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Semilinear degenerate elliptic boundary value problems via Morse theory

Dedicated to Professor Koichi Uchiyama on the occasion of his 70th birthday

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Abstract. The purpose of this paper is to study a class of semilinear elliptic boundary value problems with *degenerate* boundary conditions which include as particular cases the Dirichlet and Robin problems. By making use of the Morse and Ljusternik–Schnirelman theories of critical points, we prove existence theorems of non-trivial solutions of our problem. The approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of semilinear elliptic boundary value problems with degenerate boundary conditions. The results here extend earlier theorems due to Ambrosetti–Lupo and Struwe to the degenerate case.

1. Statement of main results.

Let Ω be a bounded domain of Euclidean space $\boldsymbol{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an N-dimensional, compact smooth manifold with boundary. Let A be a second-order, elliptic differential operator with real coefficients such that

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u. \tag{1.1}$$

Here:

(1) $a^{ij} \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq i, j \leq N$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \boldsymbol{R}^N.$$

(2) $c \in C^\infty(\overline{\Omega})$ and $c(x) \geq 0$ in Ω.

Let B be a first-order, boundary condition with real coefficients such that

$$Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u. \tag{1.2}$$

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Here:

(3) $a \in C^\infty(\partial\Omega)$ and $a(x') \geq 0$ on $\partial\Omega$.
(4) $b \in C^\infty(\partial\Omega)$ and $b(x') \geq 0$ on $\partial\Omega$.
(5) $\partial/\partial\boldsymbol{\nu} = \sum_{i,j=1}^N a^{ij}(x')\, n_j \partial/\partial x_i$ is the conormal derivative associated with the operator A, where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary $\partial\Omega$.

Our fundamental hypotheses on the boundary condition B are the following:

(H.1) $a(x') + b(x') > 0$ on $\partial\Omega$.
(H.2) $b(x') \not\equiv 0$ on $\partial\Omega$.

It is easy to see that the boundary condition B is non-degenerate if and only if either $a(x') > 0$ on $\partial\Omega$ (the Robin case) or $a(x') \equiv 0$ and $b(x') > 0$ on $\partial\Omega$ (the Dirichlet case). Therefore, our boundary condition B is a *degenerate* boundary value problem from an analytical point of view. This is due to the fact that the so-called Shapiro–Lopatinskii complementary condition is violated at each point of the set $M = \{x' \in \partial\Omega : a(x') = 0\}$ (cf. [**14**]). Amann and Zehnder [**3**] studied the boundary condition B in the non-degenerate case.

The intuitive meaning of condition (H.1) is that the absorption phenomenon occurs at each point of the set M, while the reflection phenomenon occurs at each point of the set $\partial\Omega \setminus M = \{x' \in \partial\Omega : a(x') > 0\}$ (see [**26**]). On the other hand, condition (H.2) implies that the boundary condition B is not equal to the purely Neumann condition.

In this paper we study the following semilinear homogeneous elliptic boundary value problem: Given a real-valued function $g(s)$ defined on $\boldsymbol{R}$, find a function $u(x)$ in Ω such that

$$\begin{cases} Au = \lambda u - g(u) & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where λ is a real parameter.

The approach here is based on the extensive use of the ideas and techniques characteristic of the recent developments in the theory of semilinear elliptic boundary value problems with degenerate boundary conditions ([**28**]–[**31**]). For example, in the case where $N = 3$, $a(x')$ may be a function such that, in terms of local coordinates (x_1, x_2) of $\partial\Omega$,

$$a(x') = e^{-1/x_1^2} \sin^2 \frac{1}{x_1}\, e^{-1/x_2^2} \sin^2 \frac{1}{x_2}.$$

Therefore, the crucial point in our approach is how to generalize the classical variational approach to the degenerate case (see Subsection 5.1).

In order to study the semilinear problem (1.3), we consider the linear elliptic boundary value problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

in the framework of the Hilbert space $L^2(\Omega)$. We associate with problem (1.4) a densely defined, closed linear operator

$$\mathfrak{A} : L^2(\Omega) \longrightarrow L^2(\Omega)$$

as follows:

(1) $D(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0 \text{ on } \partial\Omega\}$.
(2) $\mathfrak{A}u = Au$ for every $u \in D(\mathfrak{A})$.

Here and in the following the Sobolev space $W^{k,p}(\Omega)$ for $k \in \mathbf{N}$ and $1 < p < \infty$ is defined as follows:

$$W^{k,p}(\Omega) = \text{the space of functions } u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u, \\ |\alpha| \leq k, \text{ in the sense of distributions are in } L^p(\Omega).$$

Then we have the following fundamental spectral results (i), (ii), (iii) and (iv) of the operator $\mathfrak{A}$ (see [**27**, Theorem 5.1]):

(i) The operator $\mathfrak{A}$ is positive and selfadjoint in $L^2(\Omega)$.
(ii) Let λ_j be the eigenvalues of the operator $\mathfrak{A}$ that are arranged in an increasing sequence

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \ldots,$$

each eigenvalue being repeated according to its multiplicity. The first eigenvalue λ_1 is positive and *algebraically simple*, and its corresponding eigenfunction $\phi_1 \in C^\infty(\overline{\Omega})$ may be chosen to be *strictly positive* in Ω. Namely, we have the assertions

$$\begin{cases} A\phi_1 = \lambda_1\phi_1 & \text{in } \Omega, \\ \phi_1 > 0 & \text{in } \Omega, \\ B\phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

(iii) *No* other eigenvalues λ_j, $j \geq 2$, have positive eigenfunctions.
(iv) The family $\{\phi_j\}_{j=1}^\infty$ of eigenfunctions of $\mathfrak{A}$

$$\begin{cases} A\phi_j = \lambda_j\phi_j & \text{in } \Omega, \\ B\phi_j = 0 & \text{on } \partial\Omega \end{cases}$$

forms a *complete* orthonormal system of $L^2(\Omega)$.

In this paper we assume that the nonlinear term $g : \boldsymbol{R} \to \boldsymbol{R}$ satisfies the following two assumptions (A) and (B):

(A) $g \in C^1(\boldsymbol{R})$ and $g(0) = g'(0) = 0$.
(B) The limits $g'(\pm\infty)$ satisfy the conditions

$$g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty.$$

EXAMPLE 1.1. A simple example of the nonlinear term $g(s)$ is given by the formula

$$g(s) = \begin{cases} s^p & \text{for } s \geq 0, \\ s\,|s|^{q-1} & \text{for } s < 0, \end{cases}$$

where $p > 1$ and $q > 1$. It is easy to verify that $g(s)$ satisfies conditions (A) and (B).

Since $g(0) = 0$, then $u = 0$ is a solution of the semilinear problem (1.3) for all λ. In this paper we establish existence theorems of non-trivial solutions (i.e., $u \neq 0$) of the semilinear problem (1.3). More precisely, our main purpose is to prove the following existence theorem, which is a generalization of Ambrosetti–Lupo [**6**, Theorem] to the *degenerate* case:

THEOREM 1.1. *Assume that conditions* (A) *and* (B) *are satisfied. Then we have the following two assertions*:

(i) *For each $\lambda > \lambda_1$, the semilinear problem* (1.3) *has at least two non-trivial solutions u_1, u_2 with $u_1 > 0$ in Ω and $u_2 < 0$ in Ω.*
(ii) *For each $\lambda > \lambda_2$, the semilinear problem* (1.3) *has at least a third non-trivial solution u_3 different from u_1 and u_2.*

Rephrased, assertion (i) of Theorem 1.1 states that the semilinear problem (1.3) has at least two non-trivial solutions provided that the derivative $f'(s) = \lambda - g'(s)$ of the function

$$f(s) = \lambda s - g(s)$$

crosses the first eigenvalue λ_1 if $|s|$ goes from 0 to ∞ (see Remark 1.1 below):

$$f'(\infty) = -\infty < \lambda_1 < \lambda = f'(0).$$

Similarly, assertion (ii) of Theorem 1.1 states that the semilinear problem (1.3) has at least three non-trivial solutions provided that the derivative $f'(s) = \lambda - g'(s)$ of $f(s)$ crosses the two eigenvalues λ_1 and λ_2 if $|s|$ goes from 0 to ∞:

$$f'(\infty) = -\infty < \lambda_1 < \lambda_2 < \lambda = f'(0).$$

REMARK 1.1. It is worth pointing out that the bifurcation solution curve (λ, u) of problem (1.3) is "formally" given by the formula

$$\lambda = \lambda_1 + \frac{g(u)}{u}, \tag{1.5}$$

since the first eigenvalue λ_1 is the *unique* eigenvalue corresponding to a positive eigenfunction of the operator $\mathfrak{A}$. Indeed, if we write problem (1.3) in the form

$$\begin{cases} \mathfrak{A}u = \lambda u - g(u) = \left(\lambda - \dfrac{g(u)}{u}\right)u, \\ u > 0 \quad \text{in } \Omega, \end{cases}$$

then it follows that $\lambda_1 = \lambda - g(u)/u$. This proves formula (1.5). The situation may be represented schematically by Figure 1.1.

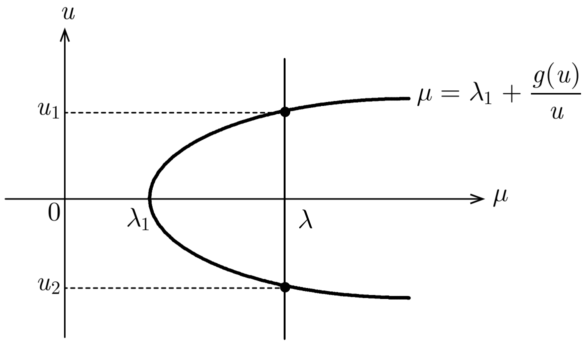

Figure 1.1.

Our proof of Theorem 1.1 is based on Morse theory on Hilbert spaces developed by Palais [**18**], Palais–Smale [**20**] and Marino–Prodi [**16**].

Remark 1.2. (a) If $\lambda \leq \lambda_1$, then the semilinear problem (1.3) in the Dirichlet case could have only the trivial solution $u = 0$. This is the case if $sg''(s) > 0$ for all $s \neq 0$ (cf. [**4**, Example 3.5]).

(b) The existence of a positive solution and a negative solution of the semilinear problem (1.3) for all $\lambda > \lambda_1$ is well known in the Dirichlet case (cf. [**21**]).

(c) Struwe [**25**] considered the Dirichlet problem under the condition that the function $g(s)$ is Lipschitz continuous. He proved assertion (ii) if the function $s \mapsto g(s)/s$ is increasing ([**25**, Propositions 1 and 2]). Hence Theorem 1.1 is a generalization of Struwe's result to the degenerate case. Moreover, under the condition that

$$\frac{g(s)}{s} < g'(s) \quad \text{for almost all } s \neq 0, \tag{1.6}$$

Ambrosetti–Mancini [**7**] proved that the semilinear problem (1.3) in the Dirichlet case has precisely two non-trivial solutions for $\lambda_1 < \lambda < \lambda_2$ ([**7**, Theorem 1.7]).

If $g(s)$ is an odd function of s, then we can improve assertion (ii) of Theorem 1.1.

In fact, the next existence theorem is a generalization of Ambrosetti [**4**, Theorem 3.1], Hempel [**13**, Theorem 2] and Thews [**32**, Theorem 3] to the *degenerate* case:

THEOREM 1.2. *Let $g(s)$ be a function as in Theorem* 1.1. *Moreover, if $g(s)$ is an* odd *function of s, then the semilinear problem* (1.3) *has at least k pairs of non-trivial solutions for all $\lambda > \lambda_k$.*

EXAMPLE 1.2. A simple example of the nonlinear term $g(s)$ is given by the formula

$$g(s) = s|s|^{p-1}, \quad p > 1.$$

Rephrased, Theorem 1.2 asserts that the semilinear problem (1.3) has at least k pairs of non-trivial solutions provided that the derivative $f'(s) = \lambda - g'(s)$ of $f(s)$ crosses the eigenvalues λ_1 through λ_k if $|s|$ goes from 0 to ∞:

$$f'(\infty) = -\infty < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k < \lambda = f'(0).$$

Our proof of Theorem 1.2 is based on the Ljusternik–Schnirelman theory on Hilbert spaces developed by Schwartz [**22**], Palais [**19**] and Clark [**10**].

The rest of this paper is organized as follows. Our proof will be carried out by looking for the solutions of the semilinear problem (1.3) as critical points of a suitable energy functional F on some Hilbert space $\mathcal{H}$, which will be studied by means of Morse theory. Section 2 is devoted to minimax methods. First, we introduce a notion of compactness due to Palais and Smale (Definition 2.1) which plays an essential role in the calculus of variations in the large. By virtue of Ekeland's variational principle and the Palais–Smale condition, we can make use of the minimization method (Theorem 2.2). In Section 3 we prove a four-solution theorem based on Morse theory on Hilbert spaces (Theorem 3.8). Theorem 3.8 is an existence theorem of critical points of an energy functional f on a Hilbert space H, which will be studied by means of Morse theory. Morse theory is concerned with relating the structure of the critical point set of f with relative homology types of the pair (H, f) (see [**17**]). Theorem 3.8 is based on the so-called Morse inequalities (Theorem 3.6). Section 4 is devoted to the Ljusternik–Schnirelman theory on Hilbert spaces which is used in the proof of Theorem 1.2. We mention that the notion of genus introduced by Krasnosel'skii is a topological invariant for the estimate of the lower bound of the number of critical points. In fact, we state an analytic version of the multiplicity theorem of the Ljusternik–Schnirelman theory specialized to the case of an even functional on a Hilbert space (Theorem 4.2). In Section 5 we introduce the notion of weak solutions of the semilinear problem (1.3) (Definition 5.1), and prove that any weak solution of the semilinear problem (1.3) is a classical solution in the usual sense (Theorem 5.3). This section is the heart of the subject. In Subsection 5.1 we introduce an underlying Hilbert space $\mathcal{H}$ for the study of the semilinear problem (1.3) (Theorems 5.1 and 5.2). The crucial point in our variational approach is how to use the theory of fractional powers of selfadjoint operators as in [**28**]. In Subsection 5.2 we prove Theorem 5.3. The proof of Theorem 5.3 is essentially based on the regularity, existence and uniqueness theorems for the linear elliptic boundary value problem (1.4) developed by [**26**] and [**27**]. In Section 6 we prove assertion (i) of Theorem 1.1. Since we have not assumed any growth condition on $g(s)$, we truncate the right-hand side in the semilinear problem (1.3) and make use of the maximum principle for the Dirichlet problem. By using Theorem 2.2, we can find a positive solution u_1 and a negative solution u_2 of problem (1.3). In Section 7 we prove assertion (ii) of Theorem 1.1. This section is divided into four subsections. To handle the general case, the proof is based on a Lyapunov–Schmidt procedure and a slight modification of the classical Morse inequalities. More precisely, the main idea of Subsections 7.1 and 7.2 is to rewrite the semilinear problem (1.3) in a suitable bifurcation system (7.8) and (7.9) (the Lyapunov–Schmidt procedure) and to solve the first (infinite-dimensional) equation (7.8), by using the global inversion theorem (Proposition 7.1). In Subsection 7.3 we deal with functionals which may have *degenerate* critical points, by using a perturbation argument and Sard's lemma (Lemma 7.2). In Subsection 7.4, by using Lemma 7.2 and applying Theorem 3.8 to our situation we can find a third non-trivial solution u_3 different from u_1 and u_2 constructed in Subsection 6.2. The last Section 8 is devoted to the proof of Theorem 1.2. By virtue of Theorem 5.3, we have only to prove Theorem 1.2 for weak solutions. The proof of Theorem 1.2 is based on the multiplicity theorem specialized to the case of an even functional on a Hilbert space (Theorem 4.2).

2. Minimax methods.

This section is devoted to minimax methods. It is known that the direct method does not work in the lack of compactness. Indeed, we can find only approximate minimizers. To do so, we make use of the following Ekeland variational principle (cf. [**9**, Theorem 4.8.1]):

THEOREM 2.1 (Ekeland). *Let (X, d) be a complete metric space and $f : X \to \boldsymbol{R} \cup \{+\infty\}$, but $f \not\equiv +\infty$. Assume that f is bounded from below and lower semi-continuous on X. If there exist a constant $\varepsilon > 0$ and a point $x_\varepsilon \in X$ such that*

$$f(x_\varepsilon) < \inf_{x \in X} f(x) + \varepsilon,$$

then we can find a point $y_\varepsilon \in X$ which satisfies the following three conditions:

(a) $f(y_\varepsilon) \leq f(x_\varepsilon)$.
(b) $d(x_\varepsilon, y_\varepsilon) \leq 1$.
(c) $f(x) > f(y_\varepsilon) - \varepsilon\, d(y_\varepsilon, x)$ *for all* $x \neq y_\varepsilon$.

First, we introduce a notion of compactness due to Palais–Smale [**20**] which plays an essential role in the calculus of variations in the large:

DEFINITION 2.1. Let H be a Hilbert space and $f \in C^1(H, \boldsymbol{R})$. We say that f satisfies $(\mathrm{PS})_c$ *condition* (the Palais–Smale condition) for a constant $c \in \boldsymbol{R}$ if every sequence $\{u_j\}_{j=1}^{\infty}$ in H such that $f(u_j) \to c$ and $\nabla f(u_j) \to 0$ as $j \to \infty$ contains a convergent subsequence. If f satisfies $(\mathrm{PS})_c$ condition for every constant $c \in \boldsymbol{R}$, then we say that f satisfies (PS) *condition.* Here and in the following ∇f denotes the gradient of f.

By virtue of Ekeland's theorem and the Palais–Smale condition, we can make use of the minimization method. In fact, we obtain the following (cf. [**9**, Corollary 4.8.4]):

Theorem 2.2. *Let H be a Hilbert space and $f \in C^1(H, \boldsymbol{R})$. Assume that f is bounded from below on H and satisfies* $(\mathrm{PS})_c$ *condition with the constant*

$$c = \inf_{x \in H} f(x).$$

Then f has a minimum.

3. Morse theory on Hilbert spaces.

In this section we state two results of Morse theory on Hilbert spaces which will be used later on. First, we establish the famous Morse inequalities between relative homology groups and critical groups (Theorem 3.6). Secondly, by using Morse inequalities we prove a four-solution theorem (Theorem 3.8). For more details, the reader might refer to Palais [**18**], Marino–Prodi [**16**, Section 2], Schwartz [**23**, Section 4] and also Chang [**9**].

3.1. Non-degenerate critical points and the splitting theorem.

Let H be a real Hilbert space with inner product $(\cdot, \cdot)_H$. If $f \in C^1(H, \boldsymbol{R})$ and $u \in H$, then its Fréchet derivative $df(u)$ at u is a bounded linear functional on H. Moreover, it follows from an application of the Riesz representation theorem ([**34**, Chapter III, Section 6, Theorem]) that there exists a unique element $\nabla f(u)$ of H such that

$$df(u)(v) = (\nabla f(u), v)_H \quad \text{for all } v \in H.$$

The element $\nabla f(u)$ of H is called the *gradient* of f at u. We can identify $df(u)$ with $\nabla f(u)$. If $\nabla f(u) \neq 0$, then u is called a *regular point* of f and if $\nabla f(u) = 0$, then u is called a *critical point* of f. If $c \in \boldsymbol{R}$, then $f^{-1}(c) = \{z \in H : f(p) = c\}$ is called a level of f and it is called a regular level of f if it contains only regular points of f and a critical level of f if it contains at least one critical point of f.

Furthermore, if $f \in C^2(H, \boldsymbol{R})$, then there is a dichotomy of the critical points of f into degenerate and non-degenerate critical points. To do this, we define the derivative $D^2 f(u)$ of ∇f at u by the formula

$$d^2 f(u)(v, w) = (D^2 f(u)v, w)_H \quad \text{for all } v,\, w \in H.$$

Then we find that the linear operator $D^2 f(u)$ is selfadjoint on H:

$$(D^2 f(u)v, w)_H = (v, D^2 f(u)w)_H \quad \text{for all } v,\, w \in H.$$

A critical point u of f is said to be *non-degenerate* if $D^2 f(u)$ has a bounded inverse; otherwise it is said to be *degenerate*. We also define the *Morse index* of $D^2 f(u)$ to be the supremum of the dimensions of linear subspaces of H on which $D^2 f(u)$ is negative definite.

The finite-dimensional version of the following canonical form theorem is due to M.

Morse (cf. [**9**, Theorem 5.1.13]):

THEOREM 3.1 (the splitting theorem). *Let H be a Hilbert space. Let U be a convex neighborhood of 0 in H and $f \in C^2(U, \boldsymbol{R})$. Assume that 0 is the only critical point of f. If $A = D^2 f(0)$ is a Fredholm operator with $N = \operatorname{Ker} A$, then there exist an open ball $B \subset U$ about 0, an origin-preserving homeomorphism φ defined on B and a C^1-map $h : B \cap N \to N^\perp$ such that*

$$(f \circ \varphi)(y + \xi) = \frac{1}{2}(A\xi, \xi)_H + f(y + h(y)) \quad \textit{for all } y \in B \cap N \textit{ and } \xi \in B \cap N^\perp.$$

3.2. Relative homology groups.

Let G be an Abelian group. The rank of G, denoted by $\operatorname{rank} G$, is the maximal number k for which

$$\sum_{i=1}^{k} n_i g_i = 0 \text{ with } n_i \in \boldsymbol{Z} \text{ and } g_i \in G \implies n_i = 0 \text{ for every } i.$$

Given a pair (X, Y) of topological spaces with $Y \subset X$ and a non-negative integer q, we consider the relative singular homology group $H_q(X, Y; G)$ where G is a coefficient Abelian group.

We let

$$\beta_q(X, Y) = \operatorname{rank} H_q(X, Y; G),$$
$$\chi(X, Y) = \sum_{q=0}^{\infty} (-1)^q \beta_q(X, Y).$$

The number $\beta_q(X, Y)$ is called the q-th *Betti number* of (X, Y) and $\chi(X, Y)$ is called the *Euler–Poincaré characteristic* of (X, Y), respectively.

3.3. Deformation retract and the non-trivial interval theorem.

In the degree theory, the excision property and the Kronecker existence theorem are useful in the study of fixed points. In the relative homology theory, the excision property is related to deformation argument.

Let X be a topological space. A *deformation* of X is a continuous map $\eta : X \times [0,1] \to X$ such that $\eta(\cdot, 0) = \text{id}$ on X. Let (X, Y) be a pair of topological spaces with $Y \subset X$ and let $i : Y \to X$ be the injection. A continuous map $r : X \to Y$ is called a *deformation retract* if it satisfies the conditions

$$r \circ i = \text{id on } Y,$$
$$i \circ r \simeq \text{id on } X,$$

where the relation $\simeq$ denotes the homotopy equivalence. In this case, Y is called a *deformation retraction* of X.

A deformation retract $r : X \to Y$ is called a *strong deformation retract* if there exists a deformation $\eta : X \times [0,1] \to X$ such that

$$\eta(\cdot, t) = \mathrm{id} \quad \text{on } Y \text{ for all } 0 \leq t \leq 1,$$
$$\eta(\cdot, 1) = i \circ r \quad \text{on } X.$$

The next theorem asserts that the excision property is related to a deformation argument (cf. [**9**, Theorem 5.1.6]):

THEOREM 3.2 (the non-critical interval theorem). *Let $f : H \to \boldsymbol{R}$ be a C^1 function satisfying* $(\mathrm{PS})_c$ *condition for all $c \in [a, b]$, and let K be the set of critical points of f. If $f^{-1}([a, b]) \cap K = \emptyset$, then $f^a = f^{-1}((-\infty, a]) = \{x \in H : f(x) \leq a\}$ is a strong deformation retraction of $f^b = f^{-1}((-\infty, b]) = \{x \in H : f(x) \leq b\}$.*

It should be emphasized that if Y is a strong deformation retraction of X, then it follows that

$$H_q(X, Y; G) = 0, \quad q = 0, 1, 2, \dots.$$

Therefore, by using the long exact sequence

$$\cdots \longrightarrow H_{r+1}(X, Y; G) \longrightarrow H_r(Y; G) \longrightarrow H_r(X; G) \longrightarrow H_r(X, Y; G) \longrightarrow \cdots,$$

we obtain the formulas

$$H_q(X; G) \cong H_q(Y; G), \quad q = 0, 1, 2, \dots.$$

The next theorem asserts that the non-triviality of $H_*(f^b, f^a; G)$ implies the existence of a critical point of f in $f^{-1}([a, b])$ (cf. [**9**, Theorem 5.1.2]):

THEOREM 3.3 (the non-trivial interval theorem). *Let $f : H \to \boldsymbol{R}$ be a C^1 function. If there exist a non-negative integer q and a pair (a, b) of numbers with $a < b$ such that the relative homology group $H_q(f^b, f^a; G)$ is non-trivial, then it follows that $f^{-1}([a, b]) \cap K \neq \emptyset$.*

3.4. Critical groups and Morse type numbers.

In this subsection we study the local behavior of non-degenerate critical points. To do this, we introduce the following (cf. [**9**, Definition 5.1.11]):

DEFINITION 3.1. Let $f \in C^1(H, \boldsymbol{R})$ and let z be an isolated critical point of f. If U is a neighborhood of z such that $U \cap K = \{z\}$, then we let

$$C_q(f, z) = H_q(f^c \cap U, (f^c \setminus \{z\}) \cap U; G), \quad q = 0, 1, 2, \dots,$$
$$f^c = f^{-1}((-\infty, c]) = \{x \in H : f(x) \leq c\}, \quad c = f(z),$$

where

$$K = \{x \in H : \nabla f(x) = 0\}$$

is the set of critical points of f.

The relative homology group $C_q(f,z)$ is called the q-th *critical group* of f at z.

By virtue of the excision property of relative homology groups, the Definition 3.1 is well-defined. Namely, the group $C_q(f,p)$ is independent of the neighborhood U of p chosen.

First, we have the following (see Remark 3.1 in Subsection 3.5):

EXAMPLE 3.1. Let $f \in C^1(H, \boldsymbol{R})$ and let z be an isolated local minimum of f. Then we have the formula

$$C_q(f,z) = \begin{cases} G & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases}$$

By using the splitting theorem (Theorem 3.1), we can study the local behavior of non-degenerate critical points (cf. [**9**, Subsection 5.1.3, Example 3]):

EXAMPLE 3.2. Let $f \in C^2(H, \boldsymbol{R})$ and let z be a non-degenerate critical point of f with Morse index j. Then we have the formula

$$C_q(f,z) = \begin{cases} G & \text{if } q = j, \\ 0 & \text{if } q \neq j. \end{cases}$$

Assume that $f \in C^1(H, \boldsymbol{R})$ has only isolated critical values c_i, and further that each value c_i corresponds to a finite number of critical points, say

$$\cdots < c_{-2} < c_{-1} < c_0 < c_1 < c_2 < \cdots, \tag{3.1}$$

$$f^{-1}(c_i) \cap K = \{z_1^i, z_2^i, \ldots, z_{m_i}^i\}, \quad i = 0, \pm 1, \pm 2, \ldots. \tag{3.2}$$

For a pair (a,b) of regular values of f with $a < b$, we let

$$M_q(a,b) = \sum_{a<c_i<b} \operatorname{rank} H_q(f^{c_i+\varepsilon_i}, f^{c_i-\varepsilon_i}; G), \quad q = 0, 1, 2, \ldots.$$

where

$$0 < \varepsilon_i < \min\{c_{i+1} - c_i, c_i - c_{i-1}\}, \quad i = 0, \pm 1, \pm 2, \ldots.$$

If the function $f(x)$ satisfies (PS) condition, then it follows from an application of the non-critical interval theorem (Theorem 3.2) that the numbers $M_*(a,b)$ are independent of $\{\varepsilon_i\}$ chosen. The number $M_q(a,b)$ is called the q-th *Morse type number* of f on (a,b).

More precisely, we have the following (cf. [**9**, Theorem 5.1.27]):

THEOREM 3.4. *Assume that $f \in C^1(H, \boldsymbol{R})$ satisfies* (PS) *condition, and has an isolated critical value c corresponding to a finite number of critical points, say*

$$f^{-1}(c) \cap K = \{z_1, z_2, \ldots, z_m\}.$$

Then we have, for $\varepsilon > 0$ sufficiently small,

$$H_q(f^{c+\varepsilon}, f^{c-\varepsilon}; G) \cong \bigoplus_{j=1}^{m} C_q(f, z_j), \quad q = 0, 1, 2, \ldots.$$

In view of Example 3.2, the next corollary asserts that the number $M_q(a, b)$ is equal to the number of critical points of f in (a, b) with Morse index q (cf. [**9**, Corollary 5.1.28]).

Corollary 3.5. *Let $f : H \to \boldsymbol{R}$ be a C^2 function satisfying* (PS) *condition all of whose critical points are given by the formulas* (3.1) *and* (3.2). *For a pair (a, b) of regular values of f with $a < b$, we have the formula*

$$M_q(a, b) = \sum_{a < c_i < b} \sum_{j=1}^{m_i} \operatorname{rank} C_q(f, z_j^i), \quad q = 0, 1, 2, \ldots.$$

3.5. Morse inequalities.

In this subsection we establish the famous Morse inequalities between relative homology groups $H_*(f^b, f^a; G)$ and critical groups $C_*(f; z)$.

Let $f : H \to \boldsymbol{R}$ be a C^2 function satisfying (PS) condition all of whose critical points are non-degenerate. Let (a, b) be a pair of regular values of f with $a < b$, and let $f^a = f^{-1}((-\infty, a])$ and $f^b = f^{-1}((-\infty, b])$, respectively. For each non-negative integer q, let $\beta_q(a, b)$ denote the q-th *Betti number* of (f^b, f^a):

$$\beta_q(a, b) = \operatorname{rank} H_q(f^b, f^a; G), \quad q = 0, 1, 2, \ldots.$$

Then we have the following ([**18**, Theorem (7)]):

Theorem 3.6 (Morse inequalities). *Let $f : H \to \boldsymbol{R}$ be a C^2 function satisfying* (PS) *condition all of whose critical points are non-degenerate. For a pair (a, b) of regular values of f with $a < b$, we have the inequalities*

$$\begin{gathered}
\beta_0(a, b) \leq M_0(a, b), \\
\beta_1(a, b) - \beta_0(a, b) \leq M_1(a, b) - M_0(a, b), \\
\sum_{m=0}^{k} (-1)^{k-m} \beta_m(a, b) \leq \sum_{m=0}^{k} (-1)^{k-m} M_m(a, b), \quad k = 2, 3, \ldots,
\end{gathered}$$

and

$$\chi(f^b, f^a) = \sum_{m=0}^{\infty} (-1)^m \beta_m(a, b) = \sum_{m=0}^{\infty} (-1)^m M_m(a, b).$$

By combining Corollary 3.5 and Example 3.1, we can obtain the following (cf. [**18**, Corollary (2)]):

Corollary 3.7. *Let $f : H \to \boldsymbol{R}$ be a C^2 function satisfying* (PS) *condition. Assume that f is bounded from below on H and further that f has only isolated local minima and non-degenerate critical points of positive Morse index. For a regular value b of f, we let*

$$\begin{aligned} \beta_k(b) &= \operatorname{rank} H_q(f^b; G) = \textit{the } k\textit{-th Betti number of } f^b, \quad k = 0, 1, 2, \ldots, \\ C_0(b) &= \textit{the number of isolated, local minima of } f, \\ C_m(b) &= \textit{the number of non-degenerate critical points of } f \\ &\qquad \textit{with Morse index } m \textit{ in } f^b, \quad m = 1, 2, \ldots. \end{aligned}$$

Then we have the inequalities

$$\begin{aligned} \beta_0(b) &\leq C_0(b), \\ \beta_1(b) - \beta_0(b) &\leq C_1(b) - C_0(b), \\ \sum_{m=0}^{k} (-1)^{k-m} \beta_m(b) &\leq \sum_{m=0}^{k} (-1)^{k-m} C_m(b), \quad k = 2, 3, \ldots, \end{aligned} \tag{3.3}$$

and

$$\chi(f^b) = \sum_{k=0}^{\infty} (-1)^k \beta_k(b) = \sum_{k=0}^{\infty} (-1)^k C_k(b).$$

Remark 3.1. Ambrosetti [**5**] observed for the first time that it is possible to include in $C_0(b)$ the possibly *degenerate*, isolated, local minima of f as in Corollary 3.7. More precisely, the justification relies on the following fact: If u_0 is a local, isolated minimum of f, then we let (see Definition 3.1)

$$U^- = \{u \in H : \|u - u_0\|_H < \varepsilon,\ f(u) \leq f(u_0)\},$$

and evaluate the relative homology groups $H_q(U^-, U^- \setminus \{u_0\}; G)$. Here it should be emphasized that u_0 need not be non-degenerate with finite Morse index (cf. [**16**, Theorem 1.2]). Indeed, it suffices to take a positive constant ε so small that $U^- = \{u_0\}$. Then we have the formula

$$\operatorname{rank} C_q(f, u_0) = \operatorname{rank} H_q(U^-, U^- \setminus \{u_0\}; G) = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases} \tag{3.4}$$

This fact may be used to show that the Leray–Schauder index of such a point is equal to one (cf. [**2**]).

By using Morse inequalities, we can prove the following *four-solution theorem* (see [**6**, Lemma 2.2]):

THEOREM 3.8. *Assume that $f \in C^2(H, \boldsymbol{R})$ is bounded from below and satisfies* (PS) *condition. Assume further that the following two conditions* (i) *and* (ii) *are satisfied*:

(i) *$u = 0$ is a non-degenerate critical point of f with Morse index $q_0 \geq 2$.*
(ii) *f has two local minima u_1 and u_2.*

Then f has at least another non-zero critical point u_3.

PROOF. Assume, to the contrary, that f has only three critical points u_1, u_2 and 0. We may assume that local minima are isolated, for otherwise we are done. Then, by applying Corollary 3.7 with

$$b > \max\{f(u_1), f(u_2), f(0)\}$$

and by using formula (3.4) with $u_0 := u_1$, u_2 and Example 3.2 with $z := 0$, we obtain from conditions (ii) and (i) that

$$C_q(b) = \begin{cases} 2 & \text{if } q = 0, \\ 0 & \text{if } q \geq 1 \text{ with } q \neq q_0, \\ 1 & \text{if } q = q_0. \end{cases}$$

This implies that $C_1(b) = 0$, since $q_0 \geq 2$. Hence we have the formula

$$C_1(b) - C_0(b) = -2. \tag{3.5}$$

On the other hand, in light of the non-critical interval theorem (Theorem 3.2) we find that f^b is a strong deformation retraction of H. Hence it follows that

$$\beta_q(b) = \operatorname{rank} H_q(f^b; G) = \operatorname{rank} H_q(H; G) = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases}$$

In particular, we have the formula

$$\beta_1(b) - \beta_0(b) = -1. \tag{3.6}$$

Therefore, we obtain from formulas (3.5) and (3.6) that

$$C_1(b) - C_0(b) = -2 < -1 = \beta_1(b) - \beta_0(b). \tag{3.7}$$

This contradicts inequality (3.3).

The proof of Theorem 3.8 is complete. □

4. Ljusternik–Schnirelman theory on Hilbert spaces.

This section is devoted to the Ljusternik–Schnirelman theory on Hilbert spaces which is used in the proof of Theorem 1.2 in Section 8. More precisely, we state an analytic version of the multiplicity theorem of the Ljusternik–Schnirelman theory specialized to the case of an even functional on a Hilbert space (Theorem 4.2). For more details, the reader might refer to Palais [**19**], Schwartz [**22**] and Chang [**9**].

4.1. The Krasnosel'skii genus.

In this subsection we introduce the notion of genus due to Krasnosel'skii.

Let H be a real Hilbert space. A subset A of H is said to be *symmetric* with respect to the origin 0 if it satisfies the condition

$$u \in A \implies -u \in A.$$

A map $f : A \to \boldsymbol{R}^n$ is said to be *odd* if it satisfies the condition

$$f(-x) = -f(x) \quad \text{for all } x \in A.$$

Definition 4.1. Let

$$\mathcal{A} = \{A \subset H \setminus \{0\} : A \text{ is symmetric}\}.$$

If $A \in \mathcal{A}$, then we define its *Krasnosel'skii genus* $\gamma(A)$ by the formula

$$\gamma(A) = \begin{cases} \text{the least integer } n \text{ such that there is an odd map } \phi \in C(A, \boldsymbol{R}^n \setminus \{0\}), \\ +\infty \quad \text{if there is no such odd map } \phi, \\ 0 \quad \text{if } A = \emptyset. \end{cases} \tag{4.1}$$

Remark 4.1. If A is a closed subset of H, then we may replace the condition that there is an odd map $\phi \in C(A, \boldsymbol{R}^n \setminus \{0\})$ in formula (4.1) by the condition that there is an odd map $\psi \in C(H, \boldsymbol{R}^n)$ such that

$$\psi(x) \neq 0 \quad \text{for all } x \in A.$$

Indeed, we can construct an extension map $\widehat{\phi} \in C(H, \boldsymbol{R}^n)$ of ϕ if we make use of the Tietze extension theorem ([**11**, Theorem 4.1]).

We list some basic properties of the Krasnosel'skii genus:

(1) If A, $B \in \mathcal{A}$ and if there exists an odd continuous map $f : A \to B$, then it follows that

$$\gamma(A) \leq \gamma(B).$$

In particular, if $A \subset B$, then we have the *monotonicity*

$$\gamma(A) \leq \gamma(B). \tag{4.2}$$

(2) If A and $B \in \mathcal{A}$, then we have the *subadditivity*

$$\gamma(A \cup B) \leq \gamma(A) + \gamma(B). \tag{4.3}$$

(3) If η is an odd continuous map of A into H, then we have the *deformation non-decreasing property*

$$\gamma(A) \leq \gamma(\eta(A)). \tag{4.4}$$

(4) If $A \in \mathcal{A}$ is compact, then it follows that $\gamma(A) < +\infty$. Furthermore, we can find an open symmetric neighborhood U_A of A such that the closure $\overline{U_A}$ of U_A belongs to $\mathcal{A}$ and satisfies the *continuity condition*

$$\gamma\big(\overline{U_A}\big) = \gamma(A). \tag{4.5}$$

(5) If p is an non-zero element of H, then $[p] = \{p, -p\} \in \mathcal{A}$ and we have the *normality*

$$\gamma([p]) = 1. \tag{4.6}$$

(6) If $A \in \mathcal{A}$ and if $\gamma(A) = m$, then there exist at least m distinct points in A.

This property follows by combining the subadditivity (4.3) and the normality (4.6).

(7) If $A \in \mathcal{A}$ and if there exists an odd homeomorphism of the n-sphere S^n onto A, then it follows that

$$\gamma(A) = \gamma(S^n) = n + 1.$$

4.2. The multiplicity theorem.

We mention that the notion of genus introduced by Krasnosel'skii is a topological invariant for the estimate of the lower bound of the number of critical points. In fact, the next multiplicity theorem is the main theorem of the Ljusternik–Schnirelman theory specialized to the case of an even functional on a Hilbert space (see [**10**, Theorem 8]; [**19**, Theorem 7.1]; [**9**, Theorem 5.2.18]):

Theorem 4.1 (the multiplicity theorem). *Let H be a real Hilbert space. If $f \in C^1(H, \boldsymbol{R})$ is an* even *function, then we let*

$$c_n(f) = \inf_{\gamma(A) \geq n} \sup_{x \in A} f(x), \quad n = 1, 2, \dots. \tag{4.7}$$

Assume that

$$c = c_{k+1}(f) = \cdots = c_{k+m}(f)$$

is finite and that $f(x)$ satisfies $(\mathrm{PS})_c$ *condition. Then it follows that*

$$\gamma(K_c) \geq m,$$

where

$$K_c = \{x \in H : f(x) = c, \ \nabla f(x) = 0\}$$

is the set of critical points of f at level c.

By virtue of assertion (6) of the Krasnosel'skii genus in Subsection 4.1, we obtain that there exist at least m distinct points in the set K_c of critical points of f at level c.

Moreover, we can obtain the following analytic version of Theorem 4.1 (see [**9**, Theorem 5.2.23]):

THEOREM 4.2. *Let H be a real Hilbert space, $f \in C^1(H, \boldsymbol{R})$ and $a < b$. Assume that $f(0) > b$ and that f is an* even *function and satisfies* (PS) *condition. Moreover, we assume that the following three conditions* (i), (ii) *and* (iii) *are satisfied*:

(i) *There exist an m-dimensional linear subspace V of H and a constant $\rho > 0$ such that*

$$\sup_{x \in V \cap S_\rho(0)} f(x) \leq b, \tag{4.8}$$

where $S_\rho(0) = \{x \in H : \|x\|_H = \rho\}$.

(ii) *There exists a j-dimensional linear subspace W of H such that*

$$\inf_{x \in W^\perp} f(x) > a, \tag{4.9}$$

where $W^\perp$ is the orthogonal complement of W in H.

(iii) $m > j$.

Then f has at least $(m - j)$ pairs of distinct critical points.

5. Regularity of weak solutions of problem (1.3).

In this section we introduce the notion of weak solutions of the semilinear problem (1.3) (Definition 5.1), and prove that any weak solution of the semilinear problem (1.3) is a classical solution in the usual sense (Theorem 5.3). This section is the heart of the subject. In Subsection 5.1 we introduce an underlying Hilbert space $\mathcal{H}$ for the study of the semilinear problem (1.3) (Theorems 5.1 and 5.2). The crucial point in our variational approach is how to use the theory of fractional powers of selfadjoint operators as in [**28**]. In Subsection 5.2 we prove Theorem 5.3. The proof of Theorem 5.3 is based on the regularity theorem and the existence and uniqueness theorem for the linear elliptic boundary value problem (1.4) ([**26**, Theorem 8.2 and Theorem 9.1]), and may be proved just as in [**31**, Theorem 3.3].

5.1. Hilbert space $\mathcal{H}$.

Since the operator $\mathfrak{A}$ is positive and selfadjoint in the Hilbert space $L^2(\Omega)$, we can define its square root

$$\mathcal{C} = \mathfrak{A}^{1/2} : L^2(\Omega) \longrightarrow L^2(\Omega)$$

as follows ([**34**, Chapter XI, Section 5, Theorem 2]):

$$\mathcal{C}u = \sum_{m=1}^{\infty} \sqrt{\lambda_m}(u, \phi_m)_{L^2(\Omega)}\phi_m \tag{5.1}$$

where the domain $D(\mathcal{C})$ is the set

$$D(\mathcal{C}) = \left\{ u \in L^2(\Omega) : \sum_{m=1}^{\infty} \lambda_m \left|(u, \phi_m)_{L^2(\Omega)}\right|^2 < \infty \right\}.$$

Moreover, we can introduce an underlying Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)_{\mathcal{H}}$ as follows:

$$\begin{aligned} &\mathcal{H} = \text{the domain } D(\mathcal{C}) \text{ with the inner product} \\ &(u, v)_{\mathcal{H}} = (\mathcal{C}u, \mathcal{C}v)_{L^2(\Omega)} \quad \text{for all } u,\, v \in D(\mathcal{C}). \end{aligned}$$

The next theorem gives a more concrete and useful characterization of the Hilbert space $\mathcal{H}$ (see [**28**, Theorem 3.1]):

THEOREM 5.1. *The Hilbert space $\mathcal{H}$ coincides with the* completion *of the domain*

$$D(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0 \text{ on } \partial\Omega\}$$

with respect to the inner product

$$\begin{aligned} (u, v)_{\mathcal{H}} &= (\mathfrak{A}u, v)_{L^2(\Omega)} \\ &= \sum_{i,j=1}^{N} \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega c(x) u \cdot v\, dx \\ &\quad + \int_{\{a(x')\neq 0\}} \frac{b(x')}{a(x')} u \cdot v\, d\sigma \quad \text{for all } u,\, v \in D(\mathfrak{A}). \end{aligned} \tag{5.2}$$

Here the last term on the right-hand side is an inner product of the Hilbert space $L^2(\partial\Omega)$ with respect to the surface measure $d\sigma$ of $\partial\Omega$.

Our approach is based on the following imbedding result for the Hilbert space $\mathcal{H}$ (see [**28**, Corollary 3.2]):

THEOREM 5.2. *We have the inclusions*

$$D(\mathfrak{A}) \subset \mathcal{H} \subset W^{1,2}(\Omega) \tag{5.3}$$

with continuous injections.

REMARK 5.1. The following diagram gives a bird's eye view of the right Hilbert space $\mathcal{H}$ for the variational approach (see [**12**, Theorems 1 and 2]):

B	$\mathcal{H}$	$a(x')$ and $b(x')$
The Dirichlet case	$W_0^{1,2}(\Omega)$	$a(x') \equiv 0$ and $b(x') > 0$
The Robin case	$W^{1,2}(\Omega)$	$a(x') > 0$ and $b(x') \not\equiv 0$
The degenerate case	$D(\mathfrak{A}^{1/2})$	(H.1) and (H.2)

First, we have, by formula (5.1),

$$(u,u)_{\mathcal{H}} = \sum_{m=1}^{\infty} \lambda_m (u,\phi_m)^2_{L^2(\Omega)}. \tag{5.4}$$

Indeed, it suffices to note the following:

$$\begin{aligned}(u,u)_{\mathcal{H}} &= (\mathcal{C}u, \mathcal{C}u)_{L^2(\Omega)} \\ &= \left(\sum_{m=1}^{\infty} \sqrt{\lambda_m}(u,\phi_m)_{L^2(\Omega)}\phi_m,\ \sum_{\ell=1}^{\infty} \sqrt{\lambda_\ell}(u,\phi_\ell)_{L^2(\Omega)}\phi_\ell \right)_{L^2(\Omega)} \\ &= \sum_{m=1}^{\infty} \lambda_m (u,\phi_m)^2_{L^2(\Omega)}. \end{aligned}\tag{5.5}$$

Secondly, since we have the Fourier series expansion formula

$$u = \sum_{m=1}^{\infty} (u,\phi_m)_{L^2(\Omega)}\,\phi_m \quad \text{in } L^2(\Omega),$$

it follows that

$$\begin{aligned}(u,u)_{L^2(\Omega)} &= \left(\sum_{m=1}^{\infty} (u,\phi_m)_{L^2(\Omega)}\phi_m,\ \sum_{\ell=1}^{\infty} (u,\phi_\ell)_{L^2(\Omega)}\phi_\ell \right)_{L^2(\Omega)} \\ &= \sum_{m=1}^{\infty} (u,\phi_m)^2_{L^2(\Omega)}. \end{aligned}\tag{5.6}$$

Thirdly, we have, by formulas (5.5) and (5.6),

$$(u,u)_{L^2(\Omega)} \leq \frac{1}{\lambda_1}(u,u)_{\mathcal{H}} \quad \text{for all } u \in L^2(\Omega). \tag{5.7}$$

If J is a positive integer, we let

$$X = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_J\},$$

and

$$Y = X^\perp = \{v \in \mathcal{H} : (v,u)_{\mathcal{H}} = 0 \text{ for all } u \in X\}.$$

From formulas (5.4) and (5.6), we obtain the inequality

$$(v,v)_{\mathcal{H}} \geq \lambda_{J+1}(v,v)_{L^2(\Omega)} \quad \text{for all } v \in Y. \tag{5.8}$$

Indeed, it follows that

$$\begin{aligned}
(v,v)_{\mathcal{H}} &= \sum_{m=1}^{\infty} \lambda_m (v,\phi_m)^2_{L^2(\Omega)} = \sum_{m=J+1}^{\infty} \lambda_m (v,\phi_m)^2_{L^2(\Omega)} \\
&\geq \lambda_{J+1} \sum_{m=N+1}^{\infty} (v,\phi_m)^2_{L^2(\Omega)} = \lambda_{J+1} \sum_{m=1}^{\infty} (v,\phi_m)^2_{L^2(\Omega)} \\
&= \lambda_{J+1}(v,v)_{L^2(\Omega)} \quad \text{for all } v \in Y.
\end{aligned}$$

Similarly, we have the inequality

$$(u,u)_{\mathcal{H}} \leq \lambda_J (u,u)_{L^2(\Omega)} \quad \text{for all } u \in X. \tag{5.9}$$

Indeed, it follows that

$$\begin{aligned}
(u,u)_{\mathcal{H}} &= \sum_{m=1}^{\infty} \lambda_m (u,\phi_m)^2_{L^2(\Omega)} = \sum_{m=1}^{J} \lambda_m (u,\phi_m)^2_{L^2(\Omega)} \\
&\leq \lambda_J \sum_{m=1}^{J} (u,\phi_m)^2_{L^2(\Omega)} = \lambda_J \sum_{m=1}^{\infty} (u,\phi_m)^2_{L^2(\Omega)} \\
&= \lambda_J (u,u)_{L^2(\Omega)} \quad \text{for all } u \in X.
\end{aligned}$$

5.2. Weak solutions of a general semilinear problem.

First, we introduce the notion of a weak solution of a general semilinear boundary value problem in the framework of the Hilbert space $\mathcal{H}$:

DEFINITION 5.1. Let $p(t)$ be a real-valued function on $\boldsymbol{R}$. We consider the following semilinear boundary value problem:

$$\begin{cases} Au = p(u) & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

A function $u \in \mathcal{H}$ is called a *weak solution* of problem (5.10) if it satisfies the condition

$$\begin{aligned} (u, w)_{\mathcal{H}} - \int_\Omega p(u) w \, dx &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx + \int_\Omega c(x) u \cdot w \, dx - \int_\Omega p(u) \cdot w \, dx \\ &\quad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} u \cdot w \, d\sigma \\ &= 0 \quad \text{for all } w \in \mathcal{H}. \end{aligned} \tag{5.11}$$

The next theorem asserts that any weak solution u of problem (5.10) is a classical solution:

THEOREM 5.3. *Assume that* $p(t)$ *is a* Lipschitz continuous *function on* $\boldsymbol{R}$. *If* $u \in \mathcal{H}$ *is a weak solution of problem* (5.10), *then it follows that*

$$u \in C^{2+\alpha}(\overline{\Omega})$$

with an exponent $0 < \alpha < 1$. *In particular, u is a* classical solution.

PROOF. The proof of Theorem 5.3 is divided into two steps. We make use of a standard “bootstrap argument”.

Step 1: First, we assume that a function $u \in \mathcal{H}$ satisfies condition (5.11). Then we have, for all $w \in D(\mathfrak{A}) \subset D(\mathfrak{A}^{1/2}) = \mathcal{H}$,

$$(u, \mathfrak{A}w)_{L^2(\Omega)} = (u, w)_{\mathcal{H}} = (p(u), w)_{L^2(\Omega)}.$$

This proves that

$$\begin{cases} u \in D(\mathfrak{A}), \\ \mathfrak{A}u = p(u), \end{cases} \tag{5.12}$$

since the operator $\mathfrak{A}$ is selfadjoint in $L^2(\Omega)$. In particular, it follows from assertion (5.3) that

$$u \in W^{1,2}(\Omega) \subset L^2(\Omega).$$

Step 2: Now we assume that $u \in L^q(\Omega)$ for some $q \geq 2$. Since $p(t)$ is Lipschitz continuous on $\boldsymbol{R}$, it follows that

$$f(x) := p(u(x)) \in L^q(\Omega).$$

Moreover, we obtain from formula (5.12) that u is a weak solution of the linear boundary value problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, it follows from an application of the regularity theorem ([**26**, Theorem 8.2]) that

$$u \in W^{2,q}(\Omega).$$

Case A: If $2q \geq N$, then it follows from an application of the Sobolev imbedding theorem ([**1**, Theorem 4.12, Part I, Case A]) that

$$u \in L^r(\Omega) \quad \text{for all } r \geq 1.$$

Case B: If $2q < N$, then it follows from an application of the Sobolev imbedding theorem ([**1**, Theorem 4.12, Part I, Case C]) that

$$u \in L^r(\Omega) \quad \text{for } r = q^* = \frac{Nq}{N-2q} > q.$$

By repeating this procedure, we have, after a finite number of steps,

$$u \in W^{2,r}(\Omega) \quad \text{for all } r > \frac{N}{1-\alpha},$$

and so

$$u \in W^{2,r}(\Omega) \subset C^{1+\beta}(\overline{\Omega})$$

with the exponent

$$\beta = 1 - \frac{N}{r} > \alpha.$$

Since $p(t)$ is Lipschitz continuous on $\boldsymbol{R}$, it follows that

$$f(x) = p(u(x)) \in C^{\alpha}(\overline{\Omega}).$$

However, by applying the existence and uniqueness theorem ([**26**, Theorem 9.1]) we can find a unique classical solution $v \in C^{2+\alpha}(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} Av = f & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

Since u and v are both solutions of problem (5.13) in $W^{2,r}(\Omega)$, by applying the uniqueness theorem ([**26**, Theorem 8.6]) we obtain that

$$u = v \in C^{2+\alpha}(\overline{\Omega}).$$

Summing up, we have proved that any weak solution u of problem (5.10) is a classical solution.

The proof of Theorem 5.3 is complete. □

6. Proof of Theorem 1.1, Part 1.

In this section we prove assertion (i) of Theorem 1.1. By virtue of Theorem 5.3, we have only to prove the existence of weak solutions of problem (1.3).

6.1. Existence of classical solutions of problem (1.3).

Since we have not assumed any growth condition on the nonlinear term $g(s)$, we truncate the right-hand side

$$f(u) = \lambda u - g(u)$$

of the semilinear problem (1.3) in the following way: By condition (B), we can find two real numbers $s^{\pm}$ such that

$$s^- < 0 < s^+,$$
$$f(s^+) = \lambda s^+ - g(s^+) \leq 0 \leq f(s^-) = \lambda s^- - g(s^-).$$

Let $p(s)$ be a C^1 function on $\boldsymbol{R}$ such that

$$p(s) \begin{cases} < 0 & \text{for } s > s^+, \\ = f(s) = \lambda s - g(s) & \text{for } s \in [s^-, s^+], \\ > 0 & \text{for } s < s^-. \end{cases} \tag{6.1}$$

Moreover, we may assume that there exists a positive constant L such that $p(s)$ satisfies the following two conditions:

(C.1) $|p(s)| \leq L$ for all $s \in \boldsymbol{R}$.
(C.2) $|p'(s)| \leq L$ for all $s \in \boldsymbol{R}$.

Example 6.1. If the nonlinear term $g(s)$ is given by the formula

$$g(s) = \begin{cases} s^p & \text{for } s \geq 0, \\ s|s|^{q-1} & \text{for } s < 0 \end{cases}$$

as in Example 1.1, then we may take

$$s^+ = \lambda^{1/(p-1)}, \quad s^- = -\lambda^{1/(q-1)}.$$

Now we consider instead of the semilinear problem (1.3) the following semilinear problem:

$$\begin{cases} Au = p(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Then, by using the maximum principle (see [**33**]) we have the following:

Claim 6.1. *Every classical solution $u(x)$ of problem* (6.2) *satisfies the condition*

$$s^- \leq u(x) \leq s^+ \quad \textit{in } \Omega.$$

In particular, it is a solution of the original problem (1.3).

Proof. First, we recall that

$$p(s) < 0 \quad \text{for } s > s^+.$$

Assume, to the contrary, that the open set

$$\Omega_+ = \{x \in \Omega : u(x) > s^+\}$$

is non-empty. Then it follows that

$$\begin{cases} Au = p(u) < 0 & \text{in } \Omega^+, \\ u = s^+ > 0 & \text{on } \partial\Omega^+. \end{cases}$$

Hence, by using the maximum principle for the Dirichlet problem we obtain that

$$u(x) \leq s^+ \quad \text{on } \overline{\Omega^+},$$

so that

$$s^+ < u(x) \leq s^+ \quad \text{in } \Omega^+.$$

This contradiction proves that $\Omega^+ = \emptyset$.

Similarly, we can prove that the open set

$$\Omega_- = \{x \in \Omega : u(x) < s^-\}$$

is empty.

The proof of Claim 6.1 is complete. □

6.1.1. Energy functionals.

In order to solve problem (6.2), we introduce an energy functional

$$F(u) = \frac{1}{2}(u,u)_{\mathcal{H}} - \int_\Omega P(u)\,dx \quad \text{for all } u \in \mathcal{H}, \tag{6.3}$$

where

$$P(s) = \int_0^s p(t)\,dt,$$

and look for the critical points of F on $\mathcal{H}$.

Step 1: First, we prove the following:

Claim 6.2. *A function $u \in \mathcal{H}$ is a* critical point *of F if and only if it is a weak solution of problem* (6.2)*.*

Proof. We have, by formula (6.3),

$$(\nabla F(u), v)_{\mathcal{H}} = (u,v)_{\mathcal{H}} - \int_\Omega p(u)v\,dx \quad \text{for all } v \in \mathcal{H}, \tag{6.4}$$

and also

$$(D^2F(u)v, w)_{\mathcal{H}} = (v,w)_{\mathcal{H}} - \int_\Omega p'(u)vw\,dx \quad \text{for all } v,\, w \in \mathcal{H}.$$

Hence it follows from formula (6.4) that $\nabla F(u) = 0$ if and only if u satisfies the condition

$$(u,v)_{\mathcal{H}} - \int_\Omega p(u)v\,dx = 0 \quad \text{for all } v \in \mathcal{H}. \tag{6.5}$$

In view of Definition 5.1, we find that u satisfies condition (6.5) if and only if it is a weak solution of problem (6.2).

Summing up, we have proved that $\nabla F(u) = 0$ if and only if u is a weak solution of problem (6.2).

The proof of Claim 6.2 is complete. □

Step 2: Secondly, we show that $F(u)$ is *bounded from below* on $\mathcal{H}$. More precisely, we prove the following:

Claim 6.3. *There exists a positive constant C_0 such that*

$$F(u) \geq -\frac{1}{2}C_0^2 \quad \textit{for all } u \in \mathcal{H}. \tag{6.6}$$

For example, we may take

$$C_0 = \frac{L|\Omega|^{1/2}}{\sqrt{\lambda_1}}$$

where $|\Omega|$ *denotes the volume of* Ω.

PROOF. Indeed, since we have, by condition (C.1),

$$|P(u)| = \left| \int_0^{u(x)} p(s)\, ds \right| \leq L|u(x)|,$$

by using Schwarz's inequality and inequality (5.7) we obtain that

$$\begin{aligned} \left| \int_\Omega P(u)\, dx \right| &\leq \int_\Omega |P(u)| dx \leq L \int_\Omega |u(x)| dx \\ &\leq L|\Omega|^{1/2} \left(\int_\Omega |u(x)|^2 dx \right)^{1/2} = L|\Omega|^{1/2} \|u\|_{L^2(\Omega)} \\ &\leq \frac{L|\Omega|^{1/2}}{\sqrt{\lambda_1}} \|u\|_{\mathcal{H}} = C_0 \|u\|_{\mathcal{H}} \quad \text{for all } u \in \mathcal{H}. \end{aligned} \tag{6.7}$$

This proves that

$$\begin{aligned} F(u) = \frac{1}{2}(u, u)_{\mathcal{H}} - \int_\Omega P(u)\, dx &\geq \frac{1}{2}\|u\|_{\mathcal{H}}^2 - C_0 \|u\|_{\mathcal{H}} \\ &\geq -\frac{1}{2} C_0^2 \quad \text{for all } u \in \mathcal{H}. \end{aligned}$$

The proof of Claim 6.3 is complete. □

6.1.2. The Palais–Smale condition.

Now we show that the energy functional $F(u)$ satisfies (PS) condition, that is, $F(u)$ satisfies $(\mathrm{PS})_c$ condition for every constant $c \in \boldsymbol{R}$. Indeed, we have the following:

CLAIM 6.4. *Let* $\{u_j\}_{j=1}^\infty$ *be an arbitrary sequence in* $\mathcal{H}$ *such that*

$$F(u_j) \longrightarrow c \quad \text{in } \boldsymbol{R} \text{ as } j \to \infty, \tag{6.8}$$

$$\nabla F(u_j) \longrightarrow 0 \quad \text{in } L(\mathcal{H}, \boldsymbol{R}) \text{ as } j \to \infty. \tag{6.9}$$

Then the sequence $\{u_j\}$ *contains a convergent subsequence.*

PROOF. The proof of Claim 6.4 is divided into four steps.

Step 1: First, by formula (6.3) and assertion (6.8) we can find a positive constant C_1 such that

$$-C_1 \leq F(u_j) = \frac{1}{2}(u_j, u_j)_{\mathcal{H}} - \int_\Omega P(u_j)\, dx \leq C_1.$$

Hence we have, by inequality (6.7),

$$\frac{1}{2}\|u_j\|_{\mathcal{H}}^2 \leq C_1 + \int_\Omega P(u_j)\,dx \leq C_1 + C_0\|u_j\|_{\mathcal{H}}.$$

This proves that

$$\|u_j\|_{\mathcal{H}} \leq C := C_0 + \sqrt{C_0^2 + 2C_1}. \tag{6.10}$$

Step 2: Secondly, we have the following three assertions (a), (b) and (c):

(a) The injections

$$\mathcal{H} \subset W^{1,2}(\Omega) \subset L^q(\Omega), \quad 1 \leq q \leq 2^* = \frac{2N}{N-2}, \tag{6.11}$$

are continuous, while the injection

$$W^{1,2}(\Omega) \subset L^q(\Omega), \quad 1 \leq q < 2^* = \frac{2N}{N-2}, \tag{6.12}$$

is *compact* (see the Rellich–Kondrachov theorem [**1**, Theorem 6.3, Part I]).

(b) The Nemytskii operator

$$\begin{aligned} N : L^q(\Omega) &\longrightarrow L^{2N/(N+2)}(\Omega) \\ u(x) &\longmapsto p(u(x)) \end{aligned} \tag{6.13}$$

is continuous (see [**8**, Chapter 1, Theorem 2.2]).

(c) It follows from an application of Hölder's inequality that

$$u(x)\cdot p(u(x)) \in L^1(\Omega)$$

and

$$\|u\cdot p(u)\|_{L^1(\Omega)} \leq \|u\|_{L^{2^*}(\Omega)} \cdot \|p(u)\|_{L^{2N/(N+2)}(\Omega)} \quad \text{for all } u \in \mathcal{H}, \tag{6.14}$$

since we have the relation

$$\frac{1}{2^*} + \frac{1}{2N/(N+2)} = 1.$$

Step 3: By inequality (6.10), it follows that the sequence $\{u_j\}$ is bounded in the Hilbert space $\mathcal{H}$. Hence, by applying the local sequential weak compactness of Hilbert spaces ([**34**, Chapter V, Section 2, Theorem 1]) we may assume that $\{u_j\}$ itself converges *weakly* to some element u in $\mathcal{H}$, that is,

$$u_j \rightharpoonup u \quad \text{in } \mathcal{H} \text{ as } j \to \infty. \tag{6.15}$$

Therefore, it follows from assertion (6.12) that $\{u_j\}$ converges *strongly* to u in $L^q(\Omega)$ for $1 \leq q < 2^*$:

$$u_j \longrightarrow u \quad \text{in } L^q(\Omega) \text{ as } j \to \infty. \tag{6.16}$$

Moreover, we have, by assertion (6.13),

$$N(u_j) = p(u_j) \longrightarrow N(u) = p(u) \quad \text{in } L^{2N/(N+2)}(\Omega) \text{ as } j \to \infty. \tag{6.17}$$

However, we obtain from formula (6.4) with $u := u_j$ that

$$(\nabla F(u_j), v)_{\mathcal{H}} = (u_j, v)_{\mathcal{H}} - \int_\Omega p(u_j) \cdot v \, dx \quad \text{for all } v \in \mathcal{H}. \tag{6.18}$$

By assertions (6.15), (6.17) and (6.9), it follows from formula (6.18) that

$$\begin{aligned} (u, v)_{\mathcal{H}} - \int_\Omega p(u) \cdot v \, dx &= \lim_{j \to \infty} \left((u_j, v)_{\mathcal{H}} - \int_\Omega p(u_j) \cdot v \, dx \right) \\ &= \lim_{j \to \infty} (\nabla F(u_j), v)_{\mathcal{H}} = 0 \quad \text{for all } v \in \mathcal{H}. \end{aligned}$$

This proves that

$$(u, v)_{\mathcal{H}} = \int_\Omega p(u) \cdot v \, dx \quad \text{for all } v \in \mathcal{H}. \tag{6.19}$$

Step 4: Finally, we can prove that

$$u_j \longrightarrow u \quad \text{in } \mathcal{H} \text{ as } j \to \infty. \tag{6.20}$$

Indeed, we have, by formulas (6.18) and (6.19),

$$(u_j - u, v)_{\mathcal{H}} = (\nabla F(u_j), v)_{\mathcal{H}} + \int_\Omega p(u_j) \cdot v \, dx - \int_\Omega p(u) \cdot v \, dx. \tag{6.21}$$

However, we obtain from inequality (6.14) and assertion (6.11) with $q := 2^*$ that, for some positive constant C_2,

$$\begin{aligned} &\|(p(u_j) - p(u))v\|_{L^1(\Omega)} \\ &\quad \leq \|p(u_j) - p(u)\|_{L^{2N/(N+2)}(\Omega)} \cdot \|v\|_{L^{2^*}(\Omega)} \\ &\quad \leq \|p(u_j) - p(u)\|_{L^{2N/(N+2)}(\Omega)} \cdot C_2 \|v\|_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H}. \end{aligned} \tag{6.22}$$

Hence, by formula (6.21) and inequality (6.22) it follows that

$$\begin{aligned}
&|(u_j - u, v)_{\mathcal{H}}| \\
&\quad \le |(\nabla F(u_j), v)_{\mathcal{H}}| + \left| \int_\Omega (p(u_j) - p(u)) v \, dx \right| \\
&\quad \le \|\nabla F(u_j)\|_{\mathcal{H}} \|v\|_{\mathcal{H}} + C_2 \|p(u_j) - p(u)\|_{L^{2N/(N+2)}(\Omega)} \|v\|_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H}.
\end{aligned}$$

In view of the Riesz representation theorem ([**34**, Chapter III, Section 6, Theorem]), we have proved that

$$\|u_j - u\|_{\mathcal{H}} \le \|\nabla F(u_j)\|_{\mathcal{H}} + C_2 \|p(u_j) - p(u)\|_{L^{2N/(N+2)}(\Omega)}. \tag{6.23}$$

Therefore, the desired assertion (6.20) follows from inequality (6.23) by using assertions (6.9) and (6.17).

The proof of Claim 6.4 is complete. □

6.1.3. Proof of existence of classical solutions of problem (1.3).

The proof of existence of classical solutions of problem (1.3) is carried out in the following way:

(I) By Claims 6.3 and 6.4, we can apply Theorem 2.2 to obtain a critical point $u \in \mathcal{H}$ of the energy functional F.
(II) By Claim 6.2, it follows that the critical point u is a weak solution of problem (6.2).
(III) By applying Theorem 5.3, we obtain that the weak solution u of problem (6.2) is a classical solution.
(IV) By Claim 6.1, it follows that the classical solution u of problem (6.2) is a classical solution of the original problem (1.3).

6.2. End of Proof of Theorem 1.1, Part 1.

To find a positive solution u_1 and a negative solution u_2 of the semilinear problem (1.3), we need another truncation of the nonlinear term

$$f(s) = \lambda s - g(s).$$

We let

$$\begin{aligned}
p^+(s) &= \max\{p(s), 0\}, \\
p^-(s) &= p(s) - p^+(s),
\end{aligned}$$

and

$$P^\pm(s) = \int_0^s p^\pm(t) \, dt.$$

It should be noticed that the functions $p^\pm(s)$ are Lipschitz continuous and satisfy the following two conditions:

(D.1) $|p^\pm(s)| \le L$ for all $s \in \boldsymbol{R}$.
(D.2) $|{p^\pm}'(s)| \le L$ for almost all $s \in \boldsymbol{R}$.

If we introduce two energy functionals $F^{\pm}$ by the formulas (cf. formula (6.3))

$$F^{\pm}(u) = \frac{1}{2}(u,u)_{\mathcal{H}} - \int_{\Omega} P^{\pm}(u)\, dx, \quad u \in \mathcal{H},$$

then it is easy to verify that the functionals $F^{\pm}(u)$ are bounded from below on $\mathcal{H}$ and satisfy (PS) condition (see Claims 6.3 and 6.4). Therefore, by applying Theorems 2.2 and 5.3 just as in Section 6.1 we obtain that the minima u_1 and u_2 of $F^{+}(u)$ and $F^{-}(u)$ exist and hence that u_1 and u_2 are classical solutions of problem (6.2) with $p(s)$ replaced by $p^{+}(s)$ and $p^{-}(s)$, respectively:

$$\begin{cases} Au_1 = p^{+}(u_1) & \text{in } \Omega, \\ Bu_1 = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} Au_2 = p^{-}(u_2) & \text{in } \Omega, \\ Bu_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by using the maximum principle just as in the proof of Claim 6.1 we find that

$$\begin{aligned} 0 \le u_1(x) \le s^{+} \quad & \text{in } \Omega, \\ s^{-} \le u_2(x) \le 0 \quad & \text{in } \Omega. \end{aligned}$$

This proves that u_1 and u_2 are solutions of problem (6.2) and hence of the original problem (1.3).

Now the proof of Theorem 1.1, Part 1 is complete. □

Remark 6.1. For the positive solution u_1, we obtain that $F(u_1) < F(u)$ for all $u > 0$ in Ω if $0 < \|u - u_1\|$ is sufficiently small. We remark also that if u_1 is not an isolated minimum, we have infinitely many solutions of the semilinear problem (1.3), and we are done. A similar remark remains valid for the negative solution u_2.

7. Proof of Theorem 1.1, Part 2.

In this section we prove assertion (ii) of Theorem 1.1. The proof is based on a Lyapunov–Schmidt procedure and a slight modification of the classical Morse inequalities. This section is divided into four subsections. The main idea of Subsections 7.1 and 7.2 is to rewrite the semilinear problem (1.3) in a suitable bifurcation system (7.8) and (7.9) and to solve the first equation (7.8), by using the global inversion theorem. In Subsection 7.3 we deal with functionals which may have *degenerate* critical points, by using a perturbation argument and Sard's lemma (Lemma 7.2). In Subsection 7.4, by using Lemma 7.2 and applying Theorem 3.8 to our situation we can find a third non-trivial solution u_3 different from u_1 and u_2 constructed in Subsection 6.2.

7.1. Lyapunov–Schmidt procedure.

Since $p'(s)$ is bounded and $\lim_{j\to\infty} \lambda_j = +\infty$, we can choose a positive integer n such that

$$p'(s) < \lambda_n \quad \text{for all } s \in \boldsymbol{R}. \tag{7.1}$$

First, we let

$$V = \operatorname{span}[\phi_1, \phi_2, \dots, \phi_n].$$

Then we have the following orthogonal decomposition in the Hilbert space $L^2(\Omega)$:

$$L^2(\Omega) = V \oplus V^\perp = \operatorname{span}[\phi_1, \phi_2, \dots, \phi_n] \oplus V^\perp, \tag{7.2}$$

where

$$V^\perp = \left\{ w \in L^2(\Omega) : \int_\Omega w(x)\phi_j(x)\, dx = 0,\ j = 1, 2, \dots, n \right\}.$$

Moreover, it follows from an application of the regularity theorem ([**26**, Theorem 8.2]) that

$$\bigoplus_{j=1}^{n} N(\mathfrak{A} - \lambda_j I) = \operatorname{span}[\phi_1, \phi_2, \dots, \phi_n] \subset C^\infty(\overline{\Omega}).$$

If we define the orthogonal projection Q from $L^2(\Omega)$ onto $V^\perp$ by the formula

$$Qu = u - \sum_{j=1}^{n} \left(\int_\Omega u(x)\phi_j(x)\, dx \right) \phi_j,$$

then we obtain the formula

$$Q(Y) = Y \cap V^\perp = \left\{ w \in Y : \int_\Omega w(x)\phi_j(x)\, dx = 0,\ j = 1, 2, \dots, n \right\}.$$

Therefore, by restricting decomposition (7.2) to the subspace $Y = C^\alpha(\overline{\Omega})$ of $L^2(\Omega)$ we obtain the orthogonal decomposition

$$Y = C^\alpha(\overline{\Omega}) = \operatorname{span}[\phi_1, \phi_2, \dots, \phi_n] \oplus (Y \cap V^\perp). \tag{7.3}$$

Similarly, if we let

$$X = C_B^{2+\alpha}(\overline{\Omega}) = \left\{ u \in C^{2+\alpha}(\overline{\Omega}) : Bu = 0 \right\},$$

then we have the formula

$$Q(X) = X \cap V^{\perp} = \left\{ w \in X : \int_{\Omega} w(x)\phi_j(x)\,dx = 0,\ j = 1, 2, \dots, n \right\}.$$

By restricting the decomposition (7.3) to the subspace $X = C_B^{2+\alpha}(\overline{\Omega})$ of Y, we obtain the orthogonal decomposition

$$X = C_B^{2+\alpha}(\overline{\Omega}) = \mathrm{span}[\phi_1, \phi_2, \dots, \phi_n] \oplus (X \cap V^{\perp}). \tag{7.4}$$

In other words, every function $u \in X$ can be written uniquely in the form

$$u = v(t) + w(t), \qquad t = (t_1, t_2, \dots, t_n) \in \boldsymbol{R}^n, \tag{7.5}$$

$$v(t) = \sum_{j=1}^{n} t_j \phi_j \in V, \quad t_j = \int_{\Omega} u(x)\phi_j(x)\,dx, \tag{7.6}$$

$$w(t) \in X \cap V^{\perp}. \tag{7.7}$$

Then, in view of formulas (7.5), (7.6) and (7.7) it is easy to verify that

$$\begin{cases} Au = p(u) & \text{in } \Omega, \\ Bu = 0 & \text{in } \partial\Omega \end{cases} \iff \begin{cases} u = v(t) + w(t), \\ Aw(t) = Q(p(v(t) + w(t))), \quad w(t) \in X \cap V^{\perp}, \\ Av(t) = (I - Q)(p(v(t) + w(t))), \quad v(t) \in V. \end{cases}$$

However, we remark the formulas

$$Av(t) = \sum_{j=1}^{n} t_j A\phi_j = \sum_{j=1}^{n} \lambda_j t_j \phi_j$$

and

$$(I - Q)(p(v(t) + w(t))) = \sum_{j=1}^{n} \left(\int_{\Omega} p(v(t) + w(t))\phi_j(x)\,dx \right) \phi_j.$$

Summing up, we are reduced to the infinite-dimensional equation

$$Aw(t) = Q(p(v(t) + w(t))), \quad w(t) \in X \cap V^{\perp}, \tag{7.8}$$

and the system of n-dimensional equations

$$\int_{\Omega} p(v(t) + w(t))\phi_j(x)\,dx = \lambda_j t_j, \quad j = 1, 2, \dots, n. \tag{7.9}$$

7.2. Infinite-dimensional equation.

In this subsection we solve the first infinite-dimensional equation (7.8), by using the global inversion theorem (Proposition 7.1). To do this, we introduce a nonlinear map

$$\Phi : \boldsymbol{R}^n \times (X \cap W) \longrightarrow W, \quad W = Y \cap V^{\perp},$$

as follows:

$$\Phi(t, w) = Aw - Q(p(v(t) + w)) \quad \text{for all } (t, w) \in \boldsymbol{R}^n \times (X \cap W), \tag{7.10}$$

where

$$v(t) = \sum_{j=1}^{n} t_j \phi_j \in V, \quad t = (t_1, t_2, \dots, t_n) \in \boldsymbol{R}^n.$$

Then it is easy to see that $\Phi \in C^1(\boldsymbol{R}^n \times (X \cap W), W)$.

The next proposition plays an essential role in the Lyapunov–Schmidt procedure:

PROPOSITION 7.1. *Assume that the function $p(s)$ satisfies conditions* (C.1) *and* (C.2). *Then, for every function $h \in Y = C^{\alpha}(\overline{\Omega})$ there exists a unique function $w(t) = w(t, Qh) \in W \cap X$ which satisfies the following two conditions*:

(i) *$\Phi(t, w(t)) = Qh$ for each $t \in \boldsymbol{R}^n$.*
(ii) *The function $w(t)$ is of class C^1 on $\boldsymbol{R}^n$.*

Proposition 7.1 can be proved just as in the proof of [**29**, Proposition 3.1].

7.3. Resolution of isolated critical points.

In some cases we can deal with functionals which may have degenerate critical points, by using a perturbation argument and Sard's theorem ([**24**]). The next lemma is essentially due to Marino–Prodi [**16**, Lemma 2.1]:

LEMMA 7.2. *Assume that $f \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$ satisfies* (PS) *condition and has x_0 as an isolated, possibly degenerate, critical point. Then, for any given small constant $\varepsilon > 0$ we can construct a function $g \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$ which satisfies the following four conditions*:

(a) *The function $g(x)$ satisfies* (PS) *condition.*
(b) *$g(x) = f(x)$ for $\|x - x_0\| \geq \varepsilon$.*
(c) *The function $g(x)$ has a finite number of* non-degenerate *critical points in the open ball $\{\|x - x_0\| < \varepsilon\}$.*
(d) *The Hessians $D^2 f(x)$ and $D^2 g(x)$ satisfy the inequality*

$$\|D^2 g(x) - D^2 f(x)\| < \varepsilon \quad \textit{for all } x \in \boldsymbol{R}^n. \tag{7.11}$$

PROOF. The proof of Lemma 7.2 is divided into four steps.

Step 1: The construction of the function $g(x)$. Without loss of generality, we may assume that $x_0 = 0$. Let $\theta(t)$ be a C^∞ function on the closed interval $[0, \infty)$ such that

$$\theta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \varepsilon/2, \\ 0 & \text{for } t \geq \varepsilon. \end{cases} \tag{7.12}$$

We remark that the function $\omega(x)$, defined by the formula

$$\omega(x) = \theta(\|x\|) = \theta\Big(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\Big), \quad x \in \boldsymbol{R}^n,$$

is of class C^∞.

For a point $y \in \boldsymbol{R}^n$, we consider a function

$$\begin{aligned} g(x) &= f(x) - \omega(x)(x, y) \\ &= f(x) - \theta\Big(\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\Big) \sum_{j=1}^{n} x_j y_j, \quad x \in \boldsymbol{R}^n. \end{aligned} \tag{7.13}$$

Then we have the assertions

$$g \in C^\infty(\boldsymbol{R}^n),$$

and

$$g(x) = \begin{cases} f(x) - (x, y) & \text{for } \|x\| \leq \varepsilon/2, \\ f(x) & \text{for } \|x\| \geq \varepsilon. \end{cases} \tag{7.14}$$

This verifies condition (b). The point y will be chosen later on (see inequality (7.20) below).

Step 2: The verification of condition (d). The proof is divided into three steps.

(1) First, it follows from formulas (7.12) and (7.13) that

$$|f(x) - g(x)| \leq |(x, y)| \leq \varepsilon \|y\| \quad \text{for all } x \in \boldsymbol{R}^n.$$

This proves that

$$\sup_{x \in \boldsymbol{R}^n} |f(x) - g(x)| < \varepsilon \quad \text{provided that } \|y\| \leq \frac{1}{2}. \tag{7.15}$$

(2) For the gradient $\nabla g(x)$ of g, we have the formula

$$\begin{aligned} \nabla g(x) &= \nabla f(x) - \nabla(\omega(x)(x, y)) \\ &= \nabla f(x) - \theta'(\|x\|)(x, y) \frac{x}{\|x\|} - \theta(\|x\|) y. \end{aligned} \tag{7.16}$$

Hence it follows from formula (7.12) that

$$\begin{aligned} \|\nabla g(x) - \nabla f(x)\| &= \left\| -\theta'(\|x\|)(x, y) \frac{x}{\|x\|} - \theta(\|x\|) y \right\| \\ &\leq |\theta'(\|x\|)| \|x\| \|y\| + \|y\| \end{aligned}$$

$$\leq \left(\varepsilon \sup_{\varepsilon/2 \leq t \leq \varepsilon} |\theta'(t)| + 1\right) \|y\| \quad \text{for all } x \in \boldsymbol{R}^n. \tag{7.17}$$

If we let

$$\alpha_1(\varepsilon) = \frac{\varepsilon}{2(\varepsilon \sup_{\varepsilon/2 \leq t \leq \varepsilon} |\theta'(t)| + 1)},$$

then we have, by inequality (7.17),

$$\sup_{x \in \boldsymbol{R}^n} \|\nabla g(x) - \nabla f(x)\| < \varepsilon \quad \text{provided that } \|y\| \leq \alpha_1(\varepsilon). \tag{7.18}$$

(3) For the Hessian $D^2 g(x)$ of g, we have the formula

$$\begin{aligned}
&(D^2 g(x)h, k) = (D^2(f(x) - \omega(x)(x,y))h, k) \\
&\quad = (D^2 f(x)h, k) - \theta''(\|x\|)\left(\frac{x}{\|x\|}, h\right)\left(\frac{x}{\|x\|}, k\right)(x,y) \\
&\quad - \theta'(\|x\|)\left(\frac{x}{\|x\|}, h\right)(k,y) - \theta'(\|x\|)\left(\frac{x}{\|x\|}, k\right)(h,y) \\
&\quad - \theta'(\|x\|)(h,k)\left(\frac{x}{\|x\|}, y\right) + \theta'(\|x\|)\left(\frac{x}{\|x\|}, h\right)\left(\frac{x}{\|x\|}, k\right)\left(\frac{x}{\|x\|}, y\right)
\end{aligned}$$

for all $h,\ k \in \boldsymbol{R}^n$.

Therefore, just as in Step 2 we can find a constant $\alpha_2(\varepsilon) > 0$ such that

$$\left\|(D^2 g(x)h, k) - (D^2 f(x)h, k)\right\| \leq \frac{\varepsilon}{2}\|h\|\|k\| \quad \text{provided that } \|y\| \leq \alpha_2(\varepsilon).$$

This proves that

$$\sup_{x \in \boldsymbol{R}^n} \|D^2 g(x) - D^2 f(x)\| < \varepsilon \quad \text{provided that } \|y\| \leq \alpha_2(\varepsilon). \tag{7.19}$$

The desired inequality (7.11) follows by combining inequalities (7.15), (7.18) and (7.19) if the point y is chosen so small that

$$\|y\| \leq \min\left\{\frac{1}{2}, \alpha_1(\varepsilon), \alpha_2(\varepsilon)\right\}. \tag{7.20}$$

Step 3: The proof of condition (c). Since the point 0 is an isolated critical point of f, we can find a positive constant m such that

$$\|\nabla f(x)\| \geq m \quad \text{on the annulus } \left\{\frac{\varepsilon}{2} \leq \|x\| \leq \varepsilon\right\}. \tag{7.21}$$

Then it follows from formula (7.16) and inequality (7.21) that

$$\begin{aligned}
\|\nabla g(x)\| &\geq \|\nabla f(x)\| - \left\| -\theta'(\|x\|)(x,y)\frac{x}{\|x\|} - \theta(\|x\|)y \right\| \\
&\geq \|\nabla f(x)\| - |\theta'(\|x\|)|\|x\|\|y\| - \|y\| \\
&\geq \left(m - \left[\varepsilon \sup_{\varepsilon/2 \leq t \leq \varepsilon} |\theta'(t)| + 1 \right] \|y\| \right) \quad \text{on the annulus } \left\{ \frac{\varepsilon}{2} \leq \|x\| \leq \varepsilon \right\}. \quad (7.22)
\end{aligned}$$

Hence, if we let

$$\alpha_3(\varepsilon) = \frac{m}{2\big(\varepsilon \sup_{\varepsilon/2 \leq t \leq \varepsilon} |\theta'(t)| + 1\big)},$$

we obtain from inequality (7.22) that

$$\max_{\frac{\varepsilon}{2} \leq \|x\| \leq \varepsilon} \|\nabla g(x)\| \geq \frac{m}{2} \quad \text{provided that } \|y\| \leq \alpha_3(\varepsilon). \tag{7.23}$$

This proves that the function $g(x)$ does not have any critical point in the annulus $\{\varepsilon/2 \leq \|x\| \leq \varepsilon\}$. More precisely, the function $g(x)$ has its critical points only in the closed ball $B_{\varepsilon/2}(0) = \{\|x\| \leq \varepsilon/2\}$ and we have, by formula (7.14),

$$\begin{aligned}
\nabla g(x) &= \nabla f(x) - y \quad \text{on the ball } B_{\varepsilon/2}(0), \\
D^2 g(x) &= D^2 f(x) \qquad\quad \text{on the ball } B_{\varepsilon/2}(0),
\end{aligned}$$

where

$$\|y\| \leq \min\left\{ \frac{1}{2}, \alpha_1(\varepsilon), \alpha_2(\varepsilon), \alpha_3(\varepsilon) \right\}. \tag{7.24}$$

Therefore, we have the following equivalent assertions:

$$\begin{aligned}
& x \in \boldsymbol{R}^n \text{ is a } \textit{degenerate} \text{ critical point of } g \text{ in the ball } B_{\varepsilon/2}(0) \\
\Longleftrightarrow \; & \begin{cases} \nabla g(x) = 0 \quad \text{for } \|x\| \leq \varepsilon/2, \\ \text{The Hessian } D^2 g(x) \text{ at } x \text{ is not invertible} \end{cases} \\
\Longleftrightarrow \; & \begin{cases} \nabla f(x) = y \quad \text{for } \|x\| \leq \varepsilon/2, \\ \text{The Hessian } D^2 f(x) \text{ at } x \text{ is not invertible} \end{cases} \\
\Longleftrightarrow \; & y \in \boldsymbol{R}^n \text{ is a critical value of } \nabla f \text{ in the ball } B_{\varepsilon/2}(0).
\end{aligned}$$

However, by virtue of Sard's lemma ([**24**]) we can find a point $y \in \boldsymbol{R}^n$ which satisfies condition (7.24) and is not a critical value of ∇f. Namely, the critical points of $g(x) = f(x) - (x, y)$ in the closed ball $B_{\varepsilon/2}(0)$ are all *non-degenerate*. We remark that the non-degenerate critical points of g are isolated.

Summing up, we have constructed the function $g(x)$ which has a *finite number* of non-degenerate critical points in the closed ball $B_{\varepsilon/2}(0)$, since its critical points are isolated. This verifies condition (c).

Step 4: The proof of condition (a). Finally, it remains to show that the function $g(x)$ satisfies $(\mathrm{PS})_c$ condition for every constant $c \in \boldsymbol{R}$.

Let $\{x_j\}_{j=1}^{\infty}$ be an arbitrary sequence in $\boldsymbol{R}^n$ such that

$$\begin{aligned} g(x_j) &\longrightarrow c \quad \text{in } \boldsymbol{R} \text{ as } j \to \infty, \\ \nabla g(x_j) &\longrightarrow 0 \quad \text{in } \boldsymbol{R}^n \text{ as } j \to \infty. \end{aligned}$$

However, we remark that $f(x)$ satisfies (PS) condition and that $g(x) = f(x)$ for $\|x\| \geq \varepsilon$. Moreover, it follows from inequality (7.23) that

$$\|\nabla g(x)\| \geq \frac{m}{2} \quad \text{on the annulus } \left\{ \frac{\varepsilon}{2} \leq \|x\| \leq \varepsilon \right\}.$$

Hence, without loss of generality we may assume that

$$\|x_j\| \leq \frac{\varepsilon}{2}.$$

Then, by applying the Bolzano–Weierstrass theorem we can find a subsequence $\{x_{j'}\}$ of $\{x_j\}$ and a point $x_0 \in \boldsymbol{R}^n$ such that

$$x_{j'} \longrightarrow x_0 \quad \text{in } \boldsymbol{R}^n.$$

This verifies condition (a).

The proof of Lemma 7.2 is complete. □

7.4. End of Proof of Theorem 1.1, Part 2.

The proof of Theorem 1.1, Part 2 is divided into four steps.

Step 1: First, by applying Proposition 7.1 with $h = 0$ we can solve equation (7.8). Namely, for any given function

$$v(t) = \sum_{j=1}^{n} t_j \phi_j \in V, \quad t = (t_1, t_2, \ldots, t_n) \in \boldsymbol{R}^n,$$

we can find a function $w(t) \in X \cap W$ such that

$$Aw(t) = Q(p(v(t) + w(t))), \quad w(t) \in X \cap W.$$

By substituting $w = w(t)$ into equation (7.9), we obtain the following equations:

$$\lambda_j t_j = \int_\Omega p(v(t)+w(t))\phi_j(x)dx, \quad j=1,2,\dots,n. \tag{7.25}$$

We introduce a function $\Psi : \boldsymbol{R}^n \to \boldsymbol{R}$ by the formula

$$\Psi(t) = \frac{1}{2}(w(t),w(t))_{\mathcal{H}} + \frac{1}{2}\sum_{j=1}^n \lambda_j t_j^2 - \int_\Omega P(v(t)+w(t))dx. \tag{7.26}$$

Since $w(t)$ is a solution of equation (7.8) for $v(t)$, it is easy to see that

$$\frac{\partial \Psi}{\partial t_j} = \lambda_j t_j - \int_\Omega p(v(t)+w(t))\phi_j(x)dx, \quad 1 \le j \le n. \tag{7.27}$$

Therefore, we have proved that $\nabla\Psi(t) = 0$ if and only if equations (7.25) are satisfied.

By formulas (6.3), (7.5), (7.6) and (7.7), we find that the function Ψ is of class C^2 and further that

$$\begin{aligned}\Psi(t) &= \frac{1}{2}(w(t)+v(t),w(t)+v(t))_{\mathcal{H}} - \int_\Omega P(v(t)+w(t))\,dx \\ &= F(v(t)+w(t)) \quad \text{for all } t \in \boldsymbol{R}^n. \end{aligned}\tag{7.28}$$

Moreover, we have the following:

Claim 7.1. *The function $\Psi(t)$ is bounded from below on $\boldsymbol{R}^n$ and satisfies* (PS) *condition.*

Proof. (i) Indeed, by formula (7.28) and inequality (6.6) it follows that

$$\Psi(t) = F(v(t)+w(t)) \ge -\frac{1}{2}C_0^2 = -\frac{L^2|\Omega|}{2\lambda_1} \quad \text{for all } t \in \boldsymbol{R}^n.$$

This proves that $\Psi(t)$ is bounded from below on $\boldsymbol{R}^n$.

(ii) Now let $\{t^{(k)}\}_{k=1}^\infty = \{(t_1^{(k)}, t_2^{(k)}, \dots, t_n^{(k)})\}_{k=1}^\infty$ be an arbitrary sequence in $\boldsymbol{R}^n$ such that

$$\begin{aligned}\Psi(t^{(k)}) &\longrightarrow c \quad \text{in } \boldsymbol{R} \text{ as } k \to \infty, \\ \nabla\Psi(t^{(k)}) &\longrightarrow 0 \quad \text{in } \boldsymbol{R}^n \text{ as } k \to \infty.\end{aligned}$$

Then it follows from formula (7.27) that, as $k \to \infty$,

$$\lambda_j t_j^{(k)} - \int_\Omega p\big(v(t^{(k)}) + w(t^{(k)})\big)\phi_j(x)\,dx \longrightarrow 0, \quad 1 \le j \le n. \tag{7.29}$$

On the other hand, we have, by condition (C.1) and Schwarz's inequality,

$$
\begin{aligned}
\left| \int_\Omega p\big(v(t^{(k)}) + w(t^{(k)})\big) \cdot \phi_j(x)\,dx \right| &\le \int_\Omega \big|p\big(v(t^{(k)}) + w(t^{(k)})\big)\big| \cdot |\phi_j(x)|dx \\
&\le L \int_\Omega |\phi_j(x)|dx \le L|\Omega|^{1/2} \left(\int_\Omega |\phi_j(x)|^2 dx \right)^{1/2} \\
&= L|\Omega|^{1/2}. \qquad (7.30)
\end{aligned}
$$

Therefore, we obtain from assertion (7.29) and (7.30) that the sequence

$$
\begin{aligned}
t_j^{(k)} = &\frac{1}{\lambda_j} \left(\lambda_j t_j^{(k)} - \int_\Omega p\big(v(t^{(k)}) + w(t^{(k)})\big)\phi_j(x)\,dx \right) \\
&+ \frac{1}{\lambda_j} \int_\Omega p\big(v(t^{(k)}) + w(t^{(k)})\big)\phi_j(x)\,dx, \quad 1 \le j \le n,
\end{aligned}
$$

is bounded in $\boldsymbol{R}$. By applying the Bolzano–Weierstrass theorem, we can find a subsequence $\{t^{(k')}\}$ of $\{t^{(k)}\}$ and a point $t_0 \in \boldsymbol{R}^n$ such that

$$
t^{(k')} \longrightarrow t_0 \quad \text{in } \boldsymbol{R}^n \text{ as } k' \to \infty.
$$

This verifies (PS) condition for the function $\Psi(t)$.

The proof of Claim 7.1 is complete. □

Step 2: Now we study the function Ψ defined by formula (7.26). Let u_1 and u_2 be respectively the positive and negative solutions of the semilinear problem (1.3) constructed in Subsection 6.2. Then we have the following:

PROPOSITION 7.3. *Let t_1 be a point of $\boldsymbol{R}^n$ such that $v(t_1) + w(t_1) = u_1$ and let t_2 be a point of $\boldsymbol{R}^n$ such that $v(t_2) + w(t_2) = u_2$, respectively. Then the points t_1 and t_2 are local minima of Ψ on $\boldsymbol{R}^n$.*

PROOF. We only prove Proposition 7.3 for t_1, since the proposition is similarly proved for t_2.

As we have seen in Step 1, we have the assertions

$$
\begin{aligned}
\|w(t) - w(t_1)\|_{C^1(\overline{\Omega})} &\longrightarrow 0, \\
\|v(t) - v(t_1)\|_{C^1(\overline{\Omega})} &\longrightarrow 0,
\end{aligned}
$$

provided that

$$
|t - t_1| \longrightarrow 0 \quad \text{in } \boldsymbol{R}^n.
$$

Hence, since $u_1 = v(t_1) + w(t_1) > 0$ in Ω, we have, for $|t - t_1| < \varepsilon$,

$$
v(t) + w(t) > 0 \quad \text{in } \Omega,
$$

if ε is sufficiently small. In view of Remark 6.1, it follows that

$$F(u_1) < F(v(t) + w(t)) \quad \text{for } 0 < |t - t_1| < \varepsilon.$$

Therefore, we obtain from formula (7.28) that

$$\Psi(t_1) = F(u_1) < F(v(t) + w(t)) = \Psi(t) \quad \text{for } 0 < |t - t_1| < \varepsilon.$$

This proves that the point t_1 is a local minimum of Ψ on $\boldsymbol{R}^n$.

The proof of Proposition 7.3 is complete. $\square$

Step 3: The case where $\lambda > \lambda_2$ and $\lambda \neq \lambda_k$ for all $k \geq 3$. In order to apply Theorem 3.8, we show that $t = 0$ is a non-degenerate critical point of Ψ on $\boldsymbol{R}^n$ with Morse index $q_0 \geq 2$.

By differentiating in formula (7.27) and setting $t = 0$, we obtain that

$$\frac{\partial^2 \Psi}{\partial t_i \partial t_j}(0) = \lambda_j \delta_{ij} - \int_\Omega p'(w(0)) \phi_j(x) \left(\phi_i(x) + \frac{\partial w}{\partial t_i}(0) \right) dx, \quad i, j = 1, 2, \ldots, n, \tag{7.31}$$

where δ_{ij} is Kronecker's delta.

However, since we have the assertions

$$p'(w(0)) = p'(0) = \lambda,$$
$$\frac{\partial w}{\partial t_i}(0) \in W = V^\perp \cap Y,$$

it follows from formula (7.31) that

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial t_i \partial t_j}(0) &= \lambda_j \delta_{ij} - \lambda \int_\Omega \phi_i(x) \phi_j(x)\, dx \\ &= (\lambda_j - \lambda) \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \end{aligned} \tag{7.32}$$

Since $\lambda > \lambda_2$ and $\lambda \neq \lambda_k$ for all $k \geq 3$, we find from formula (7.32) that $t = 0$ is a non-degenerate critical point of Ψ with Morse index $q_0 \geq 2$.

Therefore, by using Proposition 7.3 and then applying Theorem 3.8 to Ψ we can find a non-zero point t_3 of $\boldsymbol{R}^n$, different from t_1 and t_2, such that $\nabla \Psi(t_3) = 0$. Then it follows that the function

$$u_3 = v(t_3) + w(t_3)$$

is a third non-trivial solution of the semilinear problem (1.3) different from $u_1 = v(t_1) + w(t_1)$ and $u_2 = v(t_2) + w(t_2)$.

Step 4: The case $\lambda = \lambda_k$ for some $k \geq 3$.

Assume, to the contrary, that the function $\Psi(t)$ has only t_1, t_2 and 0 as critical points. By Claim 7.1, we know that the function $\Psi(t)$ is bounded from below on $\boldsymbol{R}^n$ and satisfies (PS) condition. Hence, by applying Lemma 7.2 to Ψ with $x_0 := 0$ we can find a constant $\varepsilon > 0$ and a function $\widetilde{\Psi} \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$ such that:

(a) The function $\widetilde{\Psi}(t)$ is bounded from below on $\boldsymbol{R}^n$ and satisfies (PS) condition.
(b) The function $\widetilde{\Psi}(t)$ has only the two critical points t_1 and t_2 in the closed set $\{|t| \geq \varepsilon\}$.
(c) In the open ball $\{|t| < \varepsilon\}$, the function $\widetilde{\Psi}(t)$ has only a finite number of *non-degenerate* critical points, say $\beta_1, \beta_2, \ldots, \beta_\ell \in \boldsymbol{R}^n$ with finite Morse index $q_1, q_2, \ldots, q_\ell$, respectively.

By using inequality (7.11) with $f := \Psi$ and $g := \widetilde{\Psi}$ and formula (7.32), we have, for the Morse index q_j of β_j,

$$q_j \geq 2, \quad j = 1, 2, \ldots, \ell.$$

By arguing just as in the proof of Theorem 3.8, we get a contradiction (cf. inequality (3.7)).

Therefore, we can find a non-zero point t_3 of $\boldsymbol{R}^n$, different from t_1 and t_2, such that $\nabla\Psi(t_3) = 0$. Then it follows that the function

$$u_3 = v(t_3) + w(t_3)$$

is a third non-trivial solution of the semilinear problem (1.3) different from $u_1 = v(t_1) + w(t_1)$ and $u_2 = v(t_2) + w(t_2)$.

Now the proof of Theorem 1.1, Part 2 is complete. $\square$

8. Proof of Theorem 1.2.

This last section is devoted to the proof of Theorem 1.2. By virtue of Theorem 5.3, we have only to prove Theorem 1.2 for weak solutions. The proof of Theorem 1.2 is based on the Ljusternik–Schnirelman theory of critical points specialized to the case of an even functional on a Hilbert space (Theorem 4.2). The proof of Theorem 1.2 is divided into four steps.

Step 1: First, we recall (see formula (6.3) and inequality (6.6)) that the energy functional

$$F(u) = \frac{1}{2}(u, u)_{\mathcal{H}} - \int_\Omega P(u)\, dx = \frac{1}{2}\|u\|_{\mathcal{H}}^2 - \int_\Omega \int_0^{u(x)} p(t)\, dt\, dx$$

is bounded from below on $\mathcal{H}$:

$$F(u) \geq -\frac{1}{2}C_0^2 = -\frac{L^2|\Omega|}{2\lambda_1} \quad \text{for all } u \in \mathcal{H}.$$

This verifies condition (4.9) with $W := 0$ and $W^\perp := \mathcal{H}$.

Moreover, by Claim 6.4 it follows that $F(u)$ satisfies (PS) condition.

Step 2: Secondly, if $g(t)$ is an odd function of t, then we can construct the function

$p(t)$ as an odd function of t (see formula (6.1)). Hence we may assume that the energy functional $F(u)$ is an *even* function of u:

$$F(-u) = F(u) \quad \text{for all } u \in \mathcal{H}.$$

Step 3: If $\lambda > \lambda_k$, we let

$$V = \text{span}[\phi_1, \phi_2, \dots, \phi_k], \quad \dim V = k.$$

Let u be an arbitrary element of V such that

$$u = \sum_{j=1}^{k} c_j \phi_j, \quad c_j \in \boldsymbol{R},$$

$$\|u\|_{\mathcal{H}}^2 = \sum_{j=1}^{k} \lambda_j c_j^2 = \rho^2.$$

Then we have, for ρ sufficiently small,

$$\begin{aligned} F(u) &= \frac{1}{2}(u, u)_{\mathcal{H}} - \int_\Omega P(u)\, dx \\ &= \frac{1}{2}\|u\|_{\mathcal{H}}^2 - \int_\Omega \int_0^{u(x)} (\lambda t - g(t)) dt\, dx \\ &= \frac{1}{2}\|u\|_{\mathcal{H}}^2 - \frac{\lambda}{2}\int_\Omega u(x)^2\, dx + \int_\Omega \int_0^{u(x)} g(t)\, dt\, dx \\ &= \frac{1}{2}\left(\sum_{j=1}^{k} \lambda_j c_j^2 - \lambda \sum_{j=1}^{k} c_j^2\right) + \int_\Omega \int_0^{u(x)} g(t)\, dt\, dx \\ &\le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) \sum_{j=1}^{k} \lambda_j c_j^2 + \int_\Omega \int_0^{u(x)} g(t)\, dt\, dx \\ &= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\rho^2 + \int_\Omega \int_0^{u(x)} g(t)\, dt\, dx. \end{aligned} \tag{8.1}$$

Moreover, since we have the assertion

$$g(t) = o(t) \quad \text{as } t \to 0,$$

it follows that

$$\int_\Omega \int_0^{u(x)} g(t)\, dt\, dx = o(\|u\|_{\mathcal{H}}^2) = o(\rho^2) \quad \text{as } \rho \to 0. \tag{8.2}$$

Indeed, it suffices to note that all norms on the finite-dimensional space V are equivalent.

Step 4: By combining inequality (8.1) and assertion (8.2), we obtain that

$$F(u) \leq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_k}\right)\rho^2 + o(\rho^2) \quad \text{as } u \in V \text{ and } \|u\|_{\mathcal{H}} = \rho \to 0. \tag{8.3}$$

However, we remark that

$$1-\frac{\lambda}{\lambda_k} < 0 \quad \text{for } \lambda > \lambda_k.$$

In view of assertion (8.3), we can find a small constant $\rho > 0$ such that

$$\sup_{u \in V \cap S_\rho(0)} F(u) \leq \frac{1}{4}\left(1-\frac{\lambda}{\lambda_k}\right)\rho^2.$$

This verifies condition (4.8) with $V := V$ and

$$b := \frac{1}{4}\left(1-\frac{\lambda}{\lambda_k}\right)\rho^2.$$

Therefore, Theorem 1.2 follows by applying Theorem 4.2 with

$$H := \mathcal{H}, \quad f := F, \quad V := V, \quad W := 0, \quad W^\perp := \mathcal{H},$$
$$m := k, \quad j := 0, \quad b := \frac{1}{4}\left(1-\frac{\lambda}{\lambda_k}\right)\rho^2, \quad a := -\frac{1}{2}C_0^2 + b.$$

The proof of Theorem 1.2 is now complete. $\square$

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