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Ginibre-type point processes and their asymptotic behavior

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Abstract. We introduce Ginibre-type point processes as determinantal point processes associated with the eigenspaces corresponding to the so-called Landau levels. The Ginibre point process, originally defined as the limiting point process of eigenvalues of the Ginibre complex Gaussian random matrix, can be understood as a special case of Ginibre-type point processes. For these point processes, we investigate the asymptotic behavior of the variance of the number of points inside a growing disk. We also investigate the asymptotic behavior of the conditional expectation of the number of points inside an annulus given that there are no points inside another annulus.

1. Introduction.

The Ginibre point process arises as the limiting point process of the eigenvalues of a non-hermitian complex Gaussian matrix ensemble and also in physics context as charged particles of two-dimensional one-component plasma at the special temperature [**5**], [**10**]. It is known that the Ginibre point process is the determinantal point process on $\mathbb{C}$ associated with the exponential kernel $\widetilde{K}_0(z,w) = e^{z\overline{w}}$ and the complex Gaussian measure $\lambda(dz) = \pi^{-1}e^{-|z|^2}m(dz)$. A determinantal point process on a space R is characterized by a kernel $K : R \times R \to \mathbb{C}$ (with some conditions) and a Radon measure λ on R; the correlation functions with respect to λ are given by $\rho_n(z_1,\dots,z_n) = \det(K(z_i,z_j))_{i,j=1}^n$. Many important examples of determinantal point processes arise as those associated with reproducing kernel Hilbert spaces of functions on R with reproducing kernel K. For example, the zeroes of hyperbolic Gaussian analytic function studied by Peres–Virág [**15**] are the determinantal point process associated with the so-called Bergman space, i.e., the reproducing kernel Hilbert space of the L^2-analytic functions in the unit disk with respect to the Lebesgue measure with the Bergman kernel $K_{\mathrm{Berg}}(z,w) = \pi^{-1}(1-z\overline{w})^{-2}$. In this sense, we may say that the Ginibre point process is associated with the Bargmann–Fock space, that is, the reproducing kernel Hilbert space of L^2-entire functions with the reproducing kernel $\widetilde{K}_0$ above (cf. [**1**], [**2**]).

The Landau Hamiltonian on $L^2(\mathbb{R}^2)$ is the Schrödinger operator with constant magnetic field in $\mathbb{R}^2$. It is well-known that its spectrum consists of only eigenvalues with infinite multiplicity, which are called Landau levels. The eigenspace corresponding to the lowest eigenvalue, the first Landau level, can be identified with the Bargmann–Fock

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space; besides the other eigenspaces corresponding to higher Landau levels can also be regarded as reproducing kernel Hilbert spaces and they are obtained from the first one by applying a creation operator repeatedly. We may also say that the Ginibre point process is associated with the first Landau level. As naturally expected, one can define the determinantal point processes associated with higher Landau levels; here we call them *Ginibre-type point processes*, which are indexed by $n = 0, 1, 2, \ldots$. We remark that they are invariant under translations and rotations in $\mathbb{C}$ as the original Ginibre point process is. The definition of Ginibre-type point processes will be given in Section 3.

In this paper, we focus on the random variable $\xi(A)$ which counts the number of points inside a set $A \subset \mathbb{C}$ for Ginibre-type point processes and investigate its asymptotic behavior as A tends to the whole space.

First we show that the variance of $\xi(A)$ is of the same order as the square root of the expectation for Ginibre-type point processes while that is the same as the expectation for Poisson point processes. This result has been shown for the Ginibre point process in [**14**], [**17**].

Theorem 1.1. *Let $\mu_{K_n,m}$ be the Ginibre-type point process associated with $K_n(z,w) = L_n(|z-w|^2)K_0(z,w)$ and the Lebesgue measure m, where $K_0(z,w) = \pi^{-1}\exp\{z\overline{w} - (1/2)(|z|^2+|w|^2)\}$ and $L_n(\cdot)$ is the Laguerre polynomial of degree n. Then, for the disk D_r of radius r, we have*

$$\begin{aligned}\mathrm{Var}_{K_n,m}(\xi(D_r)) &= \frac{r}{\pi}\int_0^\infty dt|L_n(t)|^2 e^{-t}\int_0^{t\wedge 4r^2}\left(1-\frac{x}{4r^2}\right)^{1/2} x^{-1/2}dx \\ &\sim C_n r \quad (r\to\infty).\end{aligned}$$

The asymptotic constant C_n is given by

$$C_n = \frac{2\Gamma(n+3/2)}{\pi n!}\,{}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, -n; 1, -\frac{1}{2}-n; 1\right) \sim \frac{8}{\pi^2}n^{1/2} \quad (n\to\infty).$$

We can also prove the following more general result, which recovers the asymptotics part of Theorem 1.1.

Theorem 1.2. *Let $\mu_{K,m}$ be a determinantal point process on $\mathbb{R}^d$ associated with a kernel K and the Lebesgue measure m. Suppose that K is a locally trace class, self-adjoint operator on $L^2(\mathbb{R}^d, m)$ with the property $K^2 = K$ and that there exists a continuous function $k : [0,\infty) \to [0,\infty)$ such that $|K(z,w)|^2 = k(|z-w|^2)$ and $v^{(d-1)/2}k(v) \in L^1([0,\infty))$. Then, for the ball B_r of radius r, we have*

$$\mathrm{Var}_{K,m}(\xi(B_r)) \sim \left(\frac{(2\pi)^{d-1}}{(d-1)!}\int_0^\infty v^{(d-1)/2}k(v)dv\right)r^{d-1}$$

as $r\to\infty$.

We note that the reproducing property of K implies that $K^2 = K$. In the case of the Ginibre-type point process, namely $d = 2$ and $k(v) = \pi^{-2} L_n(v)^2 e^{-v}$, the limiting coefficient of the variance turns out to be $C_n = (2/\pi) \int_0^\infty v^{1/2} L_n(v)^2 e^{-v} dv$ as it follows from Theorem 1.1.

From Theorem 1.2, we see that the variance is of the same order as the surface volume of B_r while the expectation is of the same order as its volume. It is different from the Poisson (with constant intensity) case, in which both expectation and variance are equal to constant multiple of the volume.

Remark 1.3. We note that the condition $K^2 = K$ implicitly assumes $\int_0^\infty v^{(d-2)/2} k(v) dv < \infty$. If the integrability condition for k in Theorem 1.2 fails, we need to find an appropriate scaling. For example, we consider the case where $R = \mathbb{R}^1$ and $K(x, y) = \sin \pi(x - y)/\pi(x - y)$. It appears as the limiting eigenvalues process of Gaussian Unitary Ensemble. In this case, the integrability condition fails since $k(v) = O(v^{-1})$ as $v \to \infty$; indeed, the variance behaves as $O(\log r)$ instead of $O(1)$ as $r \to \infty$. More details have been investigated for $\mathbb{R}^1$ in [**22**].

The resemblance between the Ginibre point process and the zero set of the planar Gaussian analytic function (PGAF) defined as a random power series $\sum_{n=0}^\infty \zeta_n z^n/\sqrt{n!}$ $(z \in \mathbb{C})$ has been emphasized and discussed, where $\{\zeta_n\}_{n \geq 0}$ are i.i.d. standard complex Gaussian random variables. Both are point processes on $\mathbb{C}$ that are of repulsive nature and invariant under translations and rotations. The hole probability that there are no zeroes of PGAF inside the disk D_r decays like $\exp(-Cr^4)$ as $r \to \infty$ [**13**], [**20**], and the large deviations for the number of points inside D_r for the Ginibre point process are discussed in [**17**]. A related problem which is called overcrowding estimates for both point processes is considered in [**11**]. The large deviations for the zeroes of time dependent PGAF are discussed in [**7**], where a simulation is also given which shows an accumulation phenomenon of zeroes in a typical realization of the zero set of PGAF conditioned that there are no zeroes inside a disk. This phenomenon was also mentioned for the Ginibre point process as a two-dimensional one-component plasma model in [**10**]. In the present paper, we show more quantitative evidence of the occurrence of a similar accumulation phenomenon for the Ginibre-type point process instead of the zeroes of PGAF. To this end, we consider two annuli with common center and compute the expectation of the number of points within one annulus conditioned that there are no points inside the other annulus.

We denote the annulus $\{z \in \mathbb{C}; x \leq |z| < y\}$ by A_x^y. The symbol $\mathbb{E}$ denotes the expectation with respect to the law of the Ginibre-type point process. We consider two disjoint annuli $A_{\sqrt{a}r}^{\sqrt{b}r}$ and $A_{\sqrt{\alpha}r}^{\sqrt{\beta}r}$. Conditional expectations differ considerably according to whether two annuli have the common boundary or not.

Theorem 1.4. *Suppose* $0 \leq \alpha < \beta \leq a < b < \infty$ *or* $0 \leq a < b \leq \alpha < \beta < \infty$. *Then,*

$$\lim_{r\to\infty} \frac{1}{r^2} \mathbb{E}\Big[\xi\Big(A_{\sqrt{a}r}^{\sqrt{b}r}\Big) \,\Big|\, \xi\Big(A_{\sqrt{\alpha}r}^{\sqrt{\beta}r}\Big) = 0\Big] = \begin{cases} b - a & \alpha < \beta < a < b \quad \text{or} \quad a < b < \alpha < \beta \\ b - \gamma_{\alpha\beta} & \alpha < \beta = a < b \\ \gamma_{\alpha\beta} - a & a < b = \alpha < \beta \end{cases}$$

where $\gamma_{\alpha\beta} = (\beta - \alpha)/(\log \beta - \log \alpha) \in [\alpha, \beta]$ *and* $\gamma_{0\beta}$ *is understood to be* 0.

Since $\mathbb{E}[\xi(A_{\sqrt{a}r}^{\sqrt{b}r})] = (b-a)r^2$, this theorem implies that the effect of the conditioning asymptotically vanishes in the limit if the observation point is away from the vacant annulus, however, it also says that there must be $O(r^2)$ number of points close to the boundary of the vacant annulus. The following theorem clarifies more precisely the situation that $O(r^2)$ number of points accumulate within a thin annulus of finite area on the boundary of the vacant annulus.

Theorem 1.5. *Let* $\kappa > 0$. *If* $0 \le \alpha < \beta$, *then*

$$\lim_{r\to\infty} \frac{1}{r^2}\mathbb{E}\Big[\xi\Big(A_{\sqrt{\beta}r}^{\sqrt{\beta r^2+\kappa}}\Big) \,\Big|\, \xi\Big(A_{\sqrt{\alpha}r}^{\sqrt{\beta}r}\Big) = 0\Big] = \int_{\gamma_{\alpha\beta}}^{\beta} (1 - e^{-\kappa(1-s/\beta)})ds$$

and if $0 < \alpha < \beta$, *then*

$$\lim_{r\to\infty} \frac{1}{r^2}\mathbb{E}\Big[\xi\Big(A_{\sqrt{\alpha r^2-\kappa}}^{\sqrt{\alpha}r}\Big) \,\Big|\, \xi\Big(A_{\sqrt{\alpha}r}^{\sqrt{\beta}r}\Big) = 0\Big] = \int_{\alpha}^{\gamma_{\alpha\beta}} (1 - e^{-\kappa(s/\alpha-1)})ds.$$

Remark 1.6. The constant $\gamma_{\alpha\beta}$ which appears in Theorem 1.4 and Theorem 1.5 is the unique solution to the equation $\mathcal{I}(\alpha/x) = \mathcal{I}(\beta/x)$ $(x > 0)$, where $\mathcal{I}(x) = x - 1 - \log x$ is the rate function for the large deviations of sum of i.i.d. exponential random variables with mean 1.

The organization of this paper is as follows. In Section 2 we recall some basic properties of Landau Hamiltonian, in particular, its eigenspaces corresponding to Landau levels and the projection integral kernels on them. In Section 3 we introduce the Ginibre-type point processes after recalling the definition of determinantal point processes and their properties. In Section 4 we introduce Gamma-like and Poisson-like random variables related to the eigenvalues of K_n in Theorem 1.1 and compute their Laplace transforms. Also we give a limit theorem for them. We prove Theorem 1.1 in Section 5 and Theorem 1.2 in Section 6. In Section 7, we recall the notion of simultaneous observability and give an expression for conditional expectation. We give proofs of Theorems 1.4 and 1.5 in Section 8.

2. Landau Hamiltonian and Landau levels.

In this section, we discuss the eigenspaces of the Landau Hamiltonian and the corresponding orthogonal projections. Although the content of this section is well-known, we briefly review it here. See for related topics (cf. [**9**], [**16**]).

Let us consider the Schrödinger operator with magnetic field in two dimension. The operator

$$H = \frac{1}{2}(i\nabla - a(\boldsymbol{x}))^2 \quad \boldsymbol{x} = (x, y) \in \mathbb{R}^2$$

acts on the Hilbert space $L^2(\mathbb{R}^2)$. When we put $a(\boldsymbol{x}) = (by, -bx)$ for $b \in \mathbb{R}$, it describes the Schrödinger operator with uniform magnetic field:

$$H = \frac{1}{2}(i\partial_x - by)^2 + \frac{1}{2}(i\partial_y + bx)^2.$$

In what follows, we assume, for simplicity, that $b > 0$ and identify $L^2(\mathbb{R}^2)$ with $L^2(\mathbb{C})$ with respect to the Lebesgue measure $m(dz)$ as usual by the mapping $(x, y) \mapsto z = x+iy$. We define creation and annihilation operators by

$$s^\dagger = \partial_z - \frac{b}{2}\bar{z}, \quad s = -\partial_{\bar{z}} - \frac{b}{2}z,$$

where $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$. These are also expressed as

$$s^\dagger = e^{b|z|^2/2}\partial_z e^{-b|z|^2/2}, \quad s = -e^{-b|z|^2/2}\partial_{\bar{z}} e^{b|z|^2/2}. \tag{2.1}$$

Note that $[s, s^\dagger] := ss^\dagger - s^\dagger s = b$. It is easy to see that s and $s^\dagger$ are mutually adjoint as operators acting on $L^2(\mathbb{C})$. Using these operators, the operator H can be represented as

$$H = 2s^\dagger s + b = 2ss^\dagger - b.$$

From this expression we easily see that the spectrum of H is given by $\sigma(H) = \{(2n + 1)b,\ n = 0, 1, 2, \dots\}$ using the standard argument. For example, for any entire function h, putting $\phi_h = e^{-b|z|^2/2}h$, we have $s\phi_h = 0$, which implies that $H\phi_h = (2s^\dagger s + b)\phi_h = b\phi_h$. Since $[H, s^\dagger] = 2bs^\dagger$, we also have $H(s^\dagger\phi_h) = 3b(s^\dagger\phi_h)$. Similarly, we can show that $H((s^\dagger)^n\phi_h) = (2n + 1)b(s^\dagger)^n\phi_h$.

Let $\mathcal{H}_n$ be the eigenspace corresponding to the eigenvalue $(2n + 1)b$. Then, we can decompose $L^2(\mathbb{C})$ into the direct sum of $\mathcal{H}_n, n = 0, 1, 2, \dots$ as $L^2(\mathbb{C}) \cong \bigoplus_{n=0}^\infty \mathcal{H}_n$. It is well-known that $\mathcal{H}_0$ coincides with the space of square integrable functions of the form $h(z)e^{-b|z|^2/2}$ with h being entire. In fact, we have the following.

LEMMA 2.1. *Let $\phi_j(z) = \sqrt{b^{j+1}/\pi j!}z^j e^{-(b/2)|z|^2}$ for $j = 0, 1, 2, \dots$. Then, the system $\{\phi_j, j = 0, 1, 2, \dots\}$ forms an orthonormal basis of $\mathcal{H}_0$. The projection operator $K_{0,b}$ from $L^2(\mathbb{C})$ onto $\mathcal{H}_0$ admits an integral kernel given by*

$$K_{0,b}(z, w) = \sum_{j=0}^\infty \phi_j(z)\overline{\phi_j(w)} = \frac{b}{\pi}e^{bz\overline{w}-(b/2)(|z|^2+|w|^2)}. \tag{2.2}$$

In particular, $K_{0,b}(z, z) = \pi^{-1}b$.

PROOF. See (cf. [**9**]). □

LEMMA 2.2. *Let $T_n = s^n(s^\dagger)^n$. Then, $T_n = (1/2^n)\prod_{k=1}^n\{H + (2k - 1)b\}$. In particular, $T_n = n!b^n$ on $\mathcal{H}_0$.*

Proof. Since $[s, H] = 2bs$ and $T_1 = ss^\dagger = (H+b)/2$, we see that

$$2^n T_n = 2^n s T_{n-1} s^\dagger = \left(\prod_{k=2}^{n} \{H + (2k-1)b\} \right) \cdot 2ss^\dagger = \prod_{k=1}^{n} \{H + (2k-1)b\}.$$

In particular, since $H = b$ on $\mathcal{H}_0$, we have $T_n = n! b^n$ on $\mathcal{H}_0$. □

Now we consider the eigenspaces corresponding to the higher Landau levels. Before proceeding, we recall the definition and properties of the generalized Laguerre polynomials

$$L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} \left(\frac{d}{dx} \right)^n (x^{n+\alpha} e^{-x}) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \tag{2.3}$$

for $\alpha \in \mathbb{R}$ and $n = 0, 1, 2, \dots$, where $\binom{\alpha}{k} = \alpha(\alpha-1)\cdots(\alpha-k+1)/k!$. Note that $L_n^{(\alpha)}(0) = 1$ and $L_0^{(\alpha)}(x) \equiv 1$. In particular, $L_n^{(0)}(x)$ is the ordinary Laguerre polynomial and simply denoted by $L_n(x)$. The following formula is also useful (cf. Problem 20 on page 96 [**12**]):

$$L_n^{(\beta)}(x) = \sum_{r=0}^{n} \frac{(\beta-\alpha)_r}{r!} L_{n-r}^{(\alpha)}(x), \tag{2.4}$$

where $(\alpha)_k$ is the Pochhammer symbol defined by $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. If we define an inner product by $\langle f, g \rangle_\alpha := \int_0^\infty f(x) g(x) x^\alpha e^{-x} dx$ for $\alpha > -1$, then we have

$$\left\langle L_k^{(\alpha)}, L_l^{(\alpha)} \right\rangle_\alpha = \frac{\Gamma(k+\alpha+1)}{k!} \delta_{k,l} \quad (\alpha > -1). \tag{2.5}$$

In other words, $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ are the orthogonal polynomials with respect to the measure $x^\alpha e^{-x} dx$ on $(0, \infty)$.

Remark 2.3. Let $p \in \mathbb{N}$ and set $\alpha = -p$. The function $x^{-p} L_k^{(-p)}(x)$ is a polynomial for $k \geq p$ and the formula (2.5) still makes sense for $k \wedge l \geq p$ even when $\alpha \leq -1$.

Proposition 2.4. *Let* $\psi_j^{(n)} = (b^n n!)^{-1/2} (s^\dagger)^n \phi_j$. *Then, the system* $\{\psi_j^{(n)}, j = 0, 1, \dots\}$ *forms an orthonormal basis of* $\mathcal{H}_n$. *More explicitly,*

$$\psi_j^{(n)}(z) = \sqrt{\frac{b^{j+1-n} n!}{\pi j!}} L_n^{(j-n)}(b|z|^2) z^{j-n} e^{-b|z|^2/2}. \tag{2.6}$$

Proof. From Lemma 2.2, we see that

$$\left\langle \psi_j^{(n)}, \psi_k^{(n)} \right\rangle = (b^n n!)^{-1} \langle T_n \phi_j, \phi_k \rangle = \delta_{j,k}.$$

The explicit expression (2.6) follows from (2.1) and (2.3). We omit the proof for completeness of the system. □

PROPOSITION 2.5. *The projection operator $K_{n,b}$ onto $\mathcal{H}_n$ admits an integral kernel given by*

$$K_{n,b}(z,w) = K_{0,b}(z,w)L_n(b|z-w|^2), \tag{2.7}$$

where $L_n(x)$ is the Laguerre polynomial of degree n defined by

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k} x^k.$$

PROOF. By (2.6), we have

$$K_{n,b}(z,w) = \sum_{j=0}^{\infty} \psi_j^{(n)}(z)\overline{\psi_j^{(n)}(w)} = (b^n n!)^{-1} e^{b(|z|^2+|w|^2)/2}(\partial_z \partial_{\overline{w}})^n g(z,w),$$

where $g(z,w) = (b/\pi)e^{b\{z\overline{w}-(|z|^2+|w|^2)\}}$. Formal computation is justified as the convergence is uniform in z and w on compacts. Since $\partial_z g(z,w) = -b\overline{(z-w)}g(z,w)$ and $\partial_{\overline{w}} g(z,w) = b(z-w)g(z,w)$, by using Leibniz's rule, we have $\partial_z^n(\partial_{\overline{w}}^n g(z,w)) = b^n n! L_n(b|z-w|^2)g(z,w)$. This implies (2.7). □

REMARK 2.6. The space $\mathcal{H}_n$ is the reproducing kernel Hilbert space with reproducing kernel $K_{n,b}$. In particular, $(\mathcal{H}_0, K_{0,b})$ is the Bargmann–Fock space as mentioned in the introduction.

3. Ginibre-type point processes.

In this section, we introduce Ginibre-type point processes. Before doing this, we recall the definition and some well-known facts about determinantal point processes (cf. [**21**], [**19**]).

Let R be a locally compact Hausdorff space with countable basis and $\mathcal{B}(R)$ the topological Borel σ-field. We fix a Radon measure $\lambda(dx)$ on $(R, \mathcal{B}(R))$. The configuration space $Q = Q(R)$ over R is the totality of non-negative integer-valued Radon measures on R equipped with the topology which is generated by the functions $Q \ni \xi \mapsto \xi(A) \in \mathbb{Z}_{\geq 0}$ for every $A \in \mathcal{B}(R)$, where $\xi(A)$ is equal to the number of points that fall in the subset A. Every element ξ of Q can be written as $\xi = \sum_i \delta_{z_i}$ and understood as a point configuration of $\{z_i\}_i \subset R$ without accumulation points. We call a pair (Q, μ) a point process on R, where μ is a Borel probability measure on Q. If there exists a non-negative measurable function $\rho_1(x)$ so that

$$E[\langle \xi, \phi \rangle] = E\left[\int_R \phi(x)\xi(dx)\right] = \int_R \phi(x)\rho_1(x)\lambda(dx)$$

for every $\phi \in C_c(R)$, the set of all continuous functions on R of compact support, we say that $\rho_1(x)$ is the first correlation function with respect to λ. By the definition, $\rho_1(x)$ is the mean density of points at $x \in R$. We define a Radon measure ξ_n on R^n from ξ by $\xi_n = \sum_{\substack{x_1,\dots,x_n \in \xi \\ \text{distinct}}} \delta_{(x_1,\dots,x_n)}$. If there exists a symmetric, non-negative measurable function ρ_n on R^n so that

$$E[\langle \xi_n, \phi \rangle] = \int_{R^n} \phi(x_1,\dots,x_n)\rho_n(x_1,\dots,x_n)\lambda^{\otimes n}(dx_1\dots dx_n)$$

for every $\phi \in C_c(R^n)$, we say that ρ_n is the n-th correlation function with respect to $\lambda^{\otimes n}$.

We summarize the existence and uniqueness result for a determinantal point process associated with kernel K and Radon measure λ as follows.

Theorem 3.1 ([**21**], [**19**]). *Let K be a self-adjoint integral operator on $L^2(R,\lambda)$ with continuous kernel $K(x,y)$. Suppose that $O \le K \le I$ and K is of locally trace class, i.e., for any compact set $\Lambda \subset R$, $K_\Lambda = I_\Lambda K I_\Lambda$ is of trace class, where I_Λ is the multiplication operator of the indicator function of a set Λ. Then there exists a unique Borel probability measure $\mu_{K,\lambda}$ on Q such that for any non-negative bounded measurable function f with compact support Λ*

$$\int_Q \mu_{K,\lambda}(d\xi)\exp(-\langle \xi, f\rangle) = \det(I - (1-e^{-f})K_\Lambda), \tag{3.1}$$

where det *denotes the Fredholm determinant for trace class operators. Moreover, the n-th correlation function with respect to λ is given by*

$$\rho_n(x_1,\dots,x_n) = \det(K(x_i,x_j))_{i,j=1}^n. \tag{3.2}$$

The resultant point process $\mu_{K,\lambda}$ is called a *determinantal* (*or fermion*) *point process* on R associated with kernel K and λ. We sometimes omit the base measure λ if there is no confusion. General properties of determinantal point processes are also found in (cf. [**8**], [**21**], [**19**]).

Now we introduce Ginibre-type point processes. We consider the projection operator $K_{n,b}$ on $L^2(\mathbb{C})$ discussed in the previous section. It is easy to check that $K_{n,b}$ satisfies the assumptions in Theorem 3.1. Therefore we have a determinantal point process $\mu_{K_{n,b}}$ on $\mathbb{C}$ associated with the operator $K_{n,b}$.

Definition 3.2 (Ginibre-type point processes). For $n = 0,1,2,\dots$ and $b > 0$, let $K_{n,b}$ be the projection operator defined as in (2.7) and m the Lebesgue measure on $\mathbb{C}$. We call the determinantal point process on $\mathbb{C}$ associated with $(K_{n,b}, m)$ a *Ginibre-type point process* (*with index* (n,b)).

Remark 3.3. When $n = 0$, the resultant point process is known as the Ginibre point process. Similar point processes that are called polyanalytic Ginibre ensembles are discussed in [**6**]. They are finite determinantal point processes associated with the reproducing kernel Hilbert spaces $\mathrm{Pol}_{b,n,q}$ of the complex polynomials of degree $\le n-1$ in z and of degree $\le q-1$ in $\bar{z}$. In [**6**], they investigate the asymptotic behavior of the Berezin measure as $n \to \infty$ for fixed q.

REMARK 3.4. We have already mentioned that (K, λ) defines a determinantal point process. However, this correspondence is not one-to-one. Suppose $\lambda(dx) = f(x)\nu(dx)$ and $K_f(x,y) = \sqrt{f(x)}K(x,y)\sqrt{f(y)}$, it is easy to check that correlation measures of $\mu_{K,\lambda}$ and $\mu_{K_f,\nu}$ are the same, which implies that the two point processes $\mu_{K,\lambda}$ and $\mu_{K_f,\nu}$ are the same. For example, if we consider $L^2(\mathbb{C}, \lambda_b)$ with $\lambda_b(dz) = (b/\pi)e^{-b|z|^2}m(dz)$ as a base L^2-space, then $(\tilde{K}_{0,b}(z,w) = e^{bz\overline{w}}, \lambda_b)$ defines the same determinantal point process as that associated with $(K_{0,b}, m)$.

By the formula (3.2), the first and second correlation functions for $\mu_{K_{n,b},m}$ are given by $\rho_1(z) = \pi^{-1}b$ and $\rho_2(z,w) = \pi^{-2}b^2L_n(b|z-w|^2)^2(1-e^{-b|z-w|^2})$, respectively. They are translation and rotation invariant. The n-th correlation function has the same invariance property for every n.

PROPOSITION 3.5. *For any $n = 0, 1, 2, \ldots$, the determinantal process $\mu_{K_{n,b}}$ is invariant under the action $z \mapsto \alpha z + \beta$ with $|\alpha| = 1$, α, $\beta \in \mathbb{C}$.*

PROOF. The proof is almost the same as in that of Proposition 3.1 in [**17**]. □

4. Poisson-like and Gamma-like random variables.

First we recall a remarkable fact that the number of points inside a set Λ under $\mu_{K,\lambda}$ can be expressed as the sum of independent Bernoulli random variables with parameters being the eigenvalues of K_Λ. Indeed, if we put $f = \alpha I_\Lambda$ for a compact set Λ in (3.1), it is easy to see that

$$\int_Q \exp(-\alpha\xi(\Lambda))\mu_{K,\lambda}(d\xi) = \det(I-(1-e^{-\alpha})K_\Lambda) = \prod_{j=0}^{\infty}\{1-\kappa_j(\Lambda)+e^{-\alpha}\kappa_j(\Lambda)\},$$

where $\{\kappa_j(\Lambda)\}_{j\ge 0}$ are the eigenvalues of K_Λ on $L^2(R,\lambda)$. This implies that the random variable $\xi(\Lambda)$ is equal in law to $\sum_{j=0}^{\infty} X_j$, the sum of independent Bernoulli random variables X_j's, where $P(X_j = 1) = \kappa_j(\Lambda)$ and $P(X_j = 0) = 1-\kappa_j(\Lambda)$. We note that the condition $O \le K \le I$ in Theorem 3.1 guarantees that the eigenvalues of the restriction K_Λ, the parameters of Bernoulli random variables, are contained in $[0,1]$. For probabilistic analysis of the number of points (cf. [**8**], [**17**]), all we have to do is analyze the eigenvalues $\{\kappa_j(\Lambda)\}_{j=0}^{\infty}$. We will use an extension (Lemma 7.2) of this fact in Sections 7 and 8.

Now we consider the Ginibre-type point processes. Let $D_r \subset \mathbb{C}$ be the disk of radius r. The restricted operator $(K_{n,b})_{D_r}$ admits an orthogonal basis $\{\psi_j^{(n)} I_{D_r}, j = 0, 1, \ldots\}$ of eigenfunctions, where $\psi_j^{(n)}$ is the same as in (2.6). The squares of their norms are equal to the eigenvalues $\{\kappa_j^{(n,b)}(r), j = 0, 1, 2, \ldots\}$ of $(K_{n,b})_{D_r}$. Hence we easily see that

$$\kappa_j^{(n,b)}(r) = \|\psi_j^{(n)} I_{D_r}\|^2 = \frac{b^{j+1-n}n!}{j!}\int_0^{r^2} \big|L_n^{(j-n)}(bt)\big|^2 t^{j-n}e^{-bt}dt.$$

When $n = 0$, we have two probabilistic representations of $\kappa_j^{(0,b)}(r)$:

$$\kappa_j^{(0,b)}(r) = P\big(S_j^{(0,b)} \le r^2\big) = P\big(Z_{r^2}^{(0,b)} \ge j+1\big), \tag{4.1}$$

where $S_j^{(0,b)}$ is the sum of $j+1$ of independent exponential random variables with mean b^{-1}, namely, the Gamma random variable with mean $b^{-1}(j+1)$, and $Z_t^{(0,b)}$ is the Poisson random variable with mean bt. See Remark 3.3 in [**17**]. For $n \ge 1$, we can also define a random variable $S_j^{(n,b)}$ such that

$$\kappa_j^{(n,b)}(r) = P\big(S_j^{(n,b)} \le r^2\big) \tag{4.2}$$

by setting

$$\frac{P(S_j^{(n,b)} \in dt)}{dt} = \frac{b^{j+1-n}n!}{j!}\big|L_n^{(j-n)}(bt)\big|^2 t^{j-n}e^{-bt}. \tag{4.3}$$

However, we cannot expect the second type of expression in (4.1) since we can easily see that $\kappa_j^{(n,b)}(r)$ is not monotone decreasing in j unless $n = 0$. Nevertheless, for later use we discuss non-negative integer-valued random variables $Z_t^{(n,b)}$ $(t,\ b > 0,\ n = 0,1,2,\dots)$ defined by

$$P\big(Z_t^{(n,b)} = j\big) = \frac{b^{j-n}n!}{j!}\big|L_n^{(j-n)}(bt)\big|^2 t^{j-n}e^{-bt} \quad (j = 0,1,2,\dots). \tag{4.4}$$

The fact that $Z_t^{(n,b)}$ is a random variable is equivalent to the formula $K_{n,b}(z,z) = \sum_{j=0}^{\infty}|\psi_j^{(n)}(z)|^2 = \pi^{-1}b$ by setting $t = |z|^2$. By (4.3) and (4.4), we have the following duality: for any Borel set $A \subset [0,\infty)$ and $I \subset \{0,1,2,\dots\}$,

$$\sum_{j\in I} P\big(S_j^{(n,b)} \in A\big) = b\int_A P\big(Z_t^{(n,b)} \in I\big)dt. \tag{4.5}$$

It is easy to see that $S_j^{(n,b)} \overset{d}{=} b^{-1}S_j^{(n,1)}$ and $Z_t^{(n,b)} \overset{d}{=} Z_{bt}^{(n,1)}$. So in what follows, we always assume that $b = 1$ and simply write $S_j^{(n)}$ and $Z_t^{(n)}$ for $S_j^{(n,1)}$ and $Z_t^{(n,1)}$, respectively.

We will compute the Laplace transforms of $S_j^{(n)}$ and $Z_t^{(n)}$.

Proposition 4.1. *For $n, j = 0,1,2,\dots$ and $\lambda < 1$,*

$$E\big[e^{\lambda S_j^{(n)}}\big] = (1-\lambda)^{-(j+n+1)}\sum_{p=0}^{n\wedge j}\binom{n}{p}\binom{j}{p}\lambda^{2p}.$$

In particular, $S_j^{(n)} \stackrel{d}{=} S_n^{(j)}$, $E[S_j^{(n)}] = j+n+1$ *and* $\mathrm{Var}(S_j^{(n)}) = (2n+1)j+n+1$.

PROOF. Let $\tilde{L}_n^{(\alpha)}(t) = (1-\lambda)^n L_n^{(\alpha)}(t/(1-\lambda))$. Since $\tilde{L}_n^{(\alpha)}$ is a polynomial of degree n, it is a linear combination of the polynomials $L_k^{(\alpha)}$, $k=0,1,\dots,n$. Indeed, we see that

$$\begin{aligned}\tilde{L}_n^{(\alpha)}(t) &= \sum_{k=0}^{n} \frac{(-t)^k}{k!}\left\{\sum_{p=0}^{n-k}(-\lambda)^p\binom{n-k}{p}\right\}\binom{n+\alpha}{n-k} \\ &= \sum_{p=0}^{n}\sum_{k=0}^{n-p}\frac{(-t)^k}{k!}(-\lambda)^p\binom{n+\alpha}{p}\binom{n-p+\alpha}{n-p-k} \\ &= \sum_{p=0}^{n}(-\lambda)^p\binom{n+\alpha}{p}L_{n-p}^{(\alpha)}(t).\end{aligned}$$

In particular, when $\alpha = j-n$,

$$\tilde{L}_n^{(j-n)}(t) = \sum_{p=0}^{n\wedge j}(-\lambda)^p\binom{j}{p}L_{n-p}^{(j-n)}(t). \tag{4.6}$$

By (4.3), (4.6) and Remark 2.3, we see that

$$\begin{aligned}E\big[e^{\lambda S_j^{(n)}}\big] &= (1-\lambda)^{-(j+n+1)}\frac{n!}{j!}\big\langle \tilde{L}_n^{(j-n)}, \tilde{L}_n^{(j-n)}\big\rangle_{j-n} \\ &= (1-\lambda)^{-(j+n+1)}\frac{n!}{j!}\sum_{p=0}^{n\wedge j}\lambda^{2p}\binom{j}{p}^2\big\langle L_{n-p}^{(j-n)}, L_{n-p}^{(j-n)}\big\rangle_{j-n}.\end{aligned}$$

Hence, we obtain the assertion by (2.5). □

REMARK 4.2. The limiting logarithmic moment generating function of the random variables $(S_j^{(n)})_{j=0,1,\dots}$ is

$$\Lambda_1^{(n)}(\lambda) := \lim_{j\to\infty}\frac{1}{j}\log E\big[e^{\lambda S_j^{(n)}}\big] = -\log(1-\lambda) \quad (\lambda<1),$$

which is independent of n. The rate function for the random variables $(S_j^{(n)})_{j=0,1,\dots}$ is given by the Legendre transform of $\Lambda_1^{(n)}(\lambda)$ and it is equal to $\mathcal{I}(x) = x-1-\log x$, which is the same as that of Poisson random variables.

Next we compute the Laplace transform of $Z_t^{(n)}$.

PROPOSITION 4.3. *For* $t>0$ *and* $n=0,1,2,\dots,$

$$E\big[e^{\lambda Z_t^{(n)}}\big] = \exp(t(e^\lambda - 1))e^{\lambda n} L_n\bigg(-4t\sinh^2\frac{\lambda}{2}\bigg) \quad (\lambda \in \mathbb{R}). \tag{4.7}$$

In particular, $E[Z_t^{(n)}] = t + n$ *and* $\mathrm{Var}(Z_t^{(n)}) = (2n+1)t$.

Proof. By duality relation (4.5) and Proposition 4.1, it is easy to verify that

$$\int_0^\infty E\big[e^{\lambda Z_t^{(n)}}\big]e^{\mu t}dt = \sum_{j=0}^\infty e^{\lambda j} E\big[e^{\mu S_j^{(n)}}\big] = \frac{(1-e^\lambda - \mu e^\lambda)^n}{(1-e^\lambda-\mu)^{n+1}} \tag{4.8}$$

whenever $1-e^\lambda-\mu > 0$. Here we used the formula $\sum_{j=p}^\infty \binom{j}{p} z^j = z^p(1-z)^{-(p+1)}$ ($|z|<1$). On the other hand, we easily see that the Laplace transform of the right-hand side of (4.7) is equal to the right-hand side of (4.8) by using the formula $\int_0^\infty e^{-\alpha t}L_n(\beta t)dt = \alpha^{-(n+1)}(\alpha-\beta)^n$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. Therefore, we obtain (4.7). □

Remark 4.4. The limiting logarithmic moment generating function of the random variables $(Z_t^{(n)})_{t>0}$ is

$$\Lambda_2^{(n)}(\lambda) := \lim_{t\to\infty} \frac{1}{t}\log E\big[e^{\lambda Z_t^{(n)}}\big] = e^\lambda - 1,$$

which is independent of n. The rate function for the random variables $(Z_t^{(n)})_{t>0}$ is given by the Legendre transform of $\Lambda_2^{(n)}(\lambda)$ and is equal to $I(x) = 1 - x + x\log x$, which is that for the Gamma random variables.

Remark 4.5. From Proposition 4.3, we can formally decompose $Z_t^{(n)}$ as

$$Z_t^{(n)} \overset{d}{=} Z_t^{(0)} + N_t^{(n)} + n.$$

Here $N_t^{(n)}$ is, in general, a "signed" random variable in the sense that $P(N_t^{(n)} \in \cdot)$ is a signed measure given by

$$P\big(N_t^{(n)} = j\big) = (-1)^j \sum_{k=|j|}^n \frac{(-t)^k}{k!}\binom{n}{k}\binom{2k}{k+|j|}.$$

It is independent of $Z_t^{(0)}$ and takes values in $[-n,n]\cap\mathbb{Z}$. There exists $t_0 > 0$ such that $P(N_t^{(n)} = j) \geq 0$ ($\forall j \in [-n,n]\cap\mathbb{Z}$) for any $t \in [0,t_0]$. In other words, $N_t^{(n)}$ is indeed a random variable for any small enough $t > 0$.

It is clear that the central limit theorem holds for $S_j^{(0)}$ and $Z_t^{(0)}$, however, it is not the case for $S_j^{(n)}$ and $Z_t^{(n)}$ for $n \geq 1$. We need a slight modification.

Let A_n be a random variable whose distribution is $(1/n!)H_n(t)^2(1/\sqrt{2\pi})e^{-t^2/2}dt$, where $H_n(t)$ is the n-th Hermite polynomial defined by

$$H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}.$$

It is easy to see that $E[A_n] = 0$ since its distribution is symmetric about the origin and $\mathrm{Var}(A_n) = 2n+1$. Moreover, from the formula (cf. Problem 9 on page 95 [**12**], in which the definition of $H_n(t)$ is slightly different from the above), the characteristic function of A_n is given by

$$E[e^{i\lambda A_n}] = \int_{\mathbb{R}} \frac{1}{n!} H_n(t)^2 \frac{1}{\sqrt{2\pi}} e^{-t^2/2+i\lambda t} dt = L_n(\lambda^2) e^{-\lambda^2/2}. \tag{4.9}$$

Then, we obtain the following limit theorem.

PROPOSITION 4.6. *The normalized random variables* $\widetilde{Z}_t^{(n)} = (Z_t^{(n)} - E[Z_t^{(n)}])/\sqrt{t}$ *and* $\widetilde{S}_j^{(n)} = (S_j^{(n)} - E[S_j^{(n)}])/\sqrt{j}$ *converge in law to* A_n *as* $t \to \infty$ *and* $j \to \infty$, *respectively.*

PROOF. By Proposition 4.3, we see that

$$E\big[e^{\lambda(Z_t^{(n)} - E[Z_t^{(n)}])}\big] = \exp(t(e^\lambda - 1 - \lambda)) L_n\Big(-4t \sinh^2 \frac{\lambda}{2}\Big).$$

Plugging $i\lambda/\sqrt{t}$ into λ and taking the limit $t \to \infty$, we have $L_n(\lambda^2)e^{-\lambda^2/2}$. From (4.9) we show the first assertion. The proof for the second one is also easy, so we omit it here. □

5. Asymptotic behavior of variance for Ginibre-type point processes.

In what follows, we always assume $b = 1$. We simply write K_n for $K_{n,1}$. In this section, we compute the variance of $\xi(D_r)$ and show that it behaves like $C_n r$ as $r \to \infty$. See [**14**], [**17**] in the case of $n = 0$. Now we recall the following representation of the variance, which is crucial for computing the exact asymptotics.

LEMMA 5.1. *Let* $\mu_{K,\lambda}$ *be the determinantal point process on* R *associated with* (K, λ) *and suppose* $K^2 = K$. *Then, the variance of linear statistics* $\langle \xi, f\rangle$ *with respect to* $\mu_{K,\lambda}$ *is given by*

$$\mathrm{Var}_{K,\lambda}(\langle \xi, f\rangle) = \frac{1}{2} \int_{R^2} |f(z) - f(w)|^2 |K(z,w)|^2 \lambda(dz)\lambda(dw).$$

In particular, for a measurable set D *in* R,

$$\begin{aligned}\mathrm{Var}_{K,\lambda}(\xi(D)) &= \int_D K(z,z)\lambda(dz) - \int_{D^2} |K(z,w)|^2 \lambda(dz)\lambda(dw) \\ &= \int_D \lambda(dz) \int_{D^c} \lambda(dw) |K(z,w)|^2. \end{aligned} \tag{5.1}$$

Proof. See [**14**], for example. □

Lemma 5.2. *Let* $M_{2n} = \int_{D_r^2} |z-w|^{2n} |K_0(z,w)|^2 m(dz)m(dw)$ *for non-negative integer* n. *Then,*

$$M_{2n} = n!\left(r^2 - \frac{r}{\pi}\sum_{k=0}^{n} \frac{\alpha_k(r)}{k!}\right), \tag{5.2}$$

where $\alpha_k(r) = \int_0^{4r^2} (1-y/4r^2)^{1/2} y^{k-1/2} e^{-y} dy$.

Proof. Let us consider the generating function of $2n$-th moments M_{2n}, that is,

$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} M_{2n} = \pi^{-2} \int_{D_r^2} e^{-(1-t)|z-w|^2} m(dz)m(dw)$$

for $|t| < 1$. By change of variables, we find that

$$g(t) = (1-t)^{-2}\{\tilde{r}^2 - \mathrm{Var}_{K_0,m}(\xi(D_{\tilde{r}}))\}$$

by (5.1), where $\tilde{r} = r\sqrt{1-t}$. By Theorem 1.3 in [**17**], we have

$$\mathrm{Var}_{K_0,m}(\xi(D_r)) = \frac{r}{\pi}\alpha_0(r). \tag{5.3}$$

Therefore,

$$g(t) = (1-t)^{-1}\left\{r^2 - \frac{r}{\pi}\int_0^{4r^2} \left(1 - \frac{y}{4r^2}\right)^{1/2} y^{-1/2} e^{-(1-t)y} dy\right\}.$$

By computing the n-th derivative at $t=0$, we obtain (5.2). □

Proof of Theorem 1.1. First we note that for each $n = 0, 1, 2, \ldots$,

$$\sum_{i,j=0}^{n} (-1)^{i+j} \binom{i+j}{i}\binom{n}{i}\binom{n}{j} = 1, \tag{5.4}$$

which is equivalent to (2.5) for $\alpha = 0$. By Lemma 5.2, we see that

$$\begin{aligned}&\int_{D_r^2} L_n(|z-w|^2)^2 |K_0(z,w)|^2 m(dz)m(dw)\\ &\quad = \sum_{i,j=0}^{n} \frac{(-1)^{i+j}}{i!j!}\binom{n}{i}\binom{n}{j}(i+j)!\left\{r^2 - \frac{r}{\pi}\sum_{k=0}^{i+j}\frac{\alpha_k(r)}{k!}\right\}.\end{aligned}$$

Since $\int_{D_r} K_n(z,z)m(dz) = r^2$, from (5.1) and (5.4), we obtain

$$\mathrm{Var}_{K_n,m}(\xi(D_r)) = \frac{r}{\pi}\sum_{i,j=0}^{n}\frac{(-1)^{i+j}}{i!j!}\binom{n}{i}\binom{n}{j}(i+j)!\sum_{k=0}^{i+j}\frac{\alpha_k(r)}{k!}.$$

By using duality relation (4.1), we see that

$$\begin{aligned}\sum_{k=0}^{n}\frac{\alpha_k(r)}{k!} &= \int_0^{4r^2}\left(1-\frac{y}{4r^2}\right)^{1/2} y^{-1/2}P(Z_y^{(0)}\le n)dy\\ &= \int_0^{4r^2}\left(1-\frac{y}{4r^2}\right)^{1/2} y^{-1/2}P(S_n^{(0)}\ge y)dy\\ &= \int_0^{\infty}\frac{t^n e^{-t}}{n!}dt\int_0^{t\wedge 4r^2}\left(1-\frac{y}{4r^2}\right)^{1/2} y^{-1/2}dy.\end{aligned}$$

Therefore, we obtain

$$\mathrm{Var}_{K_n,m}(\xi(D_r)) = \frac{r}{\pi}\int_0^{\infty}L_n(t)^2e^{-t}dt\int_0^{t\wedge 4r^2}\left(1-\frac{y}{4r^2}\right)^{1/2} y^{-1/2}dy.$$

By the monotone convergence theorem, we have

$$C_n = \frac{2}{\pi}\int_0^{\infty}L_n(t)^2e^{-t}t^{1/2}dt = \frac{2}{\pi}E\big[(S_n^{(n)})^{1/2}\big].$$

The asymptotics of C_n follows from Lemma 5.3 below. □

LEMMA 5.3. *For $\alpha > 0$, let $C_n(\alpha) := \int_0^\infty L_n(t)^2t^\alpha e^{-t}dt$. Then, we have*

$$C_n(\alpha) = \frac{\Gamma(n+\alpha+1)}{n!}\,{}_3F_2(-\alpha,-\alpha,-n;1,-n-\alpha;1) \sim \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}n^\alpha$$

as $n\to\infty$.

PROOF. By using (2.4) and (2.5), we see that

$$\begin{aligned}C_n(\alpha) &= \sum_{r=0}^{n}\frac{(-\alpha)_r^2}{(r!)^2}\frac{\Gamma(n-r+\alpha+1)}{(n-r)!}\\ &= \frac{\Gamma(n+\alpha+1)}{n!}\,{}_3F_2(-\alpha,-\alpha,-n;1,-\alpha-n;1).\end{aligned}$$

Since ${}_3F_2(-\alpha,-\alpha,-n;1,-\alpha-n;1) \nearrow {}_2F_1(-\alpha,-\alpha;1;1)$ as $n\to\infty$ when $\alpha>0$, the last asymptotics follows from Stirling's formula and the well-known formula ${}_2F_1(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)$ for $\Re(c-a-b)>0$. □

6. Proof of Theorem 1.2.

In this section, we give a proof of Theorem 1.2.

PROOF OF THEOREM 1.2. It follows from Lemma 5.1 that

$$\begin{aligned}\mathrm{Var}_{K,m}(\xi(B_r)) &= \int_{B_r} m(dz) \int_{B_r^c} m(dw) k(|z-w|^2) \\ &= A_{d-1} \int_0^r s^{d-1} ds \int_{B_r^c} m(dw) k(|x(s)-w|^2),\end{aligned}$$

where $x(s) = (s, 0, \dots, 0) \in \mathbb{R}^d$ and $A_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of $(d-1)$-dimensional unit sphere. We remark that the condition $K^2 = K$ implies $\int_0^\infty v^{(d-2)/2} k(v) dv < \infty$ as mentioned in Remark 1.3. Now we divide the integral in s into two intervals $[0, r-M]$ and $[r-M, r]$ for fixed $M \in (0, r)$. For the first integral,

$$\begin{aligned}I_1(r, M) &= \int_0^{r-M} s^{d-1} ds \int_{B_r^c} m(dw) k(|x(s)-w|^2) \\ &\le \int_0^{r-M} s^{d-1} ds \int_{B_{r-s}^c} m(dw) k(|w|^2) \\ &\le r^{d-1} \int_M^r dt \int_{B_t^c} m(dw) k(|w|^2).\end{aligned}$$

So we have

$$\limsup_{r \to \infty} \frac{I_1(r, M)}{r^{d-1}} \le \int_M^\infty dt \int_{B_t^c} m(dw) k(|w|^2) \le \frac{A_{d-1}}{2} \int_{M^2}^\infty v^{(d-1)/2} k(v) dv.$$

For the second integral,

$$\begin{aligned}I_2(r, M) &= \int_{r-M}^r s^{d-1} ds \int_{B_r^c} m(dw) k(|x(s)-w|^2) \\ &= \int_0^M (r-t)^{d-1} dt \int_{B_r^c} m(dw) k(|x(r-t)-w|^2).\end{aligned}$$

It is easy to see that

$$\int_{B_r^c} m(dw) k(|x(r-t)-w|^2) \le \frac{A_{d-1}}{2} \int_0^\infty v^{(d-2)/2} k(v) dv < \infty$$

for any $r > 0$ and that for fixed $t > 0$,

$$\int_{B_r^c} m(dw)k(|x(r-t)-w|^2) \searrow \int_{[t,\infty)\times\mathbb{R}^{d-1}} k(|w|^2)m(dw)$$

as $r \to \infty$. Hence, by the bounded convergence theorem, we obtain

$$\begin{aligned}\lim_{M\to\infty}\lim_{r\to\infty}\frac{I_2(r,M)}{r^{d-1}} &= \int_0^\infty dt \int_{[t,\infty)\times\mathbb{R}^{d-1}} k(|w|^2)m(dw) \\ &= \frac{A_{d-2}}{2(d-1)}\int_0^\infty v^{(d-1)/2}k(v)dv.\end{aligned}$$

Consequently, we obtain

$$\lim_{r\to\infty}\frac{\mathrm{Var}_K(\xi(B_r))}{r^{d-1}} = \frac{A_{d-1}A_{d-2}}{2(d-1)}\int_0^\infty v^{(d-1)/2}k(v)dv.$$

A simple calculation shows that $A_{d-1}A_{d-2}/2(d-1)$ is equal to $(2\pi)^{d-1}/(d-1)!$. □

7. Simultaneous observability.

We recall the definition of simultaneously observable sets [**8**]. Given an integral kernel K acting on $L^2(R,\lambda)$, the subsets $\Lambda_1,\dots,\Lambda_k$ with $\Lambda = \cup_{i=1}^k \Lambda_i$ are called simultaneously observable if the eigenfunctions of K_Λ restricted onto Λ_i are also the eigenfunctions of K_{Λ_i} for every $1 \le i \le k$.

Example 7.1. For the integral operator with kernel K_n, annuli $\{A_{x_i}^{y_i}\}_{i=1}^k$ with $0 \le x_i < y_i \le \infty$ are simultaneously observable subsets. Indeed, the functions $\{\psi_j^{(n)}\}_{j=0}^\infty$ is a complete orthonormal basis of the closed subspace $\mathcal{H}_n$ in $L^2(\mathbb{C})$ and they are the common eigenfunctions of any operator $(K_n)_{A_x^y}$. It is easy to see that the eigenvalue $\kappa_n(A_x^y)$ of the operator $(K_n)_{A_x^y}$ is given by the formula

$$\kappa_j^{(n)}(A_x^y) = P\big(x^2 \le S_j^{(n)} < y^2\big) \tag{7.1}$$

for $j = 0,1,2,\dots$ as was discussed in Section 4.

For simplicity and later use, we only consider the case of two simultaneously observable sets and give a proof by using Laplace transform.

Lemma 7.2 ([**8**]). *Let $\mu_{K,\lambda}$ be the determinantal point process associated with K and λ. Let A and B be disjoint subsets that are simultaneously observable. Set independent random vectors*

$$(X_j,Y_j) = \begin{cases}(0,0) & \textit{with prob. } 1-\kappa_j(A)-\kappa_j(B)\\ (1,0) & \textit{with prob. } \kappa_j(A)\\ (0,1) & \textit{with prob. } \kappa_j(B)\end{cases}$$

for $j = 0, 1, 2, \dots,$ where $\kappa_j(A)$ is the j-th eigenvalue of K_A. Then, the random vector $(\xi(A), \xi(B))$ under $\mu_{K,\lambda}$ is equal in law to the sum of the independent random vectors given as above, i.e.,

$$(\xi(A), \xi(B)) \stackrel{d}{=} \sum_{j=0}^{\infty} (X_j, Y_j). \tag{7.2}$$

Proof. Take $\alpha I_A + \beta I_B$ as f in Theorem 3.1. Since $1 - e^{-\alpha I_A - \beta I_B} = (1-e^{-\alpha})I_A + (1-e^{-\beta})I_B$ and two operators K_A and K_B admit the common eigenfunctions, we have

$$\begin{aligned}
&\int_Q \exp(-\alpha\xi(A) - \beta\xi(B))\mu_{K,\lambda}(d\xi) \\
&\quad = \det(I - \{(1-e^{-\alpha})K_A + (1-e^{-\beta})K_B\}) \\
&\quad = \prod_{j=0}^{\infty} \{1 - \kappa_j(A) - \kappa_j(B) + e^{-\alpha}\kappa_j(A) + e^{-\beta}\kappa_j(B)\}.
\end{aligned}$$

The last infinite product is nothing but the Laplace transform of the right-hand side of (7.2). □

We can apply Lemma 7.2 to the case of Ginibre-type point processes and annuli.

Lemma 7.3. *Let $0 \le p \le q \le x \le y \le \infty$. Then,*

$$\mathbb{E}[\xi(A_x^y) | \xi(A_p^q) = 0] = \sum_{j=0}^{\infty} Q_j, \quad \mathrm{Var}[\xi(A_x^y) | \xi(A_p^q) = 0] = \sum_{j=0}^{\infty} Q_j(1 - Q_j),$$

where

$$Q_j = Q_j(p, q; x, y) = \frac{\kappa_j^{(n)}(A_x^y)}{1 - \kappa_j^{(n)}(A_p^q)} = \frac{P(x^2 \le S_j^{(n)} < y^2)}{1 - P(p^2 \le S_j^{(n)} < q^2)}.$$

Proof. We use Lemma 7.2 for $A = A_x^y$ and $B = A_p^q$. The event $\xi(A_p^q) = 0$ is equivalent to the event that $Y_j = 0$ for all $j = 0, 1, \dots$. Then,

$$(\xi(A_x^y) \mid \xi(A_p^q) = 0) \stackrel{d}{=} \sum_{j=0}^{\infty} (X_j \mid Y_j = 0)$$

and by Lemma 7.2 we get Bernoulli random variables

$$(X_j | Y_j = 0) = \begin{cases} 0 & \text{with prob. } \dfrac{1 - \kappa_j^{(n)}(A_p^q) - \kappa_j^{(n)}(A_x^y)}{1 - \kappa_j^{(n)}(A_p^q)} \\[2ex] 1 & \text{with prob. } \dfrac{\kappa_j^{(n)}(A_x^y)}{1 - \kappa_j^{(n)}(A_p^q)}. \end{cases}$$

The assertion immediately follows from these two facts and (7.1). □

REMARK 7.4. It is obvious that $\mathbb{E}[\xi(A_x^y) \mid \xi(A_p^q) = 0] > \mathbb{E}[\xi(A_x^y)]$ from Lemma 7.3. This implies that the condition $\xi(A_p^q) = 0$ makes the number of points within A_x^y increase.

8. Proofs of Theorems 1.4 and 1.5.

The following lemma is a consequence of large deviations result for $S_j^{(n)}$ mentioned in Remark 4.2. In this section, we set $\rho = r^2$.

LEMMA 8.1. *Let $0 \le x < y \le \infty$ and let $I, J \subset (0,\infty)$ be closed intervals.*

(i) *If $I \cap [x,y] = \emptyset$, then*

$$\lim_{\rho\to\infty} \sum_{j\in\rho I} P\big(x\rho \le S_j^{(n)} < y\rho\big) = 0.$$

(ii) *If $I \subset (x,y)$, then for any $\delta > 0$ there exists a $\rho_\delta > 0$ such that*

$$\inf_{j\in\rho I} P\big(x\rho \le S_j^{(n)} < y\rho\big) \ge 1-\delta$$

for any $\rho \ge \rho_\delta$. If $I \subset [x,y]$, $1-\delta$ should be replaced by $1/2-\delta$ $(\ge 1/3)$.

(iii) *Suppose $I \subset (0,x)$ and $J \subset (y,\infty)$. Then there exists $C > 0$ such that*

$$\max\left(\sup_{j\in\rho I} \frac{P(S_j^{(n)} \ge y\rho)}{P(x\rho \le S_j^{(n)} < y\rho)},\ \sup_{j\in\rho J} \frac{P(S_j^{(n)} < x\rho)}{P(x\rho \le S_j^{(n)} < y\rho)}\right) \le e^{-C\rho}$$

for any sufficiently large ρ.

(iv) *For $x < y$, put $\gamma_{xy} := (y-x)/(\log y - \log x) \in [x,y]$. Suppose $I \subset [x,\gamma_{xy})$ and $J \subset (\gamma_{xy}, y]$. Then there exists $C > 0$ such that*

$$\max\left(\sup_{j\in\rho I} \frac{P(S_j^{(n)} \ge y\rho)}{P(S_j^{(n)} < x\rho)},\ \sup_{j\in\rho J} \frac{P(S_j^{(n)} < x\rho)}{P(S_j^{(n)} \ge y\rho)}\right) \le e^{-C\rho}$$

for any sufficiently large ρ.

PROOF. For (iii) and (iv), we only show the first inequalities corresponding to the interval I. The other inequalities for J can be shown in the same manner. We write $I = [p,q]$ with $p < q$. We understand $p/\infty = 0$ and $q/0 = \infty$ below.

(i) Let $Z_t^{(n)}$ be a random variable defined in (4.4) with $b = 1$. Then by the duality relation (4.5) we have

$$\sum_{j\in\rho I} P\big(x\rho \le S_j^{(n)} < y\rho\big) = \int_{x\rho}^{y\rho} P\big(Z_t^{(n)} \in \rho I\big) dt$$
$$\le \sup_{t\in[x\rho,y\rho]} P\big(Z_t^{(n)} \in \rho I\big) \cdot (y-x)\rho$$
$$\le \sup_{t\in[x\rho,y\rho]} P\bigg(\frac{Z_t^{(n)}}{t} \in [p/y, q/x]\bigg) \cdot (y-x)\rho.$$

Since $I \cap [x,y] = \emptyset$, the interval $[p/y, q/x]$ does not contain 1, and hence $\sup_{t\in[x\rho,y\rho]} P(Z_t^{(n)}/t \in [p/y,q/x])$ converges to 0 exponentially fast as $\rho \to \infty$ by the large deviations result for $Z_t^{(n)}$.

(ii) For any $j \in \rho I$, $P(x\rho \le S_j^{(n)} < y\rho) \ge P(x/p \le S_j^{(n)}/j < y/q)$. Since $1 \in [x/p, y/q]$, by the weak law of large numbers when 1 is strictly contained in the interval, by Proposition 4.6 when either x/p or y/q is 1, we obtain the assertion.

(iii) Suppose $I \subset (0,x)$. Let $\mathcal{I}(x) = x - 1 - \log x$, which is the rate function for the large deviations of $S_j^{(n)}$. First assume that $1 < q/p < y/x$. Then $\mathcal{I}(x/p) < \mathcal{I}(y/q)$ since $1 < x/p < y/q$. Take $\delta > 0$ such that $\mathcal{I}(y/q) - \mathcal{I}(x/p) > 2\delta$. For such δ there exists $j_0 \in \mathbb{N}$ such that $P(S_j^{(n)}/j \ge y/q) \le \exp(-j(\mathcal{I}(y/q) - \delta))$ and $P(x/p \le S_j^{(n)}/j < y/q) \ge \exp(-j(\mathcal{I}(x/p) + \delta))$ for any $j \ge j_0$. Hence we see that

$$\sup_{j\in\rho I} \frac{P(S_j^{(n)} \ge y\rho)}{P(x\rho \le S_j^{(n)} < y\rho)} \le \sup_{j\in\rho I} \frac{P(S_j^{(n)}/j \ge y/q)}{P(x/p \le S_j^{(n)}/j < y/q)}$$
$$\le \exp(-j(\mathcal{I}(y/q) - \mathcal{I}(x/p) - 2\delta))$$
$$\le e^{-C\rho}$$

for any sufficiently large ρ, where $C = p(\mathcal{I}(y/q) - \mathcal{I}(x/p) - 2\delta) > 0$. For the general case, take $c > 1$ such that $1 < c < y/x$ and set $I_k = [c^{k-1}p, c^k p]$ for $k = 1, 2, \dots, M-1$ and $I_M = [c^{M-1}p, q]$, where M satisfies $c^{M-1}p < q \le c^M p$. For each interval I_k, there exists $C_k > 0$ such that the above inequality holds. Hence, putting $C = \min_{k=1}^M C_k$, we have

$$\sup_{j\in\rho I} \frac{P(S_j^{(n)} \ge y\rho)}{P(x\rho \le S_j^{(n)} < y\rho)} \le e^{-C\rho},$$

for any sufficiently large ρ.

(iv) Note that when $x < y$ and $t > 0$, $\mathcal{I}(x/t) < \mathcal{I}(y/t)$ is equivalent to $t < \gamma_{xy}$, where γ_{xy} is the unique solution to the equation $\mathcal{I}(x/t) = \mathcal{I}(y/t)$. Assume $I \subset (x, \gamma_{xy})$. Since $x/q < 1 < y/q$ and $q < \gamma_{xy}$, one can take $\delta > 0$ so that $\mathcal{I}(y/q) - \mathcal{I}(x/q) > 2\delta$. If we put $C = p(\mathcal{I}(y/q) - \mathcal{I}(x/q) - 2\delta)$, as in the proof of (iii), we get

$$\sup_{j\in\rho I} \frac{P(S_j^{(n)} \ge y\rho)}{P(S_j^{(n)} < x\rho)} \le \frac{P(S_j^{(n)}/j \ge y/q)}{P(S_j^{(n)}/j < x/q)} \le e^{-C\rho}$$

for any sufficiently large ρ. □

PROOF OF THEOREM 1.4. We only show the case where $0 \le \alpha < \beta \le a < b$. Let

$$Q_j(\rho) = \frac{P(a\rho \le S_j^{(n)} < b\rho)}{P(S_j^{(n)} < \alpha\rho) + P(S_j^{(n)} \ge \beta\rho)}.$$

By Lemma 7.3, we have

$$\mathbb{E}\Big[\xi\Big(A^{\sqrt{b}r}_{\sqrt{a}r}\Big) \,\Big|\, \xi\Big(A^{\sqrt{\beta}r}_{\sqrt{\alpha}r}\Big) = 0\Big] = \sum_{j=0}^{\infty} Q_j(\rho). \tag{8.1}$$

Before going into details, we see the idea of the estimates given below. The numerator of $Q_j(\rho)$ is almost 1 when $j \in \rho(a,b)$ and exponentially small otherwise by the large deviations result. Similarly, the denominator of $Q_j(\rho)$ is exponentially small when $j \in \rho(\alpha,\beta)$ and almost 1 otherwise. Therefore, $Q_j(\rho)$ is almost 1 when $j \in \rho(a,b)$ and exponentially small when $j \in \rho([\alpha,\beta]^c \cap [a,b]^c)$. Since both the numerator and the denominator are exponentially small when $j \in \rho(\alpha,\beta)$, whether $Q_j(\rho)$ is 1 or exponentially small depends on values of the rate function $\mathcal{I}(x)$ for $S_j^{(n)}$. When $\beta = a$, this balance is more subtle and the quantity $\gamma_{\alpha\beta}$ appears. This heuristic is also valid for the proof of Theorem 1.5.

We divide the sum into four pieces to estimate the right-hand-sides. Let $I_1 = [0,\alpha] \cup [\beta, a-\epsilon] \cup [b+\epsilon, \infty)$, $I_2 = [a+\epsilon, b-\epsilon]$, $I_3 = [\alpha, \beta-\epsilon]$ and $I_4 = [0,\infty) \setminus (I_1 \cup I_2 \cup I_3)$ with sufficiently small $\epsilon > 0$. When $\beta = a$, $I_1 = [0,\alpha] \cup [b+\epsilon, \infty)$.

By Lemma 8.1 (ii), the denominator of $Q_j(\rho)$ is bounded below by $1/2 - \delta$ for any sufficiently large ρ. Since $I_1 \cap [a,b] = \emptyset$, by Lemma 8.1 (i), we see that

$$\sum_{j\in\rho I_1} Q_j(\rho) \le (1/2-\delta)^{-1} \sum_{j\in\rho I_1} P\big(a\rho \le S_j^{(n)} < b\rho\big) \to 0.$$

Since $I_2 \subset (a,b) \subset (\beta,\infty)$, by Lemma 8.1 (ii) and the large deviations result for $S_j^{(n)}$,

$$\begin{aligned}\sum_{j\in\rho I_2} (1 - Q_j(\rho)) &\le (1-\delta)^{-1} \sum_{j\in\rho I_2} \big\{P\big(S_j^{(n)} < a\rho\big) + P\big(S_j^{(n)} \ge b\rho\big)\big\} \\ &\le (1-\delta)^{-1} \cdot 2(\rho|I_2|+1)e^{-C\rho}.\end{aligned}$$

Hence, $\rho^{-1}\sum_{j\in\rho I_2} Q_j(\rho) \to |I_2| = b - a - 2\epsilon$.

For I_3, we consider two cases: $a > \beta$ and $a = \beta$. If $a > \beta$, since $I_3 \subset (0,\beta)$, by Lemma 8.1 (iii), we have

$$\sum_{j\in\rho I_3} Q_j(\rho) \le \sum_{j\in\rho I_3} \frac{P(S_j^{(n)} \ge a\rho)}{P(S_j^{(n)} \ge \beta\rho)} \le (\rho|I_3|+1)e^{-C\rho} \to 0.$$

If $a = \beta$, we subdivide I_3 into $I_3^- = [\alpha, \gamma_{\alpha\beta} - \epsilon]$, $I_3^0 = [\gamma_{\alpha\beta} - \epsilon, \gamma_{\alpha\beta} + \epsilon]$ and $I_3^+ = [\gamma_{\alpha\beta} + \epsilon, \beta - \epsilon]$. Since $I_3^- \subset [\alpha, \gamma_{\alpha\beta})$, by Lemma 8.1 (iv),

$$\sum_{j \in \rho I_3^-} Q_j(\rho) \le \sum_{j \in \rho I_3^-} \frac{P(S_j^{(n)} \ge \beta\rho)}{P(S_j^{(n)} < \alpha\rho)} \le (\rho |I_3^-| + 1) e^{-C\rho} \to 0 \tag{8.2}$$

and since $I_3^+ \subset (\gamma_{\alpha\beta}, \beta) \subset (0, \beta)$, by Lemma 8.1 (iii) and (iv),

$$\begin{aligned} \sum_{j \in \rho I_3^+} (1 - Q_j(\rho)) &= \sum_{j \in \rho I_3^+} \frac{P(S_j^{(n)} < \alpha\rho) + P(S_j^{(n)} \ge b\rho)}{P(S_j^{(n)} < \alpha\rho) + P(S_j^{(n)} \ge \beta\rho)} \\ &\le \sum_{j \in \rho I_3^+} \left(\frac{P(S_j^{(n)} < \alpha\rho)}{P(S_j^{(n)} \ge \beta\rho)} + \frac{P(S_j^{(n)} \ge b\rho)}{P(S_j^{(n)} \ge \beta\rho)} \right) \\ &\le 2(\rho |I_3^+| + 1) e^{-C\rho} \to 0. \end{aligned}$$

Hence, $\rho^{-1} \sum_{j \in \rho I_3^+} Q_j(\rho) \to |I_3^+| = \beta - \gamma_{\alpha\beta} - 2\epsilon$.

For $I_4 \cup I_3^0$, since $0 \le Q_j(\rho) \le 1$, we have that $\rho^{-1} \sum_{j \in \rho I_4 \cup \rho I_3^0} Q_j(\rho) \le 6\epsilon$. Since $\epsilon > 0$ is arbitrary, by putting it all together, we have

$$\frac{1}{\rho} \mathbb{E}\Big[\xi\Big(A_{\sqrt{a}r}^{\sqrt{b}r}\Big) \,\Big|\, \xi\Big(A_{\sqrt{\alpha}r}^{\sqrt{\beta}r}\Big) = 0\Big] = \frac{1}{\rho} \sum_{j=0}^{\infty} Q_j(\rho) \to \begin{cases} b - a, & \beta < a \\ b - \gamma_{\alpha\beta}, & \beta = a. \end{cases} \qquad \square$$

We give a uniform estimate for the ratio of Laguerre polynomials.

Lemma 8.2. *Let $\kappa \ge 0$ and $n \in \{0, 1, 2, \dots\}$ be fixed. Then, for any $\delta \in (0, 1)$ there exist positive real numbers C_δ and j_δ such that*

$$\left| \frac{L_n^{(j-n)}(t + \kappa)}{L_n^{(j-n)}(t)} - 1 \right| \le \frac{C_\delta}{t}$$

for any $j \ge j_\delta$ and $t \ge (1 - \delta)^{-1} j$.

Proof. We note that the function $F_{n,j}(t) := (-1)^n L_n^{(j-n)}(t)$ is positive and increasing for sufficiently large $t > 0$. We see that

$$\begin{aligned} n! |F_{n,j}(t + \kappa) - F_{n,j}(t)| &= n! \left| \sum_{k=0}^{n} \frac{(-t)^k}{k!} \binom{j}{n-k} \left\{ \left(1 + \frac{\kappa}{t}\right)^k - 1 \right\} \right| \\ &\le \left| \left(1 + \frac{\kappa}{t}\right)^n - 1 \right| (t + j)^n \\ &\le \frac{A_{n,\kappa}}{t} (t + j)^n \end{aligned}$$

for any $t \geq 1$ and $j \geq n$, and that

$$\begin{aligned} |n!F_{n,j}(t) - (t-j)^n| &= \left| n! \sum_{k=0}^{n} \frac{(-t)^k}{k!} \binom{j}{n-k} - \sum_{k=0}^{n} (-t)^k j^{n-k} \binom{n}{n-k} \right| \\ &\leq \sum_{k=0}^{n} t^k \binom{n}{n-k} |j(j-1)\cdots(j-(n-k)+1) - j^{n-k}| \\ &\leq (t+j)^n \left| 1 - \prod_{l=0}^{n-1} \left(1 - \frac{l}{j}\right) \right| \leq \frac{B_n}{j}(t+j)^n \end{aligned}$$

for any $t \geq 0$ and $j \geq n$. Given $\delta \in (0,1)$, there exists $j_\delta > 0$ such that $(t-j)^n > B_n j^{-1}(t+j)^n$ is satisfied whenever $j \geq j_\delta$ and $\gamma := j/t \leq 1-\delta$. Then, we obtain

$$\left| \frac{L_n^{(j-n)}(t+\kappa)}{L_n^{(j-n)}(t)} - 1 \right| \leq \frac{A_{n,\kappa} t^{-1}(t+j)^n}{(t-j)^n - B_n j^{-1}(t+j)^n} \leq \frac{A_{n,\kappa}}{((1-\gamma)/(1+\gamma))^n - B_n j_\delta^{-1}} \frac{1}{t}. \qquad \square$$

Proof of Theorem 1.5. We only show the first case. Note that

$$\mathbb{E}\Big[\xi\Big(A_{\sqrt{\beta} r}^{\sqrt{\beta r^2+\kappa}}\Big) \,\Big|\, \xi\Big(A_{\sqrt{\alpha} r}^{\sqrt{\beta} r}\Big) = 0\Big] = \sum_{j=0}^{\infty} \frac{P(\beta\rho \leq S_j^{(n)} < \beta\rho + \kappa)}{P(S_j^{(n)} < \alpha\rho) + P(S_j^{(n)} \geq \beta\rho)} =: \sum_{j=0}^{\infty} R_j(\rho).$$

Let $I_1 = [0,\alpha] \cup [\beta, \infty)$, $I_2 = [\alpha+\epsilon, \gamma_{\alpha\beta} - \epsilon]$, $I_3 = [\gamma_{\alpha\beta}+\epsilon, \beta-\epsilon]$ and $I_4 = [0,\infty) \setminus (I_1 \cup I_2 \cup I_3)$. Since $P(S_j^{(n)} < \alpha\rho) + P(S_j^{(n)} \geq \beta\rho) \geq 1/3$, $\forall j \in \rho I_1$ for sufficiently large ρ by Lemma 8.1 (ii) and the duality relation (4.5), we get

$$\frac{1}{\rho} \sum_{j \in \rho I_1} R_j(\rho) \leq \frac{3}{\rho} \sum_{j \in \rho I_1} P\big(\beta\rho \leq S_j^{(n)} \leq \beta\rho + \kappa\big) \leq \frac{3\kappa}{\rho} \to 0.$$

For $I_2 \subset (\alpha, \gamma_{\alpha\beta})$, we see that $\sum_{j \in \rho I_2} R_j(\rho) \to 0$ in the same manner as for (8.2). For I_4, we have $\rho^{-1} \sum_{n \in \rho I_4} R_j(\rho) \leq 4\epsilon$. For $j \in \rho I_3$, by Lemma 8.1 (iv), we have

$$\frac{P(S_j^{(n)} \geq \beta\rho + \kappa)}{P(S_j^{(n)} \geq \beta\rho)} \leq 1 - R_j(\rho) \leq e^{-C\rho} + \frac{P(S_j^{(n)} \geq \beta\rho + \kappa)}{P(S_j^{(n)} \geq \beta\rho)}$$

for any sufficiently large ρ. Suppose $[s, s'] \subset I_3$. For sufficiently large ρ, by Lemma 8.2, we obtain a uniform upper bound

$$\begin{aligned} P\big(S_j^{(n)} \geq \beta\rho + \kappa\big) &= \int_{\beta\rho}^{\infty} \left(\frac{L_n^{(j-n)}(t+\kappa)}{L_n^{(j-n)}(t)} \right)^2 (1+\kappa/t)^{j-n} e^{-\kappa} P\big(S_j^{(n)} \in dt\big) \\ &\leq \left(1 + \frac{C}{\beta\rho}\right)^2 \left(1 + \frac{\kappa}{\beta\rho}\right)^{s'\rho} e^{-\kappa} P\big(S_j^{(n)} \geq \beta\rho\big) \end{aligned}$$

for $j \in \rho[s, s']$. On the other hand, by using Lemma 8.2 and Lemma 8.1 (iii), we obtain a uniform lower bound

$$\begin{aligned} P\big(S_j^{(n)} \geq \beta\rho + \kappa\big) &\geq P\big(\beta\rho + \kappa \leq S_j^{(n)} < \beta'\rho + \kappa\big) \\ &= \int_{\beta\rho}^{\beta'\rho} \left(\frac{L_n^{(j-n)}(t+\kappa)}{L_n^{(j-n)}(t)} \right)^2 (1+\kappa/t)^{j-n} e^{-\kappa} P\big(S_j^{(n)} \in dt\big) \\ &\geq \left(1 - \frac{C}{\beta\rho}\right)^2 \left(1 + \frac{\kappa}{\beta'\rho}\right)^{s\rho} e^{-\kappa} P\big(\beta\rho \leq S_j^{(n)} < \beta'\rho\big) \\ &\geq \left(1 - \frac{C}{\beta\rho}\right)^2 \left(1 + \frac{\kappa}{\beta'\rho}\right)^{s\rho} e^{-\kappa} (1 + e^{-C\rho})^{-1} P\big(S_j^{(n)} \geq \beta\rho\big) \end{aligned}$$

for $j \in \rho[s, s']$ and any fixed $\beta'(> \beta)$. Therefore, by dividing I_3 into a partition $\{J_i\}_{i=1}^N$ and letting ρ to ∞, we get

$$\begin{aligned} \sum_{i=1}^N |J_i| e^{-\kappa(1-s_i/\beta')} &\leq \liminf_{\rho\to\infty} \frac{1}{\rho} \sum_{j\in\rho I_3} (1 - R_j(\rho)) \\ &\leq \limsup_{\rho\to\infty} \frac{1}{\rho} \sum_{j\in\rho I_3} (1 - R_j(\rho)) \leq \sum_{i=1}^N |J_i| e^{-\kappa(1-s_i'/\beta)}, \end{aligned}$$

where $J_i = [s_i, s_i']$ $(1 \leq i \leq N)$. Since $\beta'(> \beta)$ is arbitrary, we obtain

$$\lim_{\rho\to\infty} \frac{1}{\rho} \sum_{j\in\rho I_3} R_j(\rho) = \int_{\gamma_{\alpha\beta}+\epsilon}^{\beta-\epsilon} \big(1 - e^{-\kappa(1-s/\beta)}\big) ds.$$

Since $\epsilon > 0$ is arbitrary, we get the assertion by putting it all together. □

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