J. Math. Soc. Japan
Vol. 66, No. 3 (2014) pp. 997–1016
doi: 10.2969/jmsj/06630997

On Jacobian Kummer surfaces

By Kenji Koike

(Received Sep. 7, 2012)

Abstract. We give explicit equations of smooth Jacobian Kummer surfaces of degree 8 in $\mathbb{P}^5$ by theta functions. As byproducts, we can write down Rosenhain's 80 hyperplanes and 32 lines on these Kummer surfaces explicitly. Moreover we study the fibration of Kummer surfaces over the Satake compactification of the Siegel modular 3-fold of level $(2,4)$. The total space is a smooth projective 5-fold which is regarded as a higher-dimensional analogue of Shioda's elliptic modular surfaces.

1. Introduction.

A 1-dimensional complex torus $E_\tau = \mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$ can be embedded in $\mathbb{P}^3$ by fourth order theta functions. Explicitly, the holomorphic map

$$E_\tau \longrightarrow \mathbb{P}^3, \qquad z \mapsto [\theta_{00} : \theta_{01} : \theta_{10} : \theta_{11}](2z,\tau) = [x_{00} : x_{01} : x_{10} : x_{11}]$$

is an isomorphism from E_τ to a complete intersection

$$(*1)\ \begin{cases} a_0^2 x_{00}^2 = a_1^2 x_{01}^2 + a_2^2 x_{10}^2 \\ a_0^2 x_{11}^2 = a_2^2 x_{01}^2 - a_1^2 x_{10}^2 \end{cases}$$

with coefficients

$$a_0^2 = \theta_{00}(0,\tau)^2, \quad a_1^2 = \theta_{01}(0,\tau)^2, \quad a_2^2 = \theta_{10}(0,\tau)^2$$

(see [**Mu**]). In the higher dimensional case, defining equations of Abelian varieties are very complicated. For example, fourth order theta functions embed principally polarized Abelian surfaces into $\mathbb{P}^{15}$. Flynn showed that the Jacobian variety $\mathrm{Jac}(C)$ of a curve C of genus two is defined by 72 quadrics in $\mathbb{P}^{15}$, and he gave explicit 72 equations in terms of coefficients of the equation of C ([**Fl**]). On the other hand, the Kummer surface $\mathrm{Jac}(C)/\{\pm 1\}$ is given as a quartic surface in $\mathbb{P}^3$, and its minimal desingularization $\mathrm{Km}(C)$ is given as a complete intersection of three quadrics in $\mathbb{P}^5$. In [**Kl**], Klein gave Kummer surfaces

$$\begin{cases} x_1^2 + \cdots + x_6^2 = 0 \\ k_1 x_1^2 + \cdots + k_6 x_6^2 = 0 \\ k_1^2 x_1^2 + \cdots + k_6^2 x_6^2 = 0 \end{cases}$$

2010 *Mathematics Subject Classification.* Primary 14K25; Secondary 14J28.
Key Words and Phrases. theta functions, Kummer surfaces.

as singular surfaces of quadratic line complexes. These surfaces are desingularized Kummer surfaces of curves

$$y^2 = (x - k_1) \cdots (x - k_6)$$

of genus two (see [**GH**]).

In this paper, we rewrite Klein's models by Riemann's theta functions. Namely, we study a rational map from a principally polarized Abelian surface X to $\mathbb{P}^5$ given by six odd theta functions of order 4. Applying Riemann's theta relations, we obtain defining equations of $\mathrm{Km}(X)$:

$$\begin{array}{llllll}
\text{(E1)} & A_{10}^2 X_1^2 & & & +A_1^2 X_4^2 - A_2^2 X_5^2 - A_5^2 X_6^2 = 0, \\
\text{(E2)} & & A_{10}^2 X_2^2 & & +A_3^2 X_4^2 - A_4^2 X_5^2 - A_8^2 X_6^2 = 0, \\
\text{(E3)} & & & A_{10}^2 X_3^2 & +A_6^2 X_4^2 - A_9^2 X_5^2 - A_7^2 X_6^2 = 0,
\end{array}$$

where $A_1, \dots, A_{10}$ are ten even theta constants (Proposition 3.3). These equations are considered as a two-dimensional analogue of $(*1)$. Note that coefficients a_0^2, a_1^2, a_2^2 in $(*1)$ are modular forms of level 4. In fact, the graded ring of modular forms of level 4 is given by

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma(4)) = \mathbb{C}\big[\theta_{00}^2,\ \theta_{01}^2,\ \theta_{10}^2\big], \qquad \theta_{00}^4 - \theta_{01}^4 - \theta_{10}^4 = 0,$$

and $(*1)$ gives an elliptic fibration over $\mathbb{H}/\Gamma(4)$. More precisely, $(*1)$ together with

$$(*2) \begin{cases} a_1^2 x_{00}^2 + a_2^2 x_{11}^2 = a_0^2 x_{01}^2 \\ a_2^2 x_{00}^2 - a_1^2 x_{11}^2 = a_0^2 x_{10}^2 \end{cases}$$

define the elliptic modular surfaces $S(4)$, which is isomorphic to the Fermat quartic surface ([**BH**], [**Mu**], [**Sh**]). In our case, there are 15 quadratic equations (E1), ..., (E15) in $X_1, \dots, X_6$ with coefficients $A_1^2, \dots, A_{10}^2$, defining a fibration of Kummer surfaces over the Siegel modular 3-fold $\mathcal{A}_2(2,4)$ (Theorem 4.8). These 15 quadrics are those of rank 4 in the net spanned by (E1), (E2) and (E3). They are 4-terms theta relations, and we can find such relations in classical literature (e.g. [**Co**] and [**Kr**]). In this paper, we determine singular fibers over boundaries of $\mathcal{A}_2(2,4)$ (Section 4), and we write down 80 Rosenhain's hyperplanes that cut out 32 lines on Jacobian Kummer surfaces (Theorem 5.4).

2. Theta functions and Siegel modular varieties.

2.1. We denote the Siegel upper half space of degree g by $\mathfrak{S}_g$, the symplectic group $Sp_{2g}(\mathbb{Z})$ by Γ_g, and Igusa's congruence subgroups by $\Gamma_g(2n, 4n)$:

$$\mathfrak{S}_g = \{\Omega \in M_g(\mathbb{C}) \mid {}^t\Omega = \Omega,\ \operatorname{Im}\Omega > 0\},$$

$$\Gamma_g = \{\gamma \in GL_{2g}(\mathbb{Z}) \mid {}^t\gamma J \gamma = J\}, \quad J = \begin{bmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{bmatrix},$$

$$\Gamma_g(n) = \{\gamma \in \Gamma_2 \mid \gamma \equiv \mathbf{1}_g \mod n\},$$

$$\Gamma_g(2n, 4n) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g(2n) \mid \operatorname{diag}(B) \equiv \operatorname{diag}(C) \equiv 0 \mod 4n \right\}.$$

(Usually the symbol $\Gamma_1(n)$ denotes another group, but here we use it for the principal congruence subgroup of level n for $g = 1$.) In the case of $g = 1$, we have $\Gamma_1(2,4)/\Gamma_1(4) = \{\pm 1\}$. For $g = 2$, we have

$$\Gamma_2/\Gamma_2(2) \cong Sp_4(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{S}_6, \quad \Gamma_2(2)/\Gamma_2(2,4) \cong (\mathbb{Z}/2\mathbb{Z})^4, \quad \Gamma_2(2,4)/\Gamma_2(4,8) \cong (\mathbb{Z}/2\mathbb{Z})^9.$$

The group Γ_g acts on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (z, \Omega) = ({}^t(C\Omega + D)^{-1} z,\ (A\Omega + B)(C\Omega + D)^{-1}),$$

and acts on $(\mathbb{Q}^g/\mathbb{Z}^g)^2$ by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} Da - Cb + \frac{1}{2}\operatorname{diag}(C^tD) \\ -Ba + Ab + \frac{1}{2}\operatorname{diag}(A^tB) \end{bmatrix}, \quad a, b \in \mathbb{Q}^g/\mathbb{Z}^g.$$

Theta functions with characteristics $a, b \in \mathbb{Q}^g$ are defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp[\pi i\, {}^t(n+a)\Omega(n+a) + 2\pi i\, {}^t(n+a)(z+b)]$$

and they satisfy the automorphy property (the theta transformation formula):

$$\vartheta \begin{bmatrix} a^\sharp \\ b^\sharp \end{bmatrix} (z^\sharp, \Omega^\sharp) = \kappa(\gamma) \det(C\Omega + D)^{1/2} F(a, b, g, \Omega, z) \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$$

where $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_g$, $\left[\begin{smallmatrix} a^\sharp \\ b^\sharp \end{smallmatrix}\right] = \gamma \cdot \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $(z^\sharp, \Omega^\sharp) = \gamma \cdot (z, \Omega)$, $\kappa(\gamma)^8 = 1$ and

$$F(a, b, \gamma, \Omega, z) = \exp\left[\pi i \{ {}^t(Da - Cb)(-Ba + Ab + (A^tB)_0) - {}^tab + {}^tz(C\Omega + D)^{-1}Cz \}\right]$$

(see [**BL**, Section 8.5 and Section 8.6]). We can embed Siegel modular 3-folds

$$\mathcal{A}_2(2n, 4n) = \mathfrak{S}_2/\Gamma_2(2n, 4n)$$

for $n = 1, 2$ into projective spaces by theta constants with half integer characteristics. For the simplicity, we denote theta constants

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0, \Omega) \quad \text{with} \quad a = \frac{1}{2} \begin{bmatrix} i \\ j \end{bmatrix},\ b = \frac{1}{2} \begin{bmatrix} k \\ l \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^2$$

by $\theta[{}^{ij}_{kl}](\Omega)$. Then we have

2.2. Proposition ([**vGN**], [**vGvS**], [**Ig2**], [**Sa**])

(1) *The holomorphic map*

$$\Theta_{2,4} : \mathfrak{S}_2 \longrightarrow \mathbb{P}^3, \quad \Omega \mapsto \left[\theta[{}^{00}_{00}](2\Omega) : \theta[{}^{01}_{00}](2\Omega) : \theta[{}^{10}_{00}](2\Omega) : \theta[{}^{11}_{00}](2\Omega)\right] = [B_0 : B_1 : B_2 : B_3]$$

gives an isomorphism of the Satake compactification $\overline{\mathcal{A}_2(2,4)}$ *and* $\mathbb{P}^3$.

(2) *The holomorphic map*

$$\Theta_{4,8} : \mathfrak{S}_2 \longrightarrow \mathbb{P}^9, \quad \Omega \mapsto [A_1 : \cdots : A_{10}]$$

by 10 *even theta constants*

$$\begin{gathered} A_1 = \theta[{}^{00}_{00}](\Omega), \quad A_2 = \theta[{}^{00}_{01}](\Omega), \quad A_3 = \theta[{}^{00}_{10}](\Omega), \quad A_4 = \theta[{}^{00}_{11}](\Omega), \quad A_5 = \theta[{}^{01}_{00}](\Omega), \\ A_6 = \theta[{}^{10}_{00}](\Omega), \quad A_7 = \theta[{}^{11}_{00}](\Omega), \quad A_8 = \theta[{}^{01}_{10}](\Omega), \quad A_9 = \theta[{}^{10}_{01}](\Omega), \quad A_{10} = \theta[{}^{11}_{11}](\Omega), \end{gathered}$$

gives an isomorphism of the Satake compactification $\overline{\mathcal{A}_2(4,8)}$ *and the closure of the image of the map* $\Theta_{4,8}$.

(3) *We have quadratic relations*

$$\begin{gathered} A_1^2 = B_0^2 + B_1^2 + B_2^2 + B_3^2, \quad A_2^2 = B_0^2 - B_1^2 + B_2^2 - B_3^2, \\ A_3^2 = B_0^2 + B_1^2 - B_2^2 - B_3^2, \quad A_4^2 = B_0^2 - B_1^2 - B_2^2 + B_3^2, \\ A_5^2 = 2(B_0B_1 + B_2B_3), \quad A_6^2 = 2(B_0B_2 + B_1B_3), \quad A_7^2 = 2(B_0B_3 + B_1B_2), \\ A_8^2 = 2(B_0B_1 - B_2B_3), \quad A_9^2 = 2(B_0B_2 - B_1B_3), \quad A_{10}^2 = 2(B_0B_3 - B_1B_2) \end{gathered}$$

and the following diagram

$$\begin{tikzcd} \overline{\mathcal{A}_2(4,8)} \arrow[rr, "\Theta_{4,8}"] \arrow[d, "\pi"'] & & \mathbb{P}^9 \arrow[d, "\mathrm{Sq}"] \\ \overline{\mathcal{A}_2(2,4)} \arrow[r, "\Theta_{2,4}"] & \mathbb{P}^3 \arrow[r, "\mathrm{Ver}"] & \mathbb{P}^9 \end{tikzcd}$$

where π *is the natural covering map with the covering group* $\Gamma_2(2,4)/\Gamma_2(4,8) \cong (\mathbb{Z}/2\mathbb{Z})^9$, *the map* Sq *is the squaring of coordinates* $[A_1 : \cdots : A_{10}] \mapsto [A_1^2 : \cdots : A_{10}^2]$ *and the map*

Ver *is the Veronese map defined by the above quadratic relations.*

(4) *The* 10 *smooth quadrics* $Q_i = \{A_i = 0\}$ *of* $\mathbb{P}^3$ *correspond to the closure of the locus of decomposable principally polarized Abelian surfaces. Therefore* $\mathcal{U} = \mathbb{P}^3 - \bigcup_{i=1}^{10} Q_i$ *parametrize Jacobians of curves of genus two. The intersection* $Q_i \cap Q_j$ *consists of four lines. There are* 30 *such lines* $L_1, \ldots, L_{30}$*. They form the* 1*-dimensional boundaries of* $\mathcal{A}_2(2,4)$;

$$\overline{\mathcal{A}_2(2,4)} - \mathcal{A}_2(2,4) = \bigcup_{i=1}^{30} L_i.$$

There are 60 *intersection points of* L_i*'s. These points are* 0*-dimensional boundaries.*

3. Quadratic relations of odd theta functions.

3.1. Let C be a curve of genus two, $X = \mathrm{Jac}(C)$ be its Jacobian and $\theta \cong C$ be the theta divisor on X. Let $\pi : \tilde{X} \to X$ be the blow up of 2-torsion points $p_1, \ldots, p_{16}$, and $E_1, \ldots, E_{16}$ be the exceptional divisors. The linear system $|4\pi^*\theta - \sum E_i|$ gives a morphism of degree 2 from $\tilde{X}$ to a complete intersection Y of three quadrics in $\mathbb{P}^5$. In fact, the image Y is the desingularized Kummer surface $\tilde{X}/\{\pm 1\}$ ([**GH**, Chapter 6]). Let $N \cong \mathbb{P}^2$ be the net spanned by these quadrics. It is known that the discriminant locus $\Delta \subset N$ corresponding to singular quadrics is a union of 6 lines. There are 15 intersection points of these 6 lines corresponding to quadrics of rank 4 (see e.g. [**CR**, Theorem 3.3]).

Now let X be a complex torus $\mathbb{C}^2/\Omega\mathbb{Z}^2 + \mathbb{Z}^2$, and θ be given by $\theta[^{00}_{00}](z,\Omega) = 0$. Then $\Gamma(X, \mathcal{O}_X(4\theta))$ is identified with the vector space of 4-th order theta functions, and a basis is given by 16 theta functions $\theta[^{ij}_{kl}](2z,\Omega)$. The linear system $|4\theta - \sum p_i|$ corresponds to the subspace spanned by 6 odd theta functions

$$X_1 = \theta[^{01}_{01}](2z,\Omega), \quad X_2 = \theta[^{01}_{11}](2z,\Omega), \quad X_3 = \theta[^{11}_{01}](2z,\Omega),$$
$$X_4 = \theta[^{10}_{10}](2z,\Omega), \quad X_5 = \theta[^{10}_{11}](2z,\Omega), \quad X_6 = \theta[^{11}_{10}](2z,\Omega).$$

Therefore the defining equations of Y are given by theta relations of $X_1, \ldots, X_6$. We can find 4-terms quadratic relations of theta functions in [**Co**, Section 30] and [**Kr**, Chapter VII, Section 14], and they will give 15 quadrics of rank 4. Here we write down 15 equations in our terms.

3.2. To get quadratic relations, we apply Riemann's relation [**Mu**, p. 214, ($\mathrm{R_{CH}}$)]

$$\vartheta\begin{bmatrix} \dfrac{a+b+c+d}{2} \\ \dfrac{e+f+g+h}{2} \end{bmatrix}\left(\frac{x+y+u+v}{2}, \Omega\right) \cdots \vartheta\begin{bmatrix} \dfrac{a-b-c+d}{2} \\ \dfrac{e-f-g+h}{2} \end{bmatrix}\left(\frac{x-y-u+v}{2}, \Omega\right)$$
$$= \frac{1}{4}\sum_{\alpha,\beta\in(1/2)\mathbb{Z}^2/\mathbb{Z}^2} \exp[-2\pi i\, {}^t\beta(a+b+c+d)]\vartheta\begin{bmatrix} a+\alpha \\ e+\beta \end{bmatrix}(x,\Omega) \cdots \vartheta\begin{bmatrix} d+\alpha \\ h+\beta \end{bmatrix}(v,\Omega).$$

Putting $x = y$, $u = v = 0$, $a = b + p$, $e = f + q$, $c = d = g = h = 0$ with $p, q \in \mathbb{Z}^2$, $b, f \in (1/2)\mathbb{Z}^2$, we have

$$\begin{aligned}
&\vartheta \begin{bmatrix} b + \frac{1}{2}p \\ f + \frac{1}{2}q \end{bmatrix} (x, \Omega)^2 \vartheta \begin{bmatrix} \frac{1}{2}p \\ \frac{1}{2}q \end{bmatrix} (0, \Omega)^2 \\
&\quad = \frac{1}{4} \sum_{\alpha, \beta \in (1/2)\mathbb{Z}^2/\mathbb{Z}^2} \exp[-2\pi i \, {}^t\beta(2b + p)] \vartheta \begin{bmatrix} b + p + \alpha \\ f + q + \beta \end{bmatrix} (x, \Omega) \vartheta \begin{bmatrix} b + \alpha \\ f + \beta \end{bmatrix} (x, \Omega) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \Omega)^2 \\
&\quad = \frac{1}{4} \sum_{\alpha, \beta \in (1/2)\mathbb{Z}^2/\mathbb{Z}^2} (-1)^{2({}^t\beta p + {}^t\alpha q)} (-1)^{2 {}^t b q} (-1)^{4 {}^t\beta b} \vartheta \begin{bmatrix} b + \alpha \\ f + \beta \end{bmatrix} (x, \Omega)^2 \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \Omega)^2.
\end{aligned}$$

Now put $b = f = \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right]$ and $S = \left\{ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right\}$. We have ${}^t bq = 0$ for $q \in S$, and therefore

$$\begin{aligned}
&\sum_{p, q \in S} (-1)^{{}^t pp + {}^t qq} \vartheta \begin{bmatrix} b + \frac{1}{2}p \\ f + \frac{1}{2}q \end{bmatrix} (x, \Omega)^2 \vartheta \begin{bmatrix} \frac{1}{2}p \\ \frac{1}{2}q \end{bmatrix} (0, \Omega)^2 \\
&\quad = \sum_{p, q \in S} (-1)^{{}^t pp + {}^t qq} \frac{1}{4} \sum_{\alpha, \beta \in (1/2)\mathbb{Z}^2/\mathbb{Z}^2} (-1)^{2({}^t\beta p + {}^t\alpha q)} (-1)^{4 {}^t\beta b} \vartheta \begin{bmatrix} b + \alpha \\ f + \beta \end{bmatrix} (x, \Omega)^2 \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \Omega)^2 \\
&\quad = \frac{1}{4} \sum_{\alpha, \beta \in (1/2)\mathbb{Z}^2/\mathbb{Z}^2} \left(\sum_{p, q \in S} (-1)^{{}^t(2\beta + p)p + {}^t(2\alpha + q)q} \right) (-1)^{4 {}^t\beta b} \vartheta \begin{bmatrix} b + \alpha \\ f + \beta \end{bmatrix} (x, \Omega)^2 \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \Omega)^2,
\end{aligned}$$

where

$$\sum_{p, q \in S} (-1)^{{}^t(2\beta + p)p + {}^t(2\alpha + q)q} = \begin{cases} 4 & \left(\alpha, \beta = \left(*, \frac{1}{2} \right) \right) \\ 0 & (\text{others}) \end{cases}.$$

Since odd theta constants vanish, the result is

$$\begin{aligned}
&\theta\left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right](x, \Omega)^2 \theta\left[\begin{smallmatrix} 00 \\ 00 \end{smallmatrix}\right](0, \Omega)^2 - \theta\left[\begin{smallmatrix} 11 \\ 10 \end{smallmatrix}\right](x, \Omega)^2 \theta\left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix}\right](0, \Omega)^2 - \theta\left[\begin{smallmatrix} 10 \\ 11 \end{smallmatrix}\right](x, \Omega)^2 \theta\left[\begin{smallmatrix} 00 \\ 01 \end{smallmatrix}\right](0, \Omega)^2 \\
&\quad = -\theta\left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}\right](x, \Omega)^2 \theta\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right](0, \Omega)^2.
\end{aligned}$$

Putting $x = 2z$, we have a quadratic relation

$$\text{(E1)} \quad A_{10}^2 X_1^2 + A_1^2 X_4^2 - A_2^2 X_5^2 - A_5^2 X_6^2 = 0.$$

Since Γ_2 acts on the projective coordinates $[A_1 : \cdots : A_{10}]$ and $[X_1 : \cdots : X_6]$ via the transformation formula of theta functions, we can easily derive other 14 quadrics of rank

4 from (E1). We summarize the action of Γ_2 and 15 equations in Appendix. Now we have

3.3. PROPOSITION. *Let $X = \mathbb{C}^2/\Omega\mathbb{Z}^2 + \mathbb{Z}^2$ be a principally polarized Abelian surface with $A_i \neq 0$ for $i = 1, \dots, 10$ (hence X is the Jacobian of a curve of genus 2). Then the image Y of a rational map $\Phi : X \longrightarrow \mathbb{P}^5$*

$$z \mapsto [\theta[\begin{smallmatrix}01\\01\end{smallmatrix}] : \theta[\begin{smallmatrix}01\\11\end{smallmatrix}] : \theta[\begin{smallmatrix}11\\01\end{smallmatrix}] : \theta[\begin{smallmatrix}10\\10\end{smallmatrix}] : \theta[\begin{smallmatrix}10\\11\end{smallmatrix}] : \theta[\begin{smallmatrix}11\\10\end{smallmatrix}]](2z, \Omega) = [X_1 : X_2 : X_3 : X_4 : X_5 : X_6]$$

is a smooth complete intersection given by

$$\begin{aligned}
&(\mathrm{E1}) \quad A_{10}^2X_1^2 && && &&+A_1^2X_4^2 - A_2^2X_5^2 - A_5^2X_6^2 = 0,\\
&(\mathrm{E2}) && && A_{10}^2X_2^2 && &&+A_3^2X_4^2 - A_4^2X_5^2 - A_8^2X_6^2 = 0,\\
&(\mathrm{E3}) && && && A_{10}^2X_3^2 &&+A_6^2X_4^2 - A_9^2X_5^2 - A_7^2X_6^2 = 0,
\end{aligned}$$

and the net spanned by (E1), (E2) *and* (E3) *contains* 15 *quadrics of rank* 4 *defined by* (E1), ..., (E15) *in Appendix.*

3.4. REMARK. Translations by 2-torsion points $(1/2)\Omega\mathbb{Z}^2 + (1/2)\mathbb{Z}^2$ acts on the projective coordinate of $\mathbb{P}^5$ as sign changes:

	X_1	X_2	X_3	X_4	X_5	X_6
$\frac{1}{2}\Omega[\begin{smallmatrix}1\\0\end{smallmatrix}]$	$+$	$-$	$+$	$-$	$-$	$-$
$\frac{1}{2}\Omega[\begin{smallmatrix}0\\1\end{smallmatrix}]$	$-$	$-$	$-$	$+$	$-$	$+$
$\frac{1}{2}[\begin{smallmatrix}1\\0\end{smallmatrix}]$	$+$	$+$	$-$	$-$	$-$	$-$
$\frac{1}{2}[\begin{smallmatrix}0\\1\end{smallmatrix}]$	$-$	$-$	$-$	$+$	$+$	$-$

3.5. REMARK. The Rosenhain normal form ([**Ig1**]) of a curve of genus 2 is given by

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

with

$$\lambda_1 = \frac{A_7^2A_6^2}{A_5^2A_1^2}, \quad \lambda_2 = \frac{A_9^2A_7^2}{A_2^2A_5^2}, \quad \lambda_3 = \frac{A_9^2A_6^2}{A_2^2A_1^2}.$$

4. Universal Jacobian Kummer surface.

4.1. Let us recall that coefficients $A_1^2, \dots, A_{10}^2$ of 15 equations (E1), ..., (E15) are quadric polynomials in B_0, B_1, B_2, B_3 (see Proposition 2.2). Therefore these equations define a projective variety $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^5$, and we have the projection

$$\pi_{2,4} : \mathcal{X} \longrightarrow \mathbb{P}^3 \cong \overline{\mathcal{A}_2(2,4)}, \quad [B_0 : \cdots : B_3] \times [X_1 : \cdots : X_6] \mapsto [B_0 : \cdots : B_3].$$

Over $\mathcal{U} = \mathbb{P}^3 - \bigcup_{i=1}^{10} Q_i$, this is a $K3$-fibration and $\pi_{2,4}^{-1}(\mathcal{U})$ is smooth. Let us investigate fibers over $\bigcup_{i=1}^{10} Q_i$. Since Γ_2 acts on 10 quadrics $\{Q_1, \ldots, Q_{10}\}$ doubly transitive, we look at fibers only over $Q_{10} = \{B_0B_3 - B_1B_2 = 0\}$. We identify Q_{10} with $\mathbb{P}^1 \times \mathbb{P}^1$ by the Segre embedding

$$\mathrm{Seg} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s_0 : s_1] \times [t_0 : t_1] \mapsto [s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1].$$

Then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{H}/\Gamma_1(4) \times \mathbb{H}/\Gamma_1(4) \arrow[r, "\psi"] \arrow[d, "\varepsilon"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\mathrm{Seg}"] \\
\mathfrak{S}_2/\Gamma_2(2,4) \arrow[r, "\Theta_{2,4}"] & \mathbb{P}^3
\end{tikzcd}$$

where $\varepsilon(\tau_1, \tau_2) = \left[\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right]$ and ψ is given by

$$(\tau_1, \tau_2) \mapsto [\theta_{00}(2\tau_1) : \theta_{10}(2\tau_1)] \times [\theta_{00}(2\tau_2) : \theta_{10}(2\tau_2)].$$

4.2. Remark. The duplication formula for $g = 1$ are

$$\theta_{00}(\tau)^2 = \theta_{00}(2\tau)^2 + \theta_{10}(2\tau)^2, \quad \theta_{10}(\tau)^2 = 2\theta_{00}(2\tau)\theta_{10}(2\tau),$$
$$\theta_{01}(\tau)^2 = \theta_{00}(2\tau)^2 - \theta_{10}(2\tau)^2.$$

4.3. Since we have $\mathbb{H}/\Gamma_1(4) \cong \mathbb{P}^1 - \{6 \text{ points}\}$ ([**Mu**]), the boundary $\mathbb{P}^1 \times \mathbb{P}^1 - \operatorname{Im}\psi$ is decomposed into $6 + 6$ lines (these 12 lines are obtained as intersections of Q_{10} and Q_i);

$$[B_0 : B_1 : B_2 : B_3] = \begin{cases} [t_0 : t_1 : 0 : 0] & ([s_0 : s_1] = [1 : 0]) \\ [0 : 0 : t_0 : t_1] & ([s_0 : s_1] = [0 : 1]) \\ [t_0 : t_1 : \pm t_0 : \pm t_1] & ([s_0 : s_1] = [1 : \pm 1]) \\ [t_0 : t_1 : \pm\sqrt{-1}t_0 : \pm\sqrt{-1}t_1] & ([s_0 : s_1] = [1 : \pm\sqrt{-1}]) \end{cases}$$

$$[B_0 : B_1 : B_2 : B_3] = \begin{cases} [s_0 : 0 : s_1 : 0] & ([t_0 : t_1] = [1 : 0]) \\ [0 : s_0 : 0 : s_1] & ([t_0 : t_1] = [0 : 1]) \\ [s_0 : \pm s_0 : s_1 : \pm s_1] & ([t_0 : t_1] = [1 : \pm 1]) \\ [s_0 : \pm\sqrt{-1}s_0 : s_1 : \pm\sqrt{-1}s_1] & ([t_0 : t_1] = [1 : \pm\sqrt{-1}]) \end{cases}$$

and they intersect at 6×6 points.

4.4. For $\Omega = \varepsilon(\tau_1, \tau_2)$, we have

$$X = \mathbb{C}^2/(\Omega\mathbb{Z}^2 + \mathbb{Z}^2) = E_1 \times E_2, \qquad E_i = \mathbb{C}/(\tau_i\mathbb{Z} + \mathbb{Z}).$$

The rational map $\Phi : X \dashrightarrow \mathbb{P}^5$ is the composition of an embedding

$$\begin{aligned} &E_1 \times E_2 \longrightarrow \mathbb{P}^3 \times \mathbb{P}^3, \\ &\quad (z_1, z_2) \mapsto [\theta_{00} : \theta_{01} : \theta_{10} : \theta_{11}](2z_1, \tau_1) \times [\theta_{00} : \theta_{01} : \theta_{10} : \theta_{11}](2z_2, \tau_2) \end{aligned}$$

and a rational map

$$\begin{aligned} &\mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow \mathbb{P}^5, \\ &\quad [x_0 : x_1 : x_2 : x_3] \times [y_0 : y_1 : y_2 : y_3] \mapsto [x_0y_3 : x_1y_3 : x_2y_3 : x_3y_0 : x_3y_1 : x_3y_2]. \end{aligned}$$

The corresponding fiber $\mathcal{X}_{(s,t)}$ over $[s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1] \in Q_{10} - \{12 \text{ lines}\}$ is defined by

$$\begin{cases} T_{00}X_4^2 - T_{01}X_5^2 - T_{10}X_6^2 = 0 & \text{(E1)}, \\ S_{00}X_1^2 - S_{01}X_2^2 - S_{10}X_3^2 = 0 & \text{(E4)}, \\ T_{10}\big(S_{01}X_1^2 - S_{00}X_2^2\big) - S_{10}\big(T_{01}X_4^2 - T_{00}X_5^2\big) = 0 & \text{(E7)} \end{cases}$$

where

$$\begin{aligned} &S_{00} = s_0^2 + s_1^2, \quad S_{10} = 2s_0s_1, \quad S_{01} = s_0^2 - s_1^2, \\ &T_{00} = t_0^2 + t_1^2, \quad T_{10} = 2t_0t_1, \quad T_{01} = t_0^2 - t_1^2 \end{aligned}$$

(compare with Remark 4.2). It is easily shown that $\mathcal{X}_{(s,t)}$ is birational to the Kummer surface of $E_1 \times E_2$. More precisely, 8 curves

$$E_1 \times \{2\text{-torsions of } E_2\}, \quad \{2\text{-torsions of } E_1\} \times E_2$$

are corresponding to 8 nodes

$$[X_1 : \cdots : X_6] = \begin{cases} \left[\pm\sqrt{\dfrac{S_{00}}{S_{10}}} : \pm\sqrt{\dfrac{S_{01}}{S_{10}}} : 1 : 0 : 0 : 0 \right] \\ \left[0 : 0 : 0 : \pm\sqrt{\dfrac{T_{00}}{T_{10}}} : \pm\sqrt{\dfrac{T_{01}}{T_{10}}} : 1 \right] \end{cases},$$

and 4×4 lines

$$\left\{ \left[\pm z_0\sqrt{\frac{S_{00}}{S_{10}}} : \pm z_0\sqrt{\frac{S_{01}}{S_{10}}} : z_0 : \pm z_1\sqrt{\frac{T_{00}}{T_{10}}} : \pm z_1\sqrt{\frac{T_{01}}{T_{10}}} : z_1 \right] \middle| [z_0 : z_1] \in \mathbb{P}^1 \right\}$$

joining $4 + 4$ nodes are exceptional divisors obtained by blowing up at 2-torsions. Resolving 8 nodes, we obtain the smooth Kummer surface.

4.5. REMARK. The 8 curves on $E_1 \times E_2$ are fixed loci by

$$\pm \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix} \in \Gamma_2(2,4), \qquad \alpha = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4.6. 1-DIMENSIONAL BOUNDARY. Next we investigate fibers over a 1-dimensional boundary

$$[B_0 : B_1 : B_2 : B_3] = [t_0 : t_1 : 0 : 0] \qquad ([s_0 : s_1] = [1 : 0]).$$

Outside six 0-dimensional boundaries

$$t = t_0/t_1 = 0,\ \infty,\ \pm 1,\ \pm\sqrt{-1},$$

the fiber $\pi_{2,4}^{-1}(t)$ is defined by

$$\begin{cases} X_1^2 - X_2^2 = 0 & (\mathrm{E}4,5,6,7,8,9), \\ T_{00}X_4^2 - T_{01}X_5^2 - T_{10}X_6^2 = 0 & (\mathrm{E}1,2), \\ T_{10}X_3^2 - T_{01}X_4^2 + T_{00}X_5^2 = 0 & (\mathrm{E}10,13), \\ T_{01}X_3^2 + T_{10}X_4^2 - T_{00}X_6^2 = 0 & (\mathrm{E}11,14), \\ T_{00}X_3^2 + T_{10}X_5^2 - T_{01}X_6^2 = 0 & (\mathrm{E}12,15) \end{cases}$$

((E3) vanish identically). Because there are linear relations,

$$(\mathrm{E}10) = -\frac{T_{01}}{T_{00}}(\mathrm{E}1) + \frac{T_{10}}{T_{00}}(\mathrm{E}12), \qquad (\mathrm{E}11) = \frac{T_{10}}{T_{00}}(\mathrm{E}1) + \frac{T_{01}}{T_{00}}(\mathrm{E}12),$$

we have $\pi_{2,4}^{-1}(t) = Y_+ \cup Y_-$ with

$$Y_+ = \begin{cases} X_1 = X_2 \\ T_{00}X_4^2 - T_{01}X_5^2 - T_{10}X_6^2 = 0 \\ T_{00}X_3^2 + T_{10}X_5^2 - T_{01}X_6^2 = 0 \end{cases}, \qquad Y_- = \begin{cases} X_1 = -X_2 \\ T_{00}X_4^2 - T_{01}X_5^2 - T_{10}X_6^2 = 0 \\ T_{00}X_3^2 + T_{10}X_5^2 - T_{01}X_6^2 = 0 \end{cases}.$$

These surfaces are cones over an elliptic curve $E = Y_+ \cap Y_-$, and singular at $[1 : \pm 1 : 0 : 0 : 0 : 0]$.

4.7. 0-DIMENSIONAL BOUNDARY. Let us investigate the fiber at a 0-dimensional boundary

$$P_0 = [B_0 : B_1 : B_2 : B_3] = [1 : 0 : 0 : 0]$$

corresponding to $[s_0 : s_1] \times [t_0 : t_1] = [1 : 0] \times [1 : 0]$. Then 15 equations are

$$\begin{cases} X_1^2 - X_2^2 = 0 & (\text{E4, 5, 8, 9}), \\ X_3^2 - X_6^2 = 0 & (\text{E11, 12, 14 , 15}), \\ X_4^2 - X_5^2 = 0 & (\text{E1, 2, 10, 13}) \end{cases}$$

((E3), (E6) and (E7) vanish identically). The fiber $\pi_{2,4}^{-1}(P_0)$ is a union of 8 projective plane:

$$\mathbb{P}_{+++} = \begin{cases} X_1 = X_2 \\ X_3 = X_6 \\ X_4 = X_5 \end{cases}, \quad \mathbb{P}_{++-} = \begin{cases} X_1 = X_2 \\ X_3 = X_6 \\ X_4 = -X_5 \end{cases}, \ldots, \ \mathbb{P}_{---} = \begin{cases} X_1 = -X_2 \\ X_3 = -X_6 \\ X_4 = -X_5 \end{cases},$$

and the dual graph (see [**Pe**]) is a cube (vertices, edges and faces represent irreducible components, double lines, points of order 4 respectively).

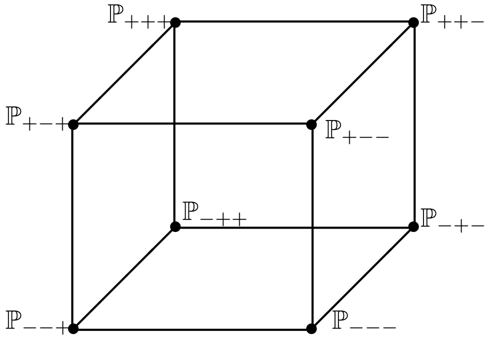

Figure 1.

4.8. Theorem. (1) *The* 5*-dimensional projective variety $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^5$ is smooth and simply connected.*

(2) *The projection $\pi_{2,4} : \mathcal{X} \to \mathbb{P}^3$ is the half-anticanonical map, and hence $\mathcal{X}$ has the Kodaira dimension $\kappa(\mathcal{X}) = -\infty$.*

Proof. (1) The smoothness is shown by the Jacobian criterion. Namely, we show that not all of 3-minors of the projective Jacobian matrix

$$\mathcal{J} = \frac{\partial(\text{E1}, \ldots, \text{E15})}{\partial(B_0, \ldots, B_3, X_1, \ldots, X_6)}$$

of the equations (E1), ..., (E15) vanish.

We first consider fibers over the 1-dimensional boundary

$$L = \{[t_0 : t_1 : 0 : 0] \mid [t_0 : t_1] \in \mathbb{P}^1, \ T_{00}T_{10}T_{01} \neq 0\}.$$

Since we have

$$\left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_2, B_3, X_1)}\right| = 8T_{00}T_{01}X_1^5$$

at points of $\pi_{2,4}^{-1}(L)$, the variety $\mathcal{X}$ is smooth at points of $\{X_1 \neq 0\} \cap \pi_{2,4}^{-1}(L)$. On the other hand, we have

$$\begin{aligned}\left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_0, B_2, X_4)}\right| &= -8X_4\big(t_0X_3^2 + t_1X_4^2\big)\big\{-t_1T_{01}X_4^2 + t_0\big(T_{00}X_3^2 + T_{10}X_5^2 - T_{01}X_6^2\big)\big\} \\ &= 8t_1T_{01}X_4^3\big(t_0X_3^2 + t_1X_4^2\big), \\ \left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_1, B_3, X_6)}\right| &= 8X_6\big(t_0X_4^2 - t_1X_3^2\big)\big\{-t_0T_{01}X_6^2 + t_1\big(T_{10}X_3^2 - T_{01}X_4^2 + T_{00}X_5^2\big)\big\} \\ &= -8t_0T_{01}X_6^3\big(t_0X_4^2 - t_1X_3^2\big)\end{aligned}$$

at points of $\{X_1 = 0\} \cap \pi_{2,4}^{-1}(L)$. We see that these values do not vanish at the same time on $\{X_1 = 0\} \cap \pi_{2,4}^{-1}(L)$ (see 4.6), and $\mathcal{X}$ is smooth along $\pi_{2,4}^{-1}(L)$.

Similarly, we have the following 3-minors

$$\left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_2, B_3, X_1)}\right| = 8X_1^5, \quad \left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_1, B_2, X_3)}\right| = -8X_3^5, \quad \left|\frac{\partial(\mathrm{E1},\ \mathrm{E4},\ \mathrm{E11})}{\partial(B_1, B_3, X_4)}\right| = 8X_4^5$$

at points of $\pi_{2,4}^{-1}(P_0)$, that is, at points

$$[X_1 : \pm X_1 : X_3 : X_4 : \pm X_4 : \pm X_3] \times [1 : 0 : 0 : 0] \in \mathcal{X} \subset \mathbb{P}^5 \times \mathbb{P}^3.$$

Therefore the Jacobian matrix $\mathcal{J}$ has rank 3 at points of fibers over 0-dimensional and 1-dimensional boundaries, and $\mathcal{X}$ is smooth there.

Now let $\operatorname{Sing}\mathcal{X}$ be the set of singular points of $\mathcal{X}$. Then $\pi_{2,4}(\operatorname{Sing}\mathcal{X}) \subset \mathbb{P}^3$ is closed subvariety since $\pi_{2,4}$ is proper. As we have seen, it holds

$$\pi_{2,4}(\operatorname{Sing}\mathcal{X}) \cap Q_{10} \subset Q_{10} - \{\text{boundaries}\} \cong (\mathbb{H}/\Gamma_1(4)) \times (\mathbb{H}/\Gamma_1(4))$$

and $\pi_{2,4}(\operatorname{Sing}\mathcal{X}) \cap Q_{10}$ must be isolated points. However, the fibration over $(\mathbb{H}/\Gamma_1(4)) \times (\mathbb{H}/\Gamma_1(4))$ is locally trivial as a topological space, and we see that $\pi_{2,4}(\operatorname{Sing}\mathcal{X}) \cap Q_{10} = \phi$. By the Γ_2-symmetry, we have $\pi_{2,4}(\operatorname{Sing}\mathcal{X}) \cap Q_{10} = \phi$ for $i = 1, \dots, 10$, and $\mathcal{X}$ is smooth.

Since we have an exact sequence

$$\pi_1(\text{a general fiber}) \longrightarrow \pi_1(\mathcal{X}) \longrightarrow \pi_1(\mathbb{P}^3) \longrightarrow 1$$

of fundamental groups ([**No**, Lemma 1.5]), $\mathcal{X}$ is simply connected.

(2) Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^5$. By the adjunction formula, we have

$$\omega_{\mathcal{X}} \cong \omega_{\mathbb{P}^3 \times \mathbb{P}^5} \otimes \wedge^3(\mathcal{I}/\mathcal{I}^2)^*$$

where p_1 (resp. p_2) is the projection to $\mathbb{P}^3$ (resp. $\mathbb{P}^5$). Hence it is enough to show that

$$\wedge^3(\mathcal{I}/\mathcal{I}^2) \cong p_1^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^5}(-6).$$

Now the problem is to show equivalence of two divisors on a smooth variety, we may ignore subvarieties of codimension 2. So we restrict ourselves to an open set $\pi_{2,4}^{-1}(U_9 \cup U_{10})$ where $U_i = \mathbb{P}^3 - Q_i$. Note that (E1), (E2) and (E3) generate $\mathcal{I}$ over $\pi_{2,4}^{-1}(U_{10})$ since they are given by

$$A_{10}^2 \begin{bmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{bmatrix} + M \begin{bmatrix} X_4^2 \\ X_5^2 \\ X_6^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad M = \begin{bmatrix} A_1^2 & -A_2^2 & -A_5^2 \\ A_3^2 & -A_4^2 & -A_8^2 \\ A_6^2 & -A_9^2 & -A_7^2 \end{bmatrix}$$

and $\det M = A_{10}^6$ as polynomials of $B_0, \dots, B_3$. Similarly, (E5), (E7) and (E8) generate $\mathcal{I}$ over $\pi_{2,4}^{-1}(U_9)$ (see the action of g_4 in Appendix). These basis are connected on $\pi_{2,4}^{-1}(U_9 \cap U_{10})$ by

$$\begin{bmatrix} (\mathrm{E5}) \\ (\mathrm{E7}) \\ (\mathrm{E8}) \end{bmatrix} = \frac{1}{A_{10}^2} \begin{bmatrix} A_2^2 & -A_4^2 & -A_9^2 \\ A_8^2 & -A_5^2 & 0 \\ A_3^2 & -A_1^2 & 0 \end{bmatrix} \begin{bmatrix} (\mathrm{E1}) \\ (\mathrm{E2}) \\ (\mathrm{E3}) \end{bmatrix}, \qquad \det \begin{bmatrix} A_2^2 & -A_4^2 & -A_9^2 \\ A_8^2 & -A_5^2 & 0 \\ A_3^2 & -A_1^2 & 0 \end{bmatrix} = -8A_{10}^2 A_9^4,$$

namely,

$$(\mathrm{E5}) \wedge (\mathrm{E7}) \wedge (\mathrm{E8}) = -8\frac{A_9^4}{A_{10}^4}(\mathrm{E1}) \wedge (\mathrm{E2}) \wedge (\mathrm{E3}).$$

Taking a standard affine open cover $V_{i,j} = \{B_i \neq 0,\ X_j \neq 0\}$ of $\mathbb{P}^3 \times \mathbb{P}^5$, and considering coordinates changes for open sets $V_{i,j} \cap \pi_{2,4}^{-1}(U_9)$ and $V_{k,l} \cap \pi_{2,4}^{-1}(U_{10})$, we see that

$$\wedge^3(\mathcal{I}/\mathcal{I}^2) \cong p_1^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^5}(-6). \qquad \square$$

5. The 80 Rosenhain hyperplanes.

5.1. Let X and Y be as in Proposition 3.3. As classically known, Kummer surface Y contains two families of disjoint 16 lines in $\mathbb{P}^5$. One is 16 exceptional curves obtained by blowing up 16 nodes, and another is 16 tropes, that are the images of translated theta divisors by 2-torsions (that is, tropes are given by $\vartheta[^a_b](z,\Omega) = 0$ with $a, b \in (1/2)\mathbb{Z}^2$). Each line intersects with 6 lines in the opposite family. These lines are cut out by special hyperplanes called Rosenhain's hyperplanes. They cut out 8 lines consisting of 4 exceptional curves and 4 tropes, and there are 80 such hyperplanes (see [**GH**, Chapter 6], [**Hu**, Chapter VII]). We can write down them by Riemann's theta relations. Putting $x = 2z$, $y = u = v = 0$ in [**Mu**, p. 214, ($\mathrm{R_{CH}}$)], we have

$$\vartheta\begin{bmatrix}\dfrac{a+b+c+d}{2}\\ \dfrac{e+f+g+h}{2}\end{bmatrix}(z)\vartheta\begin{bmatrix}\dfrac{a+b-c-d}{2}\\ \dfrac{e+f-g-h}{2}\end{bmatrix}(z)\vartheta\begin{bmatrix}\dfrac{a-b+c-d}{2}\\ \dfrac{e-f+g-h}{2}\end{bmatrix}(z)\vartheta\begin{bmatrix}\dfrac{a-b-c+d}{2}\\ \dfrac{e-f-g+h}{2}\end{bmatrix}(z)$$

$$= \frac{1}{4}\sum_{\alpha,\beta\in(1/2)\mathbb{Z}^2/\mathbb{Z}^2}\exp[-2\pi i\,{}^t\beta(a+b+c+d)]$$

$$\times\,\vartheta\begin{bmatrix}a+\alpha\\ e+\beta\end{bmatrix}(2z)\vartheta\begin{bmatrix}b+\alpha\\ f+\beta\end{bmatrix}(0)\vartheta\begin{bmatrix}c+\alpha\\ g+\beta\end{bmatrix}(0)\vartheta\begin{bmatrix}d+\alpha\\ h+\beta\end{bmatrix}(0).$$

If the left hand side is a 4-th order odd theta function, then the right hand side must be a linear combination of $X_1,\dots,X_6$. If this is the case, the above equation represents a hyperplane in $\mathbb{P}^5$ cutting 4 tropes.

5.2. For example, let us consider four functions

$$f_1(z)=\theta[\begin{smallmatrix}00\\00\end{smallmatrix}](z,\Omega),\quad f_2(z)=\theta[\begin{smallmatrix}00\\01\end{smallmatrix}](z,\Omega),\quad f_3(z)=\theta[\begin{smallmatrix}01\\00\end{smallmatrix}](z,\Omega),\quad f_4(z)=\theta[\begin{smallmatrix}01\\01\end{smallmatrix}](z,\Omega).$$

The product $F(z)=f_1(z)\cdots f_4(z)$ has the same periodicity with $\theta[\begin{smallmatrix}00\\00\end{smallmatrix}](z,\Omega)^4$, and it satisfies $F(-z)=-F(z)$. Namely, $F(z)$ is a 4-th order odd theta function. In fact, we have

$$F(z)=-\frac{1}{4}(A_1A_2A_5X_1+A_3A_4A_8X_2+A_6A_7A_9X_3)$$

by putting

$$a=b=e=g=\frac{1}{2}\begin{bmatrix}0\\0\end{bmatrix},\quad c=d=f=h=\frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}$$

in the above theta relation. Denoting 2-torsion points $(1/2)\Omega[\begin{smallmatrix}i\\j\end{smallmatrix}]+(1/2)[\begin{smallmatrix}k\\l\end{smallmatrix}]\in X$ by ${}^{ij}_{kl}$, we have the following table for the vanishing property of F.

	$\substack{00\\00}$	$\substack{00\\01}$	$\substack{00\\10}$	$\substack{00\\11}$	$\substack{01\\00}$	$\substack{10\\00}$	$\substack{11\\00}$	$\substack{01\\10}$
$\theta[\begin{smallmatrix}00\\00\end{smallmatrix}](z)$								
$\theta[\begin{smallmatrix}00\\01\end{smallmatrix}](z)$					•		•	•
$\theta[\begin{smallmatrix}01\\00\end{smallmatrix}](z)$		•		•				
$\theta[\begin{smallmatrix}01\\01\end{smallmatrix}](z)$	•		•			•		

	$\substack{10\\01}$	$\substack{11\\11}$	$\substack{01\\01}$	$\substack{01\\11}$	$\substack{11\\01}$	$\substack{10\\10}$	$\substack{10\\11}$	$\substack{11\\10}$
$\theta[\begin{smallmatrix}00\\00\end{smallmatrix}](z)$			•	•	•	•	•	•
$\theta[\begin{smallmatrix}00\\01\end{smallmatrix}](z)$		•				•	•	
$\theta[\begin{smallmatrix}01\\00\end{smallmatrix}](z)$	•	•				•		•
$\theta[\begin{smallmatrix}01\\01\end{smallmatrix}](z)$		•					•	•

Namely, F vanishes to order 3 at ${}^{11}_{11}$, ${}^{10}_{10}$, ${}^{10}_{11}$ and ${}^{11}_{10}$, and cuts out 4 exceptional curves corresponding to these points.

5.3. In general, a product of four theta function with characteristics in $(1/2)\mathbb{Z}^2/\mathbb{Z}^2$

$$\vartheta\begin{bmatrix}a'\\a''\end{bmatrix}(z,\Omega)\vartheta\begin{bmatrix}b'\\b''\end{bmatrix}(z,\Omega)\vartheta\begin{bmatrix}c'\\c''\end{bmatrix}(z,\Omega)\vartheta\begin{bmatrix}d'\\d''\end{bmatrix}(z,\Omega)$$

has the same periodicity with $\theta[{}^{00}_{00}](z)^4$ if and only if

$$a'+b'+c'+d',\ a''+b''+c''+d''\in\mathbb{Z},$$

and it is an odd function if and only if

$$2(a'\cdot a''+b'\cdot b''+c'\cdot c''+d'\cdot d'')\notin\mathbb{Z}.$$

There are only 80 such combinations, and we can find them immediately by computer. To state the result, we introduce a few notations. We put numbering codes $1,\dots,16$ for sixteen characteristics:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
${}^{00}_{00}$	${}^{00}_{01}$	${}^{00}_{10}$	${}^{00}_{11}$	${}^{01}_{00}$	${}^{10}_{00}$	${}^{11}_{00}$	${}^{01}_{10}$	${}^{10}_{01}$	${}^{11}_{11}$	${}^{01}_{01}$	${}^{01}_{11}$	${}^{11}_{01}$	${}^{10}_{10}$	${}^{10}_{11}$	${}^{11}_{10}$

We denote exceptional curves by $E_1,\dots,E_{16}$, and tropes by $D_1,\dots,D_{16}$ according to this numbering. Finally, we write $A_{i,j,k}$ instead of $A_iA_jA_k$ and we denote divisors $D_i+\cdots+D_j+E_k+\cdots+E_l$ by $D_{i,\dots,j}+E_{k,\dots,l}$.

5.4. THEOREM. *Let X and Y be as in Proposition* 3.3, *and $X[2]$ be the* 2*-torsion subgroup of X.*

(1) *Rosenhain's hyperplanes for Y are given by* 3*-terms linear equation in $X_1,\dots,X_6$ with coefficients $A_{i,j,k}$, that is, equations are written as*

$$A_{i_i,i_2,i_3}X_i+A_{j_1,j_2,j_3}X_j+A_{k_1,k_2,k_3}X_k=0.$$

The group $X[2]$ acts on eighty Rosenhain's hyperplanes by sign changes of coordinates (*see Remark* 3.4), *and they are divided into twenty orbits* (*these orbits are corresponding to twenty subset $\{i,j,k\}\subset\{1,2,3,4,5,6\}$*). *Each orbit consists of* 4 *hyperplanes*:

$$\begin{aligned}A_{i_i,i_2,i_3}X_i+A_{j_1,j_2,j_3}X_j+A_{k_1,k_2,k_3}X_k=0,\\A_{i_i,i_2,i_3}X_i+A_{j_1,j_2,j_3}X_j-A_{k_1,k_2,k_3}X_k=0,\\A_{i_i,i_2,i_3}X_i-A_{j_1,j_2,j_3}X_j+A_{k_1,k_2,k_3}X_k=0,\\A_{i_i,i_2,i_3}X_i-A_{j_1,j_2,j_3}X_j-A_{k_1,k_2,k_3}X_k=0.\end{aligned}$$

Representatives from twenty orbits are given explicitly in Appendix.

(2) *The trope D_1 is the intersection of four hyperplanes*

$$\begin{aligned} H1: &\ A_{1,3,6}X_4 + A_{2,4,9}X_5 + A_{5,7,8}X_6 = 0, \\ H2: &\ A_{1,10,3}X_3 + A_{5,8,9}X_5 - A_{2,4,7}X_6 = 0, \\ H5: &\ A_{1,6,10}X_2 + A_{4,5,7}X_5 - A_{2,8,9}X_6 = 0, \\ H14: &\ A_{5,6,9}X_1 + A_{1,2,7}X_3 - A_{3,4,10}X_6 = 0, \end{aligned}$$

and the exceptional curve E_{11} is the intersection of four hyperplanes

$$\begin{aligned} H1: &\ A_{1,3,6}X_4 + A_{2,4,9}X_5 + A_{5,7,8}X_6 = 0, \\ H2: &\ A_{1,10,3}X_3 + A_{5,8,9}X_5 - A_{2,4,7}X_6 = 0, \\ H5: &\ A_{1,6,10}X_2 + A_{4,5,7}X_5 - A_{2,8,9}X_6 = 0, \\ H11: &\ A_{3,6,10}X_1 + A_{2,7,8}X_5 - A_{4,5,9}X_6 = 0. \end{aligned}$$

Note that other topes and exceptional curves are given by 16 *translations.*

5.5. We have studied Kummer surfaces over complex numbers until now. However, our family is defined over $\mathbb{Z}$, and the result is applied for a field k of $\mathrm{char}(k) \neq 2$. In fact, we can easily show that (E1), (E2), (E3) define a smooth complete intersection if $A_i \neq 0$ for $i = 1, \ldots, 10$. It is a interesting problem to construct Kummer surfaces with 32 lines over small fields. For example, Kuwata and Shioda asked the problem finding all elliptic fibrations on given $K3$ surfaces in [**KS**], and they proposed to find 32 lines on Kummer surfaces to attack this problem in the case of Kummer surfaces. (The problem for Jacobian Kummer surfaces were solved by Kumar in [**Ku**] over algebraically closed fields of characteristic 0.)

Now we can construct a Kummer surface with 32 lines defined over k if we have

$$[B_0 : \cdots : B_3] \times [A_1 : \cdots : A_{10}] \in \mathbb{P}^3(k) \times \mathbb{P}^9(k)$$

satisfying $A_i \neq 0$ ($i = 1, \ldots, 10$) and the quadric relations in Proposition 2.2. The author does not know whether such a point exists for $k = \mathbb{Q}$. For finite prime fields $\mathbb{F}_p$, we do not have such a point if $p = 3, 5, 7, 11, 13, 17$.

5.6. EXAMPLE. Let us consider a finite prime field $\mathbb{F}_{19}$, and $b = [1 : 3 : 3 : 3] \in \mathbb{P}^3(\mathbb{F}_{19})$. The image of b by the Veronese map Ver in Proposition 2.2 is

$$\begin{aligned} &[9 : 11 : 11 : 11 : 5 : 5 : 5 : 7 : 7 : 7] \\ &\quad = [4 : 7 : 7 : 7 : -2 : -2 : -2 : 1 : 1 : 1] \\ &\quad = [2^2 : 8^2 : 8^2 : 8^2 : 6^2 : 6^2 : 6^2 : 1 : 1 : 1] \in \mathbb{P}^9(\mathbb{F}_{19}), \end{aligned}$$

and we have a smooth Kummer surface Y_b over $\mathbb{F}_{19}$ defined by

$$\begin{aligned} X_1^2 \qquad\qquad &+4X_4^2 - 7X_5^2 + 2X_6^2 = 0,\\ X_2^2 \qquad &+7X_4^2 - 7X_5^2 - \ \ X_6^2 = 0,\\ X_3^2 \ &-2X_4^2 - \ \ X_5^2 + 2X_6^2 = 0 \end{aligned}$$

with

$$D_1 : \begin{cases} X_4 + 7X_5 - 2X_6 = 0,\\ 3X_3 - 6X_5 + 4X_6 = 0,\\ 7X_2 - 3X_5 + 8X_6 = 0,\\ 2X_1 - X_3 + 7X_6 = 0, \end{cases} \qquad E_{11} : \begin{cases} X_4 + 7X_5 - 2X_6 = 0,\\ 3X_3 - 6X_5 + 4X_6 = 0,\\ 7X_2 - 3X_5 + 8X_6 = 0,\\ X_1 + X_3 - X_6 = 0. \end{cases}$$

The Rosenhain normal form (Remark 3.5) of the corresponding curve of genus two is

$$y^2 = x(x-1)(x-4)(x-9)(x-11).$$

Appendix A.

A.1. We look at the action of Γ_2 on the projective coordinates of $\mathcal{A}_2(4,8)$ and $Y = \mathrm{Km}(X)$. For unimodular transformations

$$g_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & {}^t\alpha_i^{-1} \end{bmatrix}, \qquad \alpha_1 = \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right],\ \alpha_2 = \left[\begin{smallmatrix}10\\11\end{smallmatrix}\right],\ \alpha_3 = \left[\begin{smallmatrix}11\\01\end{smallmatrix}\right],\ \alpha_4 = \left[\begin{smallmatrix}11\\10\end{smallmatrix}\right],\ \alpha_5 = \left[\begin{smallmatrix}01\\11\end{smallmatrix}\right],$$

translations

$$h_i = \begin{bmatrix} \mathbf{1} & \beta_i \\ 0 & \mathbf{1} \end{bmatrix}, \qquad \beta_1 = \left[\begin{smallmatrix}10\\00\end{smallmatrix}\right],\ \beta_2 = \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right],\ \beta_3 = \left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]$$

and $J = \left[\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right]$, we have the following table, where $\zeta = \exp(2\pi i/8)$.

1	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
g_1	A_1	A_3	A_2	A_4	A_6	A_5	A_7	A_9	A_8	A_{10}
g_2	A_1	A_2	A_4	A_3	A_7	A_6	A_5	A_{10}	A_9	$-A_8$
g_3	A_1	A_4	A_3	A_2	A_5	A_7	A_6	A_8	A_{10}	$-A_9$
g_4	A_1	A_3	A_4	A_2	A_7	A_5	A_6	A_{10}	A_8	$-A_9$
g_5	A_1	A_4	A_2	A_3	A_6	A_7	A_5	A_9	A_{10}	$-A_8$
h_1	A_3	A_4	A_1	A_2	A_8	$\zeta^7 A_6$	$\zeta^7 A_7$	A_5	$\zeta^7 A_9$	$\zeta^7 A_{10}$
h_2	A_1	A_2	A_3	A_4	A_8	A_9	iA_{10}	A_5	A_6	iA_7
h_3	A_2	A_1	A_4	A_3	$\zeta^7 A_5$	A_9	$\zeta^7 A_7$	$\zeta^7 A_8$	A_6	$\zeta^7 A_{10}$
J	A_1	A_5	A_6	A_7	A_2	A_3	A_4	A_9	A_8	$-A_{10}$

1	X_1	X_2	X_3	X_4	X_5	X_6
g_1	X_4	X_5	X_6	X_1	X_2	X_3
g_2	X_3	$-X_6$	X_1	X_5	X_4	X_2
g_3	X_2	X_1	X_5	X_6	$-X_3$	X_4
g_4	X_6	$-X_3$	X_4	X_2	X_1	X_5
g_5	X_5	X_4	X_2	X_3	$-X_6$	X_1
h_1	X_2	X_1	$\zeta^7 X_3$	$\zeta^7 X_4$	$\zeta^7 X_5$	$\zeta^7 X_6$
h_2	X_2	X_1	$-iX_6$	X_5	X_4	$-iX_3$
h_3	$\zeta^7 X_1$	$\zeta^7 X_2$	$\zeta^7 X_3$	X_5	X_4	$\zeta^7 X_6$
J	iX_1	iX_3	iX_2	iX_4	iX_6	iX_5

A.2. From the above table, we see that the equation (E1) is transformed in the following equations:

$$
\begin{array}{ll}
\text{(E1)} & A_{10}^2X_1^2 + A_1^2X_4^2 - A_2^2X_5^2 - A_5^2X_6^2 = 0 \\
\text{(E2)} = h_1\text{(E1)} & A_{10}^2X_2^2 + A_3^2X_4^2 - A_4^2X_5^2 - A_8^2X_6^2 = 0 \\
\text{(E3)} = J\text{(E2)} & A_{10}^2X_3^2 + A_6^2X_4^2 - A_9^2X_5^2 - A_7^2X_6^2 = 0 \\
\text{(E4)} = g_1\text{(E1)} & A_1^2X_1^2 - A_3^2X_2^2 - A_6^2X_3^2 + A_{10}^2X_4^2 = 0 \\
\text{(E5)} = g_1\text{(E2)} & A_2^2X_1^2 - A_4^2X_2^2 - A_9^2X_3^2 + A_{10}^2X_5^2 = 0 \\
\text{(E6)} = g_1\text{(E3)} & A_5^2X_1^2 - A_8^2X_2^2 - A_7^2X_3^2 + A_{10}^2X_6^2 = 0 \\
\text{(E7)} = g_4\text{(E3)} & A_8^2X_1^2 - A_5^2X_2^2 - A_9^2X_4^2 + A_6^2X_5^2 = 0 \\
\text{(E8)} = g_4\text{(E1)} & A_3^2X_1^2 - A_1^2X_2^2 + A_7^2X_5^2 - A_9^2X_6^2 = 0 \\
\text{(E9)} = h_3\text{(E8)} & A_4^2X_1^2 - A_2^2X_2^2 + A_7^2X_4^2 - A_6^2X_6^2 = 0 \\
\text{(E10)} = J\text{(E9)} & A_7^2X_1^2 - A_5^2X_3^2 + A_4^2X_4^2 - A_3^2X_5^2 = 0 \\
\text{(E11)} = g_5\text{(E2)} & A_9^2X_1^2 - A_2^2X_3^2 - A_8^2X_4^2 + A_3^2X_6^2 = 0 \\
\text{(E12)} = g_5\text{(E1)} & A_6^2X_1^2 - A_1^2X_3^2 - A_8^2X_5^2 + A_4^2X_6^2 = 0 \\
\text{(E13)} = g_2\text{(E1)} & A_7^2X_2^2 - A_8^2X_3^2 + A_2^2X_4^2 - A_1^2X_5^2 = 0 \\
\text{(E14)} = g_3\text{(E1)} & A_9^2X_2^2 - A_4^2X_3^2 - A_5^2X_4^2 + A_1^2X_6^2 = 0 \\
\text{(E15)} = h_3\text{(E14)} & A_6^2X_2^2 - A_3^2X_3^2 - A_5^2X_5^2 + A_2^2X_6^2 = 0
\end{array}
$$

A.3. TWENTY OF EIGHTY ROSENHAIN'S HYPERPLANES. By the action of 2-torsion points (Remark 3.4), we can get 80 Rosenhain's hyperplanes from the following 20 ones.

$$H1: A_{1,3,6}X_4 + A_{2,4,9}X_5 + A_{5,7,8}X_6 = 0, \quad D_{1,3,6,14} + E_{10,11,12,13}$$

$$H2: A_{1,10,3}X_3 + A_{5,8,9}X_5 - A_{2,4,7}X_6 = 0, \quad D_{1,3,10,13} + E_{6,11,12,14}$$

$$
\begin{aligned}
H3 &: A_{2,4,10}X_3 - A_{5,6,8}X_4 - A_{1,3,7}X_6 = 0, && D_{1,3,7,16} + E_{9,11,12,15}\\
H4 &: A_{5,8,10}X_3 + A_{2,4,6}X_4 + A_{1,3,9}X_5 = 0, && D_{1,3,9,15} + E_{7,11,12,16}\\
H5 &: A_{1,6,10}X_2 + A_{4,5,7}X_5 - A_{2,8,9}X_6 = 0, && D_{1,6,10,12} + E_{3,11,13,14}\\
H6 &: A_{2,9,10}X_2 - A_{3,5,7}X_4 - A_{1,6,8}X_6 = 0, && D_{1,6,8,16} + E_{4,11,13,15}\\
H7 &: A_{5,7,10}X_2 + A_{2,3,9}X_4 + A_{1,4,6}X_5 = 0, && D_{1,4,6,15} + E_{8,11,13,16}\\
H8 &: A_{6,8,9}X_2 - A_{3,4,7}X_3 + A_{1,2,10}X_6 = 0, && D_{1,2,10,16} + E_{5,11,14,15}\\
H9 &: A_{4,6,7}X_2 - A_{3,8,9}X_3 + A_{1,5,10}X_5 = 0, && D_{1,5,10,15} + E_{2,11,14,16}\\
H10 &: A_{3,7,9}X_2 - A_{4,6,8}X_3 + A_{2,5,10}X_4 = 0, && D_{2,5,10,14} + E_{2,5,10,14}\\
H11 &: A_{3,6,10}X_1 + A_{2,7,8}X_5 - A_{4,5,9}X_6 = 0, && D_{3,6,10,11} + E_{3,6,10,11}\\
H12 &: A_{4,9,10}X_1 - A_{1,7,8}X_4 - A_{3,5,6}X_6 = 0, && D_{3,5,6,16} + E_{4,9,10,11}\\
H13 &: A_{7,8,10}X_1 + A_{1,4,9}X_4 + A_{2,3,6}X_5 = 0, && D_{2,3,6,15} + E_{7,8,10,11}\\
H14 &: A_{5,6,9}X_1 + A_{1,2,7}X_3 - A_{3,4,10}X_6 = 0, && D_{1,2,7,13} + E_{8,12,14,15}\\
H15 &: A_{2,6,7}X_1 + A_{1,5,9}X_3 - A_{3,8,10}X_5 = 0, && D_{1,5,9,13} + E_{4,12,14,16}\\
H16 &: A_{1,7,9}X_1 + A_{2,5,6}X_3 - A_{4,8,10}X_4 = 0, && D_{2,5,6,13} + E_{4,8,10,14}\\
H17 &: A_{3,4,5}X_1 + A_{1,2,8}X_2 + A_{6,9,10}X_6 = 0, && D_{1,2,8,12} + E_{7,13,14,15}\\
H18 &: A_{2,3,8}X_1 + A_{1,4,5}X_2 + A_{6,7,10}X_5 = 0, && D_{1,4,5,12} + E_{9,13,14,16}\\
H19 &: A_{1,4,8}X_1 + A_{2,3,5}X_2 + A_{7,9,10}X_4 = 0, && D_{2,3,5,12} + E_{7,9,10,14}\\
H20 &: A_{1,2,5}X_1 + A_{3,4,8}X_2 + A_{6,7,9}X_3 = 0, && D_{1,2,5,11} + E_{10,14,15,16}
\end{aligned}
$$

References

[BH] W. Barth and K. Hulek, Projective models of Shioda modular surfaces, Manuscripta Math., **50** (1985), 73–132.

[BL] Ch. Birkenhake and H. Lange, Complex Abelian Varieties (2nd ed.), Grundlehren Math. Wiss., **302**, Springer-Verlag, Berlin, 2004.

[Co] A. B. Coble, Algebraic Geometry and Theta Functions, Amer. Math. Soc. Colloq. Publ., **10**, Amer. Math. Soc., Providence, RI, 1929.

[CR] S. Cynk and S. Rams, On a map between two $K3$ surfaces associated to a net of quadrics, Arch. Math. (Basel), **88** (2007), 109–122.

[Fl] E. V. Flynn, The Jacobian and formal group of a curve of genus 2 over an arbitrary ground field, Math. Proc. Cambridge Philos. Soc., **107** (1990), 425–441.

[vGN] B. van Geemen and N. Nygaard, On the geometry and arithmetic of some Siegel modular threefolds, J. Number Theory, **53** (1995), 45–87.

[vGvS] B. van Geemen and D. van Straten, The cusp forms of weight 3 on $\Gamma_2(2,4,8)$, Math. Comp., **61** (1993), 849–872.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Pure Appl. Math. (Hoboken), Wiley, Hoboken, NJ, 1978.

[Hu] R. Hudson, Kummer's Quartic Surface, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1990.

[Ig1] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175–200.

[Ig2] J. Igusa, On Siegel modular forms of genus two. II, Amer. J. Math., **86** (1964), 392–412.

[Ig3] J. Igusa, Theta Functions, Grundlehren Math. Wiss., **194**, Springer-Verlag, New York, Heidelberg, 1972.

[Kl] F. Klein, Zur Theorie der Liniencomplexe des ersten und zweiten Grades, Math. Ann., **2** (1870), 198–226.

[Kr] A. Krazer, Lehrbuch der Thetafunktionen, Teubner, Leipzig, 1903.

[Kul] V. Kulikov, Degenerations of $K3$ surfaces and Enriques surfaces, Math. USSR Izv., **11** (1977), 957–989.

[KS] M. Kuwata and T. Shioda, Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface, In: Algebraic Geometry in East Asia - Hanoi 2005, Adv. Stud. Pure Math., **50**, Math. Soc. Japan, Tokyo, 2008, pp. 177–215.

[Ku] A. Kumar, Elliptic fibrations on a generic Jacobian Kummer surface, arXiv:1105.1715.

[Mu] D. Mumford, Tata Lectures on Theta. I, Progr. Math., **28**, Birkhäuser, Boston, MA, 1983.

[No] M. V. Nori, Zariski's conjecture and related problems, Ann. Sci. École Norm. Sup. (4), **16** (1983), 305–344.

[Pe] U. Persson, On Degenerations of Algebraic Surfaces, Mem. Amer. Math. Soc., **11**, no. 189, Amer. Math. Soc., Providence, RI, 1977.

[Sa] R. Sasaki, Moduli space of hyperelliptic curves of genus two with level $(2, 4)$ structure and the special orthogonal group of degree three, Kyushu J. Math., **53** (1999), 333–361.

[Sh] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, **24** (1972), 20–59.

Kenji Koike
Faculty of Education
University of Yamanashi
Takeda 4-4-37, Kofu
Yamanashi 400-8510, Japan
E-mail: kkoike@yamanashi.ac.jp