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On the distribution of polynomials with bounded roots, I. Polynomials with real coefficients

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Abstract. Let $v_d^{(s)}$ denote the set of coefficient vectors of contractive polynomials of degree d with $2s$ non-real zeros. We prove that $v_d^{(s)}$ can be computed by a multiple integral, which is related to the Selberg integral and its generalizations. We show that the boundary of the above set is the union of finitely many algebraic surfaces. We investigate arithmetical properties of $v_d^{(s)}$ and prove among others that they are rational numbers. We will show that within contractive polynomials, the 'probability' of picking a totally real polynomial decreases rapidly when its degree becomes large.

1. Introduction.

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{R}[X]$. In this note, we often have to switch between $P(X)$ and the vector of its coefficients $v_P = (p_0, \ldots, p_{d-1}) \in \mathbb{R}^d$. To simplify the notation, we identify P and v_P from now on.

Let d be a positive integer, $B \geq 1$ a real number. Denote by $\mathcal{E}_d(B)$ the set of d-dimensional vectors whose roots (by the above identification) lie within the ball of radius B centered at the origin. In this Part I of our paper we are dealing solely with the case $B = 1$ therefore we will use the abbreviation $\mathcal{E}_d$ instead of $\mathcal{E}_d(1)$. The elements of $\mathcal{E}_d$ are called *contractive polynomials.*

This set was studied by several authors. I. Schur [**12**] proved a necessary and sufficient condition for $v \in \mathcal{E}_d$, which implies that the boundary of $\mathcal{E}_d$ is the union of finitely many algebraic surfaces. A. T. Fam and J. S. Meditsch [**6**] improved this result by proving that the boundary of $\mathcal{E}_d$ is the union of two hyperplanes and one hypersurface. The two hyperplanes corresponds to roots 1 and -1 respectively. You find a thorough study of the boundary in Kirschenhofer et al. [**10**].

Later A. T. Fam [**7**] computed the volume of $\mathcal{E}_d$:

$$v_d = \lambda_d(\mathcal{E}_d) = \begin{cases} 2^{2m^2} \displaystyle\prod_{j=1}^{m} \frac{(j-1)!^4}{(2j-1)!^2}, & \text{if } d = 2m, \\ 2^{2m^2+2m+1} \displaystyle\prod_{j=1}^{m} \frac{j!^2(j-1)!^2}{(2j-1)!(2j+1)!}, & \text{if } d = 2m+1, \end{cases} \tag{1}$$

where $\lambda_d(.)$ denotes the d-dimensional Lebesgue measure.

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Given a polynomial P in $\mathbb{R}[x]$, the non-real roots of P appear in complex conjugate pairs. Thus $d = r + 2s$, where r denotes the number of real and s the number of non-real pairs of roots. The pair (r, s) is called the signature of the polynomial. However in this paper, we derive asymptotic formulas with a fixed d, we call s the signature for simplicity, because $r = d - 2s$. The set $\mathcal{E}_d$ splits naturally into $\lfloor d/2 \rfloor + 1$ disjoint subsets according to the signature s. In the sequel $\mathcal{E}_d^{(s)}$ denotes the subset of $\mathcal{E}_d$ whose elements have signature s.

This paper is organized as follows. First we show in Theorem 2.1 that $\lambda_d(\mathcal{E}_d^{(s)})$ can be computed by a certain multiple integral. It turns out that for $s = 0$ this is a simple variant of $S_d(1, 1, 1/2)$, where $S_d(\alpha, \beta, \gamma)$ denotes the Selberg integral, which is a generalization of the beta integral studied from many different points of view, see e.g. [**13**], [**3**], [**8**]. By using a generalization of it, due to K. Aomoto [**5**], we prove in Theorem 4.2 an expression for $s = 1$, which makes it possible to compute $v_d^{(1)}$ for large values of d.

Here we turn to the investigation of the arithmetic nature of $v_d^{(s)}$. We prove in Theorem 5.1, that they are rational numbers. We express $S_d(1, 1, 1/2)$ as a product of binomial coefficients, which enables us to show that they are reciprocals of integers in Corollary 5.1. After this we summarize our observations on our computations on $v_d^{(s)}$. We conjecture that $v_d^{(s)}/v_d^{(0)}$ is always an integer. In the case d even and $s = d/2$ our Conjecture 5.2 is completely explicit. Theorem 5.2 supports our conjectures.

The quotient $p_d^{(s)} = v_d^{(s)}/v_d$ may be viewed as the probability of picking an element $v \in \mathcal{E}_d$ of the signature s. By Theorem 5.1 these probabilities are rational numbers. It might be surprising to observe that totally real polynomials are very rare. If the sets $\mathcal{E}_d^{(s)}$ would have approximately the same volume then $p_d^{(s)} \sim 2/d$. However using the explicit formulae for $v_d^{(0)}$ and v_d we show $p_d^{(0)} \sim 2^{-d^2/2}$, which is much smaller than we expected.

In the last section we prove a generalization of the results of I. Schur [**12**] as well as of A. T. Fam and J. S. Meditsch [**6**], that the boundary of $\mathcal{E}_d^{(s)}$ is the union of finitely many algebraic surfaces in Theorem 7.1.

In Part II [**1**] we apply our results to estimate the distribution of polynomials with integer coefficients with given degree and signature. We are able to complete this program with respect to three natural parameters, which we will call 'measures'.

2. The volume of $\mathcal{E}_d^{(s)}$.

The aim of this section is to prove that the volume of $\mathcal{E}_d^{(s)}$ can be expressed by a multiple integral. We denote by $Res_X(P(X), Q(X))$ the resultant of the polynomials $P(X), Q(X) \in \mathbb{R}[X]$.

Theorem 2.1. *Let $d \geq 1$ and r, s non-negative integers such that $r + 2s = d$. Then the set $\mathcal{E}_d^{(s)}$ is Jordan measurable. Let $R_j(X) = X^2 - y_j X + z_j$, where $0 \leq z_j \leq 1$ and the discriminant of $R_j(X)$ is negative, $j = 1, \ldots, s$. Put*

$$D_{r,s} = [-1, 1]^r \times [-2\sqrt{z_1}, 2\sqrt{z_1}\,] \times [0, 1] \times \cdots \times [-2\sqrt{z_s}, 2\sqrt{z_s}\,] \times [0, 1].$$

Then we have

$$v_d^{(s)} = \lambda_d\big(\mathcal{E}_d^{(s)}\big) = \frac{1}{r!s!}\int_{D_{r,s}} |\Delta_r|\Delta_s\Delta_{r,s}\,dX,$$

where

$$\Delta_r = \prod_{1\le j<k\le r}(x_j - x_k),$$

$$\Delta_s = \prod_{1\le j<k\le s} Res_X(R_j(X), R_k(X)),$$

$$\Delta_{r,s} = \prod_{j=1}^{r}\prod_{k=1}^{s} R_k(x_j)$$

and $dX = dx_1 \dots dx_r dy_1 dz_1 \dots dy_s dz_s$.

To prove this theorem we need some preparation. Let the signature of $P(X) = X^d + p_{d-1}X^{d-1} + \dots + p_0 \in \mathbb{R}[X]$ be $0 \le s \le \lfloor d/2 \rfloor$. Assume that its zeroes are $x_1, \dots, x_d$. Assume further that they are ordered such that $x_1, \dots, x_r \in \mathbb{R}$ and the others belong to $\mathbb{C} \setminus \mathbb{R}$. Moreover $x_{r+2j} = \bar{x}_{r+2j-1}$, $j = 1, \dots, s$, where $\bar{x}$ denotes the complex conjugate of x. Denote $S_j(x_1, \dots, x_d)$, $j = 1, \dots, d$ the j-th elementary symmetric polynomial of $x_1, \dots, x_d$, i.e., let

$$S_j(x_1, \dots, x_d) = \sum_{1\le i_1<\dots<i_j\le d} x_{i_1}\dots x_{i_j},$$

where the sum is extended to all possible values of the indices $i_1, \dots, i_j$. For later use we define $S_0(x_1, \dots, x_d) = 1$. The classical Viéta's formulae connect the roots and coefficients of $P(X)$. With the notation $p_d = 1$ they are

$$p_j = (-1)^{d-j} S_{d-j},\ j = 0, \dots, d. \tag{2}$$

The last system of equations defines a mapping $\mathbb{R}^r \times \mathbb{C}^s \mapsto \mathbb{R}^d$. To compute the volume of $\mathcal{E}_d^{(s)}$ we need a mapping $\mathbb{R}^d \mapsto \mathbb{R}^d$. Write

$$\begin{aligned} P(X) &= \prod_{j=1}^{d}(X - x_j) \\ &= \prod_{j=1}^{r}(X - x_j)\prod_{j=1}^{s}((X - x_{r+2j-1})(X - x_{r+2j})) \\ &= \prod_{j=1}^{r}(X - x_j)\prod_{j=1}^{s}(X^2 - (x_{r+2j-1} + x_{r+2j})X + x_{r+2j-1}x_{r+2j}). \end{aligned}$$

As the coefficients of the quadratic factors are real numbers, this form together with

(2) gives the desired relations. Therefore we introduce the following new variables $y_j = x_j$ $(j = 1, \dots, r)$ and $y_{r+2j-1} = x_{r+2j-1} + x_{r+2j}$, $y_{r+2j} = x_{r+2j-1}x_{r+2j}$ $(j = 1, \dots, s)$. Under this notation, we prove a

Lemma 2.1. *Let $d = r + 2s$, $R_j(X) = X^2 - y_{r+2j-1}X + y_{r+2j}$ for $j = 1, \dots, s$ and $J = (\partial S_i(x_1, \dots, x_d)/\partial y_j)_{1 \le i,j \le d}$. Then*

$$\det(J) = \prod_{1 \le j < k \le r} (y_j - y_k) \prod_{j=1}^{r} \prod_{k=1}^{s} R_k(y_j) \prod_{1 \le j < k \le s} Res_X(R_j(X), R_k(X)).$$

Proof. Let $J_1 = (\partial S_i(x_1, \dots, x_d)/\partial x_j)_{1 \le i,j \le d}$. We easily see

$$\det(J) = \frac{\det(J_1)}{\prod_{k=1}^{s}(x_{r+2k-1} - x_{r+2k})}. \tag{3}$$

by the Jacobian computation to transform variables from x_j to y_j.

In the second step we prove

$$\det(J_1) = \prod_{j=1}^{d} \prod_{k=j+1}^{d} (x_j - x_k). \tag{4}$$

Let H be a subset of $\{1, \dots, d\}$. If $t \le d - |H|$ then denote $S_{t,H}$ the t-th elementary symmetric polynomial of the variables $\{x_1, \dots, x_d\} \setminus \{x_j : j \in H\}$, otherwise let $S_{t,H} = 0$.

For $1 \le t$, $k \le d$ we have

$$S_{t,\emptyset} = y_k S_{t-1,\{k\}} + S_{t,\{k\}},$$

which implies

$$\frac{\partial S_{t,\emptyset}}{\partial x_k} = S_{t-1,\{k\}}.$$

Subtract the 1-st column of J_1 from the k-th column, where $2 \le k \le d$. All the entries of the first row will be zero, except the north west entry. Let $t \ge 2$. Then the t-th entry of the k-th column is $S_{t-1,\{k\}} - S_{t-1,\{1\}}$, which we can rewrite as follows

$$\begin{aligned} S_{t-1,\{k\}} - S_{t-1,\{1\}} &= x_1 S_{t-2,\{k,1\}} + S_{t-1,\{k,1\}} - x_k S_{t-2,\{k,1\}} - S_{t-1,\{k,1\}} \\ &= (x_1 - x_k) S_{t-2,\{k,1\}}. \end{aligned}$$

Thus all entries of the k-th column are divisible by $(x_1 - x_k)$. Factoring out all these factors from the determinant we get

$$\det(J_1) = \det(J_2) \prod_{k=2}^{d} (x_1 - x_k),$$

where J_2 is a $(d-1)\times(d-1)$ matrix, which has the same structure as J_1, but without the variable x_1. With induction we get (4).

Summarizing our computation we proved so far that

$$\det(J) = \prod_{1\le j<k\le d}(x_j - x_k) \Big/ \prod_{k=1}^{s}(x_{r+2k-1} - x_{r+2k}).$$

After canceling by the denominator, split the remaining product into three factors as follows

$$\Pi_1 = \prod_{1\le j<k\le r}(x_j - x_k),$$

$$\Pi_2 = \prod_{1\le j<k\le s}(x_{r+2j-1} - x_{r+2k-1})(x_{r+2j-1} - x_{r+2k})(x_{r+2j} - x_{r+2k-1})(x_{r+2j} - x_{r+2k}),$$

$$\Pi_3 = \prod_{k=1}^{r}\prod_{j=1}^{s}(x_k - x_{r+2j-1})(x_k - x_{r+2j}).$$

Obviously, $\Pi_1 = \Delta_r$. We have $(x_k - x_{r+2j-1})(x_k - x_{r+2j}) = x_k^2 - y_{r+2j-1}x_k + y_{r+2j} = R_j(x_k)$, thus $\Pi_3 = \Delta_{r,s}$. Finally as the roots of $R_j(X)$ are x_{r+2j-1}, x_{r+2j} we get

$$\begin{aligned} Res_X(R_j(X), R_k(X)) &= (x_{r+2j-1} - x_{r+2k-1})(x_{r+2j-1} - x_{r+2k}) \\ &\quad \times (x_{r+2j} - x_{r+2k-1})(x_{r+2j} - x_{r+2k}), \end{aligned}$$

which means $\Pi_2 = \Delta_s$ and the lemma is proved. □

Remark 2.1. By equation (3) Lemma 2.1 can be written in the form

$$\det(J) = \frac{Disc(x_1,\dots,x_d)}{\prod_{j=1}^{s}(2i\Im x_{r+2j})},$$

where $Disc(x_1,\dots,x_d)$ denotes the discriminant of $P(X) = \prod_{j=1}^{d}(X - x_j)$. Thus, if $s = 0$ we obtain exactly the discriminant of $P(X)$.

Now we are in the position to prove Theorem 2.1. In the proof of Lemma 2.1 it was convenient to use the same name for the variables. To continue this notation would make our presentation unnecessarily complicated. Therefore in the sequel we use the notation: $x_i = y_i$, $i = 1,\dots,r$, $y_i = y_{r+2i-1}$, $z_i = y_{r+2i}$, $i = 1,\dots,s$ introduced in the theorem.

Proof of Theorem 2.1. Let $v \in \mathcal{E}_d^{(s)}$. The polynomial $P_v(X)$ can be expressed by its coefficients and by its roots, moreover we have the following relation between these representations

$$P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 = \prod_{j=1}^{r}(X - x_j)\prod_{k=1}^{s}(X^2 - y_kX + z_k).$$

It is clear that

$$\lambda_d\big(\mathcal{E}_d^{(s)}\big) = \int_{\mathcal{E}_d^{(s)}} dp_0 \dots dp_{d-1}.$$

Now we change the variables $p_0, \dots, p_{d-1}$ to $x_1, \dots, x_r, y_1, z_1, \dots, y_s, z_s$. This gives rise to one to $r!s!$ correspondence, up to measure zero exceptions. It is clear that $x_j \in [-1, 1]$, $j = 1 \dots, r$. The variable z_k is the absolute value of a complex number lying in the closed unit circle, thus $z_k \in [0, 1]$, $k = 1, \dots, s$. The polynomial $R_k(X)$ has two non-real roots, thus its discriminant $y_k^2 - 4z_k \le 0$, hence $y_k \in [-2\sqrt{z_k}, 2\sqrt{z_k}\,]$. Moreover the Jacobian of the variable change was computed in Lemma 2.1. Thus we obtain

$$\lambda_d\big(\mathcal{E}_d^{(s)}\big) = \frac{1}{r!s!}\int_{D_{r,s}} |\det(J)|\, dX.$$

As the polynomials $R_k(X)$ are positive definite, $R_k(x_j) \ge 0$ and we also have $Res_X(R_j(X), R_k(X)) \ge 0$, thus we may omit the absolute value sign around Δ_s and $\Delta_{r,s}$. □

3. On the Selberg integral and its generalization by Aomoto.

After expressing $v_d^{(s)}$ in Theorem 2.1 by a multiple integral, the main question is how to compute it. We will show in the next section, that it can be expressed by Selberg's integral, if $s = 0$ and by a generalization of Selberg's integral due to K. Aomoto, if $s = 1$. To prepare our results we summarize the necessary knowledge about these integrals in this section.

Let n be a positive integer, $\mathcal{C}_n = [0, 1]^n$ and

$$\Delta = \Delta(t_1, \dots, t_n) = \prod_{1\le j<k\le n}(t_j - t_k).$$

In 1944 A. Selberg [**13**] proved the beautiful formula

$$\begin{aligned} S_n(\alpha, \beta, \gamma) &= \int_{\mathcal{C}_n} \prod_{j=1}^{n} t_j^{\alpha-1}(1 - t_j)^{\beta-1}|\Delta|^{2\gamma} dt_1 \dots dt_n \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j + 1)\gamma)}{\Gamma(\alpha + \beta + (n + j - 1)\gamma)\Gamma(1 + \gamma)}, \end{aligned}$$

which is valid for complex parameters α, β, γ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > -\min\{1/n, \Re(\alpha)/(n - 1), \Re(\beta)/(n - 1)\}$.

Besides the original proof of Selberg, there are at least two more essentially different proofs of this formula, due to G. W. Anderson [**2**] and K. Aomoto [**5**]. You find a good overview of the history, the generalizations and applications of the Selberg integral in the book of G. E. Andrews and R. Askey and R. Roy [**3**] as well as in the survey paper [**8**]. We formulate here one generalization of the Selberg integral, which we will need later.

Let $w(t) = w(t_1, \dots, t_n) = \prod_{j=1}^{n} t_j^{\alpha-1}(1-t_j)^{\beta-1}|\Delta|^{2\gamma}$ and

$$B_n(j,k,\ell) = \int_{\mathcal{C}_n} \prod_{i=1}^{j} t_i \prod_{i=j+1-\ell}^{j+k-\ell} (1-t_i) w(t)\, dt_1 \dots dt_n.$$

Thus j represents the number of extra t_i factors, k the number of extra $1-t_i$ factors, and ℓ the number of variables that overlap among the extra factors. Assuming $\ell \le j$, $k \le n$ and $j+k-\ell \le n$ then by Theorem 8.3.1 of [**3**] we have

$$B_n(j,k,\ell) = \prod_{i=1}^{\ell} \frac{\alpha+\beta+(n-i-1)\gamma}{\alpha+\beta+1+(2n-i-1)\gamma} \times \frac{\prod_{i=1}^{j}(\alpha+(n-i)\gamma)\prod_{i=1}^{k}(\alpha+(n-i)\gamma)}{\prod_{i=1}^{j+k}(\alpha+\beta+(2n-i-1)\gamma)} S_n(\alpha,\beta,\gamma)$$

for all complex numbers α, β, γ satisfying the former conditions. We need only the special case $k=\ell$, $\alpha=\beta=1$, $\gamma=1/2$. Then, as is pointed out on p. 408 of [**3**], the integral defining $B_n(j,k,k)$ can be written in the form

$$B_n(j,k,k) = \int_{\mathcal{C}_n} \prod_{i=1}^{j} t_i \prod_{i=1}^{k} (1-t_i) w(t)\, dt_1 \dots dt_n.$$

Setting $(\alpha,\beta,\gamma) = (1,1,1/2)$ and $B_n(j,k) = B_n(j,k,k)$ we obtain

$$B_n(j,k) = \prod_{i=1}^{k} \frac{2+(n-i-1)/2}{3+(2n-i-1)/2} \times \frac{\prod_{i=1}^{j}(1+(n-i)/2)\prod_{i=1}^{k}(1+(n-i)/2)}{\prod_{i=1}^{j+k}(2+(2n-i-1)/2)} S_n(1,1,1/2)$$

for $n \ge j \ge k$.

4. Computation of $v_d^{(0)}$ and $v_d^{(1)}$.

In this section we prove expressions for $v_d^{(s)}$ for $s = 0, 1$.

In Remark 2.1 an explicit formula for $\det(J)$ is given in the case $s = 0$. Then our multiple integral simplifies to a transformed Selberg integral [**13**], [**8**] and we obtain

Theorem 4.1. *Let d be a positive integer. Then*

$$v_d^{(0)} = \frac{2^{d(d+1)/2}}{d!} S_d(1,1,1/2) = \frac{2^{d(d+1)/2}}{d!} \prod_{j=0}^{d-1} \frac{(\Gamma(1+j/2))^2 \Gamma(1+(j+1)/2)}{\Gamma(2+(d+j-1)/2)\Gamma(3/2)}.$$

Proof. Using the notations of Theorem 2.1 we have $r = d$, $s = 0$, hence

$$v_d^{(0)} = \frac{1}{d!} \int_{[-1,1]^d} \left| \prod_{1 \le j < k \le d} (x_j - x_k) \right| dx_1 \dots dx_d.$$

Rearranging $x_1, \dots, x_d$ in decreasing order all factors are non-negative and we may omit the absolute value. Taking in account that $x_1, \dots, x_d$ have $d!$ different orderings we obtain

$$v_d^{(0)} = \int_{-1}^{1} \int_{x_1}^{1} \dots \int_{x_{d-1}}^{1} \prod_{1 \le j < k \le d} (x_j - x_k)\, dx_1 \dots dx_d.$$

Now we change the variables by $x_i = 2X_i - 1$, $i = 1, \dots, d$ and get

$$v_d^{(0)} = 2^{d(d+1)/2} \int_{0}^{1} \int_{X_1}^{1} \dots \int_{X_{d-1}}^{1} \prod_{1 \le j < k \le d} (X_j - X_k)\, dX_1 \dots dX_d.$$

Finally we perform the first step backwards and obtain

$$v_d^{(0)} = \frac{2^{d(d+1)/2}}{d!} \int_{0}^{1} \dots \int_{0}^{1} \left| \prod_{1 \le j < k \le d} (X_j - X_k) \right| dX_1 \dots dX_d = \frac{2^{d(d+1)/2}}{d!} S_d(1,1,1/2). \qquad \square$$

This relation implicitly appears in G. W. Anderson [**2**].

For $s > 0$ we were not able to find a similar simple relation between the Selberg integral or its generalizations and our expressions for $v_d^{(s)}$, although their form and numerical investigations suggest strong connections. For $s = 1$, using Aomoto's generalization [**5**], more precisely its variant in the book [**3**], we were able to derive a bit more complicated formula, which we present now.

THEOREM 4.2. *Let $d \geq$ be an integer. Then*

$$v_d^{(1)} = 2^{(d-1)(d-2)/2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!}$$
$$\times B_{d-2}(d-2-k, d-2-k-j) \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz.$$

PROOF. As in the proof $d-2$ appears often and this makes the formulae more complicated, we use the abbreviation $\delta = d-2$. By Theorem 2.1 we have

$$v_d^{(1)} = \frac{1}{\delta!} \int_{D_{\delta,1}} \Delta |\Delta_\delta(x)| \, dX dy \, dz,$$

where

$$\Delta = \Delta_{\delta,1} = \prod_{i=1}^{\delta} (x_i^2 - yx_i + z), \quad \Delta_\delta(x) = \prod_{1 \leq j < k \leq \delta} (x_j - x_k)$$

and

$$dX = dx_1 \dots dx_\delta.$$

Remark that for simplicity we replaced y_1, z_1 by y, z. We transform the range of integration for $x_1, \dots, x_\delta$ to the range used by Selberg and Aomoto by performing the substitutions $x_i = 2X_i - 1$, $i = 1, \dots, \delta$. (These early substitutions results much simpler final formulae, than we would do them later.) After some simple calculations we obtain

$$v_d^{(1)} = \frac{2^{(\delta+1)\delta/2}}{\delta!} \int_{[0,1]^\delta} |\Delta_\delta(X)| \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} P(X,y,z) \, dy \, dz \, dX, \tag{5}$$

with

$$P(X,y,z) = \prod_{i=1}^{\delta} ((2X_i - 1)^2 - (2X_i - 1)y + z)$$
$$= \prod_{i=1}^{\delta} (-4X_i(1-X_i) - 2X_i y + (y+z+1)).$$

By performing the multiplications we can separate the variables as follows

$$P(X,y,z) = \sum_{j=0}^{\delta} (-2)^j \sum_{k=0}^{\delta-j} (-4)^{\delta-k-j} y^j (y+z+1)^k \Sigma_1 \Sigma_2,$$

where

$$\Sigma_1 = \sum_{(i_1,\dots,i_j)\in I_1} X_{i_1}\cdots X_{i_j},$$

$$\Sigma_2 = \sum_{(i_{j+1},\dots,i_{\delta-k})\in I_2} X_{i_{j+1}}(1-X_{i_{j+1}})\cdots X_{i_{\delta-k}}(1-X_{i_{\delta-k}})$$

and I_1, I_2 run through all ordered disjoint subsets of $\{1,\dots,\delta\}$ with size j and $\delta-j-k$ respectively.

Inserting these expressions into (5) we can separate the variables. Moreover it is obvious that

$$\int_{[0,1]^\delta} |\Delta_\delta(X)| X_{i_1}\cdots X_{i_j} X_{i_{j+1}}(1-X_{i_{j+1}})\cdots X_{i_{\delta-k}}(1-X_{i_{\delta-k}})\, dX$$

does not depend on the actual values of $(i_1,\dots,i_j)$, $(i_{j+1},\dots,i_{\delta-k})$ but only on the size of the (ordered) sets I_1, I_2 to which they belong. It is also clear that this integral is equal to $B_\delta(\delta-k,\delta-j-k)$. With these observations we can considerably simplify our integral, and obtain

$$v_d^{(1)} = \frac{2^{(\delta+1)\delta/2}}{\delta!}\sum_{j=0}^{\delta}(-2)^j\binom{\delta}{j}\sum_{k=0}^{\delta-j}(-4)^{\delta-k-j}\binom{\delta-j}{k}B_\delta(\delta-k,\delta-j-k)$$
$$\times\int_0^1\int_{-2\sqrt{z}}^{2\sqrt{z}} y^j(y+z+1)^k\, dy\, dz.$$

After some obvious simplification we obtain the final formula

$$v_d^{(1)} = 2^{(\delta+1)\delta/2}\sum_{j=0}^{\delta}\sum_{k=0}^{\delta-j}\frac{(-1)^{\delta-k}2^{2\delta-2k-j}}{j!k!((\delta-j-k)!}B_\delta(\delta-k,\delta-j-k)$$
$$\times\int_0^1\int_{-2\sqrt{z}}^{2\sqrt{z}} y^j(y+z+1)^k\, dy\, dz. \qquad \square$$

5. Arithmetical properties of $v_d^{(s)}$.

After expressing $v_d^{(s)}$ in Theorem 2.1 by a multiple integral, which was simplified in the cases $s=0,1$ in the last section, we investigate arithmetical properties of these numbers. We start with a general fact.

Theorem 5.1. *The numbers v_d and $v_d^{(s)}$ are rational. Moreover*

$$v_d = \sum_{s=0}^{\lfloor d/2\rfloor} v_d^{(s)}. \tag{6}$$

Proof. The first and the last statements are obvious by the formula (1) and by the definitions of the volumes respectively. We included them only for completeness.

The second assertion is true for $s = 0, 1$ by Theorems 4.1 and 4.2 respectively. Thus we may assume $s \geq 2$. By Theorem 2.1 we have

$$v_d^{(s)} = \frac{1}{r!s!} \int_{D_{r,s}} |\Delta_r| \Delta_s \Delta_{r,s} \, dX$$

with the notations explained there. In the first step we prove

$$v_d^{(s)} = \frac{1}{r!s!} \int_{D_{r,0}} W(x_1, \dots, x_r) |\Delta_r| \, dx, \tag{7}$$

where $dx = dx_1 \dots dx_r$ and $W(x_1, \dots, x_r) \in \mathbb{Q}[x_1, \dots, x_r]$ is symmetric. As Δ_r does not depend on $y_1, z_1, \dots, y_s, z_s$ we may take

$$W(x_1, \dots, x_r) = \int_{\mathcal{F}_s} \Delta_s \Delta_{r,s} \, dY,$$

where $\mathcal{F}_s = [-2\sqrt{z_1}, 2\sqrt{z_1}\,] \times [0,1] \times \dots \times [-2\sqrt{z_s}, 2\sqrt{z_s}\,] \times [0,1]$ and $dY = dy_1 dz_1 \dots dy_s dz_s$. Remark that

$$\Delta_s \Delta_{r,s} = \prod_{1 \leq j < k \leq s} Res_X(R_j(X), R_k(X)) \prod_{j=1}^{s} \prod_{k=1}^{r} R_j(x_k)$$

is obviously a symmetric polynomial with rational coefficients in $x_1, \dots, x_r$. Further we have

$$\Delta_s \Delta_{r,s} = \Delta_{s-1} \Delta_{r,s-1} W_1(y_1, z_1),$$

where

$$W_1(y_1, z_1) = \prod_{j=2}^{s} Res_X(R_j(X), R_1(X)) \prod_{k=1}^{r} R_1(x_k)$$

is a polynomial with coefficients in $\mathcal{Q} = \mathbb{Q}[x_1, \dots, x_r, y_2, \dots y_s, z_2, \dots, z_s]$, which is again symmetric in $x_1, \dots, x_r$. Now we can rewrite the formula for W as follows

$$W = \int_{\mathcal{F}_{s-1}} \Delta_{s-1} \Delta_{r,s-1} \int_0^1 \int_{-2\sqrt{z_1}}^{2\sqrt{z_1}} W_1(y_1, z_1) dy_1 dz_1 dY_1$$

with $dY_1 = dy_2 dz_2 \dots dy_s dz_s$. It is clear that

$$\int_{-2\sqrt{z_1}}^{2\sqrt{z_1}} W_1(y_1, z_1) \, dy_1$$

is a polynomial in $\sqrt{z_1}$ with coefficients from $\mathcal{Q}$. Thus the same is true for

$$\int_0^1 \int_{-2\sqrt{z_1}}^{2\sqrt{z_1}} W_1(y_1, z_1)\, dy_1 dz_1.$$

As W_1 is symmetric in $x_1, \dots, x_r$, this property remains unaffected after the two integrations.

Now we can continue the above described process with the pairs of variables $(y_2, z_2), \dots, (y_s, z_s)$, which finally leads to the proof of (7).

Performing in (7) the variable change $x_i = 2X_i - 1$, $i = 1, \dots, r$ we obtain

$$v_d^{(s)} = \frac{2^{r(r+1)/2}}{r!s!} \int_{\mathcal{C}_r} W'(X_1, \dots, X_r) |\Delta_r(X_1, \dots, X_r)|\, dX$$

with $\mathcal{C}_r = [0,1]^r$ and $dX = dX_1 \dots dX_r$. Moreover we have $W'(X_1, \dots, X_r) = W(2X_1 - 1, \dots, 2X_r - 1)$. As W is a symmetric polynomial with rational coefficients, the same is true for W'.

It follows from a very general result of K. Aomoto [**4**, p. 177] (see also [**5**, p. 545]), that the last integral divided by $S_r(1,1,1/2)$ is a rational number, which implies the assertion immediately.

In our simple situation the proof of rationality can be completed directly. Indeed, as in the proof of Theorem 4.1 we can rearrange the variables $x_1, \dots, x_r$ in (7) in decreasing order. Then $\Delta_r > 0$ and we may omit the absolute value. Thus we obtain

$$v_d^{(s)} = \frac{1}{s!} \int_{-1}^1 \int_{x_1}^1 \dots \int_{x_{r-1}}^1 W(x_1, \dots, x_r) \Delta(x_1, \dots, x_r)\, dx_r \dots dx_1.$$

As the integrand is a polynomial with rational coefficients, this property is not affected during the successive integration by $x_r, \dots, x_1$. In the last step we obtain a polynomial with rational coefficients without variables, i.e., a rational number. The theorem is proved. □

In the sequel we concentrate mainly on the case $r = 0$. First we prove a much simpler formula for $S_d(1,1,1/2)$ as given in Theorem 4.1.

Lemma 5.1. *If d is a positive integer then*

$$S_d(1,1,1/2) = d! \prod_{i=1}^{d} \frac{(i-1)!^2}{(2i-1)!}. \tag{8}$$

Proof. In this proof we use the abbreviation $S_d = S_d(1,1,1/2)$. By Theorem 4.1 we have

$$\frac{S_{d+1}}{S_d} = \frac{\Gamma(1+d/2)^2\Gamma(1+(d+1)/2)}{\Gamma(d+1)\Gamma(3/2)}\prod_{j=0}^{d-1}\frac{\Gamma(2+(d+j-1)/2)}{\Gamma(2+(d+j)/2)}$$
$$= \frac{\Gamma(1+d/2)^2\Gamma(3/2+d/2)^2}{\Gamma(d+2)\Gamma(d+3/2)\Gamma(3/2)}.$$

Using the functional equation $\Gamma(x+1) = x\Gamma(x)$ we can considerably simplify the last formula

$$\frac{S_{d+1}}{S_d} = \frac{\Gamma(1+d/2)^2\Gamma(1/2+d/2)^2}{\Gamma(d+1)\Gamma(d+1/2)\Gamma(3/2)}\frac{(1/2+d/2)^2}{(d+1)(d+1/2)}$$
$$= \frac{S_d}{S_{d-1}}\frac{d+1}{2(2d+1)}.$$

This implies

$$\frac{S_{d+1}}{S_d} = \binom{2d+1}{d}^{-1}$$

after a short computation. The quotient of consecutive values of the sequence $d!\prod_{i=1}^{d}(i-1)!\cdot(i-1)!/(2i-1)!$ satisfy the same relation. As the starting values of both sequences coincide we proved the statement. □

Corollary 5.1. *For $d \geq 1$ we have*

$$\frac{1}{S_d(1,1,1/2)} = \prod_{j=0}^{d-1}\binom{2j+1}{j},$$

i.e. the number $S_d(1,1,1/2)$ is the reciprocal of an integer.

Proof. The statement is true for $d = 1$. Further

$$\frac{1}{S_{d+1}(1,1,1/2)} = \frac{1}{S_d(1,1,1/2)}\frac{S_d(1,1,1/2)}{S_{d+1}(1,1,1/2)} = \binom{2d+1}{d}\frac{1}{S_d(1,1,1/2)},$$

which proves the assertion. □

Theorem 2.1 allows the numerical computation of $v_d^{(s)}$. We did it by using the computer algebra software Mathematica. It was able to compute $v_d^{(s)}$ for all possible signatures for $d \leq 8$ and check the formula (6). For $d = 9$ we failed for $s = 2$ by time constraint. We computed $v_9^{(2)}$ indirectly by formula (6), which is actually

$$v_9^{(2)} = v_9 - \left(v_9^{(0)} + v_9^{(1)} + v_9^{(3)} + v_9^{(4)}\right).$$

In Table 1, you find these computed values for $d \leq 9$.

d	$v_d^{(0)}$	$v_d^{(1)}$	$v_d^{(2)}$	$v_d^{(3)}$	$v_d^{(4)}$
2	$\frac{4}{3}$	$\frac{8}{3}$			
3	$\frac{16}{45}$	$\frac{224}{45}$			
4	$\frac{64}{1575}$	$\frac{1664}{525}$	$\frac{2048}{525}$		
5	$\frac{1024}{496125}$	$\frac{428032}{496125}$	$\frac{3334144}{496125}$		
6	$\frac{16384}{343814625}$	$\frac{1114112}{10418625}$	$\frac{93519872}{22920975}$	$\frac{268435456}{68762925}$	
7	$\frac{524288}{1032475318875}$	$\frac{2124414976}{344158439625}$	$\frac{379792130048}{344158439625}$	$\frac{6491843067904}{1032475318875}$	
8	$\frac{16777216}{6643978676960625}$	$\frac{1114476904448}{6643978676960625}$	$\frac{313947815149568}{2214659558986875}$	$\frac{693972225753088}{189827962198875}$	$\frac{562949953421312}{189827962198875}$
9	$\frac{4294967296}{7268180473661075718 75}$	$\frac{92376156602368}{427540027862416 21875}$	$\frac{12626155878219776}{1433566168374965625}$	$\frac{7081776901717533 65504}{7268180473661075718 75}$	$\frac{32803926951790913 78176}{7268180473661075718 75}$

Table 1.

We also computed the value $v_{10}^{(5)} = \frac{483570327845851669882470 4}{2664364983316916082328125}$. Unfortunately our method is not generalizable, therefore we do not present it here.

In Table 2 we display the quotients $v_d/v_d^{(0)}$ and $v_d^{(s)}/v_d^{(0)}$ for $2 \le d \le 8$, $1 \le s \le \lfloor d/2 \rfloor$.

d	$v_d/v_d^{(0)}$	$v_d^{(1)}/v_d^{(0)}$	$v_d^{(2)}/v_d^{(0)}$	$v_d^{(3)}/v_d^{(0)}$	$v_d^{(4)}/v_d^{(0)}$
2	3	2			
3	15	14			
4	175	78	96		
5	3675	418	3256		
6	169785	2244	85620	81920	
7	14567553	12156	2173188	12382208	
8	2678348673	66428	56138244	1447738880	1174405120
9	930152232009	365636	1490456292	164885467424	763775942656

Table 2.

Our numerical investigations lead to a

Conjecture 5.1. *The quotient $v_d^{(s)}/v_d^{(0)}$ is an integer.*

The conjecture is true for the computed values, i.e. for any signatures for $d \le 9$. It is also true for $d = 10$, $s = 5$. The formula of Theorem 4.2 makes the computation of $v_d^{(1)}$ much more efficient. Using it we computed $v_d^{(1)}$ for $d \le 100$. This range could easily be extended, but we could not expect new information, hence we stopped there. Our computation confirmed Conjecture 5.1.[1]

Conjecture 5.1 together with relation (6) implies that $v_d/v_d^{(0)}$ is an integer for $d \ge 1$. In this case the formulae (1) and (8) imply

$$\frac{v_d}{v_d^{(0)}} = \begin{cases} 2^{-m} \prod_{j=1}^{m} \frac{(j-1)!^4}{(2j-1)!^2} \prod_{j=1}^{2m} \frac{(2j-1)!}{(j-1)!^2}, & \text{if } d = 2m, \\ 2^{-m} \prod_{j=1}^{m} \frac{j!^2 (j-1)!^2}{(2j-1)!(2j+1)!} \prod_{j=1}^{2m+1} \frac{(2j-1)!}{(j-1)!^2}, & \text{if } d = 2m+1, \end{cases}$$

which can easily be transformed to a simpler form

[1] Recently P. Kirschenhofer and M. Weitzer, *A number theoretic problem on the distribution of polynomials with bounded roots, manuscript, 2014*, confirmed Conjecture 5.1 in case $s = 1$. They proved that $v_d^{(1)}/v_d^{(0)} = (P_d(3) - 2d - 1)/4$, where $P_d(x)$ are the Legendre polynomials. As a consequence they get $\log p_d^{(1)} = -(\log 2/2)d^2 + d\log(3 + 2\sqrt{2}) + O(\log d)$, c.f. Section 6.

$$\frac{v_d}{v_d^{(0)}} = \begin{cases} 2^{-m} \prod_{j=m+1}^{2m} \binom{2j}{j} \prod_{j=1}^{m-1} \binom{2j}{j}^{-1}, & \text{if } d = 2m, \\ 2^{-m-1} \prod_{j=m+1}^{2m+1} \binom{2j}{j} \prod_{j=1}^{m} \binom{2j}{j}^{-1}, & \text{if } d = 2m+1. \end{cases} \tag{9}$$

We are able to prove that these numbers are indeed integers.

Theorem 5.2. *The numbers $v_d/v_d^{(0)}$ are odd integers.*

To prove this theorem we need a simple but important lemma.

Lemma 5.2. *Let $a_n, n = 0, 1, \dots$ be a purely periodic sequence of real numbers with period length ℓ. Assume that $a_i \le a_{i+1}$, $0 \le i \le \ell - 2$. Let c, d be integers such that $d \ge 0$, $c > 0$, $\gcd(c, \ell) = 1$. If R is a non-negative integer then*

$$\sum_{i=0}^{R} a_i \le \sum_{i=0}^{R} a_{ci+d}.$$

Proof. There exist integers q, t such that $R = q\ell + t$, $0 \le t < \ell$. Then

$$\begin{aligned} \sum_{i=0}^{R} a_i &= \sum_{i=0}^{t-1} a_i + \sum_{i=t}^{R} a_i \\ &= \sum_{i=0}^{t-1} a_i + \sum_{j=0}^{q-1} \sum_{i=0}^{\ell-1} a_{j\ell+i} \\ &= \sum_{i=0}^{t-1} a_i + q \sum_{i=0}^{\ell-1} a_i. \end{aligned}$$

Similarly

$$\begin{aligned} \sum_{i=0}^{R} a_{ci+d} &= \sum_{i=0}^{t-1} a_{ci+d} + \sum_{j=0}^{q-1} \sum_{i=0}^{\ell-1} a_{cj\ell+ci+d} \\ &= \sum_{i=0}^{t-1} a_{ci+d} + q \sum_{i=0}^{\ell-1} a_{ci+d}. \end{aligned}$$

However, as c and ℓ are coprime, if i runs through a complete residue system modulo ℓ, then $ci + d$ do the same. Hence $\sum_{i=0}^{\ell-1} a_{ci+d} = \sum_{i=0}^{\ell-1} a_i$ and we only have to compare $\sum_{i=0}^{t-1} a_i$ and $\sum_{i=0}^{t-1} a_{ci+d}$. Since $(c, \ell) = 1$ the mapping $i \bmod \ell \mapsto ci + d \bmod \ell$ is injective for $0 \le i < \ell$. Thus the monotonicity assumption implies that $\sum_{i=0}^{t-1} a_i$ attains the minimum of all the sums of the shape $\sum_{i \in K} a_i$ for any subsets $K \subset \mathbb{Z}/\ell\mathbb{Z}$ of cardinality t. $\square$

Proof of Theorem 5.2. Let $f(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor$. It is a periodic function with period one. Moreover $f(x) = 0$ if $x \in [0, 1/2)$ and $f(x) = 1$ if $x \in [1/2, 1)$ thus $f(x)$ is increasing in the interval $[0, 1)$.

Let p be an odd prime and denote $\nu_p(x)$ the largest exponent e such that p^e divides x. Let R be a positive integer. Then by Legendre's formula

$$\nu_p\left(\prod_{n=0}^{R}\binom{2n}{n}\right) = \sum_{n=0}^{R}\sum_{j=1}^{\infty}\left(\left\lfloor\frac{2n}{p^j}\right\rfloor - 2\left\lfloor\frac{n}{p^j}\right\rfloor\right)$$
$$= \sum_{j=1}^{\infty}\sum_{n=0}^{R} f\left(\frac{n}{p^j}\right).$$

For $j \geq 1$ set $a_{j,n} = f(n/p^j)$. The sequence $a_{j,n}$ is purely periodic with period length p^j. The increasing property of f in $[0, 1)$ implies that $a_{j,n}$ is increasing for $0 \leq n < p^j$. Using this notation (9) implies

$$\nu_p\left(\frac{v_d}{v_d^{(0)}}\right) = \sum_{j=1}^{\infty}\left(\sum_{k=0}^{m-1} a_{j,m+k+1} - \sum_{k=0}^{m-1} a_{j,k}\right)$$

for $d = 2m$. To get the same number of factors in numerator and denominator we extended the product in the denominator with the trivial factor $\binom{0}{0} = 1$. With the choice $c = 1, d = m + 1$ the assumptions of Lemma 5.2 hold, thus all differences in the brackets are non-negative. Hence the denominator of $v_d/v_d^{(0)}$ has no odd divisors.

Finally we prove that $v_d/v_d^{(0)}$ is odd. This is true for $d = 2$. Assume that it is true for $d = 2m$. By (9) we have

$$\nu_2\left(\frac{v_{d+2}}{v_{d+2}^{(0)}}\right) = \nu_2\left(\frac{v_d}{v_d^{(0)}}\right) - 1 + \nu_2\left(\binom{4m+4}{2m+2}\right) + \nu_2\left(\binom{4m+2}{2m+1}\right)$$
$$- \nu_2\left(\binom{2m+2}{m+1}\right) - \nu_2\left(\binom{2m}{m}\right).$$

It follows from a classical result of E. E. Kummer [**11**], (see also the expository paper of Granville [**9**]) that $\nu_2\left(\binom{2n}{n}\right)$ is exactly the number of ones in the binary expansion of n. As $2m+2 = 2(m+1)$ the number of one's in the binary expansions of $2m+2$ and $m+1$ is equal, thus $\nu_2\left(\binom{4m+4}{2m+2}\right) = \nu_2\left(\binom{2m+2}{m+1}\right)$. Further $2m + 1$ has exactly one more one's in its binary expansion than m, thus $\nu_2\left(\binom{4m+2}{2m+1}\right) = \nu_2\left(\binom{2m}{m}\right) + 1$. Hence

$$\nu_2\left(\frac{v_{d+2}}{v_{d+2}^{(0)}}\right) = \nu_2\left(\frac{v_d}{v_d^{(0)}}\right)$$

holds for d even. As this number is zero for $d = 2$, it is zero for all even d.

The proof of the case d odd is similar, therefore we left it for the reader. □

We finish this part with a conjecture, which is true for $d \le 5$.

Conjecture 5.2. *We have*

$$\frac{v_{2d}^{(d)}}{v_{2d}^{(0)}} = 2^{2d(d-1)} \binom{2d}{d}. \tag{10}$$

Combining Theorems 2.1 and 4.1 this conjecture can be written in the form

Conjecture 5.3. *Let $d \ge 1$, $0 \le z_j \le 1$, $j = 1, \ldots, d$,*

$$D_{0,d} = [-2\sqrt{z_1}, 2\sqrt{z_1}\,] \times [0,1] \times \cdots \times [-2\sqrt{z_d}, 2\sqrt{z_d}\,] \times [0,1],$$

$R_k(X) = X^2 - y_j X + z_j$, $j = 1, \ldots, d$ and $dX = dy_1 dz_1 \ldots dy_d dz_d$. Then we have

$$\frac{1}{d!} \int_{D_{0,d}} \prod_{1 \le j < k \le d} Res_X(R_j(X), R_k(X))\, dX = \frac{2^{4d^2-d}}{(2d)!} \binom{2d}{d} S_{2d}(1,1,1/2).$$

All our attempts to prove this conjecture failed. It seems unlikely to have a simple formula like (10) for other ratios $v_{2d}^{(s)}/v_{2d}^{(0)}$ or $v_{2d+1}^{(s)}/v_{2d+1}^{(0)}$ with $0 < s < d$, because we find large prime factors.

6. Probability results.

It is natural to ask: what is the probability $p_d^{(0)}$ that picking $v \in \mathcal{E}_d$ the corresponding polynomial P_v is totally real, i.e. has only real roots? Let $d = r + 2s$, where r, s are non-negative integers. More generally we can ask the probability $p_d^{(s)}$ that picking $v \in \mathcal{E}_d$ such that the corresponding polynomial P_v has signature (r, s)? Notice that in this setting we pick the coefficients of the polynomial!

Of course we can express these probabilities with our former notations as

$$p_d^{(s)} = \frac{v_d^{(s)}}{v_d}.$$

By Theorem 5.1 these probabilities are rational numbers.

The next natural question is the behavior of these numbers. Are they of similar size or is there significant difference between them?

We have the complicated, but exact formula (1) for v_d, but not for $v_d^{(s)}$ for $s \ge 0$, thus not for $p_d^{(s)}$ either. However, in the case $s = 0$ Theorem 4.1 makes it possible to prove an accurate estimate for the size of $p_d^{(0)}$. We extend this with a hypothetical estimate for $p_d^{(d/2)}$, provided d is even.

Theorem 6.1. *Let $d \ge 2$ be an integer. Then*

$$\log\left(p_d^{(0)}\right) = -\frac{\log 2}{2} d^2 + \frac{1}{8} \log d + O(1).$$

Moreover, if d is even and Conjecture 5.2 is true then

$$\log\left(p_d^{(d/2)}\right) = -\frac{3}{8}\log d + O(1). \tag{11}$$

Proof. We give a non-computer assisted proof[2]. Using ideas of A. T. Fam [**7**], we have

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} = \prod_{k=1}^{n}\frac{(2k)(2k-1)}{k^2} = 2^{2n}\prod_{k=1}^{n}\frac{2k-1}{2k} \tag{12}$$

for all positive integers n.

First we prove (11). Let d be even, say $d = 2m$. As

$$p_{2m}^{(m)} = p_{2m}^{(0)}\frac{v_{2m}^{(m)}}{v_{2m}^{(0)}}$$

we get

$$p_{2m}^{(m)} = 2^{2m^2-m}\prod_{j=m+1}^{2m}\binom{2j}{j}^{-1}\prod_{j=1}^{m}\binom{2j}{j}$$

by using (9) and Conjecture 5.2. Combining this with (12) we obtain

$$\begin{aligned}
p_{2m}^{(m)} &= 2^{2m^2-m}\prod_{j=m+1}^{2m}2^{-2j}\prod_{k=1}^{j}\frac{2k}{2k-1}\prod_{j=1}^{m}2^{2j}\prod_{k=1}^{j}\frac{2k-1}{2k}\\
&= 2^{-m}\prod_{j=1}^{m}\left(\frac{2j}{2j-1}\right)^{m}\prod_{j=m+1}^{2m}\left(\frac{2j}{2j-1}\right)^{2m+1-j}\prod_{j=1}^{m}\left(\frac{2j-1}{2j}\right)^{m+1-j}\\
&= 2^{-m}\prod_{j=1}^{m}\left(\frac{4m-2j+2}{4m-2j+1}\right)^{j}\prod_{j=1}^{m}\left(\frac{2j-1}{2j}\right)^{-j+1}.
\end{aligned}$$

Taking logarithms we get

$$\begin{aligned}
\log\left(p_{2m}^{(m)}\right) &= -m\log 2 - \sum_{j=1}^{m}(j-1)\log\left(1-\frac{1}{2j}\right) + \sum_{j=1}^{m}j\log\left(1+\frac{1}{4m-2j+1}\right)\\
&= -m\log 2 - \frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{2m+1}{2m+1-j}\right) + \frac{m}{2} - \frac{3}{8}\sum_{j=1}^{m}\frac{1}{j} + O(1)\\
&= -\frac{3}{8}\log m + O(1).
\end{aligned}$$

[2]The computation is not easy even by symbolic computation programs like Mathematica, because we have to fix properly the branches of complex functions.

Here we used the three term Taylor expansion of $\log(1+x)$.

Now we compute an asymptotic estimate for $p_{2m}^{(0)}$. We start with

$$p_{2m}^{(0)} = p_{2m}^{(m)} 2^{-2m(m-1)} \binom{2m}{m}^{-1}.$$

Using (12) we get

$$p_{2m}^{(0)} = p_{2m}^{(m)} 2^{-2m(m-1)} 2^{-2m} \prod_{k=1}^{m} \left(1 - \frac{1}{2k}\right)^{-1}.$$

Taking logarithms, using (11) and the first term of the Taylor expansion for $\log(1-x)$ we obtain

$$\log\left(p_d^{(0)}\right) = -2m^2 \log 2 + \frac{1}{2} \sum_{k=1}^{m} \frac{1}{k} - \frac{3}{8} \log m + O(1),$$

which proves the first assertion for d even.

Finally we are dealing with the case d is odd, say $d = 2m+1$. By (9) we have

$$p_{2m+1}^{(0)} = p_{2m}^{(0)}\, 2 \binom{2m}{m} \binom{4m+2}{2m+1}^{-1}.$$

By applying (12) this implies

$$p_{2m+1}^{(0)} = p_{2m}^{(0)}\, 2^{-2m-1} \prod_{k=m+1}^{2m+1} \left(1 - \frac{1}{2k}\right)^{-1}.$$

Taking logarithms, using the expression for $\log p_{2m}^{(0)}$ and Taylor's formula we obtain the result. □

Theorem 6.1 means that the probability of picking a totally real polynomial of degree d is asymptotically $2^{-d^2/2}$. On the other hand there are only $[d/2]+1$ different signatures, thus the polynomials are distributed among the different signatures not equally, the totally real signature for example is a very rare one.

Using the formula in Theorem 4.2 we computed $v_d^{(1)}$ for $d \le 100$. After having these values we also computed $v_d^{(1)}/v_d^{(0)}$, which happen to be integers. Moreover we found that the growth rate of this integer sequence is monotonically increasing for $d \ge 5$ and lies in the interval $[5.358974359, 5.798986043]$. We are not able to prove, but it seems that the growth rate is bounded above, say by q. (We expect that $q \le 6$.) If this is true then $v_d^{(1)}/v_d^{(0)} \le q^d$. Combining this with Theorem 6.1 we get

$$\log p_d^{(1)} \le -\frac{\log 2}{2} d^2 + d \log q,$$

i.e., the signature $s = 1$ has a bit larger probability than the case $s = 0$, but it is still very rare comparing to the uniform distribution.

On the other hand we have some information on the totally complex case, which is the other extremum. For even d we proposed Conjecture 5.2, which is supported by our computations. Assuming it we obtained that in this case the frequency of totally complex polynomials is about $d^{-3/8}$. This number is much bigger than $2/d$, which would be the probability if the polynomials were uniformly distributed among the different signature classes. As already for $d = 6$ the most frequent signature is $(4, 1)$ one may expect that the peak of the frequency curve will move from the totally complex case. As we have no theoretical and only few computational support, we do not continue the speculation.

7. On the boundary of $\mathcal{E}_d^{(s)}$.

For the sake of completeness we now turn to the boundary of $\mathcal{E}_d^{(s)}$ and prove a generalization of the result of I. Schur [**12**] as well as of A. T. Fam and J. S. Meditsch [**6**]. Although this description is not as explicit as the cited ones, we included it, because we need it in Part II.

THEOREM 7.1. *Let $d \geq 1$ and r, s be non-negative integers such that $r + 2s = d$. Then the boundary of the set $\mathcal{E}_d^{(s)}$ is the union of finitely many algebraic surfaces.*

PROOF. Let $\boldsymbol{v}_0 \in \mathbb{R}^d$ be a boundary point of $\mathcal{E}_d^{(s)}$. If one of its coordinates is 1 or -1 then $\boldsymbol{v}_0$ satisfies the linear equation $\boldsymbol{v}_0(1, \ldots, 1) = 0$ or $\boldsymbol{v}_0(1, -1, 1, \ldots, (-1)^d) = 0$ respectively, where $\boldsymbol{vu}$ denotes the inner product of the vectors $\boldsymbol{v}$ and $\boldsymbol{u}$. If the polynomial corresponding to $\boldsymbol{v}_0$ has a non-real root lying on the unit circle then by [**6**] $\boldsymbol{v}_0$ lies on a hypersurface.

Assume in the sequel that $\boldsymbol{v}_0$ is a boundary point of $\mathcal{E}_d^{(s)}$, such that the roots of $P_{\boldsymbol{v}_0}(X)$ lie inside the unit circle. It will be called an inner boundary point. Then for any $\delta > 0$ there exist $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^d$ such that

$$\|\boldsymbol{v}_0 - \boldsymbol{v}_1\|, \ \|\boldsymbol{v}_0 - \boldsymbol{v}_2\| < \delta, \ \boldsymbol{v}_1 \in \mathcal{E}_d^{(s)} \text{ and } \boldsymbol{v}_2 \in \mathcal{E}_d^{(s_1)}, \ s_1 \neq s. \tag{13}$$

We may assume without loss of generality that $s > s_1$, whence $r < r_1$ and it is fixed.

Denote by α_i, $i = 0, 1, 2$ the vectors of roots of $P_{\boldsymbol{v}_i}(X)$ ordered such that the real roots come first, then the non-real ones such that the complex conjugates follow each other. Then there exists a coordinate, say $1 \leq j \leq s$, such that $\alpha_{1,r+2j-1} \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha_{2,r+2j-1} \in \mathbb{R}$ (α_{ij} denotes the j-t coordinate of the vector α_i). As $\boldsymbol{v}_1$ is a real vector and the corresponding polynomial has a non-real root then its conjugate is a different root of the polynomial. Thus $\bar{\alpha}_{1,r+2j-1}$ is a root of $P_{\boldsymbol{v}_1}(x)$ and because of the ordering of the roots $\bar{\alpha}_{1,r+2j-1} = \alpha_{1,r+2j}$. Notice that $\alpha_{2,r+2j} \in \mathbb{R}$ holds too. The roots are continuous functions of the coefficients, thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that if for $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathbb{R}^d$ the inequality (13) holds then

$$|\alpha_{1,r+2j-1} - \alpha_{2,r+2j-1}|, \ |\bar{\alpha}_{1,r+2j-1} - \alpha_{2,r+2j}| < \varepsilon.$$

This can only happen, if $\alpha_{0,r+2j-1} = \alpha_{0,r+2j}$, i.e., $P_{\boldsymbol{v}_0}(X)$ has a multiple real root. Thus the inner boundary points of $\mathcal{E}_d^{(s)}$ lie on the surface

$$\prod_{1\le j<k\le r}(x_j - x_k)\prod_{j=1}^{r}\prod_{k=1}^{s}(x_j^2 - y_{r+2k-1}x_j + y_{r+2k}) = 0,$$

where the y's has to be chosen such that $0 < y_{r+2k} < 1$ and $|y_{r+2k-1}| < 2\sqrt{y_{r+2k}}$. These are obviously algebraic relations.

In the opposite direction we prove that polynomials with multiple real roots lie on the inner boundary of different signature bodies. Indeed, assume that $P(X)$ has a multiple real root $|\alpha| < 1$ and signature (r,s). Then for any $\varepsilon > 0$ we can find real numbers $0 < \delta_1,\ \delta_2 < \varepsilon$ such that

$$\alpha^2 + \delta_2^2 < 1,\ |\alpha + \delta_1|,\ |\alpha + \delta_2| < 1.$$

Consider the polynomials

$$P_1(X) = P(X)(X - (\alpha + \delta_1))(X - (\alpha + \delta_1))/(X - \alpha)^2$$

and

$$P_2(X) = P(X)(X - (\alpha + i\delta_2))(X - (\alpha - i\delta_2))/(X - \alpha)^2,\ i = \sqrt{-1}.$$

Obviously $P_1(X), P_2(X) \in \mathcal{E}_d$, they have different signature and are arbitrary near to each other. This proves the claim and finishes the proof of the statement. □

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