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The Gottlieb group of a wedge of suspensions

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Abstract. We study the Gottlieb group of a wedge sum of suspension spaces. We give necessary and sufficient conditions, in terms of Hopf invariants, for an element of a homotopy group to be in the Gottlieb group. We apply our results to wedge sums of spheres and to Moore spaces.

1. Introduction.

Let X be a connected, based space with nth homotopy group $\pi_n(X)$. The *n-th Gottlieb group* $G_n(X) \subseteq \pi_n(X)$ is defined as follows: $\alpha = [f] \in G_n(X)$ if and only if there is a map $f' : S^n \times X \to X$ such that the following diagram

$$\begin{tikzcd} S^n \vee X \arrow[r, "(f,id)"] \arrow[d, "j"'] & X \\ S^n \times X \arrow[ur, "f'"'] & \end{tikzcd}$$

is homotopy-commutative, where j is the inclusion, $id = id_X$ is the identity map of X and (f, id) is the map determined by f and id. This group was introduced and studied by Gottlieb in [**Go**] and has been shown to have many topological applications. There have been recent results on the Gottlieb group of rational spaces [**FHT**, pp. 377–380] and on the Gottlieb group of spheres [**GM**].

It is well-known and easily proved that there is an isomorphism

$$G_n(X \times Y) \cong G_n(X) \oplus G_n(Y).$$

However, there appears to be no such simple result for a wedge sum which would express $G_n(X \vee Y)$ in terms of $G_n(X)$ and $G_n(Y)$ (see Example 3.7 (2)). In this paper we study $G_n(\Sigma X_1 \vee \Sigma X_2 \vee \cdots \vee \Sigma X_k)$, where ΣX_i is the suspension of the space X_i, with particular attention to the case $k = 2$ and X_i a sphere. We give necessary and sufficient conditions for an element of $\pi_n(\Sigma X_1 \vee \Sigma X_2 \vee \cdots \vee \Sigma X_k)$ to be in $G_n(\Sigma X_1 \vee \Sigma X_2 \vee \cdots \vee \Sigma X_k)$.

2. Preliminaries.

In this paper we do not usually distinguish notationally between a map and its homotopy class. We begin by recalling the generalized Whitehead product. Let

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$$\alpha \in [\Sigma X, Z] \text{ and } \beta \in [\Sigma Y, Z].$$

Then a map $k(\alpha, \beta) : \Sigma(X \wedge Y) \to Z$ is defined in [**Ar**] whose homotopy class is the generalized Whitehead product

$$[\alpha, \beta] = [k(\alpha, \beta)] \in [\Sigma(X \wedge Y), Z].$$

In particular, if $j_1 : \Sigma X \to \Sigma X \vee \Sigma Y$ and $j_2 : \Sigma Y \to \Sigma X \vee \Sigma Y$ are inclusions and $k = k(j_1, j_2) : \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y$, then k is called the generalized Whitehead product map.

Proposition 2.1 ([**Ar**, Corollary 4.3]). *If $k : \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y$ is the generalized Whitehead product map and C_k is the mapping cone of k, then there is a homotopy equivalence $\mu : C_k \to \Sigma X \times \Sigma Y$ such that $\mu \circ i = j$, where i and j are the inclusions of $\Sigma X \vee \Sigma Y$ into C_k and $\Sigma X \times \Sigma Y$ respectively.*

This gives the following criterion for an element to be in the Gottlieb group.

Proposition 2.2. *If $\alpha \in \pi_n(\Sigma X \vee \Sigma Y)$, then $\alpha \in G_n(\Sigma X \vee \Sigma Y)$ if and only if $[\alpha, id] = 0$.*

Proof. We set $W = \Sigma X \vee \Sigma Y$ and identify the homeomorphic spaces W and $\Sigma(X \vee Y)$. Let $\ell_1 \in [\Sigma S^{n-1}, \Sigma S^{n-1} \vee W]$ and $\ell_2 \in [W, \Sigma S^{n-1} \vee W]$ be the inclusions. By Proposition 2.1, we regard $\Sigma S^{n-1} \times W$ as the mapping cone of $k(\ell_1, \ell_2)$. From the diagram

$$\begin{tikzcd}
\Sigma(S^{n-1} \wedge (X \vee Y)) \arrow[r, "k(\ell_1,\ell_2)"] & \Sigma S^{n-1} \vee W \arrow[r, "(\alpha, id)"] \arrow[d] & W \\
& \Sigma S^{n-1} \times W \arrow[ur, dotted] &
\end{tikzcd}$$

it follows that an extension of (α, id) to $\Sigma S^{n-1} \times W$ exists if and only if $(\alpha, id) \circ [\ell_1, \ell_2] = 0$. Proposition 2.2 now follows from the equality

$$(\alpha, id) \circ [\ell_1, \ell_2] = [\alpha, id]. \qquad \square$$

Next let $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ be inclusions and identify $W = \Sigma X \vee \Sigma Y$ with $\Sigma(X \vee Y)$. Then the inclusions j_1 and j_2 correspond to Σi_1 and Σi_2 under this identification. There is a homeomorphism

$$\Sigma(S^{n-1} \wedge X) \vee \Sigma(S^{n-1} \wedge Y) \cong \Sigma(S^{n-1} \wedge (X \vee Y))$$

which induces an isomorphism

$$\rho : [\Sigma(S^{n-1} \wedge (X \vee Y)), W] \to [\Sigma(S^{n-1} \wedge X), W] \oplus [\Sigma(S^{n-1} \wedge Y), W],$$

defined by

$$\rho(\beta) = (\Sigma(id \wedge i_1)^*(\beta), \Sigma(id \wedge i_2)^*(\beta)).$$

Then if $\alpha \in \pi_n(W)$,

$$\Sigma(id \wedge i_1)^*[\alpha, id] = [(\Sigma id)^*(\alpha), (\Sigma i_1)^*(id)] = [\alpha, j_1],$$

and similarly, $\Sigma(id \wedge i_2)^*[\alpha, id] = [\alpha, j_2]$. Thus $[\alpha, id]$ corresponds to $([\alpha, j_1], [\alpha, j_2])$ under the isomorphism ρ.

Therefore by Proposition 2.2 we have the following result.

Proposition 2.3. *Let $\alpha \in \pi_n(\Sigma X \vee \Sigma Y)$. Then $\alpha \in G_n(\Sigma X \vee \Sigma Y)$ if and only if $[\alpha, j_1] = 0 = [\alpha, j_2]$.*

A straightforward extension of the previous argument to k suspensions then yields the following generalization of Proposition 2.3.

Proposition 2.4. *Let T be the wedge sum $\Sigma X_1 \vee \cdots \vee \Sigma X_k$ with $j_s \in [\Sigma X_s, T]$ the class of the inclusion map, $s = 1, 2, \ldots, k$. Then $\alpha \in \pi_n(T)$ is in $G_n(T)$ if and only if $[\alpha, j_s] = 0$ for $s = 1, 2, \ldots, k$.*

3. Wedge sums of spheres.

In this section we consider the wedge sum $W = S^m \vee S^l$ of spheres such that $2 \leq m \leq l$. We first state the result of Hilton [**Hi**] regarding the homotopy groups of W. Let $\iota_j \in \pi_{n_j}(W)$ be the class of the inclusion maps, $j = 1, 2$ ($n_1 = m$, $n_2 = l$), and recall the basic (Whitehead) products in the homotopy groups of W:

(1) Weight 1: ι_1, ι_2
(2) Weight 2: $[\iota_1, \iota_2]$
(3) Weight 3: $[\iota_1, [\iota_1, \iota_2]]$, $[\iota_2, [\iota_1, \iota_2]]$
(4) Weight 4: $[\iota_1, [\iota_1, [\iota_1, \iota_2]]]$, $[\iota_2, [\iota_1, [\iota_1, \iota_2]]]$, $[\iota_2, [\iota_2, [\iota_1, \iota_2]]]$

and so on. These products are ordered as displayed and we write them in order as an infinite sequence $\omega_1, \omega_2, \omega_3, \ldots$. Then $\omega_p \in \pi_{n_p}(W)$ for some integer n_p, where $n_p = |\omega_p|$, the degree of ω_p.

Hilton's Theorem asserts that for every positive integer k, there is an isomorphism

$$\theta : \bigoplus_{p=1}^{\infty} \pi_k(S^{n_p}) \longrightarrow \pi_k(W).$$

which is defined by

$$\theta|\pi_k(S^{n_p}) = \omega_{p*} : \pi_k(S^{n_p}) \to \pi_k(W).$$

The direct sum is finite for each k since $n_p \to \infty$.

Thus for $x \in \pi_k(W)$,

$$x = \sum_p \omega_p \circ \alpha_p,$$

for unique $\alpha_p \in \pi_k(S^{n_p})$.

Now let $m = l = n$, so $W = S^n \vee S^n$, and let $\varphi : S^n \to S^n \vee S^n$ be the standard comultiplication. Then as in [**Hi**], [**Wh**] the *Hopf-Hilton invariant* $H_p : \pi_k(S^n) \to \pi_k(S^{n_{p+3}})$ is defined by

$$H_p = q_{p+3*} \circ \theta^{-1} \circ \varphi_*,$$

where $p \geq 0$ and q_i is the projection onto the ith summand. For example, $H_0 : \pi_k(S^n) \to \pi_k(S^{2n-1})$, $H_1, H_2 : \pi_k(S^n) \to \pi_k(S^{3n-2})$, etc. We call H_p the Hopf-Hilton invariant corresponding to ω_{p+3}.

The following result on the Hopf-Hilton invariants appears to be well-known, but we include it for completeness.

Lemma 3.1. *If $H_p : \pi_k(S^n) \to \pi_k(S^{n_{p+3}})$ is the pth Hopf-Hilton invariant and $\alpha \in \pi_k(S^n)$ is a suspension, then $H_p(\alpha) = 0$*

Proof. The comultiplication $\varphi = \iota_1 + \iota_2$. Then, since α is a suspension, $\varphi_*(\alpha) = \iota_1 \circ \alpha + \iota_2 \circ \alpha$. But $\theta(\alpha, \alpha, 0, 0, \ldots) = \iota_1 \circ \alpha + \iota_2 \circ \alpha$. Therefore $\theta^{-1} \circ \varphi_*(\alpha) = (\alpha, \alpha, 0, 0, \ldots)$, and so $H_p(\alpha) = q_{p+3*} \circ \theta^{-1} \circ \varphi_*(\alpha) = 0$. □

We will need a result by Barcus and Barratt, and for this we recall some notation in [**BB**]. For elements γ, δ in the homotopy groups of a space X, we inductively define $\sigma_0(\gamma, \delta) = [\gamma, \delta], \ldots, \sigma_{p+1}(\gamma, \delta) = [\gamma, \sigma_p(\gamma, \delta)]$. In the case when $X = W = S^n \vee S^n$ and γ, δ are ι_1, ι_2 respectively, then the $\sigma_p(\iota_1, \iota_2)$ are basic products (but not all of them). Let B_p be the Hopf-Hilton invariant corresponding to $\sigma_p(\iota_1, \iota_2)$ so that $B_0 = H_0$, $B_1 = H_1$, $B_2 = H_2$, and so on. Thus $B_p : \pi_m(S^n) \to \pi_m(S^{(p+2)n-p-1})$, for $p \geq 0$. To obtain a compact formula below, we define $B_{-1} = id : \pi_m(S^n) \to \pi_m(S^n)$.

Lemma 3.2 ([**BB**, Corollary 7.4]). *If $\gamma \in \pi_q(S^m)$, $\alpha \in \pi_m(X)$, $\beta \in \pi_n(X)$ and $m, n \geq 2$, then*

$$[\alpha \circ \gamma, \beta] = \sum_{p=-1}^{\infty} (-1)^{(p+1)(n+1)} \sigma_{p+1}(\alpha, \beta) \circ \Sigma^{n-1} B_p(\gamma).$$

We will consider conditions involving the suspension functor and Hopf-Hilton invariants under which an element of the homotopy group of a wedge sum belongs to the Gottlieb group.

Proposition 3.3. *Let $\iota_1 \in \pi_m(S^m \vee S^l)$ and $\iota_2 \in \pi_l(S^m \vee S^l)$ be the classes of the inclusion maps. If $\gamma \in \pi_N(S^m)$ is an arbitrary element, then $\iota_1 \circ \gamma \in G_N(S^m \vee S^l)$ if* *and only if* $\gamma \in G_N(S^m)$ *and* $\Sigma^{l-1}B_p(\gamma) = 0$, *for* $p \geq -1$. *Similarly, if* $\gamma \in \pi_N(S^l)$, *then* $\iota_2 \circ \gamma \in G_N(S^m \vee S^l)$ *if and only if* $\gamma \in G_N(S^l)$ *and* $\Sigma^{m-1}B_p(\gamma) = 0$, *for* $p \geq -1$.

PROOF. By Proposition 2.3, $\iota_1 \circ \gamma \in G_N(S^m \vee S^l)$ if and only if

$$[\iota_1 \circ \gamma, \iota_1] = 0 = [\iota_1 \circ \gamma, \iota_2].$$

But $\iota_{1*}[\gamma, id] = [\iota_1 \circ \gamma, \iota_1]$, and so $[\iota_1 \circ \gamma, \iota_1] = 0$ is equivalent to $\gamma \in G_N(S^m)$ by Proposition 2.2. By Lemma 3.2,

$$[\iota_1 \circ \gamma, \iota_2] = \sum_{p=-1}^{\infty} \pm\sigma_{p+1}(\iota_1, \iota_2) \circ \Sigma^{l-1}B_p(\gamma).$$

Since the $\sigma_p(\iota_1, \iota_2)$ are basic products, $[\iota_1 \circ \gamma, \iota_2] = 0$ if and only if $\Sigma^{l-1}B_p(\gamma) = 0$, for $p \geq -1$. The second assertion of the proposition is similarly proved. $\square$

Let $\alpha_1 \in \pi_N(S^m)$ and $\alpha_2 \in \pi_N(S^l)$, let $m \leq l$ and let $m, N \geq 2$. Then Proposition 3.3 gives necessary and sufficient conditions for any element of the form $\iota_1 \circ \alpha_1$ or $\iota_2 \circ \alpha_2$ in $\pi_N(S^m \vee S^l)$ to be in $G_N(S^m \vee S^l)$. However, a typical element $\theta \in \pi_N(S^m \vee S^l)$ has the form

$$\theta = \iota_1 \circ \alpha_1 + \iota_2 \circ \alpha_2 + \sum_{p \geq 3} \omega_p \circ \alpha_p,$$

where $\alpha_p \in \pi_N(S^{|\omega_p|})$ and ω_p are basic products (as are $\iota_1 = \omega_1$ and $\iota_2 = \omega_2$). Then $\theta \in G_N(S^m \vee S^l)$ if and only if $[\omega_p \circ \alpha_p, \iota_1] = 0 = [\omega_p \circ \alpha_p, \iota_2]$ for all p. We express each of these bracket products (except $p = 1$ in the first and $p = 2$ in the second) as a sum by Lemma 3.2. In general, this would yield a very large number of conditions for $\theta \in G_N(S^m \vee S^l)$. But we can reduce the number by assuming N is less than some linear expression in m such as $N < am - a + 1$ for some integer $a \geq 2$ (and hence $N < a_1 m + a_2 l - a + 1$ for $a_1 + a_2 = a$). We illustrate this next in the case of $a = 4$.

If $\theta \in \pi_N(S^m \vee S^l)$ with $m \leq l$ and $N < 4m - 3$, then

$$\theta = \sum_{p=1}^{5} \omega_p \circ \alpha_p$$

with $\alpha_1 \in \pi_N(S^m)$, $\alpha_2 \in \pi_N(S^l)$, $\alpha_3 \in \pi_N(S^{m+l-1})$, $\alpha_4 \in \pi_N(S^{2m+l-2})$ and $\alpha_5 \in \pi_N(S^{m+2l-2})$.

THEOREM 3.4. *With the notation and conditions of the previous sentence,* $\theta \in G_N(S^m \vee S^l) \iff \alpha_1 \in G_N(S^m)$, $\alpha_2 \in G_N(S^l)$, $\Sigma^{l-1}(\alpha_1) = 0$, $\Sigma^{m-1}(\alpha_2) = 0$, $B_p(\alpha_i) = 0$, *for* $p = 0, 1$ *and* $i = 1, 2$, *and* $\alpha_3 = \alpha_4 = \alpha_5 = 0$.

PROOF. We obtain conditions for $[\theta, \iota_1] = 0 = [\theta, \iota_2]$. Now

$$[\theta, \iota_2] = [\iota_1 \circ \alpha_1, \iota_2] + [\iota_2 \circ \alpha_2, \iota_2] + [\omega_3 \circ \alpha_3, \iota_2] + [\omega_4 \circ \alpha_4, \iota_2] + [\omega_5 \circ \alpha_5, \iota_2]$$

and we examine these five terms separately using Lemma 3.2.

$$[\iota_1 \circ \alpha_1, \iota_2] = \omega_3 \circ \Sigma^{l-1}(\alpha_1) + \sum_{p=0,1} \pm \sigma_{p+1}(\iota_1, \iota_2) \circ \Sigma^{l-1} B_p(\alpha_1) \tag{3.1}$$

since $B_p(\alpha_1) \in \pi_N(S^{(p+2)m-p-1}) = 0$ if $p \geq 2$.

$$[\iota_2 \circ \alpha_2, \iota_2] = \iota_2 \circ [\alpha_2, id]. \tag{3.2}$$

$$[\omega_3 \circ \alpha_3, \iota_2] = [\omega_3, \iota_2] \circ \Sigma^{l-1}(\alpha_3) \tag{3.3}$$

since $B_p(\alpha_3) \in \pi_N(S^{(p+2)(m+l-1)-p-1}) = 0$ if $p \geq 0$.

$$[\omega_4 \circ \alpha_4, \iota_2] = [\omega_4, \iota_2] \circ \Sigma^{l-1}(\alpha_4) \tag{3.4}$$

since $B_p(\alpha_4) \in \pi_N(S^{(p+2)(2m+l-2)-p-1}) = 0$ if $p \geq 0$.

$$[\omega_5 \circ \alpha_5, \iota_2] = [\omega_5, \iota_2] \circ \Sigma^{l-1}(\alpha_5) \tag{3.5}$$

since $B_p(\alpha_5) \in \pi_N(S^{(p+2)(m+2l-2)-p-1}) = 0$ if $p \geq 0$. Therefore

$$\begin{aligned}[\theta, \iota_2] = {} & \omega_3 \circ \Sigma^{l-1}(\alpha_1) \pm \omega_4 \circ \Sigma^{l-1} B_0(\alpha_1) \pm \omega_6 \circ \Sigma^{l-1} B_1(\alpha_1) \\ & + \iota_2 \circ [\alpha_2, id] + [\omega_3, \iota_2] \circ \Sigma^{l-1}(\alpha_3) \\ & + [\omega_4, \iota_2] \circ \Sigma^{l-1}(\alpha_4) + [\omega_5, \iota_2] \circ \Sigma^{l-1}(\alpha_5).\end{aligned}$$

Each of the Whitehead products which appears on the right side of this expression is either a basic product or, using anticommutativity, the negative of a basic product. Thus we have the following equivalence.

$$[\theta, \iota_2] = 0 \Leftrightarrow \alpha_2 \in G_N(S^l);\ \Sigma^{l-1}\alpha_p = 0,\ p = 1, 3, 4, 5;\ \Sigma^{l-1} B_p(\alpha_1) = 0,\ p = 0, 1,$$

since $\iota_2 \circ [\alpha_2, id] = 0 \Leftrightarrow [\alpha_2, id] = 0 \Leftrightarrow \alpha_2 \in G_N(S^l)$. Now we consider the suspension $\Sigma^{l-1} : \pi_N(S^{m+l-1}) \to \pi_{N+l-1}(S^{m+2l-2})$. By the Freudenthal theorem, Σ^{l-1} is an isomorphism since $N < 2(m+l-1) - 1$. Thus $\Sigma^{l-1}\alpha_3 = 0$ if and only if $\alpha_3 = 0$. Similarly $\Sigma^{l-1}\alpha_p = 0$ if and only if $\alpha_p = 0$ for $p = 4, 5$ and $\Sigma^{l-1} B_p(\alpha_1) = 0$ if and only if $B_p(\alpha_1) = 0$ for $p = 0, 1$. Hence $[\theta, \iota_2] = 0$ if and only if

$$\alpha_2 \in G_N(S^l);\ \Sigma^{l-1}(\alpha_1) = 0;\ B_p(\alpha_1) = 0,\ p = 0, 1;\ \alpha_i = 0,\ i = 3, 4, 5.$$

A similar calculation can be made for $[\theta, \iota_1]$. If we assume that $[\theta, \iota_2] = 0$, so that $\alpha_i = 0$, $i = 3, 4, 5$, then

$$[\theta, \iota_1] = \iota_1 \circ [\alpha_1, id] + [\iota_2, \iota_1] \circ \Sigma^{m-1}\alpha_2 + \sum_{p=0,1} \sigma_{p+1}(\iota_2, \iota_1) \circ \Sigma^{m-1} B_p(\alpha_2).$$

The Whitehead products on the right are $\pm$ basic products. Thus $[\theta, \iota_1] = 0$ (together with $[\theta, \iota_2] = 0$ and the Freudenthal theorem) implies $\alpha_1 \in G_N(S^m)$, $\Sigma^{m-1}\alpha_2 = 0$ and $B_p(\alpha_2) = 0$ for $p = 0, 1$. This proves one of the implications in the theorem, namely, "$\Longrightarrow$". Conversely, the hypotheses of the theorem easily imply that $[\theta, \iota_1] = 0 = [\theta, \iota_2]$, and so $\theta \in G_N(S^m \vee S^l)$. This completes the proof. $\square$

REMARK 3.5. If we replace the hypothesis $B_p(\alpha_i) = 0$, for $p = 0, 1$ and $i = 1, 2$ with the hypothesis α_i is a suspension, $i = 1, 2$ and keep the other hypotheses, then we conclude that $\theta \in G_N(S^m \vee S^l)$. This follows from Lemma 3.1 and Theorem 3.4.

We wish to emphasize that we have presented the previous theorem to illustrate our method of showing that an upper bound for N leads to a smaller number of necessary and sufficient conditions for an element to be in the Gottlieb group. Clearly this method can be used for other upper bounds. In Corollary 3.6 we state this result for other upper bounds which are smaller than the one given in Theorem 3.4 and hence the corollary follows from Theorem 3.4.

COROLLARY 3.6. *Let $\theta \in \pi_N(S^m \vee S^l)$ where $m \leq l$.*

(1) *If $N < 2m - 1$, then $G_N(S^m \vee S^l) = 0$.*
(2) *If $N < 3m - 2$, then $\theta = \sum_{p=1}^{3} \omega_p \circ \alpha_p \in G_N(S^m \vee S^l) \iff \alpha_1 \in G_N(S^m)$, $\alpha_2 \in G_N(S^l)$, $\Sigma^{l-1}(\alpha_1) = 0$, $\Sigma^{m-1}(\alpha_2) = 0$, $B_0(\alpha_i) = 0$, for $i = 1, 2$, and $\alpha_3 = 0$.*

EXAMPLE 3.7.

(1) We give an example to show that $G_N(S^m \vee S^l) \neq 0$. If $\gamma \in G_N(S^m)$ is a nontrivial suspension element such that $\Sigma^{l-1}\gamma = 0$, then $\iota_1 \circ \gamma$ is a nontrivial element of $G_N(S^m \vee S^l)$ by Proposition 3.3 and Lemma 3.1. For a specific example, consider $\gamma = \Sigma(\nu' \circ \eta_6) = (\Sigma\nu') \circ \eta_7 \in \pi_8(S^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a nonzero element with $\Sigma\gamma = 0$ [**To**, Proposition 5.8]. Furthermore, $\pi_8(S^4) = G_8(S^4)$ by [**GM**, Proposition 2.1]. Therefore

$$G_8(S^4 \vee S^l) \neq 0$$

for $l \geq 2$.
(2) We give an example to show that $G_N(S^m) \neq 0$, $G_N(S^l) \neq 0$, but $G_N(S^m \vee S^l) = 0$. We set $N = n + 2$, $m = n$, and $\ell = n + 1$, with $n \geq 3$. We assume $n \equiv 3 \bmod 4$ and from this it follows that $G_{n+1}(S^n) = \pi_{n+1}(S^n) = \mathbb{Z}_2$ and $G_{n+2}(S^n) = \pi_{n+2}(S^n) = \mathbb{Z}_2$ by [**GM**, (2.1) and (2.2)]. Since $G_N(S^m \vee S^l) = 0$ by Proposition 3.3, the example follows.

REMARK 3.8. It is possible to extend our results to a wedge sum of k spheres $T = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}$, where $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and give conditions for an element $\theta = \sum_{p \geq 1} \omega_p \circ \alpha_p \in \pi_N(T)$ to be in $G_N(T)$. The conditions are that α_p are Gottlieb elements and iterated suspensions of $B_q(\alpha_p)$ are zero. The details are formidable, and we omit them.

4. Moore spaces.

In this section we calculate the nth Gottlieb group of a Moore space $M(A,n)$ of type (A,n), $n>2$. We assume that A is a finitely-generated abelian group so that $M(A,n)$ is a wedge sum of suspensions, namely, spheres and finite Moore spaces. The following is a consequence of Theorems 5.2 and 5.4 of [**Go**].

$$G_n(M(A,n)) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd, rank } A \neq 1 \\ 2\mathbb{Z} \subseteq \mathbb{Z} = \pi_n(S^n) & \text{if } A = \mathbb{Z},\ n\,\text{odd} \neq 1,3,7 \\ \mathbb{Z} = \pi_n(S^n) & \text{if } A = \mathbb{Z},\ n = 1,3,7. \end{cases}$$

This does not cover the Moore spaces $M(\mathbb{Z}\oplus T,n)$, n odd and T finite. For this we first establish some general results.

Lemma 4.1. *Let B and C be spaces such that $\pi_n(B\vee C) = i_{1*}\pi_n(B) + i_{2*}\pi_n(C)$ and $G_n(C)=0$, where $i_1 : B \to B\vee C$ and $i_2 : C \to B \vee C$ are the inclusions. Then $G_n(B\vee C) \subseteq i_{1*}G_n(B) \subseteq i_{1*}\pi_n(B)$.*

Proof. The homomorphism $p_{1*} : \pi_n(B\vee C) \to \pi_n(B)$ induced by the projection p_1 induces $\tilde{p}_{1*} : G_n(B\vee C) \to G_n(B)$ by [**Go**, Corollary 1-5]. Similarly, p_2 induces $\tilde{p}_{2*} : G_n(B\vee C) \to G_n(C)$. If $x \in G_n(B\vee C) \subseteq \pi_n(B\vee C)$, then $x = i_{1*}b + i_{2*}c$ for $b\in\pi_n(B)$ and $c \in \pi_n(C)$. Therefore $c = \tilde{p}_{2*}x \in G_n(C) = 0$, and so $x = i_{1*}b$. Since $x \in G_n(B\vee C)$ and i_1 has a left inverse, $b \in G_n(B)$ [**Go**, Corollary 1-6]. Hence $G_n(B\vee C) \subseteq i_{1*}G_n(B)$. □

Remark 4.2. The hypothesis on $\pi_n(B\vee C)$ holds if B is $(m-1)$-connected, C is $(l-1)$-connected, and $n+1 < m+l$, since then $\pi_{n+1}(B\times C, B\vee C) = 0$.

Corollary 4.3. *Assume the hypothesis of Lemma* 4.1 *and in addition $G_n(B) = 0$. Then $G_n(B\vee C) = 0$.*

Corollary 4.4. *If n is odd, then $G_n(M(\mathbb{Z}\oplus T,n))$ is infinite cyclic, where T is a finite abelian group.*

Proof. Let $M_T = M(T,n)$, let $X = M(\mathbb{Z}\oplus T,n) = S^n \vee M_T$, and let $\iota = [id] \in \pi_n(S^n)$. By Lemma 4.1 and Proposition 2.3, it suffices to show that there exists a positive integer k such that

(1) $[ki_1, i_1] = 0$ and
(2) $[ki_1, i_2] = 0$.

(1)
$$[ki_1, i_1] = ki_{1*}[\iota,\iota] = \begin{cases} 0 & \text{if } n = 3,7 \text{ for any } k>0 \\ 0 & \text{if } n \neq 3,7 \text{ for any even } k > 0 \end{cases}$$

since $k[\iota,\iota] = 0$ in these cases.

(2) The element $i_2 \in [M_T, X]$ has finite order l since the group $[M_T, X]$ has finite order by the universal coefficient theorem for homotopy groups with coefficients. Therefore $[li_1, i_2] = [i_1, li_2] = 0$. We take $k = 2l$. □

REMARK 4.5. It would be interesting to compute other Gottlieb groups of Moore spaces such as, for example, $G_{n+1}(M(A, n))$.

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