^-$ satisfies $|\hat\omega| \equiv \sqrt{2}$, but is defined only on the open set where $s^* - s \neq 0$. In the present situation, however, we also know that $F_{\mathscr{B}}^- = 0$, so $\mathring{\varrho} = (s^*-s)\hat\omega/8$, and

$$|\mathring{r}|^2 \geq \frac{(s^*-s)^2}{16} \tag{31}$$

everywhere, with equality precisely at the points where the Ricci tensor r is J-invariant.

On the other hand, the algebraic constraint on W_+ imposed by Definition 2.1 implies that

$$|W_+|^2 = \frac{(3s^* - s)^2}{96}.$$

Thus, the Gauss–Bonnet-type formula (15) tells us that

$$\begin{aligned}
4\pi^2 (c_1^{\mathrm{orb}})^2(M, \Sigma, \beta) &= \int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu \\
&= \int_M \left(\frac{s^2}{24} + \frac{2(3s^* - s)^2}{96} - \frac{|\mathring{r}|^2}{2} \right) d\mu \\
&\leq \int_M \left(\frac{s^2}{24} + \frac{2(3s^* - s)^2}{96} - \frac{(s^* - s)^2}{32} \right) d\mu \\
&= \frac{1}{32} \int_M \left(s^2 - 2ss^* + 5(s^*)^2 \right) d\mu,
\end{aligned}$$

with equality if and only if equality holds in (31). On the other hand, since $F_{\mathscr{B}} = F^+_{\mathscr{B}}$,

$$\begin{aligned}
4\pi^2 (c_1^{\mathrm{orb}})^2(M, \Sigma, \beta) &= \int_M \left(\frac{s + s^*}{8} \omega \right) \wedge \left(\frac{s + s^*}{8} \omega \right) \\
&= \frac{1}{32} \int_M \left(s^2 + 2ss^* + (s^*)^2 \right) d\mu.
\end{aligned}$$

We therefore know that

$$\int_M \left(s^2 - 2ss^* + 5(s^*)^2 \right) d\mu \geq \int_M \left(s^2 + 2ss^* + (s^*)^2 \right) d\mu,$$

or in other words that

$$\int_M 4s^*(s^* - s) d\mu \geq 0, \tag{32}$$

with equality if and only if equality holds in (31). However, since (M, g, ω) is saturated, $s^* + s$ is a negative constant, and $W_+(\omega, \omega) \leq 0$; hence $s^* \leq s/3$, and $s^* \leq (s + s^*)/4$ is therefore negative everywhere. Since $s^* - s = |\nabla \omega|^2 \geq 0$ on any almost-Kähler orbifold, we thus have

$$s^*(s^* - s) \leq 0$$

everywhere on M, with equality only at points where $s = s^*$. The inequality (32) therefore implies

$$|\nabla \omega|^2 = s^* - s \equiv 0,$$

so that (g, ω) is Kähler. But equality in (32) implies that equality also holds in (31), so $|\mathring{r}|^2 \equiv (s^* - s)^2/16$, and hence $\mathring{r} \equiv 0$. Thus g is Kähler–Einstein, with negative scalar curvature, as promised. □

Theorem B is now an immediate consequence of Theorems 3.3 and 4.3, in conjunction with the explicit formula

$$c_1^{\mathrm{orb}}(X, \Sigma, \beta) = c_1(X) + (\beta - 1)[\Sigma]$$

for the orbifold Chern class provided by Lemma 2.2.

5. Inequalities for Einstein metrics.

With the results of the previous section in hand, we are finally in a position to prove Theorem A. For this purpose, the key observation is the following:

PROPOSITION 5.1. *Let X, M, Σ, β and $\mathfrak{c}$ be as in Theorem 4.3. Then the curvature of any orbifold metric g on (M, Σ, β) satisfies*

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu > \frac{2}{3}\left[c_1^{\mathrm{orb}}(X, \Sigma, \beta)\right]^2. \tag{33}$$

PROOF. Since there is otherwise nothing to prove, we may henceforth assume that $[c_1^{\mathrm{orb}}(X, \Sigma, \beta)]^2 \geq 0$. With this proviso, the second inequality of Theorem 4.3 tells us that

$$\left\| s - \sqrt{6}|W_+| \right\|_{L^2} \geq \left(72\pi^2 [c_1^{\mathrm{orb}}(X, \Sigma, \beta)]^2\right)^{1/2}$$

and the triangle inequality therefore yields

$$\|s\|_{L^2} + \sqrt{6}\ \|W_+\|_{L^2} \geq \left(72\pi^2 [c_1^{\mathrm{orb}}(X, \Sigma, \beta)]^2\right)^{1/2},$$

with equality only if g is Kähler–Einstein, with negative scalar curvature. However, the left-hand side can be interpreted as a dot product

$$\left(1, \frac{1}{\sqrt{8}}\right) \cdot \left(\|s\|, \sqrt{48}\|W_+\|\right)$$

in $\mathbb{R}^2$, and the Cauchy-Schwarz inequality thus tells us that

$$\left(1 + \frac{1}{8}\right)^{1/2} \left(\|s\|^2_{L^2} + 48\|W_+\|^2_{L^2}\right)^{1/2} \geq \left(72\pi^2 [c_1^{\mathrm{orb}}(X, \Sigma, \beta)]^2\right)^{1/2},$$

with equality only if $(1, 1/\sqrt{8}) \propto (\|s\|, \sqrt{48}\|W_+\|)$, and g is Kähler–Einstein with negative scalar curvature. However since $\|s\| = \sqrt{24}\|W_+\|$ for any Kähler metric, equality can never actually occur. Squaring both sides and dividing by $108\pi^2$ thus yields the promised strict inequality. □

Corollary 5.2. *Let X, M, Σ, β and $\mathfrak{c}$ be as in Theorem 4.3, so that $M \approx X\#\ell\overline{\mathbb{CP}}_2$. If (M, Σ, β) admits an orbifold Einstein metric g, then*

$$\ell < \frac{1}{3}(c_1^{\rm orb})^2(X, \Sigma, \beta).$$

Proof. When the orbifold metric g on (M, Σ, β) is Einstein, (15) asserts that

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu = (c_1^{\rm orb})^2(M, \Sigma, \beta).$$

However, notice that

$$\begin{aligned}(c_1^{\rm orb})^2(M, \Sigma, \beta) &= (c_1(M) + (\beta - 1)[\Sigma])^2\\ &= c_1^2(M) + 2(\beta - 1)\ c_1(M)\cdot[\Sigma] + (\beta - 1)^2[\Sigma]^2\\ &= c_1^2(X) - \ell + 2(\beta - 1)\ c_1(X)\cdot[\Sigma] + (\beta - 1)^2[\Sigma]^2\\ &= (c_1^{\rm orb})^2(X, \Sigma, \beta) - \ell.\end{aligned}$$

If (M, Σ, β) admits an orbifold Einstein metric g, (33) therefore says that

$$(c_1^{\rm orb})^2(X, \Sigma, \beta) - \ell > \frac{2}{3}(c_1^{\rm orb})^2(X, \Sigma, \beta)$$

and it follows that $(c_1^{\rm orb})^2(X, \Sigma, \beta)/3 > \ell$, as claimed. □

Assuming the non-vanishing of a suitable Seiberg–Witten invariant, the contrapositive of Corollary 5.2 is that (M, Σ, β) does not admit an orbifold Einstein metric if

$$\ell \geq \frac{1}{3}(c_1^{\rm orb})^2(X, \Sigma, \beta).$$

On the other hand, Theorem 3.3 gives a geometric criterion sufficient to guarantee that the relevant Seiberg–Witten invariants are indeed non-zero. Moreover, Proposition 1.1 asserts that any edge-cone metric on M of edge-cone angle $2\pi\beta$, $\beta = 1/p$, is actually an orbifold Einstein metric on (M, Σ, β). Given the explicit formula

$$c_1^{\rm orb}(X, \Sigma, \beta) = c_1(X) + (\beta - 1)[\Sigma]$$

provided by Lemma 2.2, Theorem A therefore follows.

Let us now illustrate Theorem A via some concrete examples.

Example. Let $X \subset \mathbb{CP}_3$ be a quadric surface, let $Y \subset \mathbb{CP}_3$ be a cubic surface which meets X transversely, and let $\Sigma = X \cap Y$ be the genus 4 curve in which they intersect. Let $M \approx X\#\overline{\mathbb{CP}}_2$ be the blow-up of X at a point not belonging to Σ. Then

$$(c_1^{\rm orb}(X,\Sigma,\beta))^2 = (c_1(X) + (\beta-1)[\Sigma])^2 = 2(1-3\beta)^2$$

so $(c_1^{\rm orb}(X,\Sigma,\beta))^2/3 < 1 = \ell$ for all $\beta = 1/p$, $p \geq 2$. On the other hand, $(c_1^{\rm orb}(X,\Sigma,\beta))^2 < 1$ provided that $2 \leq p \leq 10$. It follows that (M,Σ,β) does not admit orbifold Einstein metrics for any p; consequently, there exist no Einstein edge-cone metrics on (M,Σ) of cone angle $2\pi\beta$ for any $\beta = 1/p$, $p \geq 2$ an integer. Indeed, when $p > 3$, inequality (3) holds if we take ω_0 to be the restriction of the Fubini–Study Kähler form to X, so in this range the claim follows from Theorem A. Proposition 2.5 rules out the existence of Einstein edge-cone metrics for the two cases $p = 2$ or 3 not covered by this argument, and moreover also applies to an interval of real-valued β, but does not suffice to prove the assertion when the cone angle is small. ♢

Example. Let $Y \subset \mathbb{CP}_3$ once again be a cubic surface, and let $\Sigma = Y \cap X$ once again be its intersection with a generic quadric. This time, however, let us instead consider orbifold versions of the manifold $M \approx Y \# \overline{\mathbb{CP}}_2$ obtained by blowing up Y at a point not belonging to Σ. Then

$$(c_1^{\rm orb}(Y,\Sigma,\beta))^2 = (c_1(Y) + (\beta-1)[\Sigma])^2 = 3(1-2\beta)^2$$

so once again $(c_1^{\rm orb}(Y,\Sigma,\beta))^2/3 < 1 = \ell$ for all $\beta = 1/p$, $p \geq 2$, while $(c_1^{\rm orb}(Y,\Sigma,\beta))^2 < 1 = \ell$ when $p = 2$, 3, or 4. Once again, there are no orbifold Einstein metrics on (M,Σ,β) for any p, and consequently no Einstein edge-cone metrics on (M,Σ) of cone angle $2\pi\beta$ for any $\beta = 1/p$, $p \geq 2$ an integer. This time, (3) holds for all $p > 2$, so we only need to use Proposition 2.5 for one single case. ♢

Lemma 5.3. *For some integer $p \geq 2$, set $\beta = 1/p$, and let (M,Σ,β) and $\mathfrak{c}$ be as in Lemma 4.1. If $b_+(M) > 1$, suppose that $SW_{\mathfrak{c}}(M,\Sigma,\beta) \neq 0$; if $b_+(M) = 1$, instead suppose that $c_1^{\rm orb}(L)$ is time-like past-pointing, and that $SW_{\mathfrak{c}}^{\downarrow}(M,\Sigma,\beta) \neq 0$. If g is an orbifold Einstein metric on (M,Σ,β), then its scalar and self-dual Weyl curvatures satisfy*

$$\int_M \frac{s^2}{24} d\mu \geq \int_M |W_+|^2 d\mu,$$

with equality only if g is Kähler–Einstein, of negative scalar curvature.

Proof. By the first inequality of Theorem 4.3, with $\ell = 0$, we have

$$\frac{3}{4\pi^2}\int_M \frac{s^2}{24} d\mu \geq (c_1^{\rm orb})^2(M,\Sigma,\beta),$$

with equality if and only if g is Kähler–Einstein. On the other hand, (13)–(14) tells us that we have

$$(c_1^{\rm orb})^2(M,\Sigma,\beta) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu$$

for the orbifold Einstein metric g. Straightforward algebraic manipulation then yields the desired result. □

This implies the following version of the generalized [**35**] Miyaoka–Yau inequality [**6**], [**51**] for Einstein 4-manifolds:

Theorem 5.4. *For some integer $p \geq 2$, set $\beta = 1/p$, and let (M, Σ, β) and $\mathfrak{c}$ be as in Lemma 4.1. If $b_+(M) > 1$, suppose that $SW_{\mathfrak{c}}(M, \Sigma, \beta) \neq 0$; if $b_+(M) = 1$, instead suppose that $c_1^{\mathrm{orb}}(L)$ is time-like past-pointing, and that $SW_{\mathfrak{c}}^{\downarrow}(M, \Sigma, \beta) \neq 0$. If (M, Σ) admits an Einstein edge-cone metric g of cone angle $2\pi\beta$, then*

$$(\chi - 3\tau)(M) \geq (1-\beta)\big(\chi(\Sigma) - (1+\beta)[\Sigma]^2\big), \tag{34}$$

with equality if and only if $[\Sigma]^2 = (p/2)\chi(\Sigma)$ and, up to constant rescaling, g is locally isometric to the standard complex-hyperbolic metric on $\mathbb{C}\mathcal{H}_2 = SU(2,1)/U(2)$.

Proof. The Gauss–Bonnet-type formulæ (13)–(14) imply that

$$\begin{aligned}(\chi - 3\tau)(M) &- (1-\beta)\big(\chi(\Sigma) - (1+\beta)[\Sigma]^2\big) \\ &= \frac{1}{8\pi^2}\int_M \left[\left(\frac{s^2}{24} - |W_+|^2\right) + 3|W_-|^2 - \frac{|\mathring{r}|^2}{2}\right] d\mu\end{aligned}$$

for any edge-cone metric g on (M, Σ) of any β. However, if the metric is Einstein and $\beta = 1/p$ for some integer p, Proposition 1.1 tells us that g extends to (M, Σ, β) as an Einstein orbifold metric, and Lemma 5.3 then tells us that the right-hand-side is non-negative. The inequality thus follows.

If the inequality is saturated, $W_- = 0$, and Lemma 5.3 moreover tells us that g is a Kähler–Einstein metric of negative scalar curvature. The $\mathrm{End}(\Lambda^+)$ block of the curvature tensor $\mathcal{R}$ in (9) is therefore a constant multiple of $\omega \otimes \omega$, and the $\mathrm{End}(\Lambda^-)$ block is a constant multiple of the identity, while the off-diagonal blocks vanish. Hence $\nabla\mathcal{R} = 0$, and g is locally symmetric. Since a globally symmetric space is uniquely determined by the value of its curvature tensor at one point, it follows that g is modeled on the complex-hyperbolic plane $\mathbb{C}\mathcal{H}_2$, up to a constant rescaling required to normalize the value of its scalar curvature. This done, the totally geodesic complex curve Σ then looks like $\mathbb{C}\mathcal{H}_1 \subset \mathbb{C}\mathcal{H}_2$ in suitably chosen local uniformizing charts. However, in a local orthonormal frame e_1, e_2, e_3, e_4 for $\mathbb{C}\mathcal{H}_2$ for which $e_2 = Je_1$ and $e_4 = Je_3$, one has $\mathcal{R}_{1234} = (1/2)\mathcal{R}_{1212}$, so the curvature of the normal bundle of $\mathbb{C}\mathcal{H}_1 \subset \mathbb{C}\mathcal{H}_2$ is half the curvature of its tangent bundle. Since this normal bundle actually locally represents a p^{th} root of the normal bundle of Σ, it follows that the normal bundle $N \to \Sigma$ satisfies $c_1(N) = (p/2)c_1(T^{1,0}\Sigma)$. In other words, we have $[\Sigma]^2 = (p/2)\chi(\Sigma)$, as claimed. □

When equality occurs, Theorem 5.2 predicts that

$$(\chi - 3\tau)(M) = -\frac{(p-1)^2}{2p}\chi(\Sigma). \tag{35}$$

It is therefore worth pointing out that this case does actually occur. One such example is provided by a beautiful construction [**15**], [**20**] due to Inoue. Let $\mathfrak{S}$ be a genus-2 complex curve on which $\mathbb{Z}_5$ acts with 3 fixed points, let $M \approx (\mathfrak{S} \times \mathfrak{S})\#3\overline{\mathbb{CP}}_2$ be the blow-up of the product at the three corresponding points of the diagonal, and let $\Sigma \subset M$ be the disjoint union of the proper transforms of the graphs of the 5 maps $\mathfrak{S} \to \mathfrak{S}$ arising from the group action. Then $[\Sigma] \in H^2(M, \mathbb{Z})$ is divisible by 5, and it follows that there is a 5-fold cyclic branched cover $Y \to M$ ramified at Σ. One checks that Y saturates the Miyaoka–Yau inequality, and so is complex hyperbolic. The uniqueness of $\lambda < 0$ Kähler–Einstein metrics then predicts that the complex hyperbolic metric is invariant under the $\mathbb{Z}_5$-action, and so descends to an orbifold Einstein metric on (M, Σ, β), where $\beta = 1/5$. As a double-check of Theorem 5.4, notice that, in this example, we have

$$(\chi - 3\tau)(M) = [(2 - 2 \cdot 2)^2 + 3] - 3(-3) = 16,$$

while

$$-\frac{(p-1)^2}{2p}\chi(\Sigma) = -\frac{(5-1)^2}{2 \cdot 5} 5\chi(\mathfrak{S}) = 4^2 = 16.$$

Thus (35) does indeed hold in this example, as predicted. We leave it as an exercise for the interested reader to reprove the existence of the Kähler–Einstein edge metric g on $(M, \Sigma, 1/5)$ directly, using the latest techniques; Theorem 5.4 can then be used to give a different proof of the fact that it is actually complex hyperbolic. Of course, infinitely many other examples exist that also saturate (34); indeed, an infinite hierarchy of these can be produced from the above example by simply taking unbranched covers of $(\mathfrak{S} \times \mathfrak{S})\#3\overline{\mathbb{CP}}_2$.

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Claude LeBrun
Department of Mathematics
SUNY, Stony Brook
NY 11794-3651, USA
E-mail: claude@math.sunysb.edu