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Knotted handle decomposing spheres for handlebody-knots

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Abstract. We show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves. As an application, we show that the handlebody-knots 6_{14} and 6_{15} are not equivalent. We also show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of 6_{14} and 6_{15} are not homeomorphic.

1. Introduction.

A *genus g handlebody-knot* is a genus g handlebody embedded in the 3-sphere S^3. Two handlebody-knots are *equivalent* if one can be transformed into the other by an isotopy of S^3. A handlebody-knot is *trivial* if it is equivalent to a handlebody standardly embedded in S^3, whose exterior is a handlebody. We denote by $E(H) = S^3 - \mathrm{int}\, H$ the exterior of a handlebody-knot H.

Definition 1.1. A 2-sphere S in S^3 is an *n-decomposing sphere* for a handlebody-knot H if

(1) $S \cap H$ consists of n essential disks in H, and
(2) $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

In some cases it might be suitable to replace the condition (2) in Definition 1.1 with the condition

(2)' $S \cap E(H)$ is an incompressible, boundary-incompressible, and not boundary-parallel surface in $E(H)$,

although we adopt the condition (2) in this paper. The two definitions are equivalent if $n = 1$, or $n = 2$ and $E(H)$ is boundary-irreducible.

For two n-decomposing spheres S and S' for a handlebody-knot H, S *is isotopic to* S' if there is an isotopy of S^3 from S to S' such that S remains being an n-decomposing sphere throughout the isotopy.

A handlebody-knot H is *reducible* if there exists a 1-decomposing sphere for H, where we remark that (2) follows from (1) when $n = 1$. A handlebody-knot is *irreducible* if it is not reducible. A handlebody-knot H is irreducible if $E(H)$ is boundary-irreducible.

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The converse is true for a genus two handlebody-knot H. In particular, for a genus two handlebody-knot H, the following are equivalent:

(1) H is irreducible.
(2) $\pi_1(E(H))$ is indecomposable with respect to free products.
(3) $E(H)$ is boundary-prime (cf. [**16**, 2.10 Definition]).
(4) $E(H)$ is boundary-irreducible.

By [**18**], we have the equivalence between (1) and (2). By [**7**], we have the equivalence between (2) and (3) for a handlebody-knot H of arbitrary genus. The conditions (3) and (4) are equivalent if $E(H)$ is not a solid torus (cf. [**16**, Proposition 2.15]). We remark that there is an irreducible genus $g \neq 2$ handlebody-knot whose exterior is not boundary-irreducible (cf. [**16**, Theorem 5.4]).

A genus two handlebody-knot [**17**] and a trivial handlebody-knot can be uniquely decomposed by 1-decomposing spheres into handlebody-knots each of which has no 1-decomposing spheres. The uniqueness is not known for genus $g \geq 3$ handlebody-knots.

Definition 1.2. A 2-sphere S in S^3 is a *knotted handle decomposing sphere* for a handlebody-knot H if

(1) $S \cap H$ consists of two parallel essential disks in H, and
(2) $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

We say that *a 2-sphere S bounds* $(B, K; H)$ if S bounds a 3-ball B so that $S \cap H$ consists of two parallel essential disks in H, and that $H \cup E(B)$ is equivalent to a regular neighborhood of a nontrivial knot K. A knotted handle decomposing sphere for H bounds $(B, K; H)$. A 2-sphere S which bounds $(B, K; H)$ is not always a knotted handle decomposing sphere for H (see the left picture of Figure 1). In this paper, we represent a handlebody-knot by a spatial trivalent graph whose regular neighborhood is the handlebody-knot as shown in Figure 1. Then the intersection of the spatial trivalent graph and the 2-sphere indicates two disks.

If H is a genus $g \geq 2$ handlebody-knot whose exterior is boundary-irreducible, then a 2-sphere S which bounds $(B, K; H)$ is a knotted handle decomposing sphere for H, where we note that $g \geq 2$ implies that $S \cap E(H)$ is not boundary-parallel in $E(H)$, and that the boundary-irreducibility implies the incompressibility of $S \cap E(H)$. A trivial handlebody-knot has no knotted handle decomposing sphere by the following lemma.

Lemma 1.3 ([**14**, Lemma 2.2]). *An incompressible surface properly embedded in a handlebody cuts it into handlebodies.*

In [**6**], Moriuchi, Suzuki and the first and second authors gave a table of genus two handlebody-knots up to six crossings, and classified them according to the crossing number and the irreducibility. There are three pairs of handlebody-knots whose fundamental groups are isomorphic in the table. S. Lee and J. H. Lee [**11**] gave inequivalent genus two handlebody-knots with homeomorphic exteriors including the two pairs 5_1, 6_4 and 5_2, 6_{13} in the table, and distinguish them by classifying essential surfaces in the exteriors. We note that Motto [**13**] gave different examples with homeomorphic exteriors which do not appear in the above table.

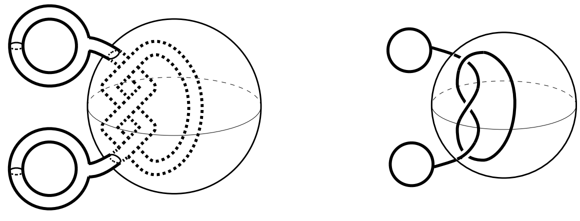

Figure 1.

The pair 6_{14}, 6_{15} is the remaining pair of handlebody-knots whose fundamental groups are isomorphic. In Section 2, we show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves (Theorem 2.2), where we note that Koda and the third author [**10**] have successfully removed the assumption that the exterior is boundary-irreducible. As an application, we show that the handlebody-knots 6_{14} and 6_{15} are not equivalent (Example 2.6). In Section 3, we show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors (Theorem 3.1). As an application, we show that the exteriors of the handlebody-knots 6_{14} and 6_{15} are not homeomorphic (Example 3.5).

2. A unique decomposition for a handlebody-knot.

Let H be a handlebody-knot in S^3, and S a knotted handle decomposing sphere for H which bounds $(B, K; H)$. Let A be an annulus properly embedded in $E(H) - \mathrm{int}\, B$ so that $A \cap S = l$ is an essential loop in the annulus $S \cap E(H)$, and that $A \cap \partial H = l'$ bounds an essential disk D in H, where $\partial A = l \cup l'$ (see Figure 2). Put $T = (S \cap E(H)) \cup (B \cap \partial H)$. Let A' be an annulus obtained from T by cutting along l and pasting two parallel copies of A, where T is slightly isotoped so that $T \cap H = \emptyset$. Then we have a new knotted handle decomposing sphere S' obtained from A' by attaching two parallel copies of D to $\partial A'$. We say that S' is obtained from S by an *annulus-move* along A. For example, in Figure 3, S' is obtained from S by an annulus-move along A.

A set $\mathcal{S} = \{S_1, \dots, S_n\}$ of knotted handle decomposing spheres for a handlebody-knot H is *unnested* if each sphere S_i bounds $(B_i, K_i; H)$ so that $B_i \cap B_j = \emptyset$ for $i \neq j$.

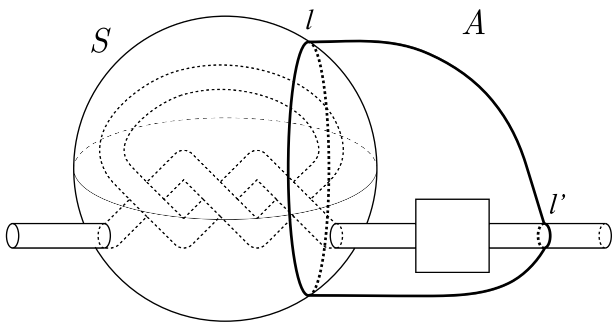

Figure 2. An annulus-move along A.

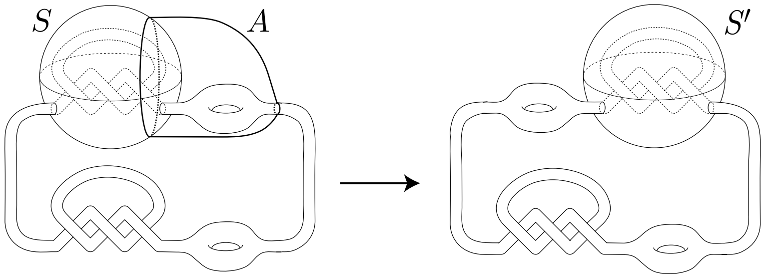

Figure 3.

An unnested set $\mathcal{S}$ is *maximal* if $n \geq m$ for any unnested set $\{S'_1, \ldots, S'_m\}$ of knotted handle decomposing spheres for H. By the Haken–Kneser finiteness theorem [**4**], [**8**], there exists a maximal unnested set of knotted handle decomposing spheres for H. By Schubert's theorem [**15**], K_i is prime for any i if $\mathcal{S}$ is maximal.

Lemma 2.1. *Let H be a handlebody-knot whose exterior is boundary-irreducible. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for H such that S_i bounds $(B_i, K_i; H)$ and that K_i is prime for any i. Let $\mathcal{S}' = \{S'_1, \ldots, S'_m\}$ be a set of 2-decomposing spheres for H. Then $\mathcal{S}$ can be deformed so that $S_i \cap S'_j = \emptyset$ for any i, j by isotopies and annulus-moves.*

Proof. Put $A_i = S_i \cap E(H)$ for $i = 1, \ldots, n$ and $A'_j = S'_j \cap E(H)$ for $j = 1, \ldots, m$. We may assume that $A_i \cap A'_j$ consists of essential arcs or loops in both A_i and A'_j, and that $|A_i \cap A'_j|$ is minimal by isotopies and annulus-moves for each pair (i, j).

Suppose that $A_i \cap A'_j$ consists of essential arcs for some i and j. Let Δ be a component of $A'_j \cap B_i$ which is cobounded by two adjacent arcs of $A_i \cap A'_j$ in A'_j. Since the arcs $\partial\Delta \cap \partial H$ are essential in the annulus $\partial H \cap B_i$ by the minimality of $|A_i \cap A'_j|$, $\partial\Delta$ winds around $B_i - \operatorname{int} H$ longitudinally twice. By attaching a 2-handle $N(\Delta)$ to the solid torus $E(B_i - \operatorname{int} H)$, we have a once punctured lens space $L(2, q)$, which contradicts Alexander's theorem [**1**]. Hence $A_i \cap A'_j$ consists of essential loops for any pair i and j.

Let F be an outermost subannulus of A'_j which is cut by $(\bigcup_{k=1}^{n} A_k) \cap A'_j$ for some j. Let A_i be the annulus such that $F \cap A_i \neq \emptyset$. If F is contained in B_i, then by the primeness of K_i, we can isotope off F from B_i. Hence F is in the outside of B_i. Then by an annulus move for S_i along the annulus F, we can reduce $|A_i \cap A'_j|$. This contradicts to the minimality of $|A_i \cap A'_j|$. □

Theorem 2.2. *A handlebody-knot H whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves.*

Proof. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$, $\mathcal{S}' = \{S'_1 \ldots, S'_n\}$ be maximal unnested sets of knotted handle decomposing spheres for H such that S_i and S'_j bound $(B_i, K_i; H)$ and $(B'_j, K'_j; H)$, respectively. By Lemma 2.1, we can deform $\mathcal{S}'$ so that $S_i \cap S'_j = \emptyset$ for any i, j by isotopies and annulus-moves. We also deform $\mathcal{S}'$ so that $B_i \cap B'_j = \emptyset$ by isotopies if $B_i \cap B'_j$ is homeomorphic to $S^2 \times I$, where I is an interval. Then we have $B_i \subset B'_j$, $B'_j \subset B_i$, or $B_i \cap B'_j = \emptyset$ for any i, j. Since $\mathcal{S}'$ is maximal, for any B_i, there exists a 3-ball

B'_j such that $B_i \subset B'_j$ or $B'_j \subset B_i$. Since K_i and K'_j are prime, S_i is parallel to S'_j. This gives a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{S}'$. Hence a maximal unnested set of knotted handle decomposing spheres for H is unique up to isotopies and annulus-moves. □

PROPOSITION 2.3. *Let H be a genus g handlebody-knot whose exterior is boundary-irreducible. Let $\{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for H such that S_i bounds $(B_i, K_i; H)$ for any i. Put $H' := H \cup B_{m+1} \cup \cdots \cup B_n$. Then $\{S_1, \ldots, S_m\}$ is an unnested set of knotted handle decomposing spheres for H', or $g = 1$ and $m = 1$.*

PROOF. Suppose that $S_i \in \{S_1, \ldots, S_m\}$ is not a knotted handle decomposing sphere for H'. If $S_i \cap E(H')$ is compressible in $E(H')$, then $S_i \cap E(H)$ is also compressible in $E(H)$, a contradiction. If $S_i \cap E(H')$ is parallel to an annulus $A \subset \partial E(H')$ in $E(H')$, then A contains some annuli of $(B_{m+1} \cup \cdots \cup B_n) \cap \partial H'$. This shows that $g = 1$ and $m = 1$. □

PROPOSITION 2.4. *Let H be a genus $g \geq 2$ handlebody-knot, S a 2-sphere which bounds $(B, K; H)$. If $E(H \cup B)$ is boundary-irreducible, then so is $E(H)$.*

PROOF. Suppose that $E(H)$ is boundary-reducible and let D be a compressing disk in $E(H)$. Since $E(H \cup B)$ is boundary-irreducible, D intersects with the annulus $A = S \cap E(H)$. Since $E(H)$ is irreducible, we may assume that $D \cap A$ consists of essential arcs in A. Since the knot K is nontrivial, an outermost disk of D gives a compressing disk in $E(H \cup B)$. This is a contradiction. □

An (*n-component*) *handlebody-link* is a disjoint union of n handlebodies embedded in the 3-sphere S^3. A *non-split handlebody-link* is a handlebody-link whose exterior is irreducible.

PROPOSITION 2.5. *Let H be a handlebody-knot, S a 2-sphere which bounds $(B, K; H)$. Suppose that $H - \operatorname{int} B$ is a non-split handlebody-link whose exterior is boundary-irreducible. If $H - \operatorname{int} B$ is 2-component handlebody-link or $E(H \cup B)$ is a handlebody, then $E(H)$ is boundary-irreducible.*

PROOF. Suppose that $E(H)$ is boundary-reducible. Let D be a compressing disk in $E(H)$. Put $A = S \cap E(H)$. If $D \cap A \neq \emptyset$, then we may assume that $D \cap A$ consists of essential arcs in A, since $E(H)$ is irreducible. Since the knot K is nontrivial, an outermost disk δ of D is contained in $E(H \cup B)$. If $H - \operatorname{int} B$ is not a handlebody-knot, then the arc $\delta \cap (H - \operatorname{int} B)$ connects the different components of $H - \operatorname{int} B$ on $\partial(H - \operatorname{int} B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then δ cuts $E(H \cup B)$ into a 3-manifold homeomorphic to $E(H - \operatorname{int} B)$, which is a handlebody by Lemma 1.3. This implies that $H - \operatorname{int} B$ is trivial, which contradicts that $E(H - \operatorname{int} B)$ is boundary-irreducible. Then $D \cap A = \emptyset$, and so D is in $E(H - \operatorname{int} B)$. Since $E(H - \operatorname{int} B)$ is boundary-irreducible, D is inessential in $E(H - \operatorname{int} B)$. Let D' be a disk in $\partial E(H - \operatorname{int} B)$ such that $\partial D' = \partial D$.

Let D_1, D_2 be the disks such that $S \cap H = D_1 \cup D_2$. If $D' \cap (D_1 \cup D_2) = \emptyset$, then $\partial D'$ is inessential in $\partial E(H)$, which contradicts that D is essential in $E(H)$. If

$D' \cap (D_1 \cup D_2) = D_1$ or $D' \cap (D_1 \cup D_2) = D_2$, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that $S' \cap (H - \operatorname{int} B) = \emptyset$, which contradicts that $H - \operatorname{int} B$ is non-split, since S' separates D_1 and D_2. Thus $D_1, D_2 \subset D'$. If $H - \operatorname{int} B$ is not a handlebody-knot, then D' connects the different components of $H - \operatorname{int} B$ on $\partial(H - \operatorname{int} B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that D' is properly embedded in $H - \operatorname{int} B$. Then S' separates a handlebody $E(H \cup B)$ into a solid torus and a handlebody which is homeomorphic to the exterior of $H - \operatorname{int} B$. This contradicts that $H - \operatorname{int} B$ is nontrivial. □

Example 2.6. We show that any two of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^* are not equivalent, where 5_4, 6_{14} and 6_{15} are the handlebody-knots depicted in Figure 4, and 5_4^*, 6_{14}^* and 6_{15}^* are their mirror images, respectively.

Let H be one of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^*. Let S be the knotted handle decomposing sphere for H depicted in Figure 4, where S bounds $(B, K; H)$ and K is a trefoil knot. By Proposition 2.5, $E(H)$ is boundary-irreducible. By Proposition 2.3, $\{S\}$ is a maximal unnested set of knotted handle decomposing spheres for H, since the trivial handlebody-knot $H \cup B$ has no knotted handle decomposing sphere. Then S is unique by Theorem 2.2, which implies that the pair $(K, H - \operatorname{int} B)$ is an invariant of H. Hence any two of the handlebody-knots 5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15} and 6_{15}^* are not equivalent.

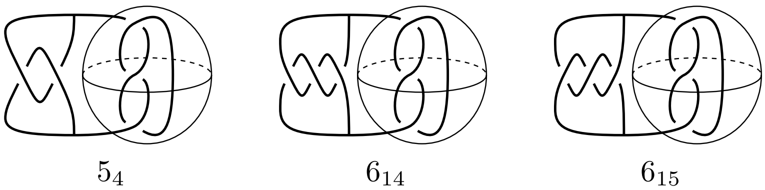

Figure 4.

Proposition 2.7. *There exists a sequence of handlebody-knots H_i ($i \in \mathbb{N} \cup \{0\}$) satisfying the following conditions.*

- *H_0 is the trivial genus two handlebody-knot, which has no knotted handle decomposing sphere.*
- *For $i \geq 1$, H_i has a unique knotted handle decomposing sphere S_i which bounds $(B_i, K_i; H_i)$.*
- *For $i \geq 1$, $H_i \cup B_i$ is equivalent to H_{i-1} as a handlebody-knot.*

Proof. Let H_0 be the trivial genus two handlebody-knot. For $i \geq 1$, let H_i be the genus two handlebody-knot with $i-1$ tangles T and a 2-sphere S_i bounding $(B_i, K_i; H_i)$ as depicted in Figure 5. Then $H_i \cup B_i$ is equivalent to H_{i-1}. We remark that H_1 is the irreducible handlebody-knot 6_{14}, whose exterior is boundary-irreducible. It follows by Proposition 2.4 that H_i is boundary-irreducible for $i \geq 1$. Then S_i is a knotted handle decomposing sphere for H_i.

We prove by induction on i that S_i is a unique knotted handle decomposing sphere for H_i. We already showed that S_1 is a unique knotted handle decomposing sphere for

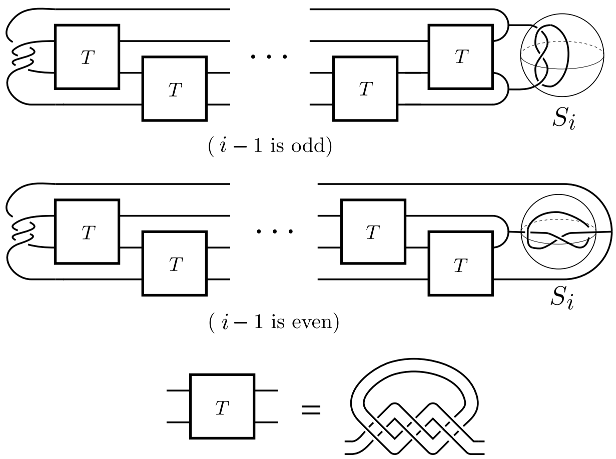

Figure 5.

H_1 in Example 2.6. Assume that S_{i-1} is a unique knotted handle decomposing sphere for H_{i-1}. Suppose that S_i is not a unique knotted handle decomposing sphere for H_i. Then, by Lemma 2.1 and Theorem 2.2, there is a knotted handle decomposing sphere S'_i for H_i which bounds $(B'_i, K'_i; H_i)$ such that the set $\{S_i, S'_i\}$ is a maximal unnested set of knotted handle decomposing spheres for H_i.

Let K_i^- be the core of $H_i - \operatorname{int} B_i$, which is a satellite knot. Let T' be the tangle obtained from T and 3 half twists as the leftmost tangle of K_i^- in Figure 5. Then T and T' are prime tangles (cf. [**5**]). Since K_i^- is obtained from T' and $i-2$ copies of T by tangle sum, K_i^- is a prime knot [**12**]. It follows by Proposition 2.3 that S'_i corresponds to S_{i-1}. Hence K'_i is the positive trefoil knot, and $(H_i \cup B_i) - \operatorname{int} B'_i$ is a regular neighborhood of K_{i-1}^-. A loop l of $S'_i \cap \partial H_i$ is in $\partial(H_i - \operatorname{int} B_i)$, since the set $\{S_i, S'_i\}$ is unnested.

If l is essential in $\partial(H_i - \operatorname{int} B_i)$, then l is a meridian loop of a solid torus $H_i - \operatorname{int} B_i$. By the primeness of K_i^-, the positive trefoil knot K'_i is equivalent to the satellite knot K_i^- for $i > 1$, a contradiction.

If l is inessential in $\partial(H_i - \operatorname{int} B_i)$, then l bounds a disk D in $\partial(H_i - \operatorname{int} B_i)$. Let D_1, D_2 be the disks such that $S_i \cap H_i = D_1 \cup D_2$. Since l is essential in ∂H_i, $D \cap (D_1 \cup D_2) \neq \emptyset$. If D contains both D_1 and D_2, then l is a separating loop in ∂H_i and ∂H_{i-1}, which contradicts that $S_{i-1} \cap \partial H_{i-1}$ consists of non-separating disks. Thus D contains either D_1 or D_2, which implies that l is parallel to the loops of $S_i \cap \partial H_i$. Then $H_i - \operatorname{int} B_i$ and $(H_i \cup B_i) - \operatorname{int} B'_i$ are equivalent as handlebody-knots. It follows that K_i^- and K_{i-1}^- are equivalent, which contradicts that K_j^- has a non-trivial Fox 3-coloring if and only if j is odd, since the replacement of the tangle T with the trivial tangle does not change the number of Fox 3-colorings.

Therefore S_i is a unique knotted handle decomposing sphere for H_i. This completes the proof. □

Proposition 2.7 suggests that the following theorem holds. Actually, the theorem is true by the recent work of Koda and the third author [**10**]. Then Proposition 2.7 gives a concrete example which has a hierarchy of any depth.

Theorem 2.8. *For any handlebody-knot H, there exists a unique sequence of handlebody-knots $H_0, \ldots, H_m = H$ satisfying the following conditions.*

- *H_0 has no knotted handle decomposing sphere.*
- *For $1 \leq i \leq m$, H_i has a unique maximal unnested set of knotted handle decomposing spheres $\{S_{i,1}, \ldots, S_{i,n_i}\}$, where each $S_{i,j}$ bounds $(B_{i,j}, K_{i,j}; H_i)$.*
- *For $1 \leq i \leq m$, $H_i \cup B_{i,1} \cup \cdots \cup B_{i,n_i}$ is equivalent to H_{i-1} as a handlebody-knot.*

3. Handlebody-knots and their exteriors.

In this section, we show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of the handlebody-knots 6_{14} and 6_{15} are not homeomorphic.

Theorem 3.1. *For $i = 1, 2$, let H_i be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere S_i bounding $(B_i, K_i; H_i)$ such that B_i contains all spheres in a maximal unnested set of knotted handle decomposing spheres for H_i. Suppose that $E(H_i \cup B_i)$ is a handlebody and that $H_i - \operatorname{int} B_i$ is a nontrivial handlebody-knot for $i = 1, 2$. Then H_1 and H_2 are equivalent if and only if there is an orientation preserving homeomorphism from $E(H_1)$ to $E(H_2)$.*

An annulus A properly embedded in a 3-manifold is *essential* if A is incompressible and not boundary-parallel. To prove Theorem 3.1, we give some lemmas.

Lemma 3.2 ([**2**, 15.26 Lemma]). *Let K be a knot in S^3. If $E(K)$ contains an essential annulus A, then either*

1. *K is a composite knot and A can be extended to a decomposing sphere for K,*
2. *K is a torus knot and A can be extended to an unknotted torus or*
3. *K is a cable knot and A is the cabling annulus.*

Lemma 3.3 ([**9**, Lemma 3.2]). *If A is an essential annulus in a genus two handlebody W, then either*

1. *A cuts W into a solid torus W_1 and a genus two handlebody W_2 and there is a complete system of meridian disks $\{D_1, D_2\}$ of W_2 such that $D_1 \cap A = \emptyset$ and $D_2 \cap A$ is an essential arc in A, or*
2. *A cuts W into a genus two handlebody W' and there is a complete system of meridian disks $\{D_1, D_2\}$ of W' such that $D_1 \cap A$ is an essential arc in A.*

We say that an annulus A is obtained from a knotted handle decomposing sphere S for a handlebody-knot H when $A = S \cap E(H)$.

Lemma 3.4. *Let H be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere S bounding $(B, K; H)$ such that B contains all spheres in a maximal unnested set of knotted handle decomposing spheres for H. Suppose that $E(H \cup B)$ is a handlebody and that $H - \operatorname{int} B$ is a nontrivial handlebody-knot. Then any essential separating annulus in $E(H)$ is isotopic to either a cabling annulus for $H - \operatorname{int} B$ or an annulus obtained from a knotted handle decomposing sphere for H.*

Proof. Let A' be an essential separating annulus in $E(H)$. Assuming that A' cannot be obtained from a knotted handle decomposing sphere for H, we show that A' is a cabling annulus for $H - \operatorname{int} B$. Put $A = S \cap E(H)$ and $W = E(H \cup B)$. We may assume that $A \cap A'$ consists of essential arcs or loops in both A and A', and that $|A \cap A'|$ is minimal by isotopies. As the proof of Lemma 2.1, we may assume that $A \cap A'$ consists of essential loops.

If $\partial A'$ is contained in B, then A' is an annulus obtained from a knotted handle decomposing sphere for H, since each loop of $\partial A'$ is parallel to $\partial(S \cap H)$. Hence there is a loop C of $\partial A'$ contained in W.

Suppose $A \cap A' \neq \emptyset$. Let F be the outermost subannulus on A' containing C, which is an annulus properly embedded in W. Since A' is incompressible in $E(H)$, F is incompressible in W. By the minimality of $|A \cap A'|$, F is not boundary-parallel in W. Let D be a disk in $E(H - \operatorname{int} B)$ such that $D \cap W = F$ and $D \cap B$ is a disk D_0 in B. If C is essential in $\partial(H - \operatorname{int} B)$, then $E(H - \operatorname{int} B)$ is boundary-reducible, which implies that $H - \operatorname{int} B$ is trivial, a contradiction. Hence C is inessential in $\partial(H - \operatorname{int} B)$. Let D' be the disk in $\partial(H - \operatorname{int} B)$ such that $\partial D' = C$. Let D_1, D_2 be the disks such that $S \cap H = D_1 \cup D_2$. If C is parallel to ∂D_0 on $\partial(H \cup B)$, then F is an annulus obtained from a knotted handle decomposing sphere for the trivial genus two handlebody-knot $H \cup B$, a contradiction. Thus $D_1, D_2 \subset D'$ or $(D_1 \cup D_2) \cap D' = \emptyset$, which contradicts that the 2-sphere $S' = D' \cup D$ separates D_1 and D_2, where S' is slightly isotoped so that D' is properly embedded in $H - \operatorname{int} B$. Hence $A \cap A' = \emptyset$, which implies that $A' \subset W$.

The annulus A' is incompressible in W, since it is incompressible in $E(H)$. If A' is boundary-parallel in W, then A' is parallel to A and is obtained from a knotted handle decomposing sphere for H, since A' is not boundary-parallel in $E(H)$. Hence A' is essential in the genus two handlebody W.

By Lemma 3.3, the separating annulus A' cuts W into a solid torus W_1 and a genus two handlebody W_2 so that A' winds around W_1 at least twice. If A is contained in $\partial W \cap W_1$, then by attaching a 2-handle $N(D)$ to the solid torus W_1, we have a once punctured lens space $L(p, q)$ $(p \geq 2)$, where D is a component of $S \cap H$. This contradicts Alexander's theorem [**1**]. Thus A is contained in $\partial W \cap W_2$ and A' cuts $W \cup B$ into W_1 and $W_2 \cup B$.

Suppose that A' is compressible in $W \cup B$. Let D be a compressing disk for A' in $W \cup B$. Then D is contained in $W_2 \cup B$, since A' is incompressible in W. By attaching a 2-handle $N(D)$ to the solid torus W_1, we have a once punctured lens space $L(p, q)$ $(p \geq 2)$, a contradiction. Thus A' is incompressible in $W \cup B$. Suppose that A' is boundary-parallel in $W \cup B$. Since A' is not boundary-parallel in W, $W_2 \cup B$ is a solid torus $A' \times I$. Then the solid torus W_1 is isotopic to $W \cup B = E(H - \operatorname{int} B)$, which implies that $H - \operatorname{int} B$ is trivial, a contradiction. Thus A' is not boundary-parallel in $W \cup B$. Therefore A' is essential in $W \cup B = E(H - \operatorname{int} B)$, which is the exterior of the tunnel number one knot represented by the core curve of $H - \operatorname{int} B$. By Lemma 3.2, A' is a cabling annulus for $H - \operatorname{int} B$, where we note that a tunnel number one knot is prime. □

Proof of Theorem 3.1. If H_1 and H_2 are equivalent, then there is an orientation preserving self-homeomorphism of S^3 which sends H_1 to H_2, which gives an orientation preserving homeomorphism from $E(H_1)$ to $E(H_2)$.

Suppose that there is an orientation preserving homeomorphism f from $E(H_1)$ to $E(H_2)$. Since any cabling annulus cuts off a solid torus from $E(H_2)$, it follows from Lemma 3.4 that $f(S_1 \cap E(H_1)) = S_2 \cap E(H_2)$. Since $E(H_i - \operatorname{int} B_i)$ and $B_i - \operatorname{int} H_i$ are exteriors of knots, by the Gordon-Luecke theorem [**3**], both of the restrictions of f to $E(H_1 - \operatorname{int} B_1)$ and $B_1 - \operatorname{int} H_1$ are extended to homeomorphisms of S^3. Hence f can be extended to a homeomorphism $\hat{f}$ of S^3 such that $\hat{f}(S_1) = S_2$ and $\hat{f}(H_1) = H_2$. □

Example 3.5. By Example 2.6, neither 6_{15} nor 6_{15}^* is equivalent to 6_{14}. We recall that each of them has a unique knotted handle decomposing sphere. By Theorem 3.1, there is no orientation preserving/reversing homeomorphism from $E(6_{14})$ to $E(6_{15})$. Hence $E(6_{14})$ and $E(6_{15})$ are not homeomorphic.

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