J. Math. Soc. Japan
Vol. 66, No. 3 (2014) pp. 693–708
doi: 10.2969/jmsj/06630693

Region crossing change is an unknotting operation

By Ayaka Shimizu

(Received June 4, 2011)
(Revised July 10, 2012)

Abstract. A region crossing change is a local transformation on a knot or link diagram. We show that a region crossing change on a knot diagram is an unknotting operation, and we define the region unknotting number for a knot diagram and a knot.

1. Introduction.

An *unknotting operation* is a local transformation of a knot diagram such that any diagram can be transformed into a diagram of the trivial knot by a finite sequence of these operations. Unknotting operations play an important role in knot theory, and many unknotting operations have been studied. For example, it is well-known that the crossing change, indicated in Figure 1, is an unknotting operation. It is also known that the ♯*-move*, indicated in Figure 1, is an unknotting operation [**6**], and that an *n-gon move*, indicated in Figure 1, is an unknotting operation [**1**], [**7**]. Let D be a link diagram

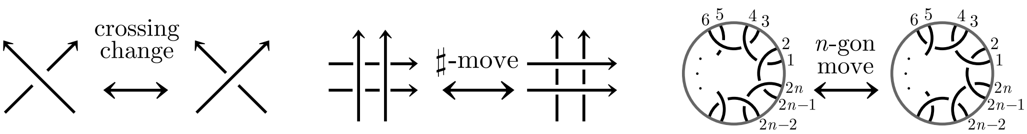

Figure 1.

on S^2, and $|D|$ be the four-valent graph obtained from D by replacing each crossing with a vertex. We call each component of $S^2 - |D|$ a *region* of D. A diagram D with c crossings has $2c$ edges and, therefore, $c+2$ regions because the Euler characteristic of S^2 is 2. For example, the diagram D with three crossings in Figure 2 has six edges and five regions $R_1, R_2, \ldots$ and R_5. A *region crossing change* at a region R of D is the local transformation on D by the changing all the crossings on ∂R. For example in Figure 3, we obtain D' (resp. E') from D (resp. E) by applying a region crossing change at R (resp. S). A ♯-move and an n-gon move on a knot diagram are examples of region crossing changes. We remark that we can apply a region crossing change on a non-alternating region even though we cannot apply an n-gon move. The region crossing change was proposed by K. Kishimoto in a seminar at Osaka City University in 2010. Kishimoto raised the following question: *Is a region crossing change on a knot diagram*

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M27.
Key Words and Phrases. crossing number, knot diagram, local move, region crossing change, unknotting operation.
This research was supported by JSPS Research Fellowships for Young Scientists.

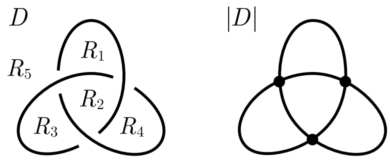

Figure 2.

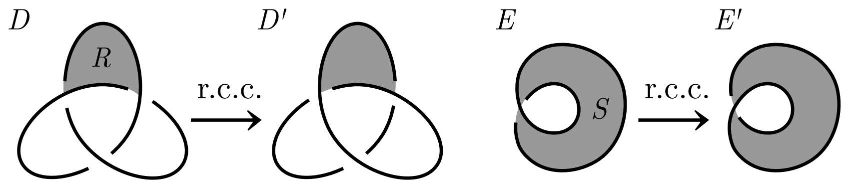

Figure 3.

an unknotting operation? This means, "Can we transform any diagram into a diagram of the trivial knot by region crossing changes". We will prove the following theorem in this paper:

Theorem 1.1. *Let D be a knot diagram, and c a crossing point of D. Let D' be the diagram obtained from D by the crossing change at c. Then, there exist region crossing changes which transform D into D'.*

The proof is given in Section 3. Since a crossing change on a knot diagram is an unknotting operation, we have the following corollary of Theorem 1.1 which answers Kishimoto's question:

Corollary 1.2. *A region crossing change on a knot diagram is an unknotting operation. Therefore, we can transform any diagram into a diagram of the trivial knot by region crossing changes.*

Remark 1.3. For a link diagram, the answer to Kishimoto's question is negative. For example, the link diagram in Figure 4 can not be transformed into a diagram of a trivial link by any number of region crossing changes.

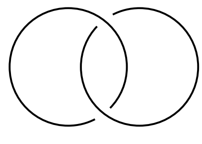

Figure 4.

We define the *region unknotting number* $u_R(D)$ of a knot diagram D to be the minimum number of region crossing changes necessary to obtain a diagram of the trivial knot from D. For example, we have $u_R(D) = 1$ for the diagram D in Figure 3. H. A. Miyazawa showed in [**1**] that for any knot K, there exists an integer n such that a diagram of K can be transformed into a diagram of the trivial knot by one n-gon move. Therefore, every knot K has a diagram D such that $u_R(D) = 1$. We define the *region unknotting number* $u_R(K)$ of a knot K to be the minimal $u_R(D)$ for all minimal crossing diagrams D of K. We have the following theorem:

THEOREM 1.4. *Let K be a knot and $c(K)$ be the crossing number of K, then*

$$u_R(K) \leq \frac{c(K)}{2} + 1.$$

The proof is given in Section 4. The rest of this paper is organized as follows: In Section 2, we develop the properties of region crossing changes that are used in proving Theorem 1.1. In Section 3, we prove Theorem 1.1. In Section 4, we consider the region unknotting number and prove Theorem 1.4. In the appendix, we discuss minimal crossing diagrams on S^2 for prime alternating knots.

2. Properties of region crossing changes.

In this section, we discuss the properties of region crossing changes on a link diagram. Let D be a link diagram and R a region of D. We denote by $D(R)$ the diagram obtained from D by the region crossing change on R. For two regions R_1 and R_2 of D, we denote by $D(R_1, R_2)$ the diagram obtained from D by the region crossing changes first on R_1, and then on R_2. We have $D(R_1, R_2) = D(R_2, R_1)$ and $D(R, R) = D$ because the result of crossing changes does not depend on the order, and two crossing changes at a crossing point cancel. For regions $R_1, R_2, \ldots$ and R_n of D, the set of regions $P = R_1 \cup R_2 \cup \cdots \cup R_n$ allows us to denote by $D(P)$ the diagram obtained from D by region crossing changes on $R_1, R_2, \ldots$ and R_n. We have the following lemma:

LEMMA 2.1. *Let D be a link diagram, and R_1, R_2 regions of D ($R_1 \neq R_2$). Let c be a crossing point of D. If c satisfies $c \in \partial R_1 \cap \partial R_2$, then the region crossing changes on R_1 and R_2 do not change c.*

A link diagram D on S^2 is *reducible* if D has a crossing as shown in Figure 5, where each square means a diagram of a tangle. A link diagram D on S^2 is *reduced* if D is not reducible. We call such a crossing a *reducible crossing*, and the set of a reducible crossing and one of the squares a *reducible part*. We have the following proposition:

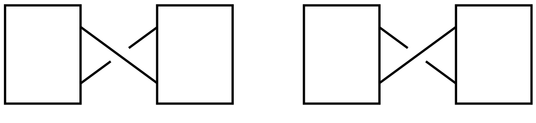

Figure 5.

PROPOSITION 2.2. *A link diagram D is a reducible link diagram if and only if there exists a crossing c of D such that the regions R_1, R_2, R_3 and R_4 around c as shown in Figure 6 satisfy $R_1 = R_3$ or $R_2 = R_4$.*

We can shade some regions of D so that each two regions which are adjacent by an edge

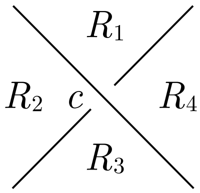

Figure 6.

of $|D|$ are shaded and unshaded. We call such shading a checkerboard coloring. From Lemma 2.1 and Proposition 2.2, we have the following corollary:

Corollary 2.3. *Let D be a reduced link diagram with a checkerboard coloring, and D' the diagram obtained from D by region crossing changes at all the shaded regions. Then, $D = D'$.*

We remark that Corollary 2.3 does not hold for a reducible link diagram (see, for example, the diagram E in Figure 3). From Corollary 2.3, we have the following corollary:

Corollary 2.4. *Let D be a reduced link diagram, and B the set of all the regions of D shaded in a checkerboard coloring. Let P be a subset of B consisting of non-empty regions of D. Then, $D(P) = D(B - P)$.*

From Corollary 2.4, we have the following corollary:

Corollary 2.5. *Let D be a reduced link diagram, and P a set of regions of D. Then there exist just one or three sets P_i $(i = 1$ or $i = 1, 2, 3)$ of regions of D such that $D(P) = D(P_i)$, where $P \neq P_i$, $P_i \neq P_j$ $(i \neq j,\ i, j = 1, 2, 3)$.*

From Lemma 2.1 and that $D(R, R) = D$, we have the following corollary:

Corollary 2.6. *Let D be a link diagram, and c a crossing point of D. Let R_1, R_2, R_3 and R_4 be regions of D around c as shown in Figure 6. If $R_1 \neq R_3$ and $R_2 \neq R_4$, then the region crossing changes at R_1, R_2, R_3 and R_4 do not change c.*

3. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let D be a knot diagram, and c a crossing point of D. We show that we can make the crossing change at c by region crossing changes by an induction on the number k of reducible crossings of D. If $k = 0$, i.e., D is a reduced diagram, we can obtain the regions of D such that we can change only c by region crossing changes at the regions by the following procedure:

Step 1: We splice D at c by giving D an orientation (see Figure 7). Then, we obtain a diagram $D_s = D^1 \cup D^2$ of a two-component link.

Step 2: We apply a checkerboard coloring for one component D^1 of D_s by ignoring another component D^2 so that the region R in Figure 7 is unshaded.

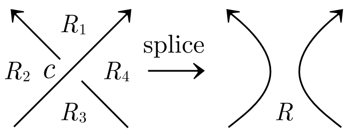

Figure 7.

Step 3: We take the regions of D corresponding to the shaded regions of D_s. An example of the above procedure is shown in Figure 8. By Lemma 2.1 and Corollary 2.6, a crossing

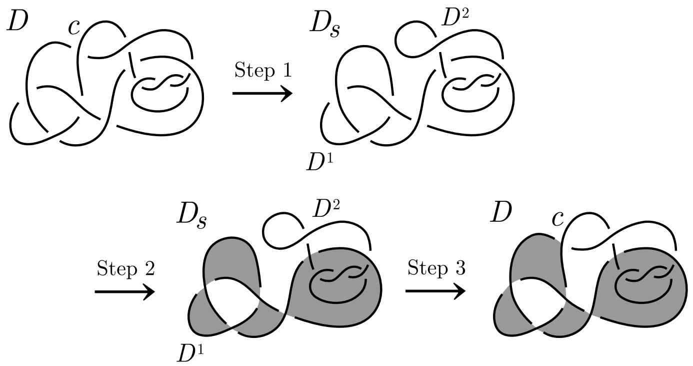

Figure 8.

point of D which corresponds to a self-crossing point of D^1 or D^2 is not changed by the region crossing changes at the regions. By Lemma 2.1, a crossing point of D which corresponds to a crossing point between D^1 and D^2 is not changed by the region crossing changes at the regions. Hence we can change only c. Therefore, the theorem holds for reduced knot diagrams.

We remark that if D_s has a reducible crossing d ($\neq c$), d corresponds to a crossing on $\partial R_1 \cap \partial R_3$ in D. That is why we apply the checkerboard coloring so that R is unshaded in Step 2.

Here, we consider a special case D has just one reducible crossing and c is the reducible crossing. Apply the checkerboard coloring to one reducible part as shown in Figure 9. Then we can change only c by the region crossing changes at the shaded regions.

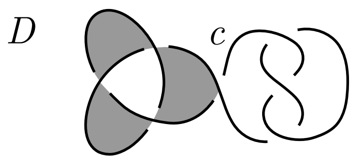

Figure 9.

We next consider the other case. We assume that the theorem holds for all the knot diagrams with k reducible crossings ($k \geq 0$). Now we consider knot diagrams with $k+1$ reducible crossings. In this case, there exists a reducible crossing p of an innermost reducible part S which does not include c. By splicing D at p, we obtain a non-connected link diagram consisting of a knot diagram D^1 with c and k reducible crossings and a reduced knot diagram D^2. By the assumption, D^1 has regions such that we can change only c by the region crossing changes at the regions. We call such set of regions P. We obtain a set Q of regions of D from P by the following rules: Let A be the region of D^1 which includes D^2, and B the opposite region of D^1 (see Figure 10). Let Q includes regions corresponding to $P \setminus A$, and

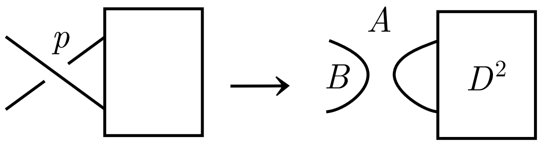

Figure 10.

(i) If $A \in P$ and $B \notin P$, all the regions of S in A is in Q.
(ii) If $A \notin P$ and $B \in P$, the shaded regions of S with the checkerboard coloring such that the outer region of S is white are in Q.
(iii) If $A \in P$ and $B \in P$, then the shaded regions of S in A with the checkerboard coloring such that the outer region of S is black are in Q.
(iv) If $A \notin P$ and $B \notin P$, all the regions of S in A is not in Q (see Figure 11).

Then, Q is the set of regions which change only c. $\square$

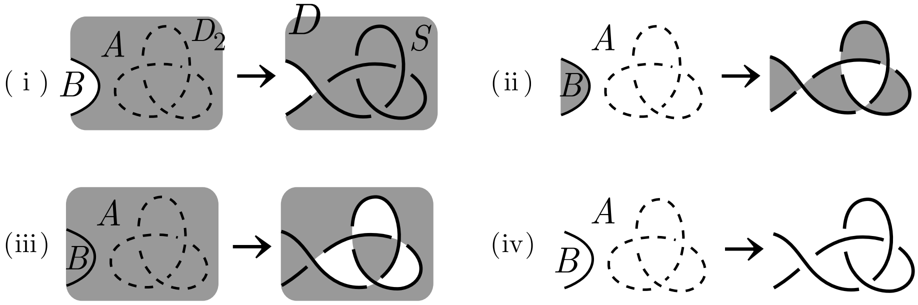

Figure 11.

4. Region unknotting number.

In this section, we discuss the region unknotting number of a knot diagram and a knot. We have the following lemma:

Lemma 4.1. *Let D be a reduced knot diagram, and $c(D)$ the crossing number of D. Then we have*

$$u_R(D) \leq \frac{c(D)}{2} + 1.$$

Proof. For a reduced knot diagram D with a checkerboard coloring, we denote by b (resp. w) the number of regions colored black (resp. white). We have

$$u_R(D) \leq \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{w}{2} \right\rfloor$$
$$\leq \frac{b+w}{2}$$

because of Corollary 2.4, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$. Since $b+w$ means the number of regions of D, we have

$$u_R(D) \leq \frac{c(D)+2}{2}. \qquad \square$$

Remark 4.2. From the proof of Lemma 4.1, it can also be said that the region unknotting number of a reduced knot diagram D is less than or equal to half the number of regions of D.

Remark 4.3. The equality in Lemma 4.1 does not hold if $c(D)$ is even and both b and w are odd, or $c(D)$ is odd.

We show an example of region unknotting numbers of knot diagrams.

Example 4.4. In Figure 32, we list all the knot diagrams based on Rolfsen's knot table [**8**] with crossing number eight or less and their region unknotting numbers. We denote by D^m_n the diagram of m_n in Rolfsen's knot table (for example, we denote by D^3_1 the diagram of 3_1).

We prove Theorem 1.4 by using Lemma 4.1:

Proof of Theorem 1.4. For a knot K and a minimal crossing diagram D of K, we have

$$u_R(K) \leq u_R(D) \leq \frac{c(D)}{2} + 1 = \frac{c(K)}{2} + 1$$

because D is a reduced knot diagram. $\square$

In the following example, we show the region unknotting numbers of all the prime knots with crossing number nine or less:

Example 4.5. The knots $7_1, 8_2, 8_7, 8_9, 8_{18}, 9_1, 9_3, 9_6, 9_{35}, 9_{40}$ have region unknotting numbers two. The other prime knots with crossing number nine or less have the region unknotting number one.

Remark 4.6. For the above knots in Example 4.5, the region unknotting numbers are realized by the diagrams in Rolfsen's knot table. We note that the knots $7_1, 8_2, 8_7, 8_9, 8_{18}, 9_1, 9_3, 9_{35}, 9_{40}$ have only one minimal crossing diagrams, respectively up to horizontal mirror image and vertical mirror image, where the horizontal mirror image of a knot diagram D is obtained from D by reflecting D across a vertical plane, and the vertical mirror image of D is obtained from D by changing all the crossings of D.

The knot 9_6 has just two minimal crossing diagrams, whose region unknotting numbers are two (see Figure 12). We will discuss how to obtain all minimal crossing diagrams on S^2 of prime alternating knots in the appendix. We remark that there exist minimal

9_6:

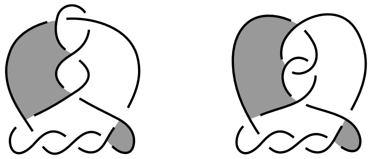

Figure 12.

crossing diagrams of prime alternating knots which do not realize the region unknotting numbers. For example, J. Banks suggested that the two minimal crossing diagrams D and E of 9_{26} in Figure 13 have $u_R(D) = 1$ and $u_R(E) = 2$.

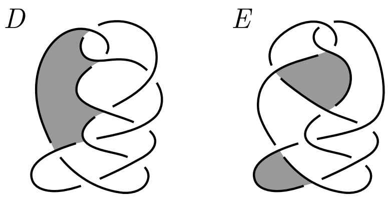

Figure 13.

For twist knots, we have the following proposition:

Proposition 4.7. *A twist knot K has $u_R(K) = 1$.*

Proof. From the minimal crossing diagram of K in Figure 14, we can obtain a diagram of the trivial knot by a region crossing change at the region P or Q. □

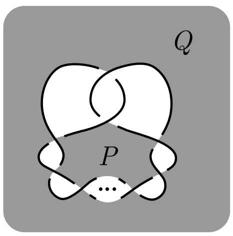

Figure 14.

For $(2, 2n+1)$-torus knots, we have the following proposition:

Proposition 4.8. *If a knot K is the $(2, 4m-1)$-torus knot or the $(2, 4m+1)$-torus knot ($m = 1, 2, \dots$), then $u_R(K) = m$.*

Proof. First of all, we remark that the $(2, 2n+1)$-torus knot has only one minimal crossing diagram on S^2 as shown in Figure 15 (see the appendix), and the region crossing change at P or Q in Figure 15 is of no use because it always transforms a diagram into just the vertical mirror image. We prove that the $(2, 4m - 1)$-torus knot has the region

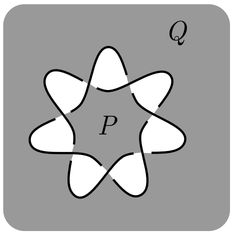

Figure 15.

unknotting number m by an induction. When $m = 1$, the trefoil knot has the region unknotting number one. We assume that in the case of $m = k$, the $(2, 4k - 1)$-torus knot K has the region unknotting number k. We shall prove for the case of $m = k + 1$, that the $(2, 4k + 3)$-torus knot K' has the region unknotting number $k + 1$.

- First, we prove that $u_R(K') \leq k + 1$. Let D be the minimal crossing diagram of K. We add two full twists to a pair of edges of D which bound a bigonal region as shown in Figure 16 so that we obtain the minimal crossing diagram D' of K'. Since we obtain D from D' by a region crossing change at any region in

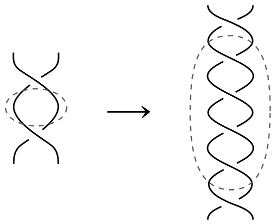

Figure 16.

the two full twists and Reidemeister moves of type II (see Figure 17), we have $u_R(K') \leq u_R(K) + 1 = k + 1$.

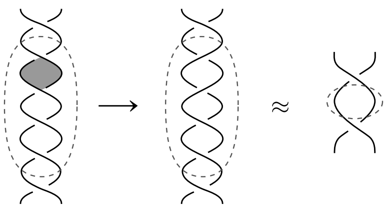

Figure 17.

- Next, we prove $u_R(K') \geq k + 1$ by an indirect proof. We assume that $u_R(K') \leq k$. Let D' be the minimal crossing diagram of K'. Let R' be a set of $u_R(K')$ regions of D' such that $D'(R')$ represents the trivial knot. Since the number of bigonal regions of D' is $4k+3$ and $u_R(K') = u_R(D') \leq k$, there exist connected four bigonal regions satisfying the following condition: One region ${R_1}'$ of them is in R', and the other three regions of them are not in R'. By applying a region crossing change at ${R_1}'$, we obtain from D' a diagram $D'({R_1}')$ which represents the knot K, and by applying Reidemeister moves of type II, we obtain from $D'({R_1}')$ a minimal crossing diagram D of K. By region crossing changes at the regions of D which corresponds to the regions $R' - {R_1}'$ of D', we obtain from D a diagram representing the trivial knot. Hence $u_R(K) \leq u_R(K') - 1 \leq k - 1$ which contradicts $u_R(K) = k$. Hence we have $u_R(K') \geq k+1$, and therefore $u_R(K') = k+1$. The $(2,5)$-torus knot has the region unknotting number one, and we can prove similarly for $(2, 4m+1)$-torus knots. □

From Proposition 4.8, we have the following corollary:

Corollary 4.9. *For an arbitrary non-negative integer n, there exists a knot K which satisfies $u_R(K) = n$.*

Appendix.

In this appendix, we explain how to obtain another minimal crossing diagram by a flyping from a minimal crossing diagram of a prime alternating knot. Then we show how to obtain all the minimal crossing diagrams of a prime alternating knot. In this appendix, a *tangle* is a portion of a knot diagram from which there emerge just four arcs pointing in the four compass directions NW, NE, SW, and SE. For a tangle T, we denote by T_h (resp. T_v) the result of rotation in a horizontal (resp. vertical) axis, and $-T$ that of crossing changes at all the crossing points of T as shown in Figure 18. We denote by

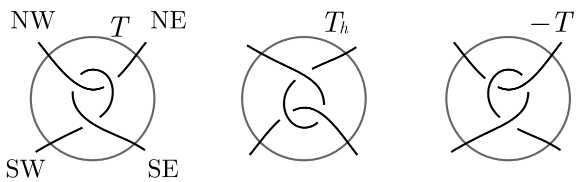

Figure 18.

1 (resp. 0) the tangle with one crossing (resp. no crossings) as shown in the left side (resp. right side) of Figure 19. For two tangles A and B, we define the *sum* $A + B$ *of* A

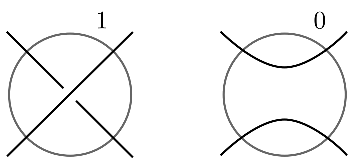

Figure 19.

and B to be the result of the operation of Figure 20. We also denote by $A - B$ the sum of A and $-B$. A tangle T is a *tangle sum* if $T = T_1 + T_2$, where neither T_1 nor T_2 is the tangle 0.

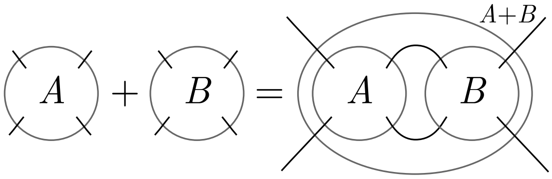

Figure 20.

Let D be a knot diagram which includes a tangle $1+T$ or $-1+T$. *Flyping* is a local transformation on D which replaces $1+T$ by T_h+1, or $-1+T$ by T_h-1 as shown in Figure 21. W. Menasco and M. Thistlethwaite showed that Tait's third conjecture is

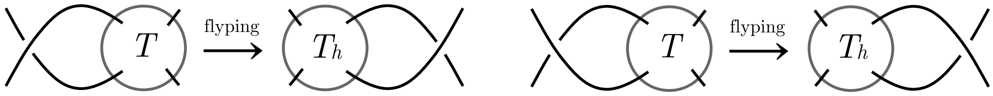

Figure 21.

true, that is, we can change D_1 into D_2 by performing a finite number of flypings for any two reduced alternating diagrams D_1 and D_2 of an alternating knot K [**4**]. Hence we can obtain all the minimal crossing diagrams of a prime alternating knot K from a minimal crossing diagram of K by flypings. Let D be a minimal crossing diagram of a non-trivial knot, and c a crossing point of D. Let T be a tangle in D whose NW arc and SW arc meet at c as shown in Figure 22. Since c can be considered as the tangle 1 or

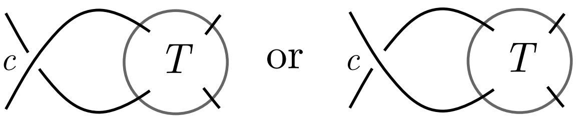

Figure 22.

-1, we can apply a flyping there. We call such a flyping *flyping at c and T*. A flyping on a knot diagram is *trivial* if we obtain D or the (vertical, horizontal, or vertical and horizontal) mirror image of D from D by the flyping. When we can apply a non-trivial flyping at a crossing point c and a tangle T of a diagram D, we say that D *admits* a non-trivial flyping at c (and T). Now we explain how to obtain all the tangles T such that D admits non-trivial flypings at a crossing point c and T for a knot diagram D.

From a knot diagram D on S^2 and a crossing point c of D, we obtain two tangles T_c^+ and T_c^- such that we obtain D from $1+T_c^+$ (resp. $-1+T_c^-$) by connecting the NW arc and the NE arc, and the SW arc and the SW arc (see Figure 23), where we remark that the tangle 1 (resp. -1) corresponds to c. We note that T_c^+ and T_c^- are unique by regarding that $\pm 1 + {T_c^\varepsilon}_{hv}$ is equivalent to $\pm 1 + T_c^\varepsilon$ on S^2 ($\varepsilon = +, -$). We remark that a flyping at c and T_c^+ or T_c^- is trivial because it comes out just horizontal mirror image of D. Hence if neither T_c^+ nor T_c^- is a tangle sum, the diagram D does not admit a non-trivial flyping at c. When T_c^ε is a tangle sum of tangles T_1 and T_2 ($\varepsilon = +, -$), we can apply a flyping at c and T_1. Remark that a flyping at c and T_1 is equivalent to a flyping at c and T_{2hv} up to horizontal mirror image (see Figure 24). Then we shall

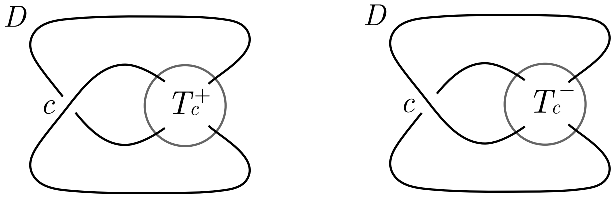

Figure 23.

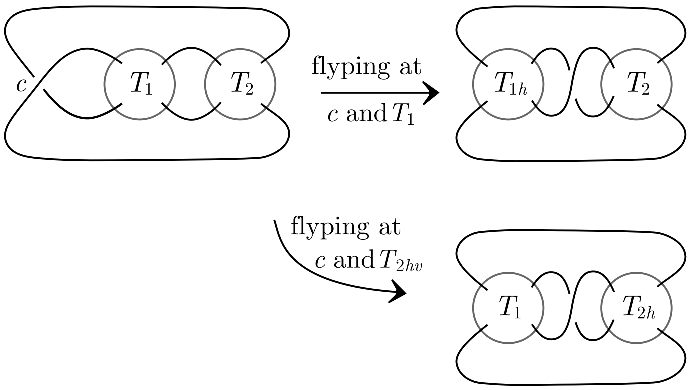

Figure 24.

consider flypings at c and only T_1. For $T_c^\varepsilon = T_1 + T_2$, the flyping at c and T_1 is trivial if T_1 or T_2 is the sum of some tangles $\varepsilon 1$ ($\varepsilon = +, -$) as shown in Figure 25. The flyping

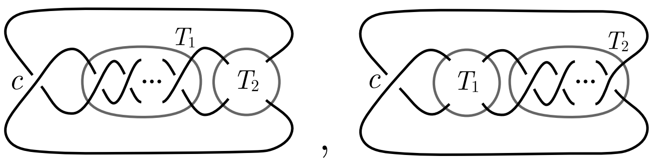

Figure 25.

at c and T_1 is also trivial if T_1 and T_2 satisfy $T_{1hv} = T_1$ and $T_{2v} = T_2$, or $T_{1v} = T_1$ and $T_{2hv} = T_2$ (see Figure 26). Then, to obtain all the minimal crossing diagrams on S^2 of a reduced alternating knot K, we will consider all the flypings at all the crossing points c and all the tangles T_1 such that $T_c^\varepsilon = T_1 + T_2$ ($\varepsilon = +, -$) of a minimal crossing diagram D of K except the following three cases:

(i): the tangle T_c^ε is not a tangle sum ($\varepsilon = +, -$),
(ii): the tangle T_1 or T_2 is the sum of some tangles $\varepsilon 1$ ($\varepsilon = +, -$),
(iii): the tangles T_1 and T_2 satisfy $T_{1hv} = T_1$ and $T_{2v} = T_2$, or $T_{1v} = T_1$ and $T_{2hv} = T_2$.

In the following examples, we will find all the minimal crossing diagrams of some knots by the above procedure.

Example 4.10. A $(2, 2n+1)$-torus knot has only one minimal crossing diagram D on S^2: Let D in Figure 27 be the minimal crossing diagram of a $(2, 2n+1)$-torus knot

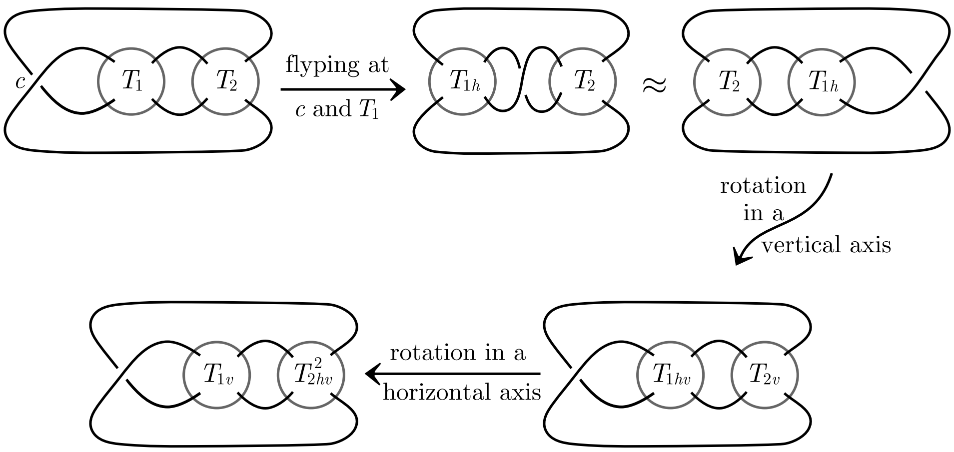

Figure 26.

$(n = 1, 2, \ldots)$. For every crossing point c, we obtain the same tangles T_c^+ and T_c^- as shown in Figure 27. The tangle T_c^+ is a sum of two tangles T_1 and T_2, where T_1 is k half

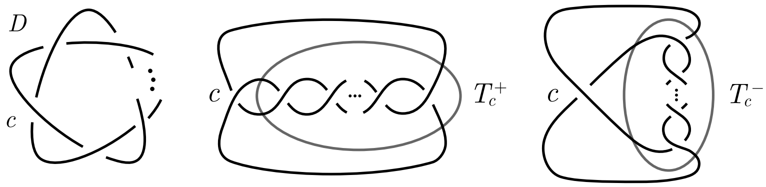

Figure 27.

twists and T_2 is $2n - k$ half twists $(k = 1, 2, \ldots, 2n - 1)$. The tangle T_c^- is not a tangle sum. Then T_c^+ and T_c^- satisfy the cases (ii) and (i), respectively. therefore, we can not apply non-trivial flypings on D.

Similarly, we have the following example:

Example 4.11. A knot K with Conway's notation mn or m, n has only one minimal crossing diagram on S^2 $(m, n \neq 0 \in \mathbb{Z}, mn > 0)$.

We show that the knot 8_2 has only one minimal crossing diagram on S^2:

Example 4.12. For the minimal crossing diagram D of the knot 8_2 in Figure 28, we call each crossing point $a, b, \ldots$ and h as shown in the figure. Neither T_a^-, T_b^-, T_c^+, T_d^-, T_e^-, T_f^-, T_g^- nor T_h^- is a tangle sum, T_a^+ and T_b^+ are the tangle sums which consist of a tangle and the tangle 1, and T_c^- is the tangle sum of the two tangles with vertical twists T_1 and T_2 satisfying $T_{1hv} = T_1$ and $T_{2v} = T_2$. The tangles T_d^+, T_e^+, T_f^+, T_g^+ and T_h^+ are always tangle sums $T_1 + T_2$ such that T_1 or T_2 is the sum of k tangles 1 $(k \geq 1)$ (see Figure 29). Hence D admits no non-trivial flypings, and therefore 8_2 has only one minimal crossing diagram D on S^2.

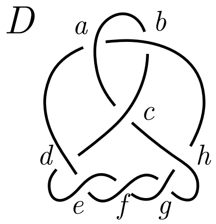

Figure 28.

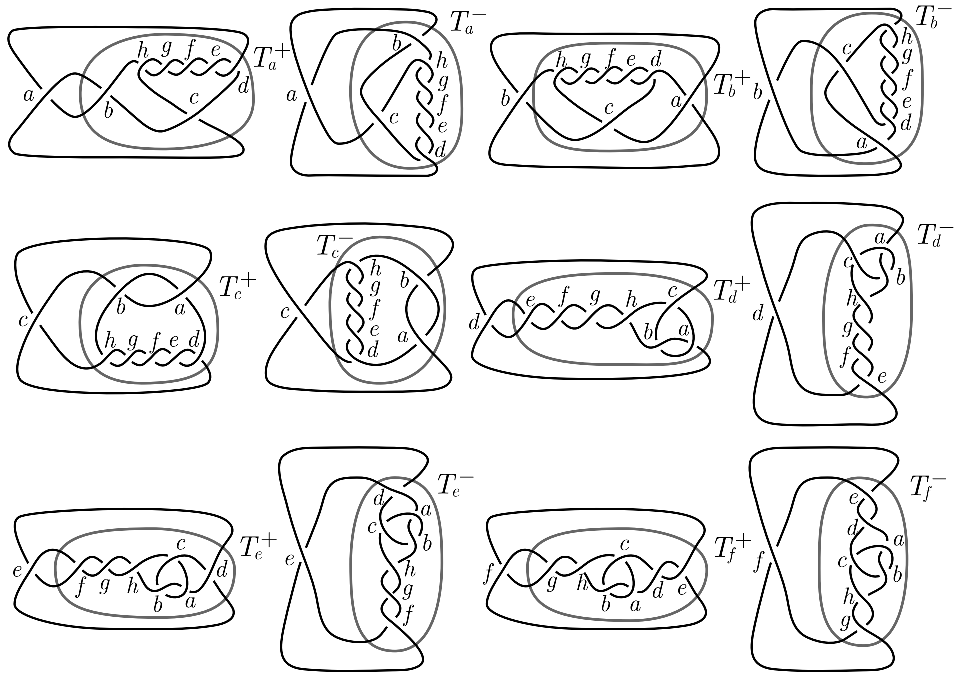

Figure 29.

We next show that the knot 9_6 has just two minimal crossing diagrams on S^2:

Example 4.13. For the minimal crossing diagram D of 9_6 in Figure 30, we call each crossing $a, b, \ldots$ and i as shown in the figure. Then D admits non-trivial flypings only at c and the tangle T_1^c and at d and T_1^d, and we obtain another diagram D' in Figure 31 by the flyping at c and T_1^c and at d and T_1^d, respectively, where we denote by $T_c^- = T_1^c + T_2^c$ and $T_d^- = T_1^d + T_2^d$ the tangles in Figure 30. The diagram D' with the crossing points $a', b', \ldots$ and i' in Figure 31 admits non-trivial flypings only at c' and ${T'}_{11}^{c'}$, c' and ${T'}_{12}^{c'}$, d' and ${T'}_{11}^{d'}$, and d' and ${T'}_{12}^{d'}$ as depicted in Figure 31, and we obtain the diagram D in Figure 30 by flyping there, respectively. Hence 9_6 has just two minimal crossing diagrams D and D' on S^2.

Acknowledgments. The author is grateful to Akio Kawauchi, Taizo Kanenobu, and the members of Friday Seminar on Knot Theory in Osaka City University who as-

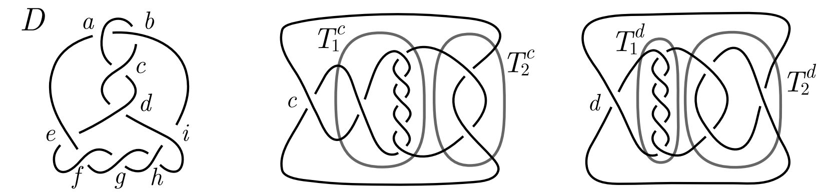

Figure 30.

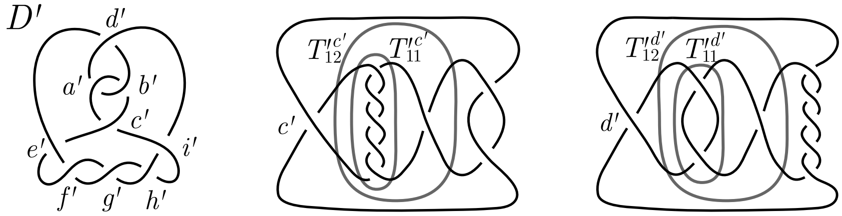

Figure 31.

sisted her in helpful advice, valuable discussions and the tender encouragement. She especially thanks Kengo Kishimoto for giving her such an interesting question and valuable advice and discussions. She also thanks Kenneth C. Millett and Makoto Ozawa for helpful comments, advice and information. She is deeply grateful to Jessica E. Banks for her valuable comments and many helpful suggestions. She is also grateful to the referee for careful reading and helpful advice.

References

[1] H. Aida, Unknotting operations of polygonal type, Tokyo J. Math., **15** (1992), 111–121.

[2] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, In: Computational Problems in Abstract Algebra, Oxford, 1967, (ed. J. Leech), Pergamon Press, New York, 1970, pp. 329–358.

[3] A. Kawauchi, A Survey of Knot Theory, Birkhäuser, Basel, 1996.

[4] W. W. Menasco and M. B. Thistlethwaite, The classification of alternating links, Ann. of Math., **138** (1993), 113–171.

[5] W. W. Menasco and M. B. Thistlethwaite, The Tait flyping conjecture, Bull. Amer. Math. Soc., **25** (1991), 403–412.

[6] H. Murakami, Some metrics on classical knots, Math. Ann., **270** (1985), 35–45.

[7] Y. Nakanishi, Replacements in the Conway third identity, Tokyo J. Math., **14** (1991), 197–203.

[8] D. Rolfsen, Knots and Links, Publish or Perish, Inc., Berkely, 1976.

Ayaka Shimizu

Department of Mathematics
Gunma National College of Technology
580 Toriba-cho, Maebashi-shi
Gunma 371-8530, Japan
E-mail: shimizu@nat.gunma-ct.ac.jp

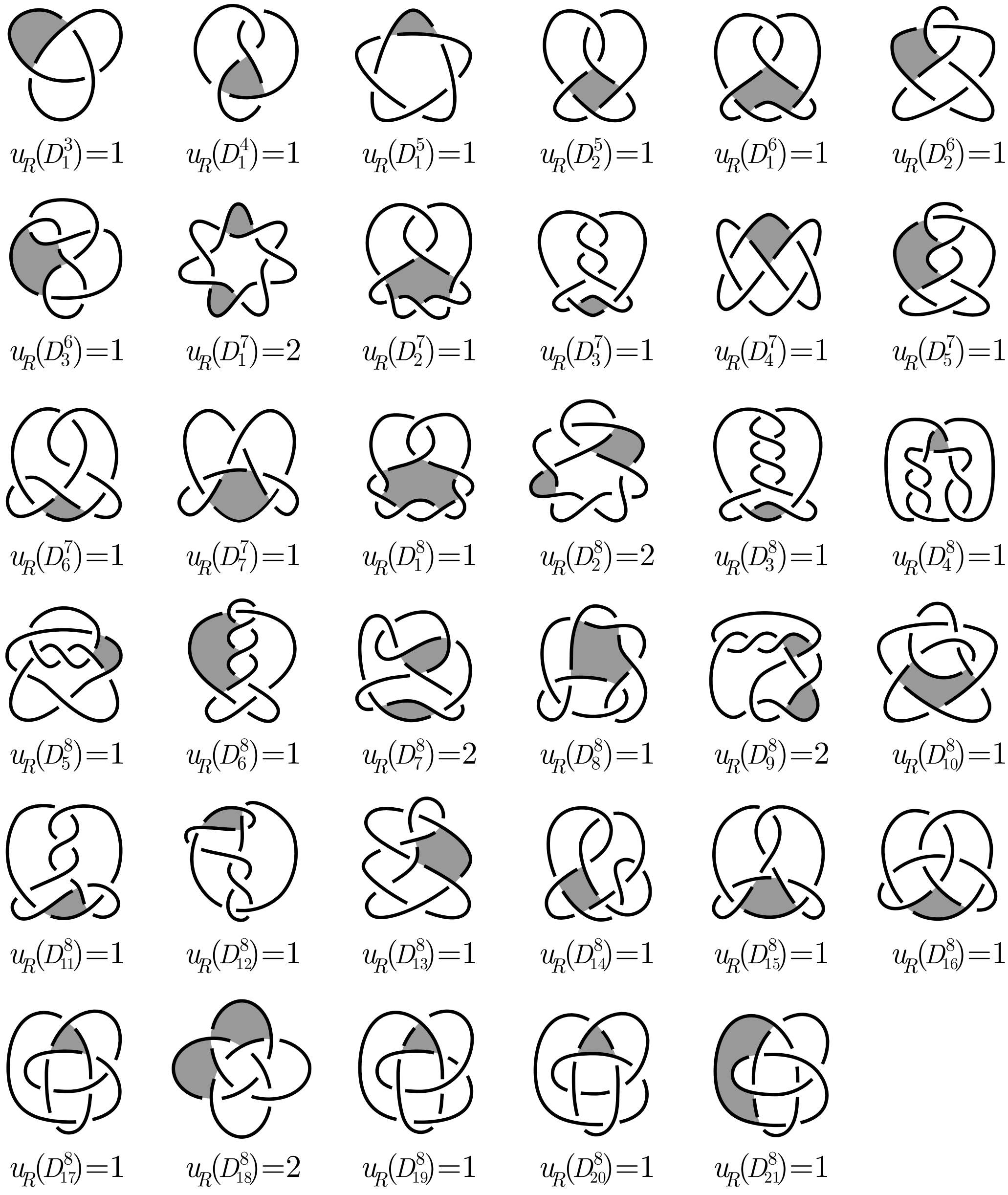
$u_R(D^3_1)=1$
$u_R(D^4_1)=1$
$u_R(D^5_1)=1$
$u_R(D^5_2)=1$
$u_R(D^6_1)=1$
$u_R(D^6_2)=1$
$u_R(D^6_3)=1$
$u_R(D^7_1)=2$
$u_R(D^7_2)=1$
$u_R(D^7_3)=1$
$u_R(D^7_4)=1$
$u_R(D^7_5)=1$
$u_R(D^7_6)=1$
$u_R(D^7_7)=1$
$u_R(D^8_1)=1$
$u_R(D^8_2)=2$
$u_R(D^8_3)=1$
$u_R(D^8_4)=1$
$u_R(D^8_5)=1$
$u_R(D^8_6)=1$
$u_R(D^8_7)=2$
$u_R(D^8_8)=1$
$u_R(D^8_9)=2$
$u_R(D^8_{10})=1$
$u_R(D^8_{11})=1$
$u_R(D^8_{12})=1$
$u_R(D^8_{13})=1$
$u_R(D^8_{14})=1$
$u_R(D^8_{15})=1$
$u_R(D^8_{16})=1$
$u_R(D^8_{17})=1$
$u_R(D^8_{18})=2$
$u_R(D^8_{19})=1$
$u_R(D^8_{20})=1$
$u_R(D^8_{21})=1$

Figure 32.