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Unique solvability of some nonlinear partial differential equations with Fuchsian and irregular singularities

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Abstract. The paper considers nonlinear partial differential equations of the form $t(\partial u/\partial t) = F(t, x, u, \partial u/\partial x)$, with independent variables $(t, x) \in \mathbb{R} \times \mathbb{C}$, and where $F(t, x, u, v)$ is a function continuous in t and holomorphic in the other variables. It is shown that the equation has a unique solution in a sectorial domain centered at the origin under the condition that $F(0, x, 0, 0) = 0$, $\mathrm{Re}F_u(0, 0, 0, 0) < 0$, and $F_v(0, x, 0, 0) = x^{p+1}\gamma(x)$, where $\gamma(0) \neq 0$ and p is any positive integer. In this case, the equation has a Fuchsian singularity at $t = 0$ and an irregular singularity at $x = 0$.

1. Introduction.

Consider first order singular nonlinear partial differential equations of the form

$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right), \tag{1.1}$$

which has a Fuchsian singularity at $t = 0$.

Suppose $F(t, x, u, v)$ is a function holomorphic in a neighborhood of the origin $(0, 0, 0, 0) \in \mathbb{C}^4$ and $F(0, x, 0, 0) \equiv 0$ near $x = 0$. Then we can write F as

$$F\left(t, x, u, \frac{\partial u}{\partial x}\right) = a(x)t + \lambda(x)u + b(x)\frac{\partial u}{\partial x} + \sum_{i+j+\alpha \geq 2} a_{i,j,\alpha}(x)t^i u^j \left(\frac{\partial u}{\partial x}\right)^{\alpha},$$

where all the coefficients $a(x)$, $\lambda(x)$, $b(x)$ and $a_{i,j,\alpha}(x)$ are holomorphic at $x = 0 \in \mathbb{C}$.

In the case $b(0) \neq 0$, we can solve (1.1) by writing it in the form

$$\frac{\partial u}{\partial x} = G\left(t, x, u, t\frac{\partial u}{\partial t}\right)$$

and then applying the Cauchy-Kowalewski theorem to this equation with data on $x = 0$. For the case $b(0) = 0$, (1.1) can be classified into two types:

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(T_1) $b(x) \equiv 0$;
(T_2) $b(x) = x^{p+1}\gamma(x)$, where $\gamma(0) \neq 0$ and $p \in \mathbb{N} := \{0, 1, 2, \dots\}$.

In [**5**], Gérard and Tahara proved that equations of type (T_1) have unique solutions when $\lambda(0) \notin \mathbb{N}^* := \{1, 2, 3, \dots\}$. Afterwards, Yamazawa [**14**] showed the existence of solutions to such equations also in the case $\lambda(0) \in \mathbb{N}^*$. Type (T_2) equations were studied in [**3**], [**4**], [**8**] and [**12**]. For $p = 0$, the equation (1.1) has a regular singularity at $x = 0$. In this case, Chen-Tahara [**3**] and Tahara [**12**] established the solvability of (1.1) whenever $\gamma(0) \in \mathbb{C} \setminus [0, \infty)$. On the other hand, the equation (1.1) has an irregular singularity at $x = 0$ when $p \geq 1$. And in this case, Chen-Luo-Tahara [**4**] studied Gevrey type estimates of formal solutions and Luo-Chen-Zhang [**8**] showed that the equation is solvable in a sectorial domain by using the Borel summation method.

The study on the equation (1.1) has been extended to the case where the function $F(t, x, u, v)$ is holomorphic in the variables (x, u, v) but only continuous in t. In this situation, Baouendi-Goulaouic [**2**] and Lope-Roque-Tahara [**7**] showed existence and uniqueness theorems for equations corresponding to (T_1)-type equations. In [**1**], the authors showed that unique solutions exist also for equations of type (T_2) in the regular singularity case $p = 0$.

In this paper, we will solve partial differential equations of type (T_2) in the irregular singularity case $p \geq 1$, under the assumption that $F(t, x, u, v)$ is holomorphic with respect to the variables (x, u, v) but only continuous in t.

2. Main Result.

Let $(t, x) \in \mathbb{R} \times \mathbb{C}$, $T_0 > 0$, $R_0 > 0$ and $\rho_0 > 0$. For any $s > 0$, denote by D_s the open disk $\{x \in \mathbb{C} : |x| < s\}$. We study the equation (1.1) under the following hypotheses:

(A_1) $F(t, x, u, v)$ is continuous on $\Delta = [0, T_0] \times D_{R_0} \times D_{\rho_0} \times D_{\rho_0}$ and holomorphic in the variables (x, u, v) for any fixed t;
(A_2) $F(0, x, 0, 0) = 0$ on D_{R_0};
(A_3) $F_v(0, x, 0, 0) = x^{p+1}\gamma(x)$ with $\gamma(0) \neq 0$ and $p \in \mathbb{N}^*$.

Set $a(t, x) = F(t, x, 0, 0)$, $\lambda(t, x) = F_u(t, x, 0, 0)$, and $b(t, x) = F_v(t, x, 0, 0) - F_v(0, x, 0, 0)$. Then the equation (1.1) can be rewritten as

$$t\frac{\partial u}{\partial t} = a(t, x) + \lambda(t, x)u + (x^{p+1}\gamma(x) + b(t, x))\frac{\partial u}{\partial x} + G_2\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{2.1}$$

where

$$G_2(t, x, u, v) = \sum_{i+j \geq 2} g_{i,j}(t, x)u^i v^j$$

represents the sum of all the terms in the Taylor expansion of $F(t, x, u, v)$ in (u, v) whose degrees with respect to (u, v) are at least 2. It is clear from our hypotheses that the functions $a(t, x)$, $\lambda(t, x)$ and $b(t, x)$ are continuous functions on $[0, T_0] \times D_{R_0}$ and holomorphic in x for any fixed t. Moreover, we have $a(0, x) \equiv 0$, $b(0, x) \equiv 0$ and

$\gamma(0) \neq 0$.

Let us introduce a weight function to describe the decreasing order that we want $a(t,x)$ and $b(t,x)$ to satisfy as t tends to 0. We say that a real-valued function $\mu(t)$ is a *weight function* on $(0,T_0]$ if it is positive, continuous, and increasing on $(0,T_0]$, and

$$\int_0^{T_0} \frac{\mu(s)}{s} ds < +\infty.$$

It follows from this definition that for any given weight function $\mu(t)$ we have $\lim_{t\to 0}\mu(t) = 0$, and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} ds \tag{2.2}$$

is well-defined on $(0,T_0]$. Moreover, we have $\lim_{t\to 0}\varphi(t) = 0$ and $\varphi'(t) = \mu(t)/t$ on $(0,T_0)$. Some examples of such weight functions are t^η and $1/(-\log t)^{\eta+1}$ for any $\eta > 0$.

We suppose that there is a weight function $\mu(t)$ such that

$$a(t,x) = O(\mu(t)) \quad \text{uniformly on } D_{R_0} \ (\text{as } t \longrightarrow 0), \tag{2.3}$$

$$b(t,x) = O(\mu(t)) \quad \text{uniformly on } D_{R_0} \ (\text{as } t \longrightarrow 0). \tag{2.4}$$

Set $\theta_0 = \arg(-\gamma(0))$. For any $R > 0$, $\epsilon > 0$, $T > 0$, and $r > 0$, we define a sector S, a distance function $d_S(x)$ on S from the boundary, and a region W_r by

$$\begin{aligned}
S &= \left\{x \in \mathbb{C} : 0 < |x| < R,\ \left|\arg x + \frac{\theta_0}{p}\right| < \frac{\pi}{2p} - \frac{\epsilon}{p}\right\},\\
d_S(x) &= \min\left\{\frac{\pi}{2p} - \frac{\epsilon}{p} - \left|\arg x + \frac{\theta_0}{p}\right|,\ \log R - \log|x|\right\} \quad \text{for } x \in S,\\
W_r &= \left\{(t,x) \in (0,T] \times S : \frac{\varphi(t)}{r} < d_S(x)\right\}.
\end{aligned}$$

If $0 < \epsilon < \pi/2$ then we have $S \neq \emptyset$.

We also define two spaces on the region $W = W_r$ or $(0,T] \times S$:

$$\begin{aligned}
X_0(W) &= \{w(t,x) \in C^0(W) : w \text{ is holomorphic in } x \text{ for any fixed } t\};\\
X_1(W) &= X_0(W) \cap C^1(W).
\end{aligned}$$

Here is our main result.

Theorem 2.1 (Main Theorem). *Suppose that* (A_1)–(A_3), (2.3), (2.4), *and the following conditions hold*:

(i) $\operatorname{Re}\lambda(0,0) < 0$,

(ii) $a(t,0) = 0$ *and* $a_x(t,0) = 0$ *on* $[0,T_0]$,

(iii) $b(t,0)=0$ *on* $[0,T_0]$.

Then for any $0<\epsilon<\pi/16$ *there exist* $R>0$*,* $r>0$*,* $M>0$ *and* $T>0$ *with* $R^2M\mu(T) < \rho_0$ *and* $RM\mu(T)<\rho_0$ *such that* (2.1) *has a unique solution* $u(t,x)$ *in* $X_1(W_r)$ *that satisfies*

$$|u(t,x)|\le M\mu(t)|x|^2 \quad \textit{and} \quad \left|\frac{\partial u}{\partial x}(t,x)\right| \le M\mu(t)|x| \quad \textit{on } W_r. \tag{2.5}$$

3. Reduction.

Set $a_1(t,x)=a(t,x)/x^2$ and $b_1(t,x)=b(t,x)/x$. The assumptions (ii) and (iii) imply that these functions $a_1(t,x)$ and $b_1(t,x)$ are continuous on $[0,T_0]\times D_{R_0}$ and holomorphic in x for any fixed t. By setting

$$u(t,x)=x^2w(t,x)$$

we can reduce the equation (2.1) in terms of the unknown function $w(t,x)$ as follows:

$$\begin{aligned} t\frac{\partial w}{\partial t} - \lambda_1(t,x)w - x^{p+1}\gamma(x)\frac{\partial w}{\partial x} \\ = a_1(t,x) + xb_1(t,x)\frac{\partial w}{\partial x} + R_2\left(t,x,w,x\frac{\partial w}{\partial x}\right), \end{aligned} \tag{3.1}$$

where $\lambda_1(t,x)=\lambda(t,x)+2x^p\gamma(x)+2b_1(t,x)$ and

$$R_2(t,x,w,w_1)=\sum_{i+j\ge 2} x^{2i+j-2}g_{i,j}(t,x)w^i(2w+w_1)^j.$$

It is easy to see that $\lambda_1(t,x)$ is continuous on $[0,T_0]\times D_{R_0}$ and holomorphic in x for any fixed t, and the function $R_2(t,x,w,w_1)$ is continuous on $\Delta_1=[0,T_0]\times D_{R_0}\times D_{\rho_1}\times D_{\rho_1}$ with $\rho_1=\min\{\rho_0/{R_0}^2, \rho_0/(3R_0)\}$ and holomorphic in (x,w,w_1) for any fixed t. It also follows from (2.3) and (2.4) that $a_1(t,x)=O(\mu(t))$ and $b_1(t,x)=O(\mu(t))$ uniformly on D_{R_0} (as $t\longrightarrow 0$). Since $p\ge 1$ and $b_1(0,x)\equiv 0$, we have $\lambda_1(0,0)=\lambda(0,0)$ and so $\mathrm{Re}\,\lambda_1(0,0)<0$. Evidently, to prove Theorem 2.1 it is enough to show the following proposition.

Proposition 3.1. *For any* $0<\epsilon<\pi/16$*, there exist* $R>0$*,* $r>0$*,* $M>0$ *and* $T>0$ *with* $M\mu(T)<\rho_1$ *such that (3.1) has a unique solution* $w(t,x)$ *in* $X_1(W_r)$ *that satisfies*

$$|w(t,x)|\le M\mu(t) \quad \textit{and} \quad \left|x\frac{\partial w}{\partial x}(t,x)\right| \le M\mu(t) \quad \textit{on } W_r. \tag{3.2}$$

We devote the rest of this paper to prove the above result. In fact, for a slight generalization, we will prove Proposition 3.1 for the equation

$$
\begin{aligned}
t\frac{\partial w}{\partial t} - \lambda_1(t,x)w - x^{p+1}c(t,x)\frac{\partial w}{\partial x} \\
= a_1(t,x) + xb_1(t,x)\frac{\partial w}{\partial x} + R_2\left(t,x,w,x\frac{\partial w}{\partial x}\right),
\end{aligned}
\tag{3.3}
$$

under the condition that $c(0,0) = \gamma(0)$. In the next section, we present a Nagumo-type lemma in a sectorial domain that will play a very important role in the proof of Proposition 3.1. Then we investigate the behavior of the solution of the equation $tdx/dt = -x^{p+1}c(t,x)$, which gives an integral curve of the vector field $t\partial/\partial t - x^{p+1}c(t,x)\partial/\partial x$. After that, we solve the equation

$$
t\frac{\partial w}{\partial t} - \lambda_1(t,x)w - x^{p+1}c(t,x)\frac{\partial w}{\partial x} = g(t,x) \tag{3.4}
$$

on the domains $(0,T_1] \times S$ and W_r. Finally, in the last section we solve the equation (3.3) by the method of Nirenberg [**10**] and Nishida [**11**] but with modification so that it also works in a sectorial domain.

4. Nagumo's lemma in a sector.

Let us recall a refined Nagumo's lemma by Walter [**13**]. Let Ω be an open set in the z-complex plane with a nonempty boundary Γ, and let $dist(z,\Gamma)$ be the distance from z to Γ. The following lemma was also called Nagumo's lemma in [**13**] (see also Nagumo [**9**]).

LEMMA 4.1 (Nagumo's lemma). *Let $f(z)$ be a holomorphic function on Ω, and let $a \geq 0$ and $C \geq 0$. Then we have*

$$
|f(z)| \leq \frac{C}{dist(z,\Gamma)^a} \quad \text{on } \Omega \implies \left|f'(z)\right| \leq \frac{\gamma_a C}{dist(z,\Gamma)^{a+1}} \quad \text{on } \Omega,
$$

where $\gamma_0 = 1$ and $\gamma_a = (1+a)(1+1/a)^a$ for $a > 0$.

For an open interval $I = (\phi_1, \phi_2)$ and $R > 0$ we define a sector $S_{I,R}$ in the x-complex plane by $S_{I,R} = \{x \in \mathbb{C} : 0 < |x| < R,\ \phi_1 < \arg x < \phi_2\}$. Under the relation $z = \log x$ the sector $S_{I,R}$ is transformed into the domain $H = \{z \in \mathbb{C} : \mathrm{Re}\, z < \log R,\ \phi_1 < \mathrm{Im}\, z < \phi_2\}$. Let us denote the boundary of H by ∂H and set the distance from $\log x$ to ∂H by

$$
d_{S_{I,R}}(x) = dist(\log x, \partial H), \quad x \in S_{I,R}.
$$

Clearly, we have

$$
d_{S_{I,R}}(x) = \min\{\phi_2 - \arg x,\ \arg x - \phi_1,\ \log R - \log|x|\}.
$$

If $u(x)$ is a holomorphic function on $S_{I,R}$, then the function $f(z) := u(e^z)$ is holomorphic on H, and we have $f'(z) = xu'(x)$. Therefore, by Lemma 4.1 we get the following result.

Lemma 4.2 (Nagumo's lemma in a sector). *Let $u(x)$ be a holomorphic function on the sector $S_{I,R}$, and let $a \geq 0$ and $C \geq 0$. Then we have*

$$|u(x)| \leq \frac{C}{d_{S_{I,R}}(x)^a} \text{ on } S_{I,R} \implies \left|xu'(x)\right| \leq \frac{\gamma_a C}{d_{S_{I,R}}(x)^{a+1}} \text{ on } S_{I,R}.$$

Let $\eta > 0$ satisfy $\eta < (\phi_2 - \phi_1)/2$. Set $S_* = \{x \in S_{I,R} : \eta < d_{S_{I,R}}(x)\}$. Then we have

$$S_* = \{x \in \mathbb{C} : 0 < |x| < Re^{-\eta},\ \phi_1 + \eta < \arg x < \phi_2 - \eta\},$$

which is also a sector, and moreover, $d_{S_{I,R}}(x) - \eta = d_{S_*}(x)$ for any $x \in S_*$.

Corollary 4.3. *Let $u(x)$ be a holomorphic function on S_*, and let $a \geq 0$ and $C \geq 0$. Then we have*

$$|u(x)| \leq \frac{C}{(d_{S_{I,R}}(x) - \eta)^a} \text{ on } S_* \implies \left|xu'(x)\right| \leq \frac{\gamma_a C}{(d_{S_{I,R}}(x) - \eta)^{a+1}} \text{ on } S_*.$$

The above corollary follows immediately from Lemma 4.2.

5. On the equation $t dx/dt = -x^{p+1}c(t,x)$.

Let $c(t,x)$ be a continuous function on $[0,T_0] \times D_{R_0}$ that is holomorphic in x for any fixed t and satisfies $c(0,0) \neq 0$. Let $0 < \epsilon_1 < \pi/8$. Then we can choose $0 < T_1 < T_0$ and $0 < R_1 < R_0$ so that the following conditions are satisfied:

(B_1) $c(t,x) \neq 0$ on $[0,T_1] \times D_{R_1}$;
(B_2) $|\arg c(t,x) - \arg c(0,0)| < \epsilon_1$ on $[0,T_1] \times D_{R_1}$.

Set $\theta_0 = \arg(-c(0,0))$ and

$$S_1 = \left\{x \in \mathbb{C} : 0 < |x| < R_1,\ \left|\arg x + \frac{\theta_0}{p}\right| < \frac{\pi}{2p} - \frac{\epsilon_1}{p}\right\}.$$

Lemma 5.1. *For any $(t_0, x_0) \in (0,T_1] \times S_1$ the initial value problem*

$$t\frac{dx}{dt} = -x^{p+1}c(t,x), \quad x|_{t=t_0} = x_0, \tag{5.1}$$

has a unique solution $x(t)$ on $(0,t_0]$ satisfying the property $x(t) \in S_1$ for any $t \in (0,t_0]$. Moreover, we have $0 < |x(t)| \leq |x_0|$ on $(0,t_0]$ and

$$\left|\arg x(t) + \frac{\theta_0}{p}\right| \leq \max\left\{\left|\arg x_0 + \frac{\theta_0}{p}\right|,\ \frac{3\epsilon_1}{p}\right\} \quad \text{on } (0,t_0].$$

Proof. Since $c(t,x)$ satisfies a *Lipschitz condition* on S_1, there exists $0 < t_1 < t_0$

such that (5.1) has a unique local solution $x(t)$ on $(t_1, t_0]$ satisfying the property $x(t) \in S_1$ for any $t \in (t_1, t_0]$. Since $x(t) \neq 0$ on $(t_1, t_0]$, (5.1) can be written as

$$\frac{1}{x(t)^{p+1}} \frac{dx(t)}{dt} = -c(t, x(t)) \frac{1}{t},$$

and integrating this equation from t to t_0 shows that the solution satisfies

$$x(t) = \frac{x_0}{\left[1 - p{x_0}^p \displaystyle\int_t^{t_0} c(s, x(s)) \frac{ds}{s}\right]^{1/p}} \quad \text{on } (t_1, t_0]. \tag{5.2}$$

We show that the local solution $x(t)$ can be continued up to the interval $(0, t_0]$. Suppose it can only be extended to a maximal interval of existence $(\alpha, t_0]$ for some $\alpha > 0$ with the property $x(t) \in S_1$ on $(\alpha, t_0]$.

Set $M_0 = \max\{|c(t, x)| : 0 \le t \le t_0, |x| \le |x_0|\}$. Let us show that the following inequalities hold for any $t \in (\alpha, t_0]$:

$$|x(t)| \le |x_0|, \tag{5.3}$$

$$|x(t)| \ge \frac{|x_0|}{[1 + p|x_0|^p M_0 \log(t_0/\alpha)]^{1/p}}, \tag{5.4}$$

$$\left|\arg x(t) + \frac{\theta_0}{p}\right| \le \max\left\{\left|\arg x_0 + \frac{\theta_0}{p}\right|, \frac{3\epsilon_1}{p}\right\}. \tag{5.5}$$

Since $x(t_0) = x_0$, it is clear that (5.3), (5.4) and (5.5) are satisfied when $t = t_0$.

Let us show that the inequalities are also true on (α, t_0). Set

$$C(t) = \int_t^{t_0} -c(s, x(s)) ds/s, \quad \alpha < t < t_0.$$

If $C(t) = 0$ we have $x(t) = x_0$, so in this case the relations (5.3), (5.4) and (5.5) are clear. Therefore, from now on we suppose that $C(t) \neq 0$.

Set $\theta(t) = \arg C(t)$. Since $\theta_0 = \arg(-c(0, 0))$, by (B_2) we have $|\theta(t) - \theta_0| \le \epsilon_1$. The solution satisfies the equation

$$x(t) = \frac{x_0}{\left[1 + p x_0^p C(t)\right]^{1/p}} \quad \text{on } (\alpha, t_0]. \tag{5.6}$$

Since $x_0 \in S_1$, which means that $|\arg x_0 + \theta_0/p| < \pi/2p - \epsilon_1/p$, and $|\theta(t) - \theta_0| \le \epsilon_1$, we have $|\arg(p{x_0}^p C(t))| < \pi/2$. Therefore, $\mathrm{Re}(p{x_0}^p C(t)) > 0$ and consequently,

$$|1 + p{x_0}^p C(t)| \ge \mathrm{Re}\left(1 + p x_0^p C(t)\right) = 1 + \mathrm{Re}\left(p x_0^p C(t)\right) > 1,$$

which implies that

$$|x(t)| = \frac{|x_0|}{\left|1 + px_0^p C(t)\right|^{1/p}} < |x_0|.$$

This proves (5.3).

The second inequality (5.4) can be easily obtained from (5.3) and the following estimate:

$$\begin{aligned}\left|1 - p{x_0}^p \int_t^{t_0} c(s, x(s)) \frac{ds}{s}\right| &\leq 1 + p|x_0|^p \int_t^{t_0} |c(s, x(s))| \frac{ds}{s} \\ &\leq 1 + p|x_0|^p \int_t^{t_0} M_0 \frac{ds}{s} \\ &\leq 1 + p|x_0|^p M_0 \log(t_0/\alpha).\end{aligned}$$

Let us now show (5.5) for $t \in (\alpha, t_0)$. We divide our proof into the following three cases:

Case 1. $\dfrac{\epsilon_1}{p} < \arg x_0 + \dfrac{\theta_0}{p} < \dfrac{\pi}{2p} - \dfrac{\epsilon_1}{p}$,

Case 2. $-\dfrac{\epsilon_1}{p} \leq \arg x_0 + \dfrac{\theta_0}{p} \leq \dfrac{\epsilon_1}{p}$,

Case 3. $-\dfrac{\pi}{2p} + \dfrac{\epsilon_1}{p} < \arg x_0 + \dfrac{\theta_0}{p} < -\dfrac{\epsilon_1}{p}$.

Since $\arg(p{x_0}^p C(t)) = p \arg x_0 + \theta_0 + (\theta(t) - \theta_0)$ and $|\theta(t) - \theta_0| \leq \epsilon_1$, it follows that in Case 1 we have $0 < \arg(p{x_0}^p C(t)) < \pi/2$, which then yields

$$0 < \arg(1 + p{x_0}^p C(t)) < \arg(p{x_0}^p C(t)).$$

As a result, we have $0 < \arg([1 + p{x_0}^p C(t)]^{1/p}) < \arg x_0 + \theta(t)/p$, and thus, by (5.6) we get

$$-\frac{\theta(t)}{p} < \arg x(t) = \arg x_0 - \arg([1 + p{x_0}^p C(t)]^{1/p}) < \arg x_0,$$

which is equivalent to

$$\frac{\theta_0}{p} - \frac{\theta(t)}{p} < \arg x(t) + \frac{\theta_0}{p} < \arg x_0 + \frac{\theta_0}{p}. \tag{5.7}$$

Furthermore, we have

$$-\left(\arg x_0 + \frac{\theta_0}{p}\right) < -\frac{\epsilon_1}{p} \leq \frac{\theta_0}{p} - \frac{\theta(t)}{p} \tag{5.8}$$

because of the inequality in Case 1 and the fact that $|\theta(t) - \theta_0| \leq \epsilon_1$. By combining (5.8) with (5.7) we arrive at

$$\left|\arg x(t)+\frac{\theta_0}{p}\right|<\left|\arg x_0+\frac{\theta_0}{p}\right|.$$

By similar arguments as in Case 1, it is easy to see that in Case 2 we have $-2\epsilon_1 \le \arg(p{x_0}^pC(t)) \le 2\epsilon_1$, which implies that

$$-2\epsilon_1 \le \arg(1+p{x_0}^pC(t)) \le 2\epsilon_1,$$

or equivalently,

$$-\frac{2\epsilon_1}{p} \le \arg\left([1+p{x_0}^pC(t)]^{1/p}\right) \le \frac{2\epsilon_1}{p}.$$

Using again equation (5.6) together with the inequality in Case 2 we obtain

$$-\frac{3\epsilon_1}{p} \le \arg x(t)+\frac{\theta_0}{p} \le \frac{3\epsilon_1}{p}.$$

In Case 3, we can show that $|\arg x(t)+\theta_0/p| < |\arg x_0+\theta_0/p|$ in the same way as in Case 1. This concludes our proof for (5.5).

The inequalities (5.3), (5.4), and (5.5), and the assumption $0<\epsilon_1<\pi/8$ show that $\{x(t) : \alpha < t \le t_0\}$ is contained in a compact subset of S_1. Therefore, the solution can be continued to the left of α (by Theorem 3.1 in Chapter 2 of [**6**]). This proves that (5.2) has a unique solution on $(0,t_0]$, that is, the continuation of the local solution $x(t)$ to $(0,t_0]$. This completes our proof for Lemma 5.1. □

Let $\psi(t)$ be a positive increasing function on $(0,T_1]$ that satisfies $\psi(T_1)<\pi/2p-\epsilon_1/p$. Set

$$d_{S_1}(x)=\min\left\{\frac{\pi}{2p}-\frac{\epsilon_1}{p}-\left|\arg x+\frac{\theta_0}{p}\right|,\ \log R_1-\log|x|\right\} \tag{5.9}$$

for $x\in S_1$, and

$$W_1^*=\left\{(t,x)\in(0,T_1]\times S_1 : \psi(t)<d_{S_1}(x)\right\}.$$

Then we have the following result, which can be obtained easily from the proof of Lemma 5.1.

Corollary 5.2. *Let $(t_0,x_0)\in W_1^*$ and let $x(t)$ be the unique solution of (5.1) on $(0,t_0]$. If $0<\epsilon_1<\pi/12$ and $\psi(T_1)\le\epsilon_1/p$ then we have*

$$d_{S_1}(x(t))-\psi(t) \ge \frac{2\epsilon_1}{\pi}\left(d_{S_1}(x_0)-\psi(t)\right) \quad \textit{on } (0,t_0]. \tag{5.10}$$

Moreover, we have $(t,x(t))\in W_1^*$ for any $t\in(0,t_0]$.

Proof. We know that if $\epsilon_1/p < |\arg x_0 + \theta_0/p| < \pi/2p - \epsilon_1/p$, we have $|x(t)| \le |x_0|$ and $|\arg x(t) + \theta_0/p| \le |\arg x_0 + \theta_0/p|$. This shows that $d_{S_1}(x(t)) \ge d_{S_1}(x_0)$. Since $0 < 2\epsilon_1/\pi < 1/6$, we obtain

$$d_{S_1}(x(t)) - \psi(t) \ge d_{S_1}(x_0) - \psi(t) > \frac{2\epsilon_1}{\pi}(d_{S_1}(x_0) - \psi(t)).$$

On the other hand, if $|\arg x_0 + \theta_0/p| \le \epsilon_1/p$, we have $|x(t)| \le |x_0|$ and $|\arg x(t) + \theta_0/p| \le 3\epsilon_1/p$, which means that

$$\begin{aligned}\log R_1 - \log|x(t)| - \psi(t) &\ge \log R_1 - \log|x_0| - \psi(t)\\ &> \frac{2\epsilon_1}{\pi}(\log R_1 - \log|x_0| - \psi(t))\end{aligned}$$

and

$$\begin{aligned}\frac{\pi}{2p} - \frac{\epsilon_1}{p} - \left|\arg x(t) + \frac{\theta_0}{p}\right| - \psi(t) &\ge \frac{\pi}{2p} - \frac{\epsilon_1}{p} - \frac{3\epsilon_1}{p} - \frac{\epsilon_1}{p}\\ \ge \frac{\epsilon_1}{p} = \frac{2\epsilon_1}{\pi} \times \frac{\pi}{2p} &\ge \frac{2\epsilon_1}{\pi} \times \left(\frac{\pi}{2p} - \frac{\epsilon_1}{p} - \left|\arg x_0 + \frac{\theta_0}{p}\right| - \psi(t)\right).\end{aligned}$$

Therefore, by (5.9) we have $d_{S_1}(x(t)) - \psi(t) \ge (2\epsilon_1/\pi)(d_{S_1}(x_0) - \psi(t))$. This proves (5.10). In additon, since $\psi(t) \le \psi(t_0)$, we also have $d_{S_1}(x_0) - \psi(t) \ge d_{S_1}(x_0) - \psi(t_0) > 0$. As a result, we have $d_{S_1}(x(t)) - \psi(t) > 0$, that is, $(t, x(t)) \in W_1^*$. □

Let us denote by $\chi(t; t_0, x_0)$ the unique solution of (5.1) and consider it as a function on

$$\Omega_1 = \{(t, t_0, x_0) : 0 < t \le t_0 \text{ and } (t_0, x_0) \in (0, T_1] \times S_1\}.$$

Note that $\chi(t; t_0, x_0) \in S_1$ and $|\chi(t; t_0, x_0)| \le |x_0|$ for any $(t, t_0, x_0) \in \Omega_1$. The fact that $\chi(t; t_0, x_0)$ belongs to $C^1(\Omega_1)$ follows from a result concerning the dependence on initial data of solutions of ordinary differential equations (see Theorem 3.1 in Chapter 5 of [**6**]). Since $c(t, x)$ is holomorphic in $x \in S_1$, it is easy to see that $\chi(t; t_0, x_0)$ is holomorphic in $x_0 \in S_1$. Moreover, the derivative of $\chi(t; t_0, x_0)$ with respect to x_0 can be estimated as follows.

Lemma 5.3. *The following two kinds of estimates hold on* Ω_1 :

$$\left|x_0 \frac{\partial \chi}{\partial x_0}(t; t_0, x_0)\right| \le \frac{R_1}{d_{S_1}(x_0)}, \tag{5.11}$$

$$\left|x_0 \frac{\partial \chi}{\partial x_0}(t; t_0, x_0)\right| \le |\chi(t; t_0, x_0)| \left(1 + \frac{|x_0|^p C_1 R_1}{d_{S_1}(x_0)} \log(t_0/t)\right), \tag{5.12}$$

where $C_1 > 0$ *is a constant satisfying* $|c_x(t, x)| \le C_1$ *on* $[0, T_1] \times D_{R_1}$.

Proof. The first estimate (5.11) is a consequence of Lemma 4.2 since $|\chi(t;t_0,x_0)| \le |x_0| \le R_1$ on Ω_1. Let us show (5.12). Note that $\chi(t;t_0,x_0)$ satisfies the equation

$$\chi(t;t_0,x_0) = \frac{x_0}{\left[1 - p{x_0}^p \displaystyle\int_t^{t_0} c(s,\chi(s;t_0,x_0))\frac{ds}{s}\right]^{1/p}} \quad \text{on } \Omega_1.$$

Differentiating both sides with respect to x_0, we get

$$x_0\frac{\partial\chi}{\partial x_0}(t;t_0,x_0) = \chi(t;t_0,x_0) \times \frac{1}{\left[1 - p{x_0}^p \displaystyle\int_t^{t_0} c(s,\chi(s;t_0,x_0))\frac{ds}{s}\right]}$$
$$\times \left(1 + {x_0}^p \int_t^{t_0} \frac{\partial c}{\partial x}(s,\chi(s;t_0,x_0)) \times x_0\frac{\partial\chi}{\partial x_0}(s;t_0,x_0)\frac{ds}{s}\right).$$

According to the proof of Lemma 5.1 we have

$$\left|1 - p{x_0}^p \int_t^{t_0} c(s,\chi(s;t_0,x_0))\frac{ds}{s}\right| \ge 1 \quad \text{on } \Omega_1,$$

and so by (5.11) and our choice of C_1 we arrive at

$$\begin{aligned} \left|x_0\frac{\partial\chi}{\partial x_0}(t;t_0,x_0)\right| &\le |\chi(t;t_0,x_0)| \times \left(1 + |x_0|^p \int_t^{t_0} C_1\frac{R_1}{d_{S_1}(x_0)}\frac{ds}{s}\right) \\ &\le |\chi(t;t_0,x_0)| \times \left(1 + |x_0|^p \frac{C_1R_1}{d_{S_1}(x_0)}\log(t_0/t)\right) \quad \text{on } \Omega_1, \end{aligned}$$

which is our second estimate. □

If we set $y(t;t_0,x_0) = t_0(\partial\chi/\partial t_0)(t;t_0,x_0)$, $z(t;t_0,x_0) = (\partial\chi/\partial x_0)(t;t_0,x_0)$ and $h(t,x) = (\partial/\partial x)(x^{p+1}c(t,x))$, it is well known (e.g. see Theorem 3.1 in Chapter 5 of [**6**]) that $y(t)$ and $z(t)$ satisfy the following initial value problems:

$$t\frac{\partial y}{\partial t} = -h(t,\chi(t;t_0,x_0))y, \quad y|_{t=t_0} = {x_0}^{p+1}c(t_0,x_0); \tag{5.13}$$
$$t\frac{\partial z}{\partial t} = -h(t,\chi(t;t_0,x_0))z, \quad z|_{t=t_0} = 1. \tag{5.14}$$

6. The equation (3.4) on $(0,T] \times S$.

Let C_1, R_1, T_1, ϵ_1, and $c(t,x)$ be as in Section 5. Suppose $\lambda(t,x)$ is a continuous function on $[0,T_1] \times D_{R_1}$ that is holomorphic in x for any fixed t, and $\mathrm{Re}\,\lambda(0,0) < 0$. Since $R_1 > 0$ and $T_1 > 0$ can be taken sufficiently small, we may suppose that

(B_3) $\mathrm{Re}\,\lambda(t,x) \le -L$ on $[0,T_1] \times D_{R_1}$ for some $L > 0$.
Let $0 < \epsilon_1 < \epsilon < \pi/8$, $0 < T < T_1$ and $0 < R < R_1$. Consider the equation

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - x^{p+1}c(t,x)\frac{\partial w}{\partial x} = g(t,x) \quad \text{on } (0,T] \times S, \tag{6.1}$$

where

$$S = \left\{ x \in \mathbb{C} : 0 < |x| < R, \ \left| \arg x + \frac{\theta_0}{p} \right| < \frac{\pi}{2p} - \frac{\epsilon}{p} \right\}. \tag{6.2}$$

We have the following result:

PROPOSITION 6.1. *Let $g(t,x) \in X_0((0,T] \times S)$. If $g(t,x)$ and $xg_x(t,x)$ are bounded on $(0,T] \times S$, then the equation (6.1) has a unique solution $w(t,x)$ in $X_1((0,T] \times S)$, which is bounded on $(0,T] \times S$. Moreover, if $|g(t,x)| \le K$ and $|xg_x(t,x)| \le K_1$ on $(0,T] \times S$, then we have*

$$|w(t,x)| \le \frac{K}{L} \ \textit{and} \ \left| x\frac{\partial w}{\partial x}(t,x) \right| \le \frac{K_1}{L} + \frac{K_1 R^p C_1 R_1 + \Lambda_1 R_1 K}{\delta L^2}$$

on $(0,T] \times S$, where $\delta = \min\{\epsilon - \epsilon_1, \log R_1 - \log R\}$ and $\Lambda_1 > 0$ is a constant satisfying $|\lambda_x(t,x)| \le \Lambda_1$ on $[0,T_1] \times D_R$.

Let $\chi(t;t_0,x_0)$ be the unique solution of the equation (5.1), and set

$$\Omega = \{(s,t,x) : 0 < s \le t \text{ and } (t,x) \in (0,T] \times S\},$$
$$\phi(s,t,x) = \chi(s;t,x) \quad \text{on } \Omega.$$

We know that $\phi(s,t,x)$ is differentiable in s and t, holomorphic in x, and $|\phi(s,t,x)| \le |x|$ and $\phi(s,t,x) \in S$ on Ω. Moreover, from Lemma 5.3 we have

$$\left| x\frac{\partial \phi}{\partial x}(s,t,x) \right| \le \frac{R_1}{\delta} \quad \text{on } \Omega, \tag{6.3}$$

$$\left| x\frac{\partial \phi}{\partial x}(s,t,x) \right| \le |\phi(s,t,x)| \left(1 + \frac{R^p C_1 R_1}{\delta} \log\left(\frac{t}{s}\right) \right) \quad \text{on } \Omega. \tag{6.4}$$

LEMMA 6.2. *The function $\phi(s,t,x)$ is the unique solution of*

$$\begin{cases} t\dfrac{\partial \phi}{\partial t} - x^{p+1}c(t,x)\dfrac{\partial \phi}{\partial x} = 0 & \textit{on } \Omega, \\ \phi(t,t,x) = x & \textit{on } (0,T] \times S. \end{cases} \tag{6.5}$$

PROOF. We can show this result in the same way as in [**1**, Lemma 3.3], but here we give another proof. Set

$$W(s,t,x) = t\frac{\partial \phi}{\partial t} - x^{p+1}c(t,x)\frac{\partial \phi}{\partial x} \quad \text{on } \Omega.$$

Then by (5.13) and (5.14) we have

$$s\frac{\partial W}{\partial s} = -h(s,\phi(s,t,x))W, \quad W|_{s=t} = 0.$$

Since this is nothing but a linear ordinary differential equation in W we can conclude that $W \equiv 0$ on Ω. This proves that $\phi(s,t,x)$ is a solution of (6.5).

Let us show the uniqueness of the solution of (6.5). It suffices to show that if $\psi(s,t,x)$ satisfies

$$t\frac{\partial \psi}{\partial t} - x^{p+1}c(t,x)\frac{\partial \psi}{\partial x} = 0 \text{ on } \Omega, \quad \text{and } \psi(t,t,x) = 0 \text{ on } (0,T] \times S, \tag{6.6}$$

then we have $\psi \equiv 0$ on Ω.

Take any $(s,t_0,x_0) \in \Omega$ and set $\xi_0 = \chi(s;t_0,x_0)$. Since $\chi(t;t_0,x_0)$ is defined on $(0,t_0]$, we may suppose that $\chi(t;s,\xi_0)$ can be extended to $(0,t_0]$. Consequently, we have $\chi(t;s,\xi_0) = \chi(t;t_0,x_0)$ on $(0,t_0]$. In particular, we have $\chi(t_0;s,\xi_0) = x_0$.

Set $f(t) = \psi(s,t,\chi(t;s,\xi_0))$ for any $t \in (0,t_0]$. Then $f(t_0) = \psi(s,t_0,x_0)$ and $f(s) = \psi(s,s,\xi_0) = 0$. By taking the derivative of $f(t)$ with respect to t and using the fact that $\chi(t;s,\xi_0)$ satisfies (5.1) we get

$$\begin{aligned} f'(t) &= \frac{\partial \psi}{\partial t}(s,t,\chi(t;s,\xi_0)) + \frac{\partial \psi}{\partial x}(s,t,\chi(t;s,\xi_0))\frac{d\chi}{dt}(t;s,\xi_0) \\ &= \left[\frac{\partial \psi}{\partial t}(s,t,x) - \frac{x^{p+1}\,c(t,x)}{t}\frac{\partial \psi}{\partial x}(s,t,x)\right]\Bigg|_{x=\chi(t;s,\xi_0)} = 0. \end{aligned} \tag{6.7}$$

Thus, $f(t)$ is constant, and as a result, we have $\psi(s,t_0,x_0) = f(t_0) = f(s) = 0$. Since $(s,t_0,x_0) \in \Omega$ is taken arbitrarily, we conclude that $\psi \equiv 0$ on Ω. □

Proof of Proposition 6.1. We set

$$w(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau,\phi(\tau,t,x))\frac{d\tau}{\tau}\right] g(s,\phi(s,t,x))\frac{ds}{s}. \tag{6.8}$$

This integral expression makes sense because $\phi(s,t,x) \in S$ for any $(s,t,x) \in \Omega$, which means that $(s,\phi(s,t,x)) \in (0,T] \times S$ for any $(t,x) \in (0,T] \times S$ and $0 < s \le t$. If $|g(t,x)| \le K$ on $(0,T] \times S$, we have

$$\begin{aligned} |w(t,x)| &\le \int_0^t \exp\left[\int_s^t \operatorname{Re}\lambda(\tau,\phi(\tau,t,x))\frac{d\tau}{\tau}\right] |g(s,\phi(s,t,x))|\frac{ds}{s} \\ &\le \int_0^t \exp\left[\int_s^t -L\frac{d\tau}{\tau}\right] K\frac{ds}{s} \end{aligned}$$

$$= \int_0^t \left(\frac{s}{t}\right)^L K \frac{ds}{s} = \frac{K}{L} \quad \text{on } (0,T] \times S.$$

This shows that $w(t,x)$ is a well-defined function in $X_0((0,T] \times S)$. Since $w(t,x)$ is holomorphic in x, it is differentiable with respect to x, and because $w(t,x)$ is given by the integral (6.8) we also have the differentiability of $w(t,x)$ with respect to t. Thus, $w(t,x)$ belongs to $X_1((0,T] \times S)$.

From (6.8), we get

$$\begin{aligned} x\frac{\partial w}{\partial x}(t,x) &= \int_0^t \exp\left[\int_s^t \lambda(\tau,\phi(\tau,t,x))\frac{d\tau}{\tau}\right] \frac{\partial g}{\partial x}(s,\phi(s,t,x)) \cdot x\frac{\partial \phi}{\partial x}(s,t,x)\frac{ds}{s} \\ &\quad + \int_0^t \exp\left[\int_s^t \lambda(\tau,\phi(\tau,t,x))\frac{d\tau}{\tau}\right]\left(\int_s^t \frac{\partial \lambda}{\partial x}(\tau,\phi(\tau,t,x)) \cdot x\frac{\partial \phi}{\partial x}(\tau,t,x)\frac{d\tau}{\tau}\right) \\ &\quad \times g(s,\phi(s,t,x))\frac{ds}{s}. \end{aligned} \tag{6.9}$$

If $|xg_x(t,x)| \le K_1$ on $(0,T] \times S$, then by (6.4) we have

$$\begin{aligned} &\left|\frac{\partial g}{\partial x}(s,\phi(s,t,x)) \cdot x\frac{\partial \phi}{\partial x}(s,t,x)\right| \\ &\quad \le \left|\frac{\partial g}{\partial x}(s,\phi(s,t,x))\right| \times |\phi(s,t,x)|\left(1 + \frac{R^p C_1 R_1}{\delta}\log\left(\frac{t}{s}\right)\right) \\ &\quad \le K_1\left(1 + \frac{R^p C_1 R_1}{\delta}\log\left(\frac{t}{s}\right)\right). \end{aligned} \tag{6.10}$$

Therefore, by applying (6.10) and (6.3) to (6.9) we have

$$\begin{aligned} \left|x\frac{\partial w}{\partial x}(t,x)\right| &\le \int_0^t \left(\frac{s}{t}\right)^L K_1\left(1 + \frac{R^p C_1 R_1}{\delta}\log\left(\frac{t}{s}\right)\right)\frac{ds}{s} \\ &\quad + \int_0^t \left(\frac{s}{t}\right)^L\left(\int_s^t \frac{\Lambda_1 R_1}{\delta}\frac{d\tau}{\tau}\right)K\frac{ds}{s} \\ &\le \frac{K_1}{L} + \frac{K_1 R^p C_1 R_1 + \Lambda_1 R_1 K}{\delta}\int_0^t \left(\frac{s}{t}\right)^L \log\left(\frac{t}{s}\right)\frac{ds}{s} \\ &= \frac{K_1}{L} + \frac{K_1 R^p C_1 R_1 + \Lambda_1 R_1 K}{\delta L^2} \quad \text{on } (0,T] \times S. \end{aligned} \tag{6.11}$$

Here we have used the fact that $\int_0^1 x^L \log(1/x)dx/x = 1/L^2$ if $L > 0$.

Similarly, if we take $C_0 > 0$ and $\Lambda_0 > 0$ such that $|c(t,x)| \le C_0$ and $|\lambda(t,x)| \le \Lambda_0$ on $(0,T_1] \times D_R$, we can show that

$$\left|t\frac{\partial w}{\partial t}(t,x)\right| \le K + \frac{\Lambda_0 K}{L} + R^p C_0\left(\frac{K_1}{L} + \frac{K_1 R^p C_1 R_1 + \Lambda_1 R_1 K}{\delta L^2}\right)$$

on $(0,T]\times S$.

The above estimates guarantee that the formal differentiations of (6.8) make sense, and therefore, a straightforward calculation together with (6.5) shows that $w(t,x)$ is a solution to the equation (6.1), which belongs to $X_1((0,T]\times S)$ and bounded on $(0,T]\times S$.

To prove the uniqueness of the solution of (6.1), we show that if $w(t,x)\in X_1((0,T]\times S)$ is bounded on $(0,T]\times S$ and

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - x^{p+1}c(t,x)\frac{\partial w}{\partial x} = 0 \quad \text{on } (0,T]\times S, \tag{6.12}$$

then $w\equiv 0$ on $(0,T]\times S$.

Choose any $(t_0,x_0)\in(0,T]\times S$. Let $\chi(t;t_0,x_0)$ be the unique solution of the equation (5.1) and set $w_0(t) = w(t,\chi(t;t_0,x_0))$ on $(0,t_0]$. Then by the same calculation as in (6.7) we get

$$t\frac{dw_0}{dt}(t) = \lambda(t,\chi(t;t_0,x_0))w_0(t).$$

Since $w_0(t_0) = w(t_0,x_0)$, it follows that

$$w_0(t) = w(t_0,x_0)\times\exp\left[-\int_t^{t_0}\lambda(\tau,\chi(\tau;t_0,x_0))\frac{d\tau}{\tau}\right], \quad t\in(0,t_0].$$

Therefore, by (B_3) and the fact that $w_0(t) = O(1)$ (as $t\longrightarrow 0$) we obtain

$$\begin{aligned}|w(t_0,x_0)| &\le |w_0(t)|\times\exp\left[\int_t^{t_0}\mathrm{Re}\,\lambda(\tau,\chi(\tau;t_0,x_0))\frac{d\tau}{\tau}\right]\\ &\le |w_0(t)|\times\exp\left[\int_t^{t_0} -L\frac{d\tau}{\tau}\right] = |w_0(t)|\left(\frac{t}{t_0}\right)^L \longrightarrow 0 \text{ (as } t\longrightarrow 0).\end{aligned}$$

This shows that $w(t_0,x_0)=0$. Since $(t_0,x_0)\in(0,T]\times S$ is chosen arbitrarily, we have $w\equiv 0$ on $(0,T]\times S$. □

7. The equation (3.4) on W_r.

Let $0<R<R_1$, L, S, Ω, $c(t,x)$, $\lambda(t,x)$, and $\phi(s,t,x)$ be the same as in Section 6. Let $\mu(t)$ be a weight function on $(0,T_1]$, and $\varphi(t)$ be the function defined by (2.2) on $(0,T_1]$. In this section, we require $0<\epsilon_1<\epsilon<\pi/16$.

For any $r>0$ we set

$$W_r = \left\{(t,x)\in(0,T]\times S : \frac{\varphi(t)}{r} < d_S(x)\right\}, \tag{7.1}$$

where

$$d_S(x) = \min\left\{\frac{\pi}{2p} - \frac{\epsilon}{p} - \left|\arg x + \frac{\theta_0}{p}\right|,\ \log R - \log|x|\right\} \quad \text{for } x \in S. \tag{7.2}$$

Also, we set $S_\tau = \{x \in S : (\tau, x) \in W_r\}$ for any $\tau > 0$. Thus, we have

$$S_\tau = \left\{x \in \mathbb{C} : 0 < |x| < Re^{-\varphi(\tau)/r},\ \left|\arg x + \frac{\theta_0}{p}\right| < \frac{\pi}{2p} - \frac{\epsilon}{p} - \frac{\varphi(\tau)}{r}\right\}.$$

Obviously, we have $S_\tau \neq \emptyset$ for any $0 < \tau \leq T$ whenever $\varphi(T)/r < \pi/2p - \epsilon/p$. We say that S' is a *subsector* of S_τ if it can be expressed as $S' = \{x \in \mathbb{C} : 0 < |x| < \eta, |\arg x + \theta_0/p| < \omega\}$ for some $0 < \eta < \mathrm{Re}^{-\varphi(\tau)/r}$ and $0 < \omega < \pi/2p - \epsilon/p - \varphi(\tau)/r$.

We define another two spaces on the region W_r. We denote by $\mathscr{X}_0(W_r)$ the set of all continuous functions on W_r that are holomorphic in x for any fixed t and bounded on $(0,\tau] \times S'$ for any $\tau \in (0,T]$ and any subsector S' of S_τ. We then set $\mathscr{X}_1(W_r) = C^1(W_r) \cap \mathscr{X}_0(W_r)$.

Choose $T > 0$ sufficiently small so that $\varphi(T)/r \leq \epsilon/p$. Then, by Corollary 5.2 we have

$$d_S(\phi(s,t,x)) - \varphi(s)/r \geq \frac{2\epsilon}{\pi}\big(d_S(x) - \varphi(s)/r\big) \quad \text{on } \Omega, \tag{7.3}$$

$$(s, \phi(s,t,x)) \in W_r \quad \text{for any } (s,t,x) \in \Omega. \tag{7.4}$$

Now, consider the equation

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - x^{p+1}c(t,x)\frac{\partial w}{\partial x} = g(t,x) \tag{7.5}$$

on the region W_r. Then we have the following result which is similar to Proposition 6.1.

Proposition 7.1. *For any given $g(t,x) \in \mathscr{X}_0(W_r)$, the equation (7.5) has a unique solution $w(t,x)$ in $\mathscr{X}_1(W_r)$, and it is given by*

$$w(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau,t,x))\frac{d\tau}{\tau}\right] g(s, \phi(s,t,x))\frac{ds}{s}. \tag{7.6}$$

Moreover, the following estimates are true on W_r given any nondecreasing, nonnegative function $\psi(t)$:

(a) *If $|g(t,x)| \leq K\psi(t)\mu(t)$, then we have $|w(t,x)| \leq K\psi(t)\varphi(t)$.*
(b) *If $|g(t,x)| \leq K\psi(t)$ and $|xg_x(t,x)| \leq K_1\psi(t)$, then we have*

$$|w(t,x)| \leq \frac{K}{L}\psi(t) \quad \textit{and} \quad \left|x\frac{\partial w}{\partial x}(t,x)\right| \leq \left(\frac{K_1}{L} + \frac{K_1R^pC_1R_1 + \Lambda_1R_1K}{\delta L^2}\right)\psi(t).$$

(c) *If* $|g(t,x)| \le K\psi(t)\mu(t)/(d_S(x)-\varphi(t)/r)$, *then we have*

$$|w(t,x)| \le \frac{(\pi/2p)K\psi(t)\,r}{d_S(x)-\varphi(t)/r} \quad \textit{and}$$
$$\left| x\frac{\partial w}{\partial x}(t,x) \right| \le \left(4\left(1+\frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 + \frac{\Lambda_1 R_1}{\delta e L}\frac{\pi}{2p} \right) \frac{K\psi(t) r}{(d_S(x)-\varphi(t)/r)}.$$

The constants C_1, Λ_1 and δ are the same as in Proposition 6.1. *Note that they are independent of T.*

Proof. Since $0<\epsilon<\pi/16$ and $\varphi(T)/r \le \epsilon/p$, we have $\epsilon + p\varphi(\tau)/r < \pi/8$. Thus, by setting

$$\Sigma(\tau) = \bigl\{ (R_2,\epsilon_2) : 0 < R_2 < Re^{-\varphi(\tau)/r},\ \epsilon + p\varphi(\tau)/r < \epsilon_2 < \pi/8 \bigr\},$$
$$S(R_2,\epsilon_2) = \left\{ x \in \mathbb{C} : 0 < |x| < R_2,\ \left| \arg x + \frac{\theta_0}{p} \right| < \frac{\pi}{2p} - \frac{\epsilon_2}{p} \right\},$$

we have

$$W_r = \bigcup_{0<\tau\le T} (0,\tau] \times S_\tau = \bigcup_{0<\tau\le T} \bigcup_{(R_2,\epsilon_2)\in\Sigma(\tau)} (0,\tau]\times S(R_2,\epsilon_2).$$

It follows from Lemma 4.2 and the fact that $g(t,x) \in \mathscr{X}_0(W_r)$ that $g(t,x)$ and $xg_x(t,x)$ are bounded on $(0,\tau]\times S(R_2,\epsilon_2)$ for any $0<\tau\le T$ and $(R_2,\epsilon_2)\in\Sigma(\tau)$. Therefore, by applying Proposition 6.1 to the equation (7.5) on $(0,\tau]\times S(R_2,\epsilon_2)$, we obtain a unique solution $w(t,x)$ of (7.5), which is defined by the integral in (7.6) and belongs to $\mathscr{X}_1(W_r)$.

The estimate (a) is verified as follows:

$$|w(t,x)| \le \int_0^t \left(\frac{s}{t}\right)^L K\psi(s)\mu(s)\frac{ds}{s} \le K\psi(t)\int_0^t \mu(s)\frac{ds}{s} = K\psi(t)\varphi(t) \quad \text{on } W_r.$$

The estimates in (b) can be proved in the same way as in Proposition 6.1.

Let us prove the estimate (c). The first estimate follows from (7.3) and the fact that $d_S(x)-\varphi(s)/r \ge d_S(x)-\varphi(t)/r$ on Ω and $\varphi(t)/r \le \epsilon/p$ on $(0,T]$:

$$\begin{aligned} |w(t,x)| &\le \int_0^t \left(\frac{s}{t}\right)^L \frac{K\psi(s)\mu(s)}{(d_S(\phi(s,t,x))-\varphi(s)/r)}\frac{ds}{s} \\ &\le \frac{(\pi/2\epsilon)K\psi(t)}{d_S(x)-\varphi(t)/r}\int_0^t \mu(s)\frac{ds}{s} = \frac{(\pi/2\epsilon)K\psi(t)\varphi(t)}{d_S(x)-\varphi(t)/r} \\ &\le \frac{(\pi/2p)K\psi(t)\,r}{d_S(x)-\varphi(t)/r} \quad \text{on } W_r. \end{aligned}$$

Let us show the second estimate of (c). By Corollary 4.3 we have

$$\left| x\frac{\partial g}{\partial x}(t,x) \right| \le \frac{4K\psi(t)\mu(t)}{(d_S(x)-\varphi(t)/r)^2} \quad \text{on } W_r.$$

Since $\varphi'(t) = \mu(t)/t$ and $\sup_{(0,1]} |x^L \log(1/x)| = 1/(eL)$, by the same argument as in (6.11) we obtain

$$\begin{aligned}
\left| x\frac{\partial w}{\partial x}(t,x) \right| &\le \int_0^t \left(\frac{s}{t}\right)^L \frac{4K\psi(s)\mu(s)}{(d_S(\phi(s,t,x))-\varphi(s)/r)^2}\left(1+\frac{R^p C_1 R_1}{\delta}\log\left(\frac{t}{s}\right)\right)\frac{ds}{s} \\
&\quad + \int_0^t \left(\frac{s}{t}\right)^L \frac{\Lambda_1 R_1}{\delta}\frac{K\psi(s)\mu(s)}{(d_S(\phi(s,t,x))-\varphi(s)/r)}\left(\log\left(\frac{t}{s}\right)\right)\frac{ds}{s} \\
&\le 4K\psi(t)\left(1+\frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 \int_0^t \frac{\varphi'(s)}{(d_S(x)-\varphi(s)/r)^2}ds \\
&\quad + \frac{\Lambda_1 R_1}{\delta e L}\frac{(\pi/2\epsilon)K\psi(t)}{(d_S(x)-\varphi(t)/r)}\int_0^t \mu(s)\frac{ds}{s} \\
&= 4K\psi(t)\left(1+\frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 \left[\frac{r}{(d_S(x)-\varphi(s)/r)}\right]_{s=0}^{s=t} \\
&\quad + \frac{\Lambda_1 R_1}{\delta e L}\frac{(\pi/2\epsilon)K\psi(t)\varphi(t)}{(d_S(x)-\varphi(t)/r)} \\
&\le \left(4\left(1+\frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 + \frac{\Lambda_1 R_1}{\delta e L}\frac{\pi}{2p}\right)\frac{K\psi(t) r}{(d_S(x)-\varphi(t)/r)}.
\end{aligned}$$

In the last inequality we used again the fact that $\varphi(t)/r \le \epsilon/p$ on $(0,T]$. □

8. Proof of Proposition 3.1.

Let $0 < R_1 < R_0$, $0 < T_1 < T_0$, $c(t,x)$ and $\lambda(t,x)$ be the same as in the previous section. Consider the equation

$$\begin{aligned}
&t\frac{\partial u}{\partial t} - \lambda(t,x)u - x^{p+1}c(t,x)\frac{\partial u}{\partial x} \\
&\quad = a(t,x) + xb(t,x)\frac{\partial u}{\partial x} + R_2\left(t,x,u,x\frac{\partial u}{\partial x}\right),
\end{aligned} \tag{8.1}$$

where

$$R_2(t,x,u,v) = \sum_{i+j\ge 2} a_{i,j}(t,x)u^i v^j.$$

Here we assume that $a(t,x)$ and $b(t,x)$ are continuous functions on $[0,T_1]\times D_{R_1}$ that are holomorphic in x for any fixed t, and $R_2(t,x,u,v)$ is a continuous function on $\Delta_1 = [0,T_1]\times D_{R_1}\times D_{\rho_1}\times D_{\rho_1}$ for some $\rho_1 > 0$ that is holomorphic in (x,u,v) for any fixed t. Because of (2.3) and (2.4), we may assume that

(B_4) $|a(t,x)| \le A\mu(t)$ on $[0,T_1] \times D_{R_1}$ for some $A > 0$,
(B_5) $|b(t,x)| \le B\mu(t)$ on $[0,T_1] \times D_{R_1}$ for some $B > 0$,

for some weight function $\mu(t)$ on $(0,T_1]$. Again, we define the function $\varphi(t)$ by (2.2) on $(0,T_1]$.

In this section, we prove Proposition 3.1 in the following form:

THEOREM 8.1. *Suppose* (B_1)*–*(B_5) *hold. Then for any* $\epsilon_1 < \epsilon < \pi/16$, $0 < \rho < \rho_1$ *and* $0 < R < R_1$, *there exist* $T > 0$, $r > 0$ *and* $M > 0$ *with* $M\mu(T) \le \rho$ *such that the equation* (8.1) *has a unique solution* $u(t,x)$ *in* $\mathscr{X}_1(W_r)$ *that satisfies*

$$|u(t,x)| \le M\mu(t) \text{ and } \left| x\frac{\partial u}{\partial x}(t,x) \right| \le M\mu(t) \quad \text{on } W_r, \tag{8.2}$$

where W_r *is the region defined in* (7.1), *which depends on* ϵ, R, T *and* r.

We write the equation (8.1) as

$$\mathcal{P}u = a(t,x) + \Phi[u], \tag{8.3}$$

where

$$\mathcal{P} = t\frac{\partial}{\partial t} - \lambda(t,x) - x^{p+1}c(t,x)\frac{\partial}{\partial x},$$
$$\Phi[u] = xb(t,x)\frac{\partial u}{\partial x} + R_2\left(t,x,u,x\frac{\partial u}{\partial x}\right).$$

We recall that in [**7**] the equation (8.3), in the case $c(t,x) \equiv 0$, was solved by the method of Nirenberg [**10**] and Nishida [**11**], while in [**1**] the case $p = 0$ was solved by using a fixed point theorem (or a contraction principle) like in Walter [**13**]. To prove Theorem 8.1, we will use similar arguments as in [**7**]. The difference is that in [**7**] the discussion was done on a disk, while in our case the discussion must be done on a sector S. The use of the distance function $d_S(x)$ is essential for our purpose.

PROOF OF THEOREM 8.1. Take any $\epsilon_1 < \epsilon < \pi/16$, $0 < \rho < \rho_1$ and $0 < R < R_1$. We define the sector S by (6.2) and the function $d_S(x)$ by (7.2). Because of (B_4), we may suppose that

$$|xa_x(t,x)| \le A_1\mu(t) \text{ on } [0,T_1] \times D_R \text{ for some } A_1 > 0. \tag{8.4}$$

We define $M > 0$ by

$$M/2 = \max\left\{\frac{A}{L}, \left(\frac{A_1}{L} + \frac{A_1R^pC_1R_1 + \Lambda_1R_1A}{\delta L^2}\right)\right\}, \tag{8.5}$$

where C_1, Λ_1 and δ are the same as in Proposition 7.1. Note that this M is independent of T.

Let $\Delta = [0, T_1] \times D_R \times D_\rho \times D_\rho$, and set

$$\begin{cases} B_{2,0} = \sup_{\Delta} |(\partial^2 R_2/\partial u^2)(t,x,u,v)|, \\ B_{1,1} = \sup_{\Delta} |(\partial^2 R_2/\partial u \partial v)(t,x,u,v)|, \\ B_{0,2} = \sup_{\Delta} |(\partial^2 R_2/\partial v^2)(t,x,u,v)|. \end{cases} \tag{8.6}$$

We also set

$$\alpha = B + (B_{2,0} + 2B_{1,1} + B_{0,2})M,$$
$$\beta = 4\left(1 + \frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 + \frac{\Lambda_1 R_1}{\delta e L}\frac{\pi}{2p}.$$

Choose $r_0 > 0$ sufficiently small so that $0 < 2\alpha\beta r_0 < 1$. Then we define the decreasing sequence $r_0 > r_1 > r_2 > \cdots$ by

$$r_k = r_0 \times \prod_{p=1}^{k} \bigl(1 - (2\alpha\beta r_0)^p\bigr), \quad k = 1, 2, \ldots.$$

This is a sequence of positive numbers converging to a positive limit r_∞. Moreover, we have

$$\frac{(\alpha\beta r_0)^k}{1 - r_k/r_{k-1}} = \left(\frac{1}{2}\right)^k, \quad k = 1, 2, \ldots. \tag{8.7}$$

Set $r = r_\infty$ and take $T > 0$ small enough so that $M\mu(T) \le \rho$ and $\varphi(T)/r \le \epsilon/p$. Clearly, we have

$$\varphi(T)/r_k \le \epsilon/p, \quad k = 0, 1, 2, \ldots, \infty.$$

In accordance with our definition in (7.1), we set

$$W_{r_k} = \{(t,x) \in (0,T] \times S : \varphi(t)/r_k < d_S(x)\}, \quad k = 0, 1, 2, \ldots,$$
$$W_{r_\infty} = W_r = \{(t,x) \in (0,T] \times S : \varphi(t)/r < d_S(x)\}.$$

Notice that

$$W_{r_0} \supset W_{r_1} \supset W_{r_2} \supset \cdots \supset W_{r_k} \supset \cdots \supset W_{r_\infty}.$$

Lemma 8.2. *Let $w_j(t,x) \in \mathscr{X}_0(W_\varepsilon)$ $(j = 1, 2)$ for some $0 < \varepsilon \le r_0$. If both $|w_j(t,x)|$ and $|x(\partial w_j/\partial x)(t,x)|$ are bounded by $M\mu(t)$ on W_ε, then we have $\Phi[w_j] \in \mathscr{X}_0(W_\varepsilon)$ $(j = 1, 2)$ and*

$$|\Phi[w_1] - \Phi[w_2]| \leq B\mu(t)\left|x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right| + (B_{2,0} + B_{1,1})M\mu(t)|w_1 - w_2| \\ + (B_{1,1} + B_{0,2})M\mu(t)\left|x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right| \quad \text{on } W_\varepsilon. \tag{8.8}$$

PROOF. By (B_5) and the definition of the function Φ, we have

$$|\Phi[w_1] - \Phi[w_2]| \\ \leq B\mu(t)\left|x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right| + \left|R_2\left(t, x, w_1, x\frac{\partial w_1}{\partial x}\right) - R_2\left(t, x, w_2, x\frac{\partial w_2}{\partial x}\right)\right|.$$

Recall from Taylor's theorem that a function $f(u,v)$ that is holomorphic in a neighborhood of $(0,0) \in \mathbb{C} \times \mathbb{C}$ may be expressed as

$$f(u,v) = f(0,0) + u\int_0^1 \frac{\partial f}{\partial u}(su, sv)ds + v\int_0^1 \frac{\partial f}{\partial v}(su, sv)ds. \tag{8.9}$$

Since $w_1 = (w_1 - w_2) + w_2$ and $x(\partial w_1/\partial x) = (x(\partial w_1/\partial x) - x(\partial w_2/\partial x)) + x(\partial w_2/\partial x)$, by (8.9) we obtain

$$R_2\left(t, x, w_1, x\frac{\partial w_1}{\partial x}\right) - R_2\left(t, x, w_2, x\frac{\partial w_2}{\partial x}\right) \\ = (w_1 - w_2)\int_0^1 \frac{\partial R_2}{\partial u}\left(t, x, s(w_1 - w_2) + w_2, s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + x\frac{\partial w_2}{\partial x}\right)ds \\ + \left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right)\int_0^1 \frac{\partial R_2}{\partial v}\left(t, x, s(w_1 - w_2) + w_2, s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + \frac{\partial w_2}{\partial x}\right)ds. \tag{8.10}$$

Again, by applying (8.9) to the first integrand in (8.10) and using the fact that $(\partial R_2/\partial u)(t, x, 0, 0) = 0$, we arrive at

$$\frac{\partial R_2}{\partial u}\left(t, x, s(w_1 - w_2) + w_2, s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + x\frac{\partial w_2}{\partial x}\right) \\ = (s(w_1 - w_2) + w_2) \\ \times \int_0^1 \frac{\partial^2 R_2}{\partial u^2}\left(t, x, \sigma(s(w_1 - w_2) + w_2), \sigma\left(s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + x\frac{\partial w_2}{\partial x}\right)\right)d\sigma \\ + \left(s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + x\frac{\partial w_2}{\partial x}\right) \\ \times \int_0^1 \frac{\partial^2 R_2}{\partial v \partial u}\left(t, x, \sigma(s(w_1 - w_2) + w_2), \sigma\left(s\left(x\frac{\partial w_1}{\partial x} - x\frac{\partial w_2}{\partial x}\right) + x\frac{\partial w_2}{\partial x}\right)\right)d\sigma. \tag{8.11}$$

Clearly, the bounds in (8.6) and the fact that both $|w_j(t,x)|$ and $|x(\partial w_j/\partial x)(t,x)|$ are bounded by $M\mu(t)$ on W_ε (for $j=1,2$) imply that (8.11) is bounded on W_ε by $(B_{2,0}+B_{1,1})M\mu(t)$. Similar argument shows that the second integrand in (8.10) is bounded on W_ε by $(B_{1,1}+B_{0,2})M\mu(t)$. Then, the desired estimate for $|\Phi[w_1-\Phi[w_2]|$ follows immediately from (8.10). □

Now, let us solve the equation (8.3). We define the approximate solutions $u_k(t,x)\in\mathscr{X}_1(W_{r_k})$ $(k=0,1,2,\dots)$ by

$$\mathcal{P}u_0=a(t,x) \tag{8.12}$$

and for $k\geq 1$,

$$\mathcal{P}u_k=a(t,x)+\Phi[u_{k-1}]. \tag{8.13}$$

By applying Proposition 7.1 to (8.12) and using the estimates (B_4) and (8.4), we obtain a unique solution $u_0\in\mathscr{X}_1(W_{r_0})$ satisfying

$$\begin{aligned} |u_0(t,x)| &\leq \frac{A}{L}\mu(t) \quad \text{on } W_{r_0}, \text{ and} \\ \left|x\frac{\partial u_0}{\partial x}(t,x)\right| &\leq \left(\frac{A_1}{L}+\frac{A_1R^pC_1R_1+\Lambda_1R_1A}{\delta L^2}\right)\mu(t) \quad \text{on } W_{r_0}. \end{aligned}$$

Thus, by our choice of M we have

$$\max\left\{|u_0|,\left|x\frac{\partial u_0}{\partial x}\right|\right\}\leq (M/2)\mu(t) \quad \text{on } W_{r_0}. \tag{8.14}$$

As we proceed, we show that (8.13) $(k=1,2,3,\dots)$ has a unique solution $u_k(t,x)\in\mathscr{X}_1(W_{r_k})$, and prove that they converge to a solution of (8.3) in W_r.

Proposition 8.3. *The following statements hold for $k\geq 1$:*

$(1)_k$ *There exists a unique $u_k\in\mathscr{X}_1(W_{r_{k-1}})$ satisfying the equation* (8.13).

$(2)_k$ *On $W_{r_{k-1}}$, we have*

$$\max\left\{|u_k-u_{k-1}|,\left|x\frac{\partial u_k}{\partial x}-x\frac{\partial u_{k-1}}{\partial x}\right|\right\}\leq\frac{\alpha\beta(\alpha\beta r_0)^{k-1}(M/2)\varphi(t)\mu(t)}{d_S(x)-\varphi(t)/r_{k-1}}.$$

$(3)_k$ *On W_{r_k}, we have*

$$\max\left\{|u_k-u_{k-1}|,\left|x\frac{\partial u_k}{\partial x}-x\frac{\partial u_{k-1}}{\partial x}\right|\right\}\leq\frac{(M/2)\mu(t)}{2^k}.$$

$(4)_k$ *On W_{r_k}, we have*

$$\max\left\{|u_k|, \left|x\frac{\partial u_k}{\partial x}\right|\right\} \le \sum_{i=0}^{k}\left(\frac{1}{2}\right)^i \times (M/2)\mu(t).$$

Proof. We prove Proposition 8.3 by induction.

By (8.14) and Lemma 8.2 we have $\Phi[u_0] \in \mathscr{X}_0(W_{r_0})$. Therefore, the initial case $(1)_1$ follows immediately from Proposition 7.1. Also from Lemma 8.2, we have $|\Phi[u_0]| \le \alpha(M/2)(\mu(t))^2$ on W_{r_0}. Thus, applying (a) of Proposition 7.1 to

$$\mathcal{P}(u_1 - u_0) = \Phi[u_0] \quad \text{on } W_{r_0}$$

gives us $|u_1 - u_0| \le \alpha(M/2)\mu(t)\varphi(t)$ on W_{r_0}, and by Nagumo's lemma (Corollary 4.3) we get

$$\left|x\frac{\partial u_1}{\partial x} - x\frac{\partial u_0}{\partial x}\right| \le \frac{\alpha(M/2)\mu(t)\varphi(t)}{d_S(x) - \varphi(t)/r_0} \quad \text{on } W_{r_0}.$$

Since $d_S(x) - \varphi(t)/r_0 < d_S(x) < \pi/2p < 2 < \beta$, we obtain

$$\max\left\{|u_1 - u_0|, \left|x\frac{\partial u_1}{\partial x} - x\frac{\partial u_0}{\partial x}\right|\right\} \le \frac{\alpha\beta(M/2)\varphi(t)\mu(t)}{d_S(x) - \varphi(t)/r_0} \quad \text{on } W_{r_0},$$

which proves $(2)_1$. Moreover, on W_{r_1} we have $(d_S(x) - \varphi(t)/r_0) = (d_S(x) - \varphi(t)/r_1) + (\varphi(t)/r_1 - \varphi(t)/r_0) > (\varphi(t)/r_1 - \varphi(t)/r_0)$ and so by (8.7) ($k = 1$) we have

$$\begin{aligned}\frac{\alpha\beta(M/2)\mu(t)\varphi(t)}{d_S(x) - \varphi(t)/r_0} &\le \frac{\alpha\beta(M/2)\mu(t)\varphi(t)}{(\varphi(t)/r_1 - \varphi(t)/r_0)} = \frac{\alpha\beta(M/2)\mu(t)r_1}{(1 - r_1/r_0)}\\ &\le \frac{(\alpha\beta r_0)(M/2)\mu(t)}{(1 - r_1/r_0)} \le \frac{1}{2} \times (M/2)\mu(t) \quad \text{on } W_{r_1},\end{aligned}$$

which proves $(3)_1$. Since $u_1 = (u_1 - u_0) + u_0$, by (8.14) and $(3)_1$ we obtain $(4)_1$.

Suppose $(1)_k - (4)_k$ hold for $0 \le k \le n$. By $(4)_n$ and Lemma 8.2 we have $\Phi[u_n] \in \mathscr{X}_0(W_{r_n})$. Thus, by Proposition 7.1 we have $(1)_{n+1}$, and consequently, we have

$$\mathcal{P}(u_{n+1} - u_n) = \Phi[u_n] - \Phi[u_{n-1}] \quad \text{on } W_{r_n}. \tag{8.15}$$

By Lemma 8.2, $(2)_n$ and the relation $0 < r_n < r_{n-1}$, we have

$$|\Phi[u_n] - \Phi[u_{n-1}]| \le \alpha \times \frac{\alpha\beta(\alpha\beta r_0)^{n-1}(M/2)\varphi(t)(\mu(t))^2}{d_S(x) - \varphi(t)/r_n} \quad \text{on } W_{r_n}.$$

Therefore, by applying (c) of Proposition 7.1 to (8.15) we get

$$|u_{n+1} - u_n| \le \alpha \times \frac{\alpha\beta(\alpha\beta r_0)^{n-1}(M/2)\varphi(t)\mu(t)r_n}{d_S(x) - \varphi(t)/r_n} \quad \text{on } W_{r_n},$$

and

$$\left| x\frac{\partial u_{n+1}}{\partial x} - x\frac{\partial u_n}{\partial x} \right| \le \left(4\left(1 + \frac{R^p C_1 R_1}{\delta e L}\right)\left(\frac{\pi}{2\epsilon}\right)^2 + \frac{\Lambda_1 R_1}{\delta e L}\frac{\pi}{2p} \right)$$
$$\times \alpha \times \frac{\alpha\beta(\alpha\beta r_0)^{n-1}(M/2)\varphi(t)\mu(t) \times r_n}{d_S(x) - \varphi(t)/r_n} \quad \text{on } W_{r_n}.$$

Hence, from our definition of β and the fact that $r_n < r_0$, we establish $(2)_{n+1}$.

Since we have $(d_S(x) - \varphi(t)/r_n) = (d_S(x) - \varphi(t)/r_{n+1}) + (\varphi(t)/r_{n+1} - \varphi(t)/r_n) > (\varphi(t)/r_{n+1} - \varphi(t)/r_n)$ on $W_{r_{n+1}}$, by $(2)_{n+1}$ and (8.7) $(k = n+1)$ we obtain

$$\max\left\{ |u_{n+1} - u_n|, \left| x\frac{\partial u_{n+1}}{\partial x} - x\frac{\partial u_n}{\partial x} \right| \right\}$$
$$\le \frac{\alpha\beta(\alpha\beta r_0)^n (M/2)\varphi(t)\mu(t)}{(\varphi(t)/r_{n+1} - \varphi(t)/r_n)} = \frac{\alpha\beta(\alpha\beta r_0)^n (M/2)\mu(t) \times r_{n+1}}{(1 - r_{n+1}/r_n)}$$
$$\le \frac{(\alpha\beta r_0)^{n+1}(M/2)\mu(t)}{(1 - r_{n+1}/r_n)} \le \left(\frac{1}{2}\right)^{n+1} \times (M/2)\mu(t) \quad \text{on } W_{r_{n+1}}.$$

This proves $(3)_{n+1}$. Finally, we obtain $(4)_{n+1}$ from $(4)_n$ and $(3)_{n+1}$. □

The existence of a solution of (8.3) follows from the preceding proposition. Note that for $k \ge 1$, we have

$$u_k(t,x) = u_0(t,x) + \sum_{j=1}^{k} (u_j - u_{j-1})(t,x)$$

and thus, it follows from $(3)_k$ and $(4)_k$ $(k = 1, 2, \ldots)$ that our approximate solutions converge to a function $u(t,x) \in \mathscr{X}_0(W_r)$ satisfying $|u(t,x)| \le M\mu(t)$ on W_r. Similarly, $x(\partial u_k/\partial x)$ converges to $x(\partial u/\partial x)$ and we have $|x(\partial u/\partial x)| \le M\mu(t)$ on W_r. Since $u_k(t,x)$ may be written in the integral form

$$u_k(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau,t,x)) \frac{d\tau}{\tau} \right] \Big(a + \Phi[u_{k-1}] \Big)(s, \phi(s,t,x)) \frac{ds}{s}$$

(with $a = a(t,x)$), by letting k approach infinity, we get

$$u(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau,t,x)) \frac{d\tau}{\tau} \right] \Big(a + \Phi[u] \Big)(s, \phi(s,t,x)) \frac{ds}{s},$$

which shows that $u(t,x)$ belongs to $\mathscr{X}_1(W_r)$ and it is indeed a solution of (8.3).

Finally, let us show the uniqueness of the solution. Suppose we have another solution $v(t,x) \in \mathscr{X}_1(W_r)$ satisfying $|v(t,x)| \le M\mu(t)$ and $|x(\partial v/\partial x)| \le M\mu(t)$ on W_r. To prove that $u \equiv v$ on W_r, we show by induction that the following estimate for $|u - v|$ and

$|x(\partial u/\partial x) - x(\partial v/\partial x)|$ holds on W_r for $k = 0, 1, 2, \ldots$:

$$\max\left\{|u - v|, \left|x\frac{\partial u}{\partial x} - x\frac{\partial v}{\partial x}\right|\right\} \le \frac{4M(\alpha\beta r)^k \mu(t)}{d_S(x) - \varphi(t)/r}. \tag{8.16}$$

The case $k = 0$ is clear due to the fact that $d_S(x) < 2$ and both u and v satisfy (8.2). Assume now that (8.16) holds for $k = n$. Then, by Lemma 8.2,

$$|\Phi[u] - \Phi[v]| \le \alpha \times \frac{4M(\alpha\beta r)^n (\mu(t))^2}{d_S(x) - \varphi(t)/r} \quad \text{on } W_r.$$

Similar to our previous computations, it follows from (c) of Proposition 7.1 that

$$\max\left\{|u - v|, \left|x\frac{\partial u}{\partial x} - x\frac{\partial v}{\partial x}\right|\right\} \le \frac{4M(\alpha\beta r)^{n+1} \mu(t)}{d_S(x) - \varphi(t)/r} \quad \text{on } W_r,$$

which is the case $k = n + 1$. Therefore, (8.16) is true for all $k \ge 0$. Since $\alpha\beta r < 1/2$, by letting k approach infinity we obtain $u \equiv v$ on W_r. This completes the proof of Theorem 8.1. □

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