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The singular locus of Lauricella's F_C

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Abstract. We determine the singular locus of the holonomic system of differential equations annihilating Lauricella's hypergeometric function F_C by the theory of D-modules and of Gröbner bases. We also study the A-hypergeometric system associated to F_C.

1. Introduction.

Lauricella's hypergeometric function F_C with parameters $a, b, c_1, \ldots, c_m$ is defined by

$$F_C(x) = F_C(a, b, c_1, \ldots, c_m; x) = \sum_{k \in \mathbb{Z}^m_{\geq 0}} \frac{(a)_{|k|}(b)_{|k|}}{k!(c_1)_{k_1} \cdots (c_m)_{k_m}} x^k,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Put $\theta_i = x_i \partial_i$ for $i = 1, \ldots, m$ and $\theta = \theta_1 + \cdots + \theta_m$. We consider the left ideal $I(m)$ generated by the operators

$$\ell_i = \theta_i(\theta_i + c_i - 1) - x_i(\theta + a)(\theta + b), \quad i = 1, \ldots, m, \tag{1}$$

where $a, b, c_i \in \mathbb{C}$ are parameters. Lauricella's function F_C is annihilated by the left ideal $I(m)$. We will show, in Theorem 11, that the singular locus of $I(m)$ agrees with the zero set of

$$\prod_{i=1}^{m} x_i \prod_{\varepsilon_i \in \{-1,1\}} (1 + \varepsilon_1\sqrt{x_1} + \cdots + \varepsilon_m\sqrt{x_m}). \tag{2}$$

The proof of this fact occupies Sections 3, 4 and 5. Note that when we expand (2), it becomes a polynomial of x. In the last section, we study the A-hypergeometric system associated to the Lauricella F_C and determine its singular locus in the complex torus by utilizing our main theorem.

We have believed that the singular locus of $I(m)$ is well-known among experts, but we find few literatures on rigorous proofs on these facts. In our knowledge, there are theses by Kaneko [**7**] and by Yoshida [**15**], who prove that the singular locus of $I(m)$ is contained in the zero set of (2) but they do not discuss the opposite inclusion.

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2. Preliminaries.

Let $D = \mathbb{C}\langle x_1, \ldots, x_m, \partial_1, \ldots, \partial_m \rangle$ be the Weyl algebra of m variables. We take a $2m$ dimensional integral vector (u, v), $u, v \in \mathbb{Z}^m$ such that $u_i + v_i > 0$ for $i = 1, \ldots, m$. For an element $p = \sum_{(\alpha,\beta)\in E} c_{\alpha\beta} x^\alpha \partial^\beta$ of D, we define its (u, v)-initial form $\mathrm{in}_{(u,v)}(p)$ by the sum of the terms in p which have the highest (u, v)-weight. In other words, we define

$$\mathrm{ord}_{(u,v)}(p) = \max_{(\alpha,\beta)\in E} (\alpha \cdot u + \beta \cdot v),$$

$$\mathrm{in}_{(u,v)}(p) = \sum_{(\alpha,\beta)\in E, \alpha\cdot u+\beta\cdot v=\mathrm{ord}_{(u,v)}(p)} c_{\alpha\beta} x^\alpha \xi^\beta.$$

Here, ξ_i is a new variable which commutes with the other ones (see, e.g., [**12**, Section 1.1]). When $u_i + v_i = 0$, we define the (u, v)-initial form analogously and ξ_i is replaced by ∂_i in the definition above. Put $\mathbf{0} = (0, \ldots, 0), \mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^m$. For $p \in D$, its $(\mathbf{0}, \mathbf{1})$-initial form is called the principal symbol of p. For a given left ideal I of D, its characteristic ideal $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I)$ is the ideal in $\mathbb{C}[x, \xi] = \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m]$ generated by all principal symbols of the elements of I.

The zero set in $\mathbb{C}^{2m}$ of the characteristic ideal is called the characteristic variety, which is denoted by $\mathrm{Ch}(I)$. When the (Krull) dimension of the characteristic variety $\mathrm{Ch}(I)$ is equal to m, D/I and I are called a holonomic D-module and a holonomic ideal, respectively. The projection of $\mathrm{Ch}(I)\setminus V(\xi_1, \ldots, \xi_m)$ to the first m-coordinates $\mathbb{C}^m = \{x\}$ is called the singular locus of I and is denoted by $\mathrm{Sing}(I)$ or $\mathrm{Sing}(D/I)$, where

$$V(q_1, \ldots, q_k) = \left\{(x, \xi) \in \mathbb{C}^{2m} \mid q_1(x, \xi) = \cdots = q_k(x, \xi) = 0\right\}$$

for elements $q_1, \ldots, q_k$ in $\mathbb{C}[x, \xi]$. As to these fundamental notions of D-modules, see, e.g., [**2**], [**9**], [**12**].

Let $R = \mathbb{C}(x_1, \ldots, x_m)\langle \partial_1, \ldots, \partial_m \rangle$ be the ring of differential operators with rational function coefficients. The holonomic rank of I is the dimension of R/RI as a $\mathbb{C}(x) = \mathbb{C}(x_1, \ldots, x_m)$ vector space, and is denoted by $\mathrm{rank}\,(I)$. The rank is equal to the multiplicity of the characteristic ideal at a generic point. In other words, we have

$$\mathrm{rank}(I) = \dim_{\mathbb{C}} \mathcal{O}_a/\mathcal{O}_a \cdot \mathrm{in}_{(\mathbf{0},\mathbf{1})}(I),$$

where a is a point in $\mathbb{C}^m \setminus \mathrm{Sing}(I)$ and $\mathcal{O}_a = \mathbb{C}\{\xi_1 - a_1, \ldots, \xi_m - a_m\}$. We also have the identity

$$\mathrm{rank}(I) = \dim_{\mathbb{C}(x)} \mathbb{C}(x)[\xi]/\mathbb{C}(x)[\xi] \cdot \mathrm{in}_{(\mathbf{0},\mathbf{1})}(I),$$

where $\mathbb{C}(x)[\xi]$ denotes $\mathbb{C}(x_1, \ldots, x_m)[\xi_1, \ldots, \xi_m]$. Let $\mathrm{Sol}(I)$ be the constructive sheaf of holomorphic solutions of the differential system given by I on $\mathbb{C}^m$. That is, $\mathrm{Sol}(I)$ is the sheafication of the presheaf $\{f \in \mathcal{O} \mid \ell \cdot f = 0 \text{ for all } \ell \in I\}$, where $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^m$. The holonomic rank $\operatorname{rank}(I)$ is equal to $\dim_{\mathbb{C}} \operatorname{Sol}(I)(U)$ for any simply connected open set U in $\mathbb{C}^m \setminus \operatorname{Sing}(I)$. As to these characterizations of the holonomic rank, see, e.g., [**9**], [**12**, Chapter 1] and the references therein.

3. A variety containing the singular locus.

The singular locus of the system $I(m)$ is $\pi(\operatorname{Ch}(I(m)) \setminus V(\xi_1, \dots, \xi_m))$ by definition. Here, $\pi : \mathbb{C}^{2m} \longrightarrow \mathbb{C}^m$ is the projection sending (x, ξ) to x. The principal symbol L_i of ℓ_i is equal to

$$L_i = x_i^2 \xi_i^2 - x_i \left(\sum_{j=1}^m x_j \xi_j \right)^2$$

for $i = 1, \dots, m$. Since $L_i \in \operatorname{in}_{(\mathbf{0},\mathbf{1})}(I(m))$, the singular locus $\operatorname{Sing}(I(m))$ is contained in $C' = \pi(V(L_1, \dots, L_m) \setminus V(\xi_1, \dots, \xi_m))$.

Let us regard $V(L_1, \dots, L_m)$ as an analytic space. When $x_i \neq 0$, L_i is factored as

$$L_i = \left(x_i \xi_i - \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) \right) \left(x_i \xi_i + \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) \right) \tag{3}$$

in the extension field $\mathbb{C}(\sqrt{x_1}, \dots, \sqrt{x_m})$ of $\mathbb{C}(x)$. Therefore, the necessary and sufficient condition that x lies in $C' \cap (\mathbb{C}^*)^m$ is that there exists $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ such that

$$x_i \xi_i + \varepsilon_i \sqrt{x_i} \left(\sum_{j=1}^m x_j \xi_j \right) = 0, \quad i = 1, \dots, m \tag{4}$$

has a non-trivial solution $\xi \neq 0$. This condition can be written in terms of the determinant of the system regarded as a system of linear equations with respect to the variables ξ.

PROPOSITION 1. *The determinant of the coefficient matrix of the system of linear equation* (4) *is equal to* $\prod_{i=1}^m x_i \bigl(1 + \sum_{j=1}^m \varepsilon_j \sqrt{x_j}\bigr)$.

PROOF. The coefficient matrix of the system (4) is $(\varepsilon_i x_j \sqrt{x_i} + \delta_{ij} x_j)_{1 \le i,j \le m}$, whose determinant is equal to $\prod_{i=1}^m x_i \cdot \det(\varepsilon_i \sqrt{x_i} + \delta_{ij})_{1 \le i,j \le m}$. The matrix

$$(E + E_{m,1} + \cdots + E_{m,m-1})(\varepsilon_i \sqrt{x_i} + \delta_{ij})(E - E_{m,1} - \cdots - E_{m,m-1})$$

is of upper half triangle with diagonal components $1, \dots, 1, 1 + \sum_{j=1}^m \varepsilon_j \sqrt{x_j}$, where $E_{ij} \in M_m(\mathbb{Z})$ is the (i,j)-matrix unit. The determinants of the matrices $E \pm (E_{m,1} + \cdots + E_{m,m-1})$ are equal to 1, therefore we get the conclusion. □

THEOREM 2 ([**7**], [**15**]). *The singular locus of $I(m)$ is contained in the zero set of* (2).

Proof. Since $x_i = 0$ are contained in the zero set of (2), we may only consider the singular locus in $(\mathbb{C}^*)^m$. If $x \in C' \cap (\mathbb{C}^*)^m$, the equation (4) must have a non-trivial solution $\xi \neq 0$. By Proposition 1, we get the conclusion. $\square$

In the sequel, we want to prove the opposite inclusion $C \subseteq \mathrm{Sing}(I(m))$, where C is the zero set of (2). If a classical solution of $I(m)$ has singularities on the all irreducible components of the zero set C, then we have the above assertion. However, as the following examples show, the singular locus of classical solutions may smaller than the zero set C.

Example 3. Assume $m = 2$. When $a = -1/2$, $b = -2$, $c_1 = c_2 = 1/2$, the solution space of the differential equations is spanned by the following functions

$$1 + 2x + 2y - 2xy - x^2/3 - y^2/3, \quad \sqrt{x}, \quad \sqrt{y}, \quad \sqrt{xy}(1 - x/3 - y/3).$$

Note that the singular locus of these solutions is contained in $xy = 0$, which is smaller than the zero set C.

Example 4. Assume $m = 2$ again. When $a = -1/2$, $b = c_1 = c_2 = 0$, the solution space is spanned by functions

$$1, \quad x, \quad y, \quad xyF_C(1, 2, 2, 2; x, y).$$

They do not have singularities along $xy = 0$.

We close this section with two preparatory propositions.

Proposition 5. *The left ideal $I(m)$ is holonomic.*

Proof. Since the Bernstein inequality $\dim \mathrm{Ch}(I(m)) \geq m$ holds (see, e.g., [**2**], [**12**]), we have only to prove $\dim \mathrm{Ch}(I(m)) \leq m$. Let $x \in (\mathbb{C}^*)^m$. If x does not belong to the zero set of (2), we have $(x, 0) \in ((\mathbb{C}^*)^m \times \mathbb{C}^m) \cap V(L_1, \ldots, L_m)$. Otherwise $(x, \xi) \in ((\mathbb{C}^*)^m \times \mathbb{C}^m) \cap V(L_1, \ldots, L_m)$ for some $\xi \neq 0$ by Proposition 1. We conclude that the dimension of $((\mathbb{C}^*)^m \times \mathbb{C}^m) \cap V(L_1, \ldots, L_m)$ is equal to m because in $(\mathbb{C}^*)^m \times \mathbb{C}^m$, the variety defined by (4) coincides with the one given by

$$\begin{cases} \left(1 + \sum_{j=1}^{m} \varepsilon_j \sqrt{x_j}\right) \xi_1 = 0, \\ \varepsilon_i \sqrt{x_i} \xi_i - \varepsilon_1 \sqrt{x_1} \xi_1 = 0, \quad i = 2, \ldots, m, \end{cases}$$

whose dimension is equal to m.

The remaining thing to do is the evaluation of the dimension at the points in $x_i = 0$. We put $I_0(m) = \langle L_1, \ldots, L_m \rangle$, which is contained in the characteristic ideal $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I(m))$. We will prove $\dim I_0(m) = m$ by induction. When $m = 1$, it is easy to see that $\dim I_0(m) = 1$. Let us assume $\dim I_0(m-1) = m - 1$. We note that

$$V(I_0(m)) \cap V(x_m) = \left\{((x', 0), (\xi', \xi_m)) \mid (x', \xi') \in V(I_0(m-1)) \subset \mathbb{C}^{2(m-1)}, \xi_m \in \mathbb{C}\right\}$$

because $x_m \xi_m = 0$ in L_i when $x_m = 0$. It follows from the induction hypothesis $\dim V(I_0(m-1)) = m-1$ that the dimension of $V(I_0(m))$ at any point in $x_m = 0$ is equal to $(m-1)+1 = m$. $\square$

Since the Galois group $\mathrm{Gal}(\mathbb{C}(\sqrt{x_1}, \ldots, \sqrt{x_m})/\mathbb{C}(x_1, \ldots, x_m))$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$, we have the following.

PROPOSITION 6. *The polynomial*

$$\prod_{\varepsilon_i \in \{+1,-1\}} (1 + \varepsilon_1 \sqrt{x_1} + \cdots + \varepsilon_m \sqrt{x_m})$$

is irreducible in $\mathbb{C}[x_1, \ldots, x_m]$.

4. The singular locus in the complex torus.

If we can show that $L_1, \ldots, L_m$ generate the characteristic ideal $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I(m))$, we conclude that the singular locus $\mathrm{Sing}(I(m))$ agrees with the zero set C of (2). However, it seems not to be easy to prove it. Instead of proving it directly, we determine the characteristic variety in the complex torus in this section.

We consider the left ideal $I'(m)$ generated by

$$\ell_i' = y_i \theta_i (\theta_i - c_i + 1) - (\theta - a)(\theta - b), \quad i = 1, \ldots, m.$$

Here, $\theta_i = y_i \partial_{y_i}$ and $\theta = \theta_1 + \cdots + \theta_m$. These operators $\ell_1', \ldots, \ell_m'$ are obtained by applying the change of the coordinates $y_i = 1/x_i$, $i = 1, \ldots, m$ to ℓ_i's and multiplying y_i to them. The ring of differential operators with respect to the variable y is also denoted by D as long as no confusion arises. The characteristic varieties of $I(m)$ and $I'(m)$ agree in the complex torus $(\mathbb{C}^*)^m$ under the change of the coordinates $y_i = 1/x_i$.

We use the order $\succ_w$ defined by the first weight vector $w^{(1)} = (\mathbf{0}, \mathbf{1})$ and the second weight vector $w^{(2)} = (\mathbf{1}, \mathbf{0})$. In other words, $y^\alpha \partial^\beta \succ_w y^{\alpha'} \partial^{\beta'}$ if and only if

$$\begin{aligned}
&(\alpha, \beta) \cdot w^{(1)} > (\alpha', \beta') \cdot w^{(1)} \\
&\quad \text{or } \big((\alpha, \beta) \cdot w^{(1)} = (\alpha', \beta') \cdot w^{(1)} \text{ and } (\alpha, \beta) \cdot w^{(2)} > (\alpha', \beta') \cdot w^{(2)}\big) \\
&\quad \text{or } \big((\alpha, \beta) \cdot w^{(j)} = (\alpha', \beta') \cdot w^{(j)} \ (j = 1, 2) \text{ and } (\alpha, \beta) >_{\mathrm{lex}} (\alpha', \beta')\big).
\end{aligned}$$

Here, $>_{\mathrm{lex}}$ is the lexicographic order. We denote by $\mathrm{in}_{\prec_w}(f)$ the leading monomial of $f \in D$ with respect to the order $\prec_w$. For two elements $f, g \in D$ with

$$\mathrm{in}_{\prec_w} f = f_{\alpha\beta} y^\alpha \xi^\beta, \qquad \mathrm{in}_{\prec_w} g = g_{\alpha'\beta'} y^{\alpha'} \xi^{\beta'},$$

we define their S-pair $\mathrm{sp}(f, g)$ by

$$\mathrm{sp}(f, g) = g_{\alpha'\beta'} y^{\gamma - \alpha} \partial^{\delta - \beta} f - f_{\alpha\beta} y^{\gamma - \alpha'} \partial^{\delta - \beta'} g,$$

where

$$\gamma = \bigl(\max\{\alpha_1, \alpha'_1\}, \ldots, \max\{\alpha_m, \alpha'_m\}\bigr), \quad \delta = \bigl(\max\{\beta_1, \beta'_1\}, \ldots, \max\{\beta_m, \beta'_m\}\bigr).$$

For a subset G of D, the relation $f = \sum c_i g_i$, $g_i \in G$ is called a standard representation of f with respect to G when $c_i g_i \preceq_w f$ holds for all i such that $c_i \neq 0$.

Proposition 7. *The characteristic ideal* $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I'(m))$ *is generated by principal symbols* $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(\ell'_i)$, $i = 1, \ldots, m$.

Proof. We use the order $\succ_w$. Since $\mathrm{in}_{\prec_w}(\ell'_i) = y_i^3 \xi_i^2$, we have

$$\mathrm{sp}(\ell'_i, \ell'_j) = y_j^3 \partial_j^2 \ell'_i - y_i^3 \partial_i^2 \ell'_j.$$

It is expressed as

$$\begin{aligned} \mathrm{sp}\bigl(\ell'_i, \ell'_j\bigr) &= \bigl(y_j^3 \partial_j^2 - \ell'_j\bigr)\ell'_i - \bigl(y_i^3 \partial_i^2 - \ell'_i\bigr)\ell'_j - \bigl(\ell'_i \ell'_j - \ell'_j \ell'_i\bigr) \\ &= \bigl(y_j^3 \partial_j^2 - \ell'_j\bigr)\ell'_i - \bigl(y_i^3 \partial_i^2 - \ell'_i\bigr)\ell'_j - (2\theta - a - b + 1)\bigl(\ell'_i - \ell'_j\bigr) \\ &= \bigl\{y_j^3 \partial_j^2 - \ell'_j - (2\theta - a - b + 1)\bigr\}\ell'_i - \bigl\{y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1)\bigr\}\ell'_j. \end{aligned} \tag{5}$$

Note that we have used commutation relations

$$\ell'_i \ell'_j - \ell'_j \ell'_i = -(2\theta - a - b + 1)\bigl(\ell'_i - \ell'_j\bigr),$$

which are obtained by a straightforward calculation. We have

$$\begin{aligned} y_j^3 \partial_j^2 \ell'_i = y_i^3 y_j^3 \partial_i^2 \partial_j^2 - y_j^3 \Biggl\{&(-c_i + 2) y_i \partial_i + \sum_{k=1}^m \theta_k^2 + 4(1 + y_j \theta_j) \\ &+ \sum_{k \neq k'} \theta_k \theta_{k'} + 4 \sum_{k \neq j} \theta_k - (a+b)\Biggl(\sum_{k=1}^m \theta_k + 2\Biggr) + ab\Biggr\}\partial_j^2, \end{aligned}$$

which implies

$$\begin{aligned} \mathrm{in}_{\prec_w} \mathrm{sp}\bigl(\ell'_i, \ell'_j\bigr) &= \mathrm{in}_{\prec_w}\Biggl\{-y_j^3\Biggl(\sum_{k,k'} y_k y_{k'} \partial_k \partial_{k'}\Biggr)\partial_j^2 + y_i^3\Biggl(\sum_{k,k'} y_k y_{k'} \partial_k \partial_{k'}\Biggr)\partial_i^2\Biggr\} \\ &= y_1^2 y_i^3 \xi_1^2 \xi_i^2 \end{aligned}$$

for $i < j$. On the other hand, we have

$$\begin{aligned} &\mathrm{in}_{\prec_w}\bigl\{y_i^3 \partial_i^2 - \ell'_i - (2\theta - a - b + 1)\bigr\} \\ &\quad = \mathrm{in}_{\prec_w}\{(c_i - 2) y_i \theta_i + (\theta - a)(\theta - b) - (2\theta - a - b + 1)\} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{in}_{\prec_w}(\theta^2)\\
&= y_1^2\xi_1^2.
\end{aligned}
$$

Note that it is independent of the index i. Hence we conclude that

$$
\begin{aligned}
\mathrm{in}_{\prec_w}\{y_j^3\partial_j^2 - \ell'_j - (2\theta - a - b + 1)\}\ell'_i &= y_1^2 y_i^3 \xi_1^2 \xi_i^2,\\
\mathrm{in}_{\prec_w}\{y_i^3\partial_i^2 - \ell'_i - (2\theta - a - b + 1)\}\ell'_j &= y_1^2 y_j^3 \xi_1^2 \xi_j^2,
\end{aligned}
$$

which imply that the expression (5) is a standard representation of $\mathrm{sp}(\ell'_i, \ell'_j)$ with respect to the set $\{\ell'_1 \dots, \ell'_m\}$ and the order $\prec_w$. It follows from the Buchberger's criterion that it is a Gröbner basis with respect to that order. Therefore the set

$$
\{\mathrm{in}_{(\mathbf{0},\mathbf{1})}(\ell'_i) \mid i = 1, \dots, m\} = \left\{ y_i(y_i\xi_i)^2 - \left(\sum_{j=1}^m y_j\xi_j\right)^2 \,\middle|\, i = 1, \dots, m \right\}
$$

is a Gröbner basis of $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I'(m))$ by the theorem stated in [**9**, Section 2] (the condition on the order can be weakened as in [**12**, Theorem 1.1.6]). In particular, it is a set of generators of the characteristic ideal $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I'(m))$. □

Let us determine the singular locus of $I'(m)$. The principal symbol L'_i of ℓ'_i is equal to

$$
L'_i = y_i^3\xi_i^2 - \left(\sum_{j=1}^m y_j\xi_j\right)^2.
$$

When $y_i \neq 0$, it is factored as

$$
L'_i = \left(y_i\sqrt{y_i}\xi_i - \sum_{j=1}^m y_j\xi_j\right)\left(y_i\sqrt{y_i}\xi_i + \sum_{j=1}^m y_j\xi_j\right)
$$

in the extension field $\mathbb{C}(\sqrt{y_1}, \dots, \sqrt{y_m}\,)$ of $\mathbb{C}(y)$. We can easily show that the determinant of the coefficient matrix of the system

$$
y_i\sqrt{y_i}\xi_i + \varepsilon_i \sum_{j=1}^m y_j\xi_j = 0, \quad i = 1, \dots, m
$$

is equal to

$$
\left(\prod_{j=1}^m y_j\sqrt{y_j}\right)\left(1 + \sum_{j=1}^m \frac{\varepsilon_j}{\sqrt{y_j}}\right). \tag{6}
$$

Therefore, the singular locus of $I'(m)$ is equal to the union of the zero sets of (6) where ε_j's run over $\{-1, +1\}$. Thus, we have the following theorem.

Theorem 8. *The singular locus of* $I(m)$ *agrees with the zero set of* (2) *in the complex torus* $(\mathbb{C}^*)^m$.

5. Singular locus and the coordinate hyperplanes.

In this section, we prove that the coordinate hyperplanes are contained in the singular locus $\mathrm{Sing}(I(m))$ of $I(m)$ by discussing the cohomological solution sheaf $\mathcal{E}xt^1_{\mathcal{D}^{an}}(\mathcal{D}^{an}/\mathcal{D}^{an}I(m), \mathcal{O}^{an})$. We need a set of generators of the syzygies of $I(m)$ to describe the first cohomological solutions (as to an algorithmic method to determine it, see, e.g., [**14**]). We utilize a Gröbner basis with the order $\succ_{(-\mathbf{1},\mathbf{1})}$, which is given by the weight vector $(-\mathbf{1},\mathbf{1}) = (-1,\dots,-1,1,\dots,1)$ and the lexicographic order $\partial_1 \succ \cdots \succ \partial_m \succ x_1 \succ \cdots \succ x_m$ as the tie-breaker, to determine the syzygies among generators of $I(m)$.

In order to use the S-pair criterion, we will work in the homogenized Weyl algebra $D^{(h)} = \mathbb{C}[h]\langle x_1,\dots,x_m,\partial_1,\dots,\partial_m\rangle$ (see, e.g., [**12**, Section 1.2]). The variable h is the homogenization variable which commutes with all other variables and we have the relation $\partial_i x_i = x_i\partial_i + h^2$. Put

$$S_i = \theta_i(\theta_i + (c_i-1)h^2), \qquad S_{ab} = \left(\sum_{i=1}^m \theta_i + ah^2\right)\left(\sum_{i=1}^m \theta_i + bh^2\right)$$

and

$$T_i = hS_i - x_iS_{ab}, \qquad T_{ij} = x_jS_i - x_iS_j.$$

They are homogeneous elements in $D^{(h)}$. The operators T_i and T_{ij} are the homogenizations of ℓ_i and $x_j\ell_i - x_i\ell_j$, respectively. For two elements in $D^{(h)}$, their S-pair with respect to the order $\succ_{(-\mathbf{1},\mathbf{1})}$ is defined similarly as in Section 4. We also use the terminology "standard representation" analogously for elements in $D^{(h)}$.

Theorem 9. *The set* $G = \{T_1,\dots,T_m,T_{12},T_{13},\dots,T_{m-1,m}\}$ *satisfies the* S*-pair criterion in the homogenized Weyl algebra* $D^{(h)}$; G *is a Gröbner basis of the ideal generated by itself with respect to the order* $\succ_{(-\mathbf{1},\mathbf{1})}$.

Proof. We have the following standard representations of S-pairs in terms of G:

$$\mathrm{sp}(T_i,T_j) = S_jT_i - S_iT_j = S_{a-1,b-1}T_{ij}, \tag{7}$$

$$\mathrm{sp}(T_i,T_{ij}) = x_jT_i - hT_{ij} = x_iT_j, \tag{8}$$

$$\begin{aligned}\mathrm{sp}(T_j,T_{ij}) &= x_i^2\partial_i^2T_j - hx_j\partial_j^2T_{ij}\\ &= \left\{x_i(x_j^{-1}S_j) - c_ih^2\theta_i\right\}T_j - (2h^2\theta_j + c_jh^4)T_i + (c_jh^3\partial_j - S_{a-1,b-1})T_{ij},\end{aligned} \tag{9}$$

$$\begin{aligned}\mathrm{sp}(T_k,T_{ij}) &= x_i^2x_j\partial_i^2T_k - hx_k^2\partial_k^2T_{ij}\\ &= hS_jT_{ki} + x_kS_iT_j - c_ih^2x_j\theta_iT_k + c_kh^3\theta_kT_{ij},\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathrm{sp}(T_{ij},T_{ik}) &= x_kT_{ij}-x_jT_{ik}=-x_iT_{jk}, && (11)\\
\mathrm{sp}(T_{ij},T_{kj}) &= x_k^2\partial_k^2T_{ij}-x_i^2\partial_i^2T_{kj}=S_jT_{ik}-c_kh^2\theta_kT_{ij}+c_ih^2\theta_iT_{kj}, && (12)\\
\mathrm{sp}(T_{ij},T_{jk}) &= x_jx_k\partial_j^2T_{ij}-x_i^2\partial_i^2T_{jk}\\
&= \{S_k+(2-c_j)h^2x_k\partial_j\}T_{ij}+(c_j-2)h^4T_{ik}\\
&\quad +\{(2-c_j)h^2x_i\partial_j+c_ih^2\theta_i-x_i\theta_j\partial_j\}T_{jk}, && (13)\\
\mathrm{sp}(T_{ij},T_{i'j'}) &= x_{i'}^2x_{j'}\partial_{i'}^2T_{ij}-x_i^2x_j\partial_i^2T_{i'j'}\\
&= x_{j'}S_jT_{ii'}-x_{i'}S_iT_{jj'}-c_{i'}h^2x_{j'}\theta_{i'}T_{ij}+c_ih^2x_j\theta_iT_{i'j'}, && (14)
\end{aligned}$$

where we assume that the indices i,j,k,i',j' satisfy $i\neq k$, $j\neq k$ and $\{i,j\}\cap\{i',j'\}=\phi$. Note in the above that we regard $T_{ji}=-T_{ij}$ for $i<j$.

Thus, we have proved that the set G is a Gröbner basis. □

By [**10**, Theorem 9.10], syzygies are generated by the dehomogenizations of the standard representations of the S-pairs. The following Corollary will be used to complete the proof of our main theorem.

Corollary 10. *The set of relations derived from the standard representations of the S-pairs gives a set of generators of the syzygies among ℓ_i, $(i=1,\dots,m)$, $\ell_{ij}=x_j\ell_i-x_i\ell_j$, $1\leq i<j\leq m$. For example,* (8) *yields the syzygy $x_j\ell_i-\ell_{ij}-x_i\ell_j=0$.*

Theorem 11. *The singular locus of $I(m)$ is the zero set of* (2).

Proof. It follows from the discussions in Section 4 that we may prove only that $x_i=0$ are contained in the singular locus. Let $g_m(x')$ be a non-zero solution of $I(m-1)$ at a generic point in $\mathbb{C}^{m-1}$, where $I(m-1)$ is a left ideal generated by

$$\theta_i(\theta_i+c_i-1)-x_i\left(\sum_{k=1}^{m-1}\theta_k+a\right)\left(\sum_{k=1}^{m-1}\theta_k+b\right),\quad i=1,\dots,m-1.$$

This function $g_m(x')$ does not depend on x_m. Put $g_1=\cdots=g_{m-1}=0$. Then, we have $\ell_i\cdot g_m=0$ for $i\neq m$ and $\ell_j\cdot g_i=0$ for $i=1,\dots,m-1$. Define $g_{ij}=x_jg_i-x_ig_j$. $\sum g_ie_i+\sum g_{ij}e_{ij}$ are annihilated by the generators of the syzygies given in Corollary 10. For instance, we have the syzygy

$$(\theta_j(\theta_j-1)+c_j\theta_j)\ell_i-(\theta_i(\theta_i-1)+c_i\theta_i)\ell_j-(\theta+a-1)(\theta+b-1)\ell_{ij}=0$$

by the equation (7). For $i<j=m$, we have

$$\begin{aligned}
&(\theta_j(\theta_j-1)+c_j\theta_j)g_i-(\theta_i(\theta_i-1)+c_i\theta_i)g_j-(\theta+a-1)(\theta+b-1)g_{ij}\\
&\quad=-(\theta_i(\theta_i-1)+c_i\theta_i)g_m-(\theta+a-1)(\theta+b-1)(-x_ig_m)\\
&\quad=-(\theta_i(\theta_i-1)+c_i\theta_i)g_m+x_i(\theta+a)(\theta+b)g_m
\end{aligned}$$

$$
\begin{aligned}
&= -\Big\{(\theta_i(\theta_i-1)+c_i\theta_i) - x_i\Big(\sum_{k=1}^{m-1}\theta_k + a\Big)\Big(\sum_{k=1}^{m-1}\theta_k + b\Big)\Big\}g_m \\
&\qquad \text{(since } g_m = g_m(x') \text{ does not depend on } x_m \text{, we have } \partial_m g_m = 0) \\
&= 0.
\end{aligned}
$$

When $i < j < m$, obviously the equation

$$(\theta_j(\theta_j-1)+c_j\theta_j)g_i - (\theta_i(\theta_i-1)+c_i\theta_i)g_j - (\theta+a-1)(\theta+b-1)g_{ij} = 0$$

holds.

Let us try to solve $\ell_i \cdot f = g_i$, $i = 1, \ldots, m$ and $\ell_{ij} \cdot f = g_{ij}$, $1 \le i < j \le m$. The second group of equation is solved when the first group is solved. Put $f = \sum_{k=0}^{\infty} f_k x_m^k$ where f_k is a function in x'. The left hand side of $\ell_m f$ can be factored by x_m. On the other hand, the right hand side g_m is nonzero and does not depend on x_m. Therefore the system $\ell_i \cdot f = g_i$ does not have a holomorphic solution along $x_m = 0$. Therefore, we have proved that $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{D}^{an}/\mathcal{D}^{an}E_C, \mathcal{O})$ is not zero at a generic point in $x_m = 0$. By Kashiwara's theorem [**8**, Theorem 4.1], $\mathcal{E}xt^1$ must be zero if $x_m = 0$ is not a singular locus. Thus, we have proved that $x_m = 0$ is contained in the singular locus. We can analogously show that other varieties $x_i = 0$ are also contained in the singular locus. □

6. The A-hypergeometric system associated to the Lauricella F_C.

The binomial D-modules [**3**] are introduced to study classical hypergeometric systems including the Lauricella F_C. The contents of the first half part of this section are implicitly or explicitly explained in [**3**], but they do not seem to be publicized to people who study classical Lauricella functions and related topics. The first part of this section explains how to apply the theory of A-hypergeometric systems and binomial D-modules to study F_C. The second part contains a new result and utilizes the first part; the last Theorem 14 describes the singular locus of the A-hypergeometric system associated to F_C in the complex torus. The singular locus is the zero set of the principal A-determinant [**5**] for the A associated to F_C.

We denote $I(m)$, of which elements annihilate the Lauricella function F_C, by E_C in this section. For a given Horn system, there exists a corresponding binomial D-module. In case of E_C, the corresponding binomial system is an A-hypergeometric system. Let us study this system.

Let $e_1, \ldots, e_{m+1}, e_{m+2}$ be the standard basis of $\mathbb{Z}^{m+2}$. Following [**11**], consider the set of points

$$
\begin{aligned}
\mathcal{A} = \{&e_1 + e_{m+2}, e_2 + e_{m+2}, \ldots, e_{m+1} + e_{m+2}, \\
&- e_1 + e_{m+2}, -e_2 + e_{m+2}, \ldots, -e_{m+1} + e_{m+2}\}.
\end{aligned}
$$

We define the matrix $A(F_C, m)$ consisting of these points as column vectors. This matrix is of type $(m+2) \times 2(m+1)$. For example, we have

$$A(F_C,2)=\begin{pmatrix}1&0&0&-1&0&0\\0&1&0&0&-1&0\\0&0&1&0&0&-1\\1&1&1&1&1&1\end{pmatrix}.$$

Let $H_A(\beta)$ be the A-hypergeometric system associated to the matrix $A(F_C,m)$, the parameter $\beta^T=(1-c_1,\dots,1-c_m,b-a,\sum_{j=1}^m c_j-a-b-m)$ and the independent variables $u_1,\dots,u_{m+1},u_{-1},\dots,u_{-(m+1)}$. The associated differential operators for u_j and u_{-j} are denoted by ∂_j and ∂_{-j}, respectively. For $A=A(F_C,m)$, its toric ideal I_A is defined by $I_A=\{\partial^u-\partial^v\mid Au=Av,u,v\in\mathbb{N}_0^{2m+2}\}$, which is generated by $\partial_j\partial_{-j}-\partial_{m+1}\partial_{-(m+1)}$ in $\mathbb{C}[\partial_1,\dots,\partial_{m+1},\partial_{-1},\dots,\partial_{-m-1}]$ $(j=1,\dots,m)$.

The left ideal $H_A(\beta)$ is generated by the row vectors of $A\theta_u-\beta$ and I_A, where $\theta_u=(u_1\partial_1,\dots,u_{m+1}\partial_{m+1},u_{-1}\partial_{-1},\dots,u_{-(m+1)}\partial_{-(m+1)})^T$. We denote the i-th row vector of $A\theta-\beta$ by $E_i-\beta_i$.

Theorem 12 ([**3**]). *The isomorphism* $\mathrm{Sol}(E_C)\simeq\mathrm{Sol}(H_A(\beta))$ *holds for any parameter. In particular, the holonomic rank of $E_C=I(m)$ is equal to 2^m.*

Proof. Let F be a solution of E_C. Following [**11**], we consider the following function

$$f(u)=u_{m+1}^{-a}u_{-(m+1)}^{-b}\prod_{j=1}^m u_{-j}^{c_j-1}F\left(\frac{u_1u_{-1}}{u_{m+1}u_{-(m+1)}},\dots,\frac{u_mu_{-m}}{u_{m+1}u_{-(m+1)}}\right).\tag{15}$$

Let us prove that the function $f(u)$ is a solution of $H_A(\beta)$. It is easy to see that $(E_i-\beta_i)\cdot f=0$. Put $\eta=u_{m+1}^{-a}u_{-(m+1)}^{-b}\prod_{j=1}^m u_{-j}^{c_j-1}$ and $z_j=u_ju_{-j}/(u_{m+1}u_{-(m+1)})$. We have

$$\begin{aligned}\theta_j\theta_{-j}\cdot f(u)&=\theta_j\cdot((c_j-1)\eta F+\eta z_jF_j)\\&=(c_j-1)\eta z_jF_j+\eta z_jF_j+\eta z_jz_jF_{jj}\\&=\eta\,\theta_{z_j}(\theta_{z_j}+c_j-1)\cdot F(z),\end{aligned}\tag{16}$$

where F_j denotes the partial derivative of $F(z_1,\dots,z_m)$ with respect to the variable z_j. Analogously, we get

$$\theta_{m+1}\theta_{-(m+1)}\cdot f(u)=\eta\left(\sum_{i=1}^m\theta_{z_i}+a\right)\left(\sum_{i=1}^m\theta_{z_i}+b\right)\cdot F(z).\tag{17}$$

Put $\tilde{\ell}_j=u_ju_{-j}u_{m+1}u_{-(m+1)}(\partial_j\partial_{-j}-\partial_{m+1}\partial_{-(m+1)})$. This is equal to

$$u_{m+1}u_{-(m+1)}\theta_j\theta_{-j}-u_ju_{-j}\theta_{m+1}\theta_{-(m+1)}.$$

It follows from (16) and (17) that we have $\tilde{\ell}_j\cdot f(u)=0$, which implies that $f(u)$ is a solution of $H_A(\beta)$. Note that the correspondence from F to f is an injection among

$\mathbb{C}$-vector spaces of solutions on a simply connected open set.

Conversely, let f be a solution of $H_A(\beta)$. We define new $2m+2$ variables z_j by

$$z_j = \frac{u_j u_{-j}}{u_{m+1}u_{-(m+1)}}, \quad z_{m+j} = u_{-j}^{-1}, \quad j = 1, \dots, m, \tag{18}$$
$$z_{2m+1} = u_{m+1}^{-1}, \quad z_{2m+2} = u_{-(m+1)}^{-1}.$$

Note that this gives an isomorphism of the complex tori $(\mathbb{C}^*)^{2m+1} = \{u\}$ and $(\mathbb{C}^*)^{2m+1} = \{z\}$. The Euler operator $\theta_{\pm u_j} = u_{\pm j}\partial_{\pm j}$ can be written as a sum of Euler operators with respect to z_i's. In fact, we have

$$u_j\partial_j = z_j\partial_{z_j}, \quad u_{-j}\partial_{-j} = z_j\partial_{z_j} - z_{m+j}\partial_{z_{m+j}},$$
$$u_{m+1}\partial_{m+1} = -\sum_{k=1}^m z_j\partial_{z_k} - z_{2m+1}\partial_{z_{2m+1}},$$
$$u_{-(m+1)}\partial_{-(m+1)} = -\sum_{k=1}^m z_j\partial_{z_k} - z_{2m+2}\partial_{z_{2m+2}}.$$

Put $f' = \eta^{-1}f$. The equations $(E_i - \beta_i)\cdot f = 0$ yield $\theta_{z_j}\cdot f' = 0$ for $j = m+1, \dots, 2(m+1)$. This implies that f' depends only on $z_1, \dots, z_m$. An analogous calculation with (16) and (17) yields the equation $\ell_i \cdot f'(u(z)) = 0$. This means that the map $F(z) \mapsto f(u)$ is surjective. Thus, we have proved $\mathrm{Sol}(E_C) \simeq \mathrm{Sol}(H_A(\beta))$.

The correspondence gives the holonomic rank of E_C by evaluating the degree of I_A [**4**]. Since $\{\underline{\partial_j\partial_{-j}} - \partial_{m+1}\partial_{-(m+1)} \mid j = 1, \dots, m\}$ is a Gröbner basis of I_A, the degree is equal to that of the monomial ideal generated by $\partial_j\partial_{-j}$, $j = 1, \dots, m$. This degree is equal to 2^m. □

An application of this isomorphism is the following irreducibility condition of E_C. We can utilize recent results by Beukers [**1**] and Schulze and Walther [**13**] on irreducibility of A-hypergeometric systems to give a condition of the irreducibility of E_C.

Theorem 13 ([**1**], [**13**]). *The system E_C is irreducible if and only if*

$$\frac{1}{2}\left(\sum_{i=1}^m c_i - a - b - m + \sum_{i=1}^m \varepsilon_i(1-c_i) + \varepsilon_{m+1}(b-a)\right) \notin \mathbb{Z}$$

for all combinations of $\varepsilon_i \in \{-1, 1\}$.

Proof. It follows from the previous theorem that the irreducibility of E_C is equivalent to that of $H_A(\beta)$. In fact, the solution spaces of them are locally isomorphic and differential operators with rational function coefficients in z is mapped to those in u by (18). The primitive integral support functions $P_J(s)$ for $A(F_C, m)$ are $(1/2)\big(s_{m+2} + \sum_{j\in J} s_j - \sum_{j\notin J} s_j\big)$, $J \subseteq [1, m+1]$, where s_j's are the dual basis for the e_i's [**11**]. It follows from [**1**] or [**13**] that the irreducibility condition is that $P_J(\beta) \notin \mathbb{Z}$

for all J, which is equivalent to the condition in the theorem. □

Finally, we discuss the singular locus of the $H_A(\beta)$ via the correspondence. The correspondence is not only for the classical solutions as we have seen, but also for some D-module invariants including the singular locus on the complex torus. In this case, we utilize our result on the singular locus for F_C to derive a result on the A-hypergeometric system.

THEOREM 14. *The singular locus of $H_A(\beta)$ in the complex torus is given by the zero set of*

$$\prod_{\varepsilon_i\in\{-1,1\}}\left(1+\varepsilon_1\sqrt{\frac{u_1u_{-1}}{u_{m+1}u_{-(m+1)}}}+\cdots+\varepsilon_m\sqrt{\frac{u_mu_{-m}}{u_{m+1}u_{-(m+1)}}}\right).$$

PROOF. We denote by D^*_{2m+2} the ring of differential operators on the complex torus

$$\mathbb{C}\langle z_1^{\pm},\dots,z_{2m+2}^{\pm},\partial_{z_1},\dots,\partial_{z_{2m+2}}\rangle.$$

Let I be a left ideal in D^*_{2m+2}. For a complex number α, we denote by $D^*z^{\alpha}_{2m+2}$ the left $\mathbb{C}\langle z^{\pm}_{2m+2}\partial_{z_{2m+2}}\rangle$-module $\mathbb{C}\langle z^{\pm}_{2m+2}\partial_{z_{2m+2}}\rangle/\langle z_{2m+2}\partial_{z_{2m+2}}-\alpha\rangle$. The outer tensor product $(D^*_{2m+2}/I)\boxed{\times}D^*z^{\alpha}_{2m+2}$ is defined by the restriction of

$$D^*_{2m+3}/\langle I, z_{2m+3}\partial_{z_{2m+3}}-\alpha\rangle$$

to $z_{2m+3}-z_{2m+2}=0$. In other words,

$$\begin{aligned}&(D^*_{2m+2}/I)\boxed{\times}D^*z^{\alpha}_{2m+2}|_{z_{2m+2}\mapsto t,\partial_{z_{2m+2}}\mapsto\partial_t}\\&\quad\simeq D^*_{2m+1}\langle t^{\pm},s,\partial_t,\partial_s\rangle/\big(\langle I,-t\partial_s+s\partial_s-\alpha\rangle+sD^*_{2m+1}\langle t^{\pm},s,\partial_t,\partial_s\rangle\big),\end{aligned}\tag{19}$$

where we make replacements

$$s=z_{2m+2}-z_{2m+3},\quad t=z_{2m+2},$$
$$z_{2m+2}\partial_{2m+2}=t\partial_s+t\partial_t,\quad z_{2m+3}\partial_{2m+3}=-t\partial_s+s\partial_s$$

in I. Let $u=(0,\dots,0,1)$ be the weight vector where 1 stands for the variable s. Then, $b=\mathrm{in}_{(-u,u)}(-t\partial_s+s\partial_s-\alpha)=-t\partial_s$. Since t is invertible, we have $D^*_{2m+1}\langle t^{\pm},s,\partial_t,\partial_s\rangle b\cap\mathbb{C}[s\partial_s]=\langle s\partial_s\rangle$. Therefore, by the restriction algorithm (see, e.g., [**10**]), we can prove that (19) is isomorphic to

$$D^*_{2m+1}\langle t^{\pm},\partial_t\rangle/\big(\big(\langle I,-t\partial_s+s\partial_s-\alpha\rangle+sD^*_{2m+1}\langle t^{\pm},s,\partial_t,\partial_s\rangle\big)\cap D^*_{2m+1}\langle t^{\pm},\partial_t\rangle\big),$$

of which denominator is called the restriction ideal.

Put $\tau_j=z_j\partial_{z_j}$. Let I be the hypergeometric ideal $H_A(\beta)$ expressed in terms of the variable in z_j (18), which is generated in D^*_{2m+2} by

$$\tau_j(\tau_j-\tau_{m+j})-z_j\Big(\sum_{j=1}^m \tau_j+\tau_{2m+1}\Big)\Big(\sum_{j=1}^m \tau_j+\tau_{2m+2}\Big),\quad j=1,\dots,m$$

and

$$\tau_{m+j}-(1-c_j),\quad j=1,\dots,m,\quad \tau_{2m+2}-\tau_{2m+1}-(b-a),\quad \tau_{2m+1}-a.$$

Note that $\tau_j(\tau_j-\tau_{m+j})-z_j\big(\sum_{j=1}^m\tau_j+\tau_{2m+1}\big)\big(\sum_{j=1}^m\tau_j+t\partial_s+t\partial_t\big)$ is in I under the change of variables from z_{2m+2}, z_{2m+3} to s,t. Subtracting $z_j\big(\sum\tau_j+\tau_{2m+1}\big)\big(-t\partial_s+s\partial_s-\alpha\big)$ from it, we conclude that the restriction ideal contains

$$\tau_j(\tau_j-\tau_{m+j})-z_j\Big(\sum\tau_j+\tau_{2m+1}\Big)\Big(\sum\tau_j+t\partial_t-\alpha\Big). \tag{20}$$

We have defined the outer tensor product by $D^*z_{2m+2}^{\alpha}$ and studied its properties. We can make analogous discussions for outer tensor products by other variables in η and we conclude from (20) that there exists a left ideal I' such that

$$(D^*_{2m+2}/I)\boxed{\times}D^*\eta^{-1}\simeq D^*_{2m+2}/I' \tag{21}$$

and $I'\supseteq\langle I(m),\partial_{z_{m+1}},\dots,\partial_{z_{2m+2}}\rangle$. Here, $I(m)$ is the left ideal generated by the Lauricella operators (1) (x_i's are replaced by z_i's respectively).

We denote by $\mathrm{Sing}^*(M)$ the singular locus of the left D^* module M in the complex torus. It follows from (21) that

$$\begin{aligned}\mathrm{Sing}^*(D^*_{2m+2}/I)&=\mathrm{Sing}^*\big((D^*_{2m+2}/I)\boxed{\times}D^*\eta^{-1}\big)\\&=\mathrm{Sing}^*(D^*_{2m+2}/I')\subseteq\mathrm{Sing}^*(D^*/I(m)).\end{aligned} \tag{22}$$

Since $\mathrm{Sing}^*(D^*/I(m))$ is irreducible by Proposition 6, the singular locus of the A-hypergeometric system $\mathrm{Sing}^*(D^*_{2m+2}/I)$ is empty or agrees with $\mathrm{Sing}^*(D^*/I(m))$. Since the toric ideal I_A is Cohen-Macaulay when $A=A(F_C,m)$, the singular locus of the A-hypergeometric system $H_A(\beta)$ does not depend on the parameter β by results in [**4**], [**12**, Section 4.3]. Then, we may suppose that $H_A(\beta)$ is irreducible by Theorem 13. By [**6**], the A-hypergeometric system is regular holonomic and the irreducibility implies that the irreducibility of the monodromy representation. If the singular locus in the complex torus is empty, then the monodromy representation is reducible. Then, the singular locus is not empty and then we obtain the conclusion. □

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