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Functional distribution for a collection of Lerch zeta functions

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Abstract. Let $0 < \alpha < 1$ be a transcendental real number and $\lambda_1, \dots, \lambda_r$ be real numbers with $0 \le \lambda_j < 1$. It is conjectured that a joint universality theorem for a collection of Lerch zeta functions $\{L(\lambda_j, \alpha, s)\}$ will hold for every numbers λ_j's which are different each other. In this paper we will prove that the joint universality theorem for the set $\{L(\lambda_j, \alpha, s)\}$ holds for almost all real numbers λ_j's.

1. Introduction.

Let α be a real number with $0 < \alpha \le 1$ and λ be a real number. The Lerch zeta function $L(\lambda, \alpha, s)$ is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e(\lambda m)}{(m+\alpha)^s}, \tag{1.1}$$

for $\sigma = \Re s > 1$, where $e(x) = e^{2\pi i x}$. If $\lambda - \lambda' \in \mathbb{Z}$, then $L(\lambda, \alpha, s) = L(\lambda', \alpha, s)$. Therefore, in the following we assume that $0 \le \lambda < 1$.

The Lerch zeta function is one of the classical and popular objects in analytic number theory, since it was introduced by Lerch [**6**]. Many analytic properties of Lerch zeta functions have been established by several mathematicians (see, for instance, Laurinčikas and Garunkstis [**4**]). Here, we are concerned with universality property. A function $f(s)$ is said to have universality property on a region U, if arbitrary analytic function can be uniformly approximated on compact subsets of U by vertical translation of f. Such universality property for Lerch zeta functions was established by Laurinčikas [**3**]. To state it, we prepare some symbols. Let μ be the Lebesgue measure on the set $\mathbb{R}$ of all real numbers. For $T > 0$ define

$$\nu_T(\cdots) = \frac{1}{T}\mu\{\tau \in [0, T] : \cdots\},$$

where in place of dots we write some conditions satisfied by a real number τ. Let D denote a strip $\{s \in \mathbb{C} \mid 1/2 < \sigma < 1\}$. Now we state the universality theorem for the Lerch zeta function.

Theorem 1 (Laurinčikas [**3**]). *Assume that $0 < \alpha < 1$ be a real transcendental number. Let λ be a real number with $0 \le \lambda < 1$. Let K be a compact subset of D with*

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connected complement and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \nu_T\Big(\max_{s\in K}|L(\lambda,\alpha,s+i\tau)-f(s)|<\varepsilon\Big)>0.$$

In this paper, we treat value distribution of a collection of Lerch zeta functions

$$\{L(\lambda_j,\alpha,s) \mid 1\le j\le r\},$$

when α is a fixed transcendental real number and λ_j's are real numbers. The first result in this direction was obtained by Laurinčikas and Matsumoto [**5**].

Theorem 2 (Laurinčikas and Matsumoto [**5**]). *Let α be a transcendental real number. Let $\lambda_j = (a_j/q_j)$ $(1\le j\le r)$ be distinct rational numbers which satisfy $(a_j,q_j)=1$ and $0\le a_j<q_j$. For each $1\le j\le r$, let K_j be a compact subset in the strip D with connected complement and $h_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then for any small positive number ε we have*

$$\liminf_{T\to\infty}\frac{1}{T}\mu\Big\{\tau\in[0,T]\ \Big|\ \max_{1\le j\le r}\max_{s\in K_j}|L(\lambda_j,\alpha,s+i\tau)-h_j(s)|<\varepsilon\Big\}>0.$$

The above inequality implies that for a collection of Lerch zeta functions the corresponding universality properties hold simultaneously. We call this type of property for a collection of zeta functions *joint universality.*

Here we give a conjecture which asserts that the joint universality theorem for the set $\{L(\lambda_j,\alpha,s)\}$ will hold without the restriction that *λ_j's are rational numbers.* As a consequence of Theorem 1, Laurinčikas [**3**] showed that for $1/2<\sigma_0\le 1$ and a positive integer N the set

$$\big\{\big(L(\lambda,\alpha,\sigma_0+it),\dots,L^{(N-1)}(\lambda,\alpha,\sigma_0+it)\big)\in\mathbb{C}^N\ \big|\ t\in\mathbb{R}\big\}$$

is dense in $\mathbb{C}^N$. Conversely, the universality theorem for $L(\lambda,\alpha,s)$ is interpreted as an extension of the above multi-dimensional denseness result for $\zeta(s)$ to the functional space. Recently, Nagoshi [**8**] obtained the similar multi-dimensional denseness result for the collection of Lerch zeta functions.

Theorem 3. *Let $0<\alpha<1$ be a transcendental real number. Let $\lambda_1,\dots,\lambda_r$ be distinct real numbers with $0\le\lambda_j<1$. Fix a real number σ_0 with $1/2<\sigma_0\le 1$ and a positive integer N. Then the set*

$$\begin{aligned}\big\{\big(&L(\lambda_1,\alpha,\sigma_0+it),\dots,L^{(N-1)}(\lambda_1,\alpha,\sigma_0+it),\\ &\dots,L(\lambda_r,\alpha,\sigma_0+it),\dots,L^{(N-1)}(\lambda_r,\alpha,\sigma_0+it)\big)\in\mathbb{C}^{rN}\ \big|\ t\in\mathbb{R}\big\}\end{aligned}$$

is dense in $\mathbb{C}^{rN}$.

From Theorem 3, we predict the following conjecture.

Conjecture 1. *Let $0 < \alpha < 1$ be a transcendental real number. Let $\lambda_1, \dots, \lambda_r$ be distinct real numbers with $0 \le \lambda_j < 1$. Then the joint universality theorem holds for the set of Lerch zeta functions $\{L(\lambda_j, \alpha, s) \mid 1 \le j \le r\}$ on the strip D.*

The purpose of this paper is to give results which assure this conjecture. The first main theorem asserts that the joint universality for the set $\{L(\lambda_j, \alpha, s)\}$ holds for almost all real numbers λ_j's.

Theorem 4. *There exists a subset Λ of $[0,1)^r$ which has the following properties.*

1. *The set Λ contains almost all real numbers λ_j's such that $1, \lambda_1, \dots, \lambda_r$ are linearly independent over $\mathbb{Q}$. Namely,*

$$\mu_r(\Lambda) = 1$$

   where μ_r is the Lebesgue measure on $\mathbb{R}^r$.
2. *Let $0 < \alpha < 1$ be a transcendental real number. Assume that $(\lambda_1, \dots, \lambda_r) \in \Lambda$. For each $1 \le j \le r$, let K_j be a compact subset in the strip D with connected complement and $h_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then for any small positive number ε we have*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\Big\{\tau \in [0,T] \;\Big|\; \max_{1\le j\le r} \max_{s\in K_j} |L(\lambda_j, \alpha, s+i\tau) - h_j(s)| < \varepsilon\Big\} > 0.$$

The definition of the set Λ will be given in the next section. Also we will give the explicit example of λ_j's for which the joint universality of Lerch zeta functions holds.

Theorem 5. *Let $0 < \alpha < 1$ be a transcendental real number. Assume that real numbers $\lambda_1, \dots, \lambda_r$ with $0 \le \lambda_j < 1$ satisfy one of the following conditions*:

1. *$\lambda_1, \dots, \lambda_r$ are algebraic irrational numbers such that $1, \lambda_1, \dots, \lambda_r$ are linearly independent over $\mathbb{Q}$.*
2. *$\lambda_j = e^{r_j}$, where r_j's are distinct rational numbers.*

For each $1 \le j \le r$, let K_j be a compact subset in the strip D with connected complement and $h_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then for any small positive number ε we have

$$\liminf_{T\to\infty} \frac{1}{T}\mu\Big\{\tau \in [0,T] \;\Big|\; \max_{1\le j\le r} \max_{s\in K_j} |L(\lambda_j, \alpha, s+i\tau) - h_j(s)| < \varepsilon\Big\} > 0.$$

The construction of the paper is as follows. In Section 2 we quote definitions and results on discrepancy estimate in uniform distribution theory. In Section 3 we prepare some lemmas for the proof of the theorems. In Section 4 we give the proof of Theorem 3 which differs from Nagoshi's original proof. We prove Theorem 4 and 5 in Section 5–7.

2. Discrepancy.

To prove our theorems, we need to know the behavior of the sequence $\{(e(\lambda_1 m), \ldots, e(\lambda_r m)) \mid m \geq 0\}$. For the purpose, we quote definitions and results in uniform distribution theory.

For an r-tuple $(\lambda_1, \ldots, \lambda_r)$ of real numbers, the Kronecker sequence $\{\omega(n) \mid n \in \mathbb{N}\}$ is defined by

$$\omega(n) = (\{n\lambda_1\}, \{n\lambda_2\}, \ldots, \{n\lambda_r\}) \in [0,1)^r$$

for $n \geq 1$, where $\{x\}$ denotes the fractional part of the real number x. The next lemma is the well-known Kronecker approximation theorem.

Lemma 1. *Suppose that the numbers $1, \lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{Q}$. Then the Kronecker sequence $\{\omega(n)\}$ is uniformly distributed in $[0,1)^r$. Namely, for an interval $I = [a_1, b_1) \times \cdots \times [a_r, b_r) \subset [0,1)^r$ and a positive integer N define*

$$A_N(I) = \sharp\{1 \leq n \leq N \mid \omega(n) \in I\}.$$

Then for any interval I,

$$\lim_{N\to\infty} \frac{A_N(I)}{N} = \mathrm{vol}(I) = \prod_{j=1}^{r} (b_j - a_j). \tag{2.1}$$

Now we define the discrepancy of the sequence $\{\omega(n)\}$ by

$$D_N = D_N(\lambda_1, \ldots, \lambda_r) = \sum_{I \subset [0,1)^r} \left| \frac{A_N(I)}{N} - \mathrm{vol}(I) \right|.$$

Then (2.1) means that

$$D_N = o(1), \quad \text{as} \quad N \to \infty. \tag{2.2}$$

Estimate (2.2) is useful enough to prove Theorem 3 only. To prove the joint universality theorems, however, we need a more precise estimate for the upper bound of the discrepancy D_N. Here we quote two classical results from uniform distribution theory omitting the proof. The next lemma is due to Schmidt [**10**].

Lemma 2. *For almost all r-tuples $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$, the discrepancy $D_N = D_N(\lambda_1, \ldots, \lambda_r)$ satisfies*

$$D_N = O\left(\frac{(\log N)^{r+1+\varepsilon}}{N} \right), \quad \textit{as} \quad N \to \infty, \tag{2.3}$$

for every $\varepsilon > 0$.

Now we put

$$\Lambda = \left\{(\lambda_1, \ldots, \lambda_r) \in [0,1)^r \;\middle|\; \text{estimate (2.3) holds for } D_N(\lambda_1, \ldots, \lambda_r).\right\}.$$

Then Lemma 2 yields that $\mu_r(\Lambda) = 1$.

The following lemma is easily deduced from Theorem 6.1 of Niederreiter [**9**].

LEMMA 3. *Assume that real numbers $\lambda_1, \ldots, \lambda_r$ with $0 < \lambda_j < 1$ satisfy one of the following conditions*:

1. *$\lambda_1, \ldots, \lambda_r$ are algebraic irrational numbers such that $1, \lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{Q}$.*
2. *$\lambda_j = e^{r_j}$, where r_j's are distinct rational numbers.*

Then the discrepancy $D_N = D_N(\lambda_1, \ldots, \lambda_r)$ satisfies

$$D_N = O(N^{-1+\varepsilon}), \quad \text{as} \quad N \to \infty,$$

for every $\varepsilon > 0$.

Combining the above lemmas, we obtain the next approximation formula.

LEMMA 4. *Assume that real numbers $\lambda_1, \ldots, \lambda_r$ with $0 < \lambda_j < 1$ satisfy one of the following conditions*:

1. *$\lambda_1, \ldots, \lambda_r \in \Lambda$, where the set Λ is given by (2.3).*
2. *$\lambda_1, \ldots, \lambda_r$ are algebraic irrational numbers such that $1, \lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{Q}$.*
3. *$\lambda_j = e^{r_j}$, where r_j's are distinct rational numbers.*

Then for any interval I,

$$A_N(I) = \mathrm{vol}(I)N + O(N^{\varepsilon}), \tag{2.4}$$

for every $\varepsilon > 0$.

3. Preliminaries.

In this section we will prepare some lemmas which we need to prove Theorems 3, 4, and 5.

LEMMA 5. *Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. Assume that a sequence $u_n \in H$ $(n \geq 1)$ satisfying*

(I) *The series*

$$\sum_n \|u_n\|^2$$

is convergent.

(II) *For any non-zero element* $e \in H$, *the series*

$$\sum_n |\langle u_n, e\rangle|$$

is divergent.

Then the set of convergent series

$$\left\{ \sum_n a_n u_n \in H \,\middle|\, |a_n| = 1 \right\}$$

is dense in H.

Proof. This is Theorem 6.1.16 in [**2**]. □

Lemma 6. *Let* $0 < \alpha \leq 1$ *and* $0 \leq \lambda < 1$. *For* $x > 0$

$$L(\lambda, \alpha, s) = \sum_{0 \leq m \leq x} \frac{e(\lambda m)}{(m+\alpha)^s} + \delta_\lambda \frac{x^{1-s}}{s-1} + O_\lambda(x^{-\sigma}),$$

holds uniformly in the region $0 < \sigma_1 \leq \sigma \leq 2$, $2\pi \leq |t| \leq \pi x$, *where*

$$\delta_\lambda = \begin{cases} 1 & (\lambda = 0), \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. This is a combination of Theorem 3.2.1 in Karatsuba and Voronin [**13**] and Theorem 3.1.2 in [**4**]. □

Lemma 7. *Let* C *and* C' *be compact subsets in* $\mathbb{C}$ *such that* C *is contained in the interior of* C'. *There exists a positive constant* $a(C, C')$ *with the following property*:

If an analytic function $f(s)$ *on* C' *satisfies the estimate*

$$\iint_{C'} |f(s)|^2 d\sigma dt < A,$$

for $A > 0$, *then*

$$\max_{s \in C} |f(s)| < a(C, C')\sqrt{A}.$$

Proof. This is Lemma 2.5 in the author and Nagoshi [**7**]. □

Lemma 8. *Let* U *be a simply connected bounded region which is included in the strip* $\sigma_1 < \Re s < \sigma_2$. *Let* $h(s)$ *be a non-zero analytic function on* U. *Define*

$$\Delta_h(z) = \iint_U e^{-sz}\overline{h(s)}d\sigma dt.$$

Then $\Delta_h(z)$ is entire, and satisfies the following properties.

(I) *The function $\Delta_h(z)$ has the series expansion*

$$\Delta_h(z) = \sum_{m=0}^{\infty} \frac{\alpha_h(m)}{m!} z^m,$$

where

$$|\alpha_h(m)| \leq 1 \quad \textit{for all } m \geq 1.$$

(II) *There exists a divergent positive sequence $R_n \to \infty$ $(n \to \infty)$ and a sequence of intervals $I_n = [x_n, x_n + y_n] \subset [R_n - 1, R_n + 1]$ such that*

$$|\Delta_h(x)| \geq \frac{1}{4} e^{-\sigma_2 x_n} \quad (x \in I_n),$$

and such that

$$y_n \sim R_n^{-8}.$$

Proof. This is essentially established in the proof of Lemma 7.1 of [**13**]. □

The next two lemmas are elementary inequalities for complex numbers.

Lemma 9. *Let $z_1, \ldots, z_r$ be complex numbers. If all real parts of z_j have the same sign, then*

$$|z_1 + \cdots + z_r| \geq |\Re z_1|.$$

Lemma 10. *For each $1 \leq k \leq 4$, define intervals*

$$A_k = \left[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}\right), \quad B_k = \left[\frac{13-3k}{12}, \frac{14-3k}{12}\right). \tag{3.1}$$

Suppose that a non-zero complex number z satisfies $\arg z \in A_k$ and that a real number t satisfies $t \in B_k$ for the same k. Then

$$\Re(e(t)z) \geq \frac{1}{2}|z| > 0.$$

Proof. Put $z = re^{i\theta}$. If $\theta \in A_k$ and $t \in B_k$ for the same k, then $\theta + 2\pi t \in [(5/3)\pi, (7/3)\pi)$. Therefore

$$\Re(e(t)z) = r\cos(\theta + 2\pi t) \geq r\cos\frac{\pi}{3} = \frac{1}{2}r. \qquad \square$$

The next lemma is classical Jensen's formula.

Lemma 11. *Let $f(z)$ be an analytic function on the disc $|z| < R$ satisfying $f(0) \neq 0$. Let ρ_k $(k \geq 1)$ be zeros of $f(z)$ in $|z| < R$ and $r_k = |\rho_k|$. Assume that $r_1 \geq r_2 \geq \cdots$. Let $0 < r < R$ satisfying $r_n \leq r \leq r_{n+1}$. Then*

$$\log\frac{r^n|f(0)|}{r_1 \ldots r_n} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta.$$

Proof. This is Theorem 3.61 in Titchmarsh [**12**]. $\square$

Lemma 12. *Let $G(z)$ be an analytic function satisfying*

$$G(z) = \sum_{m=0}^{\infty} \frac{\alpha_m}{m!}z^m, \quad |\alpha_m| \ll 1, \quad \text{and} \quad G \not\equiv 0.$$

Let $\delta = \delta(R)$ be a positive valued function satisfying $\delta \to 0$ as $R \to \infty$. There exists a positive constant $C' = C'(G)$ such that for any sufficiently large number R the interval $I(R) = [R, R+\delta]$ contains a subinterval $J'(R)$ with length $C'\delta^2 R^{-2}$ such that $G(x)$ has no zeros on $J'(R)$.

Proof. Assume that $G(z)$ has a zero at $z = 0$ with multiplicity g. Put $G(z) = z^g G_1(z)$. If the function $G_1(z)$ has no zeros on some interval included in $I(R)$, then $G(z)$ also has no zeros on this interval. Therefore we may only consider the case that $G(0) \neq 0$. Let a_i $(1 \leq i \leq n)$ be different zeros of $G(x)$ on the interval $I(R)$ and n_i be order of a_i. Let b_j $(1 \leq j \leq m)$ be other zeros of $G(x)$ in the disc $|z| < R + \delta$. Then Lemma 11 implies that

$$\begin{aligned}\log|G(0)| &+ \sum_{i=1}^{n} n_i \log\frac{R+\delta}{a_i} + \sum_{j=1}^{m}\log\frac{R+\delta}{|b_j|} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\bigl|G((R+\delta)e^{i\theta})\bigr|d\theta. \end{aligned} \tag{3.2}$$

From the assumption we have $|G(x)| \ll e^x$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}\log\bigl|G((R+\delta)e^{i\theta})\bigr|d\theta = O(R), \quad \text{and} \quad \log|G(0)| = O(1).$$

Remark that the other terms in the left hand side of (3.2) are positive. Therefore

$$\sum_{i=1}^{n}\log\frac{R+\delta}{a_i} = O(R). \tag{3.3}$$

Let N be a positive integer. We divide the interval $I(R)$ into N-subintervals

$$J_k = \left[R + \frac{k-1}{N}\delta, R + \frac{k}{N}\delta\right] \quad (1 \leq k \leq N).$$

Now we estimate the upper bound of N such that all sub intervals J_k contain at least one zero a_i. If a_i is contained in J_k, then $a_i \leq R + (k/N)\delta$. Therefore

$$\frac{R+\delta}{a_i} \geq \left(1 - \frac{(N-k)\delta}{N(R+\delta)}\right)^{-1}.$$

Put $l = N - k$. From (3.3) it follows

$$\sum_{l=1}^{N-1} \log\left(1 - \frac{l\delta}{N(R+\delta)}\right)^{-1} = O(R).$$

Applying an elementary inequality $-\log(1-x) > x$ for $0 < x < 1$, we have

$$\sum_{l=1}^{N-1} \frac{l\delta}{N(R+\delta)} = O(R).$$

Then

$$N = O\left(\frac{R^2}{\delta}\right).$$

Therefore there exists a positive constant C_1' such that if $N \geq C_1' R^2 \delta^{-1}$ then there is at least one sub interval J_k which contains no zeros of $G(x)$. The length of this interval is $\delta N^{-1} = C_1'^{-1} \delta^2 R^{-2}$. Putting $C' = C_1'^{-1}$, we have the lemma. □

Lemma 13. *Let $G(z)$ and $\delta = \delta(R)$ be functions as in Lemma 12. Let $A_k = [(k-1)\pi/2, k\pi/2)$ $(1 \leq k \leq 4)$ be intervals given by (3.1). For any sufficiently large number R, there exists a subinterval $J(R)$ of the interval $I(R) = [R, R+\delta]$ and a integer k_R with $1 \leq k_R \leq 4$ satisfying the following properties*:

1. *The length of $J(R)$ is $C\delta^4 R^{-6}$, where $C = C(G)$ is a positive constant depends only on G.*
2. *We have*

$$\arg G(x) \in A_{k_R}$$

for all $x \in J(R)$.

Proof. Put

$$\beta_m = \frac{\alpha_m + \overline{\alpha_m}}{2}, \quad G_1(z) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} z^m,$$

and

$$\gamma_m = \frac{\alpha_m - \overline{\alpha_m}}{2i}, \quad G_2(z) = \sum_{m=0}^{\infty} \frac{\gamma_m}{m!} z^m.$$

Then $G_1(z)$ and $G_2(z)$ satisfy the same condition in Lemma 12 and

$$G_1(x) = \Re G(x), \quad \text{and} \quad G_2(x) = \Im G(x) \quad \text{for} \quad x \in \mathbb{R}.$$

By Lemma 12, there exists a sub interval $J'(R)$ in $I(R)$ such that $G_1(x)$ has no zeros on $J'(R)$. The length of $J'(R)$ is $\delta' = \delta'(R) = C'\delta^2 R^{-2}$. For functions $G_2(x)$, δ' and the interval $J'(R)$ we apply Lemma 12 again. Then there exists a sub interval $J(R)$ in $J'(R)$ such that $G_2(x)$ has no zeros on $J(R)$. In particular, the argument of $G(x)$ belongs to one of A_k for all $x \in J(R)$. The length of $J(R)$ is

$$C'' \frac{\delta'^2}{R^2} = (C''C'^2) \frac{\delta^4}{R^6}.$$

Putting $C = C''C'^2$, we complete the proof of the lemma. $\square$

4. Proof of Theorem 3.

Let $\lambda_1, \ldots, \lambda_r$ be distinct real numbers with $0 \le \lambda_j < 1$ and σ_0 be real numbers with $1/2 < \sigma_0 \le 1$. For $m \ge 0$ we put

$$\boldsymbol{F}_m = \left(\frac{e(\lambda_1 m)}{(m+\alpha)^{\sigma_0}}, \ldots, \frac{e(\lambda_1 m)(-\log(m+\alpha))^{N-1}}{(m+\alpha)^{\sigma_0}}, \right.$$
$$\left. \ldots, \frac{e(\lambda_r m)}{(m+\alpha)^{\sigma_0}}, \ldots, \frac{e(\lambda_r m)(-\log(m+\alpha))^{N-1}}{(m+\alpha)^{\sigma_0}} \right) \in \mathbb{C}^{rN}.$$

First we prove the next lemma.

Lemma 14. *The set of convergent series*

$$\left\{ \sum_{m=0}^{\infty} \varepsilon_m \boldsymbol{F}_m \in \mathbb{C}^{rN} \;\middle|\; |\varepsilon_m| = 1 \right\}$$

is dense in $\mathbb{C}^{rN}$.

Proof. We will check that the sequence $\{\boldsymbol{F}_m\}$ in the complex Hilbert space $H = \mathbb{C}^{rN}$ satisfies condition (I) and (II) in Lemma 5. Since $\sigma_0 > 1/2$,

$$\sum_{m=0}^{\infty} \|\boldsymbol{F}_m\|^2 = \sum_{m=0}^{\infty} \frac{r}{(m+\alpha)^{2\sigma_0}} \left\{1 + \log^2(m+\alpha) + \cdots + \log^{N-1}(m+\alpha)\right\} < \infty.$$

Thus condition (I) is valid. Let $\boldsymbol{u} = (u_{10}, \ldots, u_{r0}, \ldots, u_{1N-1}, \ldots, u_{rN-1})$ be any non-zero element of $\mathbb{C}^{rN}$. Put

$$N_0 = \max\{0 \leq n \leq N-1 \mid u_{jn} \neq 0 \text{ for some } j\}.$$

Then we have

$$\langle \boldsymbol{F}_m, \boldsymbol{u} \rangle = \sum_{j=1}^{r} \frac{e(\lambda_j m)(-\log(m+\alpha))^{N_0} \overline{u_{jN_0}}}{(m+\alpha)^{\sigma_0}} + \sum_{n=0}^{N_0-1} \sum_{j=1}^{r} \frac{e(\lambda_j m)(-\log(m+\alpha))^{n} \overline{u_{jn}}}{(m+\alpha)^{\sigma_0}}.$$

There exists a positive integer $M_0 = M_0(\sigma_0, \boldsymbol{u})$ such that for any $m > M_0$

$$|\langle \boldsymbol{F}_m, \boldsymbol{u} \rangle| \geq \frac{1}{2} \frac{\log^{N_0}(m+\alpha)}{(m+\alpha)^{\sigma_0}} \cdot \left| \sum_{j=1}^{r} e(\lambda_j m) \overline{u_{jN_0}} \right|. \tag{4.1}$$

Put $a(m) = \sum_{j=1}^{r} e(m\lambda_j)\overline{u_{jN_0}}$. Now we show that for sufficiently large M there exist a set A_M of integers $M \leq m < 2M$ and positive constants c_1 and c_2 such that

$$|a(m)| \geq c_1 \tag{4.2}$$

holds for all $m \in A_M$, and that

$$\lim_{M \to \infty} \frac{\sharp A_M}{M} > c_2. \tag{4.3}$$

First we consider the case that all λ_j's are rational numbers a_j/q_j with $(a_j, q_j) = 1$ and $0 \leq a_j < q_j$. Let Q be the least common multiple of q_j's. Substituting $m = 0, 1, \ldots, Q-1$ into a vector $(e(\lambda_1 m), \ldots, e(\lambda_r m))$, we obtain at least r distinct vectors. If $a(m)$ becomes zero for all these vectors, then $u_{jN_0} \equiv 0$ for all $1 \leq j \leq r$, which contradicts to the definition of N_0. Therefore there exists some m_0 such that $a_{m_0} \neq 0$. Put

$$A_M = \{M \leq m < 2M \mid m \equiv m_0 \pmod{Q}\}.$$

Then $a_m = a_{m_0}$ for all $m \in A_M$. Needless to say, the set A_M has a positive density.

Next we consider the general case. Assume that $1, \lambda_1, \ldots, \lambda_{r_1}$ are linearly independent over $\mathbb{Q}$, and that $\lambda_{r_1+1}, \ldots, \lambda_{r_1+r_2}$ $(r = r_1 + r_2)$ belong to the set $\mathbb{Q}[1, \lambda_1, \ldots, \lambda_{r_1}]$. Namely,

$$\lambda_{r_1+k} = \frac{b_{0k}}{c_{0k}} + \frac{b_{1k}}{c_{1k}}\lambda_1 + \cdots + \frac{b_{r_1k}}{c_{r_1k}}\lambda_{r_1} \quad \begin{pmatrix} b_{jk}, c_{jk} \in \mathbb{Z} \\ (b_{jk}, c_{jk}) = 1 \end{pmatrix}$$

for $1 \le k \le r_2$. Let c be the least common multiple of c_{jk} and $B_{jk} = (c/c_{jk})b_{jk}$. Then

$$c\lambda_{r_1+k} = B_{0k} + B_{1k}\lambda_1 + \cdots + B_{r_1k}\lambda_{r_1} \quad (B_{jk} \in \mathbb{Z})$$

for $1 \le k \le r_2$. Define a function $F : \mathbb{R}^{r_1} \to \mathbb{C}$ by

$$F(t_1, \ldots, t_{r_1}) = \sum_{j=1}^{r_1} \overline{u_{jN_0}} e(t_j)^c + \sum_{k=1}^{r_2} \overline{u_{r_1+kN_0}} \prod_{j=1}^{r_1} e(t_j)^{B_{jk}}.$$

Then we have

$$F(cm\lambda_1, \ldots, cm\lambda_{r_1}) = a(cm) \quad \text{for} \quad m \ge 0.$$

Since F is a continuous function which is not identically zero, there exists a positive constant δ and an interval $I = [a_1, b_1) \times \cdots \times [a_{r_1}, b_{r_1}) \subset [0, 1)^{r_1}$ such that

$$|F(t_1, \ldots, t_{r_1})| > \delta \quad \text{for} \quad (t_1, \ldots, t_{r_1}) \in I.$$

Define

$$A_M = \{M \le m < 2M \mid (\{cm\lambda_1\}, \ldots, \{cm\lambda_{r_1}\}) \in I\}$$

then $|a(cm)| > \delta$ holds for any $m \in A_M$. By Lemma 1,

$$\lim_{M\to\infty} \frac{\sharp A_M}{M} = \frac{\mathrm{vol}(I)}{c} > 0.$$

Thus (4.2) and (4.3) holds for the set A_M.

From (4.1)–(4.3), we have

$$\sum_{m\in A_M} |\langle \boldsymbol{F}_m, \boldsymbol{u}\rangle| \gg \frac{\log^{N_0}(m+\alpha)}{(m+\alpha)^{\sigma_0}} \sharp A_M \gg M^{1-\sigma_0+\varepsilon}.$$

Since $1/2 < \sigma_0 \le 1$, the above sum diverges as $M \to \infty$. This completes the proof of the lemma. $\square$

Now we prove Theorem 3. For any rN-tuple $(z_{10}, \ldots, z_{1N-1}, \ldots, z_{r0}, \ldots, z_{rN-1}) \in \mathbb{C}^{rN}$ and every $\varepsilon > 0$ we will prove that there exists a real number t for which

$$\max_{1\le j\le r} \max_{0\le n\le N-1} \left| L^{(n)}(\lambda_j, \alpha, \sigma_0 + it) - z_{jn} \right| < \varepsilon. \tag{4.4}$$

From Lemma 6 and Cauchy's integral formula, for $0 \le n \le N-1$,

$$L^{(n)}(\lambda, \alpha, s) = \sum_{0\le m\le x} \left(\frac{e(\lambda m)}{(m+\alpha)^s} \right)^{(n)} + \delta_\lambda \left(\frac{x^{1-s}}{s-1} \right)^{(n)} + \left(O_\lambda(x^{-\sigma}) \right)^{(n)}$$

holds uniformly in the same region as in Lemma 6. Let T be a sufficiently large number and put $x = T$. Then we have

$$\left| L^{(n)}(\lambda_j, \alpha, \sigma_0 + it) - \sum_{0 \le m \le T} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0+it}} \right| < \frac{\varepsilon}{3}, \tag{4.5}$$

for any $T \le t < 2T$, $1 \le j \le r$ and $0 \le n \le N-1$.

Let M be a positive integer smaller than T. By the Montgomery-Vaughan estimate (Theorem 2.7.2 in [**2**]), we obtain the second mean estimate

$$\int_T^{2T} \left| \sum_{M < m \le T} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0+it}} \right|^2 dt \ll TM^{1-2\sigma_0} + T^{2-2\sigma_0+\varepsilon}.$$

Since $1 - 2\sigma_0 < 0$, there exists a positive integer M_0 such that for all $M \ge M_0$ and all sufficiently large T we have

$$\int_T^{2T} \left| \sum_{M < m \le T} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0+it}} \right|^2 dt < \frac{\varepsilon^3}{3}. \tag{4.6}$$

Now we define a set B_T of real numbers $t \in [T, 2T)$ for which

$$\left| \sum_{M < m \le T} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0+it}} \right| < \frac{\varepsilon}{3} \tag{4.7}$$

holds. Then (4.6) yields that

$$\liminf_{T \to \infty} \frac{\mu(B_T)}{T} > 1 - \varepsilon. \tag{4.8}$$

By Lemma 14, there exists a sequence $\{\varepsilon_m \in S^1 \mid m \ge 0\}$ such that

$$\sum_{m=0}^{\infty} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0}} \varepsilon_m = z_{jn}$$

holds for $1 \le j \le r$ and $0 \le n \le N-1$. Fix a positive integer $M > M_0$ for which

$$\left| \sum_{m=0}^{M} \frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0}} \varepsilon_m - z_{jn} \right| < \frac{\varepsilon}{6}$$

hold. For a positive number δ we set

$$I_\delta(M) = \left[\frac{\arg \varepsilon_0}{2\pi} - \delta, \frac{\arg \varepsilon_0}{2\pi} + \delta \right) \times \cdots \times \left[\frac{\arg \varepsilon_M}{2\pi} - \delta, \frac{\arg \varepsilon_M}{2\pi} + \delta \right) \subset [0,1)^{M+1}.$$

If δ is sufficiently small, then

$$\left|\sum_{m=0}^{M}\frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0}}e(t_m)-z_{jn}\right|<\frac{\varepsilon}{3},$$

holds for any $(t_0,\dots,t_M)\in I_\delta(M)$. Now we define

$$A_T=\left\{t\in[T,2T)\;\middle|\;\left(\left\{-t\frac{\log\alpha}{2\pi}\right\},\dots,\left\{-t\frac{\log(M+\alpha)}{2\pi}\right\}\right)\in I_\delta(M)\right\}$$

Then for any $t\in A_T$,

$$\left|\sum_{m=0}^{M}\frac{(-\log(m+\alpha))^n e(\lambda_j m)}{(m+\alpha)^{\sigma_0+it}}-z_{jn}\right|<\frac{\varepsilon}{3}. \tag{4.9}$$

Since α is a transcendental number, the numbers $1,\log\alpha,\dots,\log(M+\alpha)$ are linearly independent over $\mathbb{Q}$. Therefore, by Kronecker's approximation theorem,

$$\lim_{T\to\infty}\frac{\mu(A_T)}{T}=(2\delta)^{M+1}.$$

This and (4.8) imply that the intersection $A_T\cap B_T$ has a positive lower density. For any $t\in A_T\cap B_T$ estimates (4.5), (4.7) and (4.9) hold. Combining these estimates, we obtain (4.4). This completes the proof of the theorem.

5. A joint limit theorem for Lerch zeta functions.

Nowadays, the joint universality for a collection of zeta functions is mainly obtained as an application of the joint limit theorem on the weak convergence of probability measure associated with the set of zeta functions. This method was given by Bagchi [**1**]. To describe Bagchi's probabilistic method, we define some notations.

Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. Let $H^r(D)=H(D)\times\cdots\times H(D)$ be the product space. For a topological space S, let $\mathcal{B}(S)$ denote the family of Borel subsets of S. Assume that $0<\alpha<1$ is a transcendental real number and that $\lambda_1,\dots,\lambda_r$ are real numbers with $0\le\lambda_j<1$. For $T>0$ we define a probability measure P_T on $(H^r(D),\mathcal{B}(H^r(D)))$ by

$$P_T(A)=\nu_T\big((L(\lambda_1,\alpha,s+i\tau),\dots,L(\lambda_r,\alpha,s+i\tau))\in A\big)$$

for $A\in\mathcal{B}(H^r(D))$. Laurinčikas and Matsumoto [**5**] obtained the following joint limit theorem.

Proposition 1. *The probability measure P_T on $(H^r(D),\mathcal{B}(H^r(D)))$ converges weakly to a certain limit measure P on $(H^r(D),\mathcal{B}(H^r(D)))$ as T tends to infinity.*

The limit measure P is given as follows. Let γ be the unit circle $\{s \in \mathbb{C} \mid |s| = 1\}$ and

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for each $m \geq 0$. With the product topology and pointwise multiplication Ω is a compact Abelian group. Let m_H be the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. For $\omega = \{\omega(m) \mid m \geq 0\} \in \Omega$ define

$$L(\lambda_j, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e(\lambda_j m)\omega(m)}{(m+\alpha)^s}. \tag{5.1}$$

For almost all $\omega \in \Omega$ the series (5.1) converges uniformly on compact subsets of D. Therefore $L(\lambda_j, \alpha, s, \omega)$ is considered as an $H(D)$-valued random element. Define the $H^r(D)$-valued random element $\underline{L}(s, \omega)$ by

$$\underline{L}(s, \omega) = (L(\lambda_1, \alpha, s, \omega), \ldots, L(\lambda_r, \alpha, s, \omega)). \tag{5.2}$$

Then the limit measure P is the distribution of $\underline{L}(s, \omega)$. Namely,

$$P(A) = m_H(\{\omega \in \Omega \mid \underline{L}(s, \omega) \in A\}) \tag{5.3}$$

for $A \in \mathcal{B}(H^r(D))$.

6. A joint denseness lemma for Lerch zeta functions.

For $m \geq 0$ define

$$\underline{F}_m(s) = \left(\frac{e(\lambda_1 m)}{(m+\alpha)^s}, \ldots, \frac{e(\lambda_r m)}{(m+\alpha)^s} \right) \in H^r(D).$$

Then from (5.1) and (5.2)

$$\underline{L}(s, \omega) = \sum_{m=0}^{\infty} \omega(m) \underline{F}_m(s) \tag{6.1}$$

holds for almost all $\omega \in \Omega$ and $s \in D$. Our aim is to obtain the following joint denseness result.

PROPOSITION 2. *Assume that real numbers $\lambda_1, \ldots, \lambda_r$ with $0 \leq \lambda_j < 1$ satisfy one of the following conditions*:

1. $(\lambda_1, \ldots, \lambda_r) \in \Lambda$, *where the set Λ is given by* (2.3).
2. $\lambda_1, \ldots, \lambda_r$ *are algebraic irrational numbers such that* $1, \lambda_1, \ldots, \lambda_r$ *are linearly inde-*

pendent over $\mathbb{Q}$.
3. $\lambda_j = e^{r_j}$, *where* r_j*'s are distinct rational numbers.*

Then the set of all convergent series

$$\sum_{m=0}^{\infty} a_m \underline{F}_m(s), \quad (a_m \in \gamma)$$

is dense in the space $H^r(D)$.

Here we construct a complex Hilbert space which consists of analytic functions on D. Let U be a bounded simply connected region satisfying $\overline{U} \subset D$. Let $L^2(U)$ be the set of all $\mathbb{C}$-valued measurable functions on U which are square integrable with respect to the Lebesgue measure. Then $L^2(U)$ is a complex Hilbert space with the inner product

$$\langle g(s), h(s) \rangle = \iint_U g(s)\overline{h(s)} d\sigma dt,$$

and the norm

$$\|g(s)\| = \sqrt{\langle g(s), g(s) \rangle} = \left(\iint_U |g(s)|^2 d\sigma dt \right)^{1/2}.$$

Let H be the closure of $H(D)$ in $L^2(U)$ and $H_r = H \times \cdots \times H$ be the product space.

LEMMA 15. *Suppose that real numbers* $\lambda_1, \ldots, \lambda_r$ *with* $0 \leq \lambda_j < 1$ *satisfy the same assumption in Proposition* 2. *Then the set of convergent series*

$$\left\{ \sum_{m=0}^{\infty} a_m \underline{F}_m(s) \in H_r \;\middle|\; a_m \in \gamma \right\}$$

is dense in H_r.

Proposition 2 easily follows from this lemma and Lemma 7. For each $1 \leq j \leq r$, let K_j be a compact subset of D and $\underline{f}(s) = (f_1(s), \ldots, f_r(s)) \in H^r(D)$. Let $U \subset D$ be a bounded simply connected region such that $\overline{U} \subset D$ and that $\bigcup_{1 \leq j \leq r} K_j \subset U$. From Lemma 15, there exists a sequence $a_m \in \gamma$ such that

$$\left\| \sum_{m=0}^{\infty} a_m \underline{F}_m(s) - \underline{f}(s) \right\| < \frac{\varepsilon^2}{a(\bigcup_{1 \leq j \leq r} K_j, U)^2}$$

where $a(\bigcup_{1 \leq j \leq r} K_j, U)$ is a constant given by Lemma 7. Then it follows from Lemma 7

$$\sum_{j=1}^{r} \max_{s \in K_j} \left| \sum_{m=0}^{\infty} a_m \frac{e(\lambda_j m)}{(m+\alpha)^s} - f_j(s) \right| < \varepsilon.$$

This implies that Proposition 2 holds.

Now we prove Lemma 15 using Lemma 5. We will check that the sequence $\{\underline{F}_m(s)\}$ in H_r satisfies conditions (I) and (II) of Lemma 5. Let σ_1 and σ_2 be real numbers with $1/2 < \sigma_1 < \sigma_2 < 1$ such that the strip $\sigma_1 < \Re s < \sigma_2$ contains the region U. Then we have

$$\sum_{m=0}^{\infty} \|\underline{F}_m(s)\|^2 = \sum_{m=0}^{\infty} \iint_U \frac{r}{(m+\alpha)^{2\sigma}} d\sigma dt \ll_U \sum_p \frac{1}{(m+\alpha)^{2\sigma_1}} < \infty.$$

Therefore condition (I) holds. Next we check condition (II). Let $\underline{g}(s) = (g_1(s), \dots, g_r(s))$ be a non-zero element of H_r. Then

$$\langle \underline{F}_m(s), \underline{g}(s) \rangle = \sum_{j=1}^{r} \iint_U \frac{e(\lambda_j m)}{(m+\alpha)^s} \overline{g_j(s)} d\sigma dt.$$

Putting

$$\Delta_j(x) = \iint_U e^{-sx} \overline{g_j(s)} d\sigma dt \tag{6.2}$$

then we have

$$\langle \underline{F}_m(s), \underline{g}(s) \rangle = \sum_{j=1}^{r} e(\lambda_j m) \Delta_j(\log(m+\alpha)).$$

Now our purpose is the next lemma.

Lemma 16. *Suppose that real numbers $\lambda_1, \dots, \lambda_r$ with $0 \le \lambda_j < 1$ satisfy the same assumption in Proposition 2. Let $\underline{g}(s) = (g_1(s), \dots, g_r(s))$ be a non-zero element of H_r. Then the series*

$$\sum_{m=0}^{\infty} \left| \sum_{j=1}^{r} e(\lambda_j m) \Delta_j(\log(m+\alpha)) \right|$$

is divergent.

Proof. We may only consider the case that all g_j's are not identically equal to zero. Applying Lemma 8, there exists a divergent positive sequence $R_n \to \infty$ $(n \to \infty)$ and a sequence of intervals $I_n = [x_n, x_n + y_n] \subset [R_n - 1, R_n + 1]$ such that

$$x_n = R_n + O(1), \quad y_n \sim R_n^{-8},$$

and that

$$|\Delta_1(x)| \ge \frac{1}{4} e^{-\sigma_2 x_n} \tag{6.3}$$

for $x \in I_n$. Next we apply Lemma 13 for

$$G(z) = \Delta_1(x), \quad \delta(R) = y_n, \quad \text{and} \quad I(R) = I_n.$$

Then there exist a positive integer $N^{(1)}$, a sequence of subintervals $J_n^{(1)} = [\alpha_n^{(1)}, \alpha_n^{(1)} + \beta_n^{(1)}] \subset I_n$, and a sequence of integers $k_n^{(1)}$ with $1 \le k_n^{(1)} \le 4$ such that

$$\alpha_n^{(1)} = R_n + O(1), \quad \beta_n^{(1)} \sim R_n^{-N^{(1)}} \tag{6.4}$$

and that

$$\arg \Delta_1(x) \in A_{k_n^{(1)}} \quad \text{for all} \quad x \in J_n^{(1)}, \tag{6.5}$$

where A_k is the interval given by (3.1). Remark that inequality (6.3) also holds for $x \in J_n^{(1)}$. Again, we apply Lemma 13 for

$$G(z) = \Delta_2(x), \quad \delta(R) = R^{-N^{(1)}}, \quad \text{and} \quad I(R) = J_n^{(1)}.$$

Then there exist a positive integer $N^{(2)}$, a sequence of subintervals $J_n^{(2)} = [\alpha_n^{(2)}, \alpha_n^{(2)} + \beta_n^{(2)}] \subset J_n^{(1)}$, and a sequence of integers $k_n^{(2)}$ with $1 \le k_n^{(2)} \le 4$ which satisfy the similar properties as (6.4) and (6.5). Repeating this argument, we obtain a positive integer N, a sequence of subintervals $J_n = [\alpha_n, \alpha_n + \beta_n] \subset I_n$, and a sequence of integers $k_n^{(j)}$ $(1 \le j \le r)$ with $1 \le k_n^{(j)} \le 4$ which satisfy the following properties:

1. We have

$$\alpha_n = R_n + O(1), \quad \beta_n \sim R_n^{-N}. \tag{6.6}$$

2. For $x \in J_n$

$$|\Delta_1(x)| \gg e^{-\sigma_2 R_n}. \tag{6.7}$$

3. For $x \in J_n$ and $1 \le j \le r$

$$\arg \Delta_j(x) \in A_{k_n^{(j)}}. \tag{6.8}$$

For each $n \ge 1$ we define the set of integers X_n as follows.

$$X_n = \{m \ge 0 \mid \log(m+\alpha) \in J_n, \quad \{\lambda_j m\} \in B_{k_n^{(j)}} \ (1 \le j \le r)\}. \tag{6.9}$$

If $m \in X_n$, then from (6.8) and Lemma 10

$$\Re\big(e(\lambda_j m)\Delta_j(\log(m+\alpha))\big) \ge \frac{1}{2}|\Delta_j(\log(m+\alpha))| \ge 0,$$

for any $1 \le j \le r$. In particular, by (6.7)

$$\Re\big(e(\lambda_1 m)\Delta_1(\log(m+\alpha))\big) \gg e^{-\sigma_2 R_n}.$$

Therefore, by Lemma 9, we have

$$\left|\sum_{j=1}^{r} e(\lambda_j m)\Delta_j(\log(m+\alpha))\right| \ge \frac{1}{2}|\Delta_1(\log(m+\alpha))| \gg e^{-\sigma_2 R_n} \tag{6.10}$$

for all $m \in X_n$.

Now we calculate the the lower bound of the cardinality of the set X_n. Since the numbers $(\lambda_1, \dots, \lambda_r)$ satisfy the condition in Lemma 4, we may apply (2.4) for the set $B(n) = B_{k_n^{(1)}} \times \cdots \times B_{k_n^{(r)}} \subset [0,1)^r$. Then we have

$$\sharp A_N(B(n)) = \mathrm{vol}(B(n))M + O(M^{\varepsilon}) \tag{6.11}$$

for every $\varepsilon > 0$. Since $\mathrm{vol}(B_k) = 1/12$ for all $1 \le k \le 4$,

$$\mathrm{vol}(B(n)) = \left(\frac{1}{12}\right)^r.$$

Therefore, by (6.6) and (6.9),

$$\begin{aligned}\sharp X_n &= A_{e^{\alpha_n+\beta_n}-\alpha}(B(n)) - A_{e^{\alpha_n}-\alpha}(B(n)) \\ &= \left(\frac{1}{12}\right)^r (e^{\alpha_n+\beta_n} - e^{\alpha_n}) + O(e^{\varepsilon\alpha_n}) \gg \frac{e^{R_n}}{R_n^N}.\end{aligned}$$

From this and (6.10)

$$\sum_{m \in X_n}\left|\sum_{j=1}^{r} e(\lambda_j m)\Delta_j(\log(m+\alpha))\right| \gg \frac{e^{(1-\sigma_2)R_n}}{R_n^N}.$$

Since $\sigma_2 < 1$, this series diverges as n tends to infinity. This completes the proof of Lemma 16, and the proof of Proposition 2. □

7. Completion of the proof of joint universality theorems.

Assume that $\lambda_1, \dots, \lambda_r$ are real numbers with $0 \le \lambda_j < 1$ satisfying either the assumption in Theorem 4, or the assumption in Theorem 5. From Proposition 1 and Proposition 2 we will complete the proof of the theorems at one time.

It follows from Proposition 1 that the probability measure

$$P_T(A) = \nu_T\big((L(\lambda_1, \alpha, s+i\tau), \dots, L(\lambda_r, \alpha, s+i\tau)) \in A\big), \quad A \in \mathcal{B}(H^r(D)),$$

weakly converges to the probability measure P as $T \to \infty$, where the measure P is the distribution of the $H^r(D)$-valued random element

$$\underline{L}(s,\omega) = (L(\lambda_1,\alpha,s,\omega),\ldots,L(\lambda_r,\alpha,s,\omega)), \quad \omega \in \Omega,$$

where $L(\lambda_j,\alpha,s,\omega)$ is defined by (5.1). Now we calculate the support of the measure P, which is a minimal closed subset $S \subset H^r(D)$ such that $P(S) = 1$. For the purpose we quote Lemma 12.7 in [**11**].

Lemma 17. *Let $\{X_m\}$ be a sequence of independent $H^r(D)$-valued random elements, and suppose that the series*

$$\sum_{n=1}^{\infty} X_m$$

converges almost surely. Then the support of the sum of this series is the closure of the set of all $\underline{f} \in H^r(D)$ which may be written as a convergent series

$$\underline{f} = \sum_{m=1}^{\infty} \underline{f}_m, \quad \underline{f}_m \in S_{X_m},$$

where S_{X_m} is the support of the random element X_m.

As we saw in (6.1), the random element $\underline{L}(s,\omega)$ is the series of the random elements

$$X_m = \omega(m)\underline{F}_m(s), \quad \omega(m) \in \gamma.$$

The support of each $\omega(m)$ is the unit circle γ. Therefore the support of the random element $\omega(m)\underline{F}_m(s)$ is

$$\big\{\underline{f} \in H^r(D),\ \underline{f}(s) = a\underline{F}_m(s), \quad a \in \gamma\big\}.$$

Moreover, since $\{\omega(m) \mid m \geq 0\}$ is a sequence of independent random variable, $\{\omega(m)\underline{F}_m(s) \mid m \geq 0\}$ is also a sequence of $H^r(D)$-valued random elements. By Lemma 17, the support of P, that is, the support of $\underline{L}(s,\omega)$ is the closure of the set of all convergent series

$$\sum_{m=0}^{\infty} a_m \underline{F}_m(s) \quad (a_m \in \gamma).$$

From Proposition 2, we obtain the support of P.

Proposition 3. *Assume that $\lambda_1,\ldots,\lambda_r$ are real numbers with $0 \leq \lambda_j < 1$ satisfying either the assumption in Theorem 4, or the assumption in Theorem 5. Then the support of the probability measure P is $H^r(D)$ itself.*

Now we prove the theorems. Let arbitrary sets K_j's and functions h_j's be taken as in Theorem 4 and 5. First we suppose that all functions $h_j(s)$ $(1 \le j \le r)$ can be analytically continued to the whole of strip D. Then, by Proposition 3, the function $\underline{h}(s) = (h_1(s), \ldots, h_l(s))$ belongs to S. Put

$$G_0 = \left\{ \underline{g}(s) = (g_j(s)) \in H^r(D) \;\middle|\; \max_{1\le j\le r} \max_{s\in K_j} |g_j(s) - h_j(s)| < \frac{\varepsilon}{2} \right\}.$$

From the definition of the support, we have

$$P(G_0) > 0. \tag{7.1}$$

Since the measure P_T weakly converges to the measure P as $T \to \infty$, we have

$$\liminf_{T\to\infty} P_T(G) \ge P(G), \tag{7.2}$$

for any open subsets $G \in \mathcal{B}(H^r(D))$. From (7.1) and (7.2),

$$\liminf_{T\to\infty} P_T(G_0) > 0,$$

which implies

$$\liminf_{T\to\infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] \;\middle|\; \max_{1\le j\le r} \max_{s\in K_j} |L(\lambda_j, \alpha, s + i\tau) - h_j(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

Next we consider the general case for the functions $h_j(s)$. By Mergelyan's theorem, there exist polynomials $p_j(s)$ satisfying

$$\max_{s\in K_j} |h_j(s) - p_j(s)| < \frac{\varepsilon}{2} \tag{7.3}$$

for all $1 \le j \le r$. Remark that the polynomials $p_j(s)$ belong to the support S. According to the similar argument as above, we have

$$\liminf_{T\to\infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] \;\middle|\; \max_{1\le j\le r} \max_{s\in K_j} |L(\lambda_j, \alpha, s + i\tau) - p_j(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (7.3), we complete the proof of the theorems.

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