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Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers

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Abstract. In this article, we make use of geometry of sections of elliptic surfaces and elementary arithmetic on the Mordell-Weil group in order to study existence problem of dihedral covers with given reduced curves as the branch loci. As an application, we give some examples of Zariski pairs (B_1, B_2) for "conic-line arrangements" satisfying the following conditions:

(i) $\deg B_1 = \deg B_2 = 7$.
(ii) Irreducible components of B_i $(i = 1, 2)$ are lines and conics.
(iii) Singularities of B_i $(i = 1, 2)$ are nodes, tacnodes and ordinary triple points.

Introduction.

Let $\varphi : S \to \mathbb{P}^1$ be a relatively minimal elliptic surface over $\mathbb{P}^1$ with a distinguished section O. Let $\mathrm{MW}(S)$ be the set of sections of S. It is well-known that one can define a structure of an abelian group on $\mathrm{MW}(S)$ with identity element O and that $\mathrm{MW}(S)$ is called the Mordell-Weil group of $\varphi : S \to \mathbb{P}^1$. We denote the group law by $\dot{+}$ and the multiplication-by-m map $(m \in \mathbb{Z})$ on $\mathrm{MW}(S)$ by $[m]s$ for $s \in \mathrm{MW}(S)$. Also we identify a section with its image on S.

Take $s_1, \ldots, s_k \in \mathrm{MW}(S)$. Then $\sum_i [a_i]s_i$ gives another element of $\mathrm{MW}(S)$ and its image on S gives rise to a new curve on S. In this article, we consider p-divisibility (p: odd prime) of $\sum_i [a_i]s_i$ in $\mathrm{MW}(S)$ and a reduced divisor on S given by the union of $[a_i]s_i$ $(i = 1, \ldots, k)$ in order to study dihedral covers of the Hirzebruch surface Σ_d of degree d (d : even) or its blowing-ups $\widehat{\Sigma}_d$. As an application, we give examples of Zariski pairs of degree 7 for conic-line arrangements. This can be considered as a continuation of the author's previous articles ([**22**], [**23**], [**24**], [**25**]). Before we go on to explain our results in detail, let us first recall the definition of a Zariski pair.

Definition 1. A pair (B_1, B_2) of reduced plane curves B_i $(i = 1, 2)$ of degree n in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ (the base field of this article is always the field of complex numbers $\mathbb{C}$) is called a Zariski pair of degree n if it satisfies the following condition:

(i) B_i $(i = 1, 2)$ are curves of degree n such that the combinatorial type (see Definition 2 below) of B_1 is the same as that of B_2.
(ii) $(\mathbb{P}^2, B_1)$ is not homeomorphic to $(\mathbb{P}^2, B_2)$.

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Definition 2 ([**7**]). The *combinatorial type* of a curve B is given by a 7-tuple

$$(\mathrm{Irr}(B), \deg, \mathrm{Sing}(B), \Sigma_{\mathrm{top}}(B), \sigma_{\mathrm{top}}, \{B(P)\}_{P\in\mathrm{Sing}(B)}, \{\beta_P\}_{P\in\mathrm{Sing}(B)}),$$

where:

- $\mathrm{Irr}(B)$ is the set of irreducible components of B and $\deg : \mathrm{Irr}(B) \to \mathbb{Z}_{\geq 0}$ assigns to each irreducible component its degree.
- $\mathrm{Sing}(B)$ is the set of singular points of B, $\Sigma_{\mathrm{top}}(B)$ is the set of topological types of $\mathrm{Sing}(B)$, and $\sigma_{\mathrm{top}} : \mathrm{Sing}(B) \to \Sigma_{\mathrm{top}}(B)$ assigns to each singular point its topological type.
- $B(P)$ is the set of local branches of B at $P \in \mathrm{Sing}(B)$, and $\beta_P : B(P) \to \mathrm{Irr}(B)$ assigns to each local branch the global irreducible component containing it.

We say that two curves B_1 and B_2 have the *same combinatorial type* (or simply the *same combinatorics*) if their data of combinatorial types

$$(\mathrm{Irr}(B_i), \deg_i, \mathrm{Sing}(B_i), \Sigma_{\mathrm{top}}(B_i), \sigma_{\mathrm{top}_i}, \{\beta_{i,P}\}_{P\in\mathrm{Sing}(B_i)}, \{B_i(P)\}_{P\in\mathrm{Sing}(B_i)}), \quad i = 1, 2,$$

are equivalent, that is, if $\Sigma_{\mathrm{top}}(B_1) = \Sigma_{\mathrm{top}}(B_2)$, and there exist bijections $\varphi_{\mathrm{Sing}} : \mathrm{Sing}(B_1) \to \mathrm{Sing}(B_2)$, $\varphi_P : B_1(P) \to B_2(\varphi_{\mathrm{Sing}}(P))$ (restriction of a bijection of dual graphs) for each $P \in \mathrm{Sing}(B_1)$, and $\varphi_{\mathrm{Irr}} : \mathrm{Irr}(B_1) \to \mathrm{Irr}(B_2)$ such that $\deg_2 \circ \varphi_{\mathrm{Irr}} = \deg_1$, $\sigma_{\mathrm{top}_2} \circ \varphi_{\mathrm{Sing}} = \sigma_{\mathrm{top}_1}$, and $\beta_{2,\varphi_{\mathrm{Sing}}(P)} \circ \varphi_P = \varphi_{\mathrm{Irr}} \circ \beta_{1,P}$.

Note that when B_i ($i = 1, 2$) are irreducible, B_1 and B_2 have the same combinatorics if they have the same degree and the same local topological types for singularities. Also, for line arrangements, B_1 and B_2 have the same combinatorial type if they have the same set of incidence relations. The first example of a Zariski pair is given by Zariski ([**30**], [**31**]), which is as follows:

Example 3. Let (B_1, B_2) be a pair of irreducible sextics such that (i) both of B_1 and B_2 have six cusps as their singularities, and (ii) the six cusp of B_1 are on a conic, while no such conic for B_2 exists. Then (B_1, B_2) is a Zariski pair.

For these twenty years, Zariski pairs have been studied by many mathematicians and many examples have been found (see [**7**] and its reference). Among them, Zariski pairs for line arrangements of degrees 9 and 11 are considered by Artal Bartolo, Carmona Ruber, Cogolludo Agustin and Marco Buzunariz ([**5**], [**6**]), Rybnikov ([**19**]) and those for conic arrangements of degree 8 are considered by Namba and Tsuchihashi ([**15**]). In this article, we study Zariski pairs for conic-line arrangements.

Remark 4. Conic-line arrangements have been studied by M. Amram, M. Friedman, D. Garber, M. Teicher and A. M. Uludag. They put emphasis in studying properties of the fundamental group of the complements of conic-line arrangements ([**1**], [**2**], [**3**], [**9**]). No example of a Zariski pair, however, seems to be given.

As we explain in [**7**], the study of Zariski pairs, in general, consists of two parts:

(I) To give curves B_1 and B_2 having the same combinatorics, but some "*different property*," e.g., the location of singularities as in Example 3.
(II) To show $(\mathbb{P}^2, B_1)$ is not homeomorphic to $(\mathbb{P}^2, B_2)$.

One of our goals in this article is to consider a new method for (I). Namely we make use of elementary arithmetic and geometry of sections of the Mordell-Weil group of an elliptic surface. Let us explain how it will be done briefly.

We first recall that any elliptic surface $\varphi : S \to \mathbb{P}^1$ with section O is always obtained in the following way (see in Section 1, 1.2):

- Let Σ_d be the Hirzebruch surface of degree d (d: even).
- Let Δ_0 be the section with $\Delta_0^2 = -d$ and let T be a tri-section on Σ_d such that (i) T has at most simple singularities and (ii) $\Delta_0 \cap T = \emptyset$.
- Let $f' : S' \to \Sigma_d$ be a double cover with branch locus $\Delta_0 + T$.
- Let $\mu : S \to S'$ be the canonical resolution. By our assumption, μ is the minimal resolution and we have the following double cover diagram as in Section 1.2:

$$\begin{array}{ccc} S' & \xleftarrow{\ \mu\ } & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma_d & \xleftarrow[\ q\ ]{} & \widehat{\Sigma}_d, \end{array}$$

where morphisms q and f are those introduced in Section 1.2.

Under these circumstances, S is an elliptic surface over $\mathbb{P}^1$ such that

- the elliptic fibration $\varphi : S \to \mathbb{P}^1$ is induced by $\Sigma_d \to \mathbb{P}^1$ and
- φ has a section O which comes from Δ_0.

Let Δ_1 and Δ_2 be sections of Σ_d with $\Delta_i^2 = d$ and $\Delta_i \cap \Delta_0 = \emptyset$ $(i = 1, 2)$. Let $\overline{\Delta}_i$ $(i = 1, 2)$ be the proper transforms of Δ_i $(i = 1, 2)$ by q, respectively. We now suppose the following conditions are satisfied:

1. $f^*(\overline{\Delta}_i)$ consists of two sections $s^{\pm}_{\Delta_i}$ for each i.
2. $\widehat{\Sigma}_d$ can be blown down to $\mathbb{P}^2$, which we denote by $\overline{q} : \widehat{\Sigma}_d \to \mathbb{P}^2$.

Let $[2]s^+_{\Delta_i}$ be the duplication of $s^+_{\Delta_i}$ in $\mathrm{MW}(S)$ for $i = 1, 2$. In order to give two plane curves B_1 and B_2 with the same combinatorics, we make use of $\overline{q} \circ f(s^+_{\Delta_i})$, $\overline{q} \circ f([2]s^+_{\Delta_i})$ $(i = 1, 2)$, and $\overline{q}(\Delta(S/\widehat{\Sigma}_d))$, where $\Delta(S/\widehat{\Sigma}_d)$ is the branch locus of f. We apply this method to the case when $d = 2$ to construct examples of Zariski pairs for conic-line arrangements of degree 7 (see Proposition 4.4). The author hopes that this method adds a new viewpoint to the study of elliptic surfaces and their Mordell-Weil groups.

As for (II), we also make use of theory of dihedral covers and p-divisibility of sections of an elliptic surface as in our previous papers ([**23**], [**24**], [**25**]). Our main results of this article along this line are Theorems 3.2 and 3.3

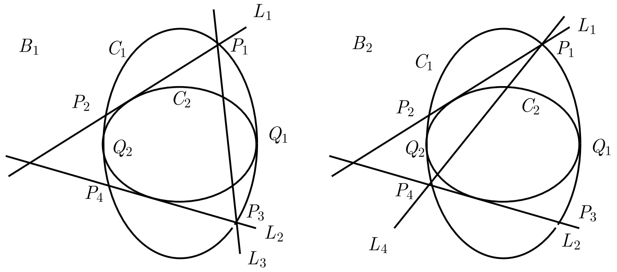
Now let us explain conic-line arrangements of degree 7 considered in this article.

Conic-line arrangement 1.

Let C_i $(i = 1, 2)$ be smooth conics and let L_j $(i = 1, 2, 3, 4)$ be lines as follows:

(i) Both L_1 and L_2 meet C_1 transversely. We put $C_1 \cap L_1 = \{P_1, P_2\}$, $C_1 \cap L_2 = \{P_3, P_4\}$.
(ii) C_2 is tangent to C_1 at two distinct points $\{Q_1, Q_2\}$ or at one point $\{Q\}$. We call the former type (a) and the latter type (b).
(iii) The tangent lines at $C_1 \cap C_2$ do not pass through $L_1 \cap L_2$.
(iv) C_2 is tangent to L_1 and L_2.
(v) L_3 passes through P_1 and P_3.
(vi) L_4 passes through P_1 and P_4.
(vii) Both L_3 and L_4 meet C_2 transversely.

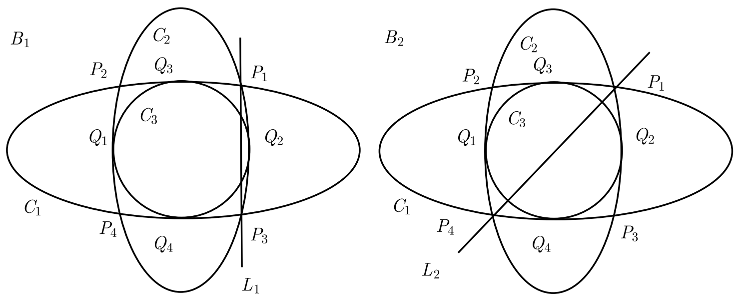
We put $B_1 := C_1 + C_2 + L_1 + L_2 + L_3$ and $B_2 := C_1 + C_2 + L_1 + L_2 + L_4$. Then B_1 and B_2 have the same combinatorics.



Conic-line arrangement 1 of type (a).

We now go on to explain Conic-line arrangement 2. It is obtained from Conic-line arrangement 1 by replacing two lines L_1 and L_2 by a smooth conic.

Conic-line arrangement 2.



Conic-line arrangement 2 of type (a).

Let C_1, C_2 and C_3 be smooth conics and L_1 and L_2 be lines as follows:

(i) C_1 and C_2 meet transversely. We put $C_1 \cap C_2 = \{P_1, P_2, P_3, P_4\}$.
(ii) C_3 is tangent to both C_1 and C_2 such that the intersection multiplicities at intersection points are all even. By exchanging C_1 and C_2 if necessary, we may assume that there are three possibilities:

(a) $C_3 \cap C_1 = \{Q_1, Q_2\}$, $C_3 \cap C_2 = \{Q_3, Q_4\}$,
(b) $C_3 \cap C_1 = \{Q_1\}$, $C_3 \cap C_2 = \{Q_2, Q_3\}$ or
(c) $C_3 \cap C_1 = \{Q_1\}$, $C_3 \cap C_2 = \{Q_2\}$.

(iii) No tangent line at Q_i is bitangent to $C_1 + C_2$.
(iv) L_1 passes through P_1 and P_3.
(v) L_2 passes through P_1 and P_4.
(vi) Both of L_1 and L_2 meet C_3 transversely.

We put $B_1 := C_1 + C_2 + C_3 + L_1$, $B_2 := C_1 + C_2 + C_3 + L_2$. Then B_1 and B_2 have the same combinatorics.

THEOREM 5. (i) *Let* (B_1, B_2) *be the pair of Conic-line arrangement* 1. *Then* (B_1, B_2) *is a Zariski pair.*

(ii) *Let* C_1 *and* C_2 *be conics intersecting at four distinct points,* P_1, P_2, P_3 *and* P_4 *and let* L_0, L_1 *and* L_2 *be lines through* $\{P_1, P_2\}, \{P_1, P_3\}$ *and* $\{P_1, P_4\}$*, respectively. Choose a point* z_o *on* C_1 *such that the tangent line at* z_o *to* C_1 *is not tangent to* C_2*. Then there exist just three conics* $C_3^{(0)}, C_3^{(1)}$ *and* $C_3^{(2)}$ *satisfying the following conditions*:

- $z_o \in C_3^{(i)}$ *for each* i.
- *Both* C_1 *and* C_2 *are tangent to* $C_3^{(i)}$ *for each* i *and the intersection multiplicities* $I_x(C_3^{(i)}, C_j)$ *are either* 2 *or* 4 *for* $\forall x \in C_3^{(i)} \cap C_j$ $(j = 1, 2)$.
- *For* $i, j = 0, 1, 2$ $(i \neq j)$*, if both of* $C_1 + C_2 + C_3^{(i)} + L_i$ *and* $C_1 + C_2 + C_3^{(i)} + L_j$ *have the combinatoric for Conic-line arrangement 2 of the same type, then* $(C_1 + C_2 + C_3^{(i)} + L_i, C_1 + C_2 + C_3^{(i)} + L_j)$ *is a Zariski pair.*

REMARK 6. The triple $(C_1 + C_2 + C_3^{(i)} + L_0, C_1 + C_2 + C_3^{(i)} + L_1, C_1 + C_2 + C_3^{(i)} + L_2)$ may be a candidate for a Zariski triple. Our method in this article, however, does not work to see whether it is or not.

This article consists of 5 sections. In Section 1 and Section 2, we summarize some facts and results for theory of elliptic surfaces and D_{2n}-covers, which we need to prove our theorem. We prove Theorem 3.2 in Section 3 and Theorem 3.3 in Section 4. In Section 5, we prove Theorem 5 and give another example of a Zariski pair by our method.

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1. Elliptic surfaces.

1.1. General facts.

We first summarize some facts from the theory of elliptic surfaces. As for details, we refer to [**11**], [**13**], [**14**], [**20**].

In this article, the term, an *elliptic surface*, always means a smooth projective surface S equipped with a structure of a fiber space $\varphi : S \to C$ over a smooth projective curve,

C, as follows:

(i) There exists a non-empty finite subset, $\mathrm{Sing}(\varphi)$, of C such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C \setminus \mathrm{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v \in \mathrm{Sing}(\varphi)$.
(ii) φ has a section $O : C \to S$ (we identify O with its image).
(iii) There is no exceptional curve of the first kind in any fiber.

Under these circumstances, we first recall the basic results on invariants of S.

Proposition 1.1. *Let $\varphi : S \to C$ be an elliptic surface as above. Then*

(i) $\chi(\mathcal{O}_S) > 0$,
(ii) $O \cdot O = -\chi(\mathcal{O}_S)$, *and*
(iii) $\dim H^1(S, \mathcal{O}_S) =$ *genus of C. In particular, the irregularity of S is 0 if $C = \mathbb{P}^1$.*

Proof. Since $\mathrm{Sing}(\varphi) \neq \emptyset$, $\chi(\mathcal{O}_S) > 0$ by [**11**, Theorem 12.2]. By [**12**, Proposition 2.3], we have (ii) and (iii). □

For $v \in \mathrm{Sing}(\varphi)$, we put $F_v = \varphi^{-1}(v)$. We denote its irreducible decomposition by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i} \Theta_{v,i},$$

where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0} \cdot O = 1$. We call $\Theta_{v,0}$ the *identity component*. The types of singular fibers are classified by [**11**]. There are two types for irreducible singular fibers. One is a rational curve with a node, and the other is a rational curve with a cusp. The former is called of type I_1, while the latter is called of type II. The following dual graphs and figures explain types of reducible singular fibers. Every vertex in dual graphs and every smooth irreducible component of Type III and IV are rational curve with self-intersection number -2.

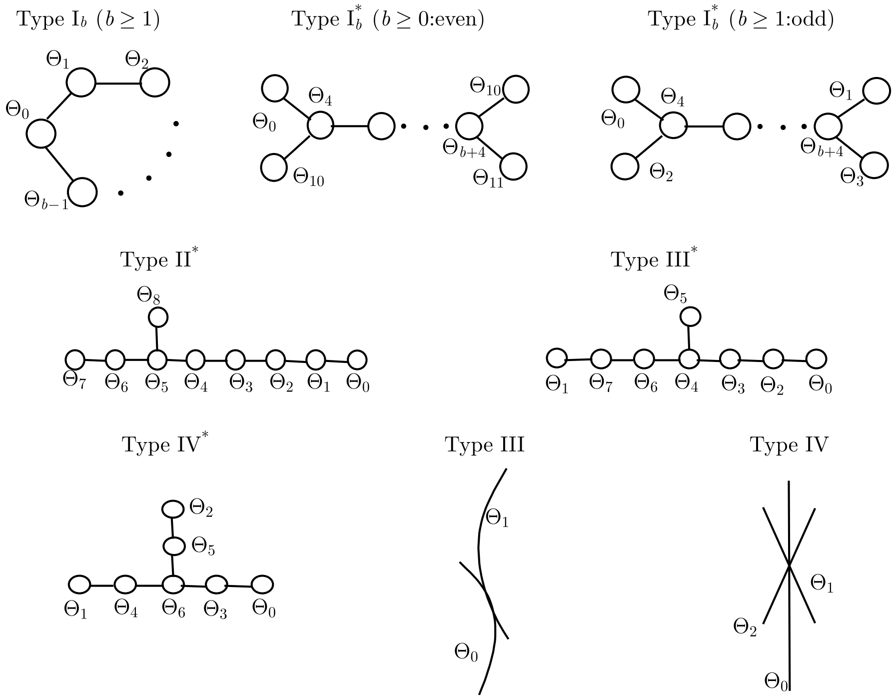
We also define a subset of $\mathrm{Sing}(\varphi)$ by $\mathrm{Red}(\varphi) := \{v \in \mathrm{Sing}(\varphi) \mid F_v \text{ is reducible}\}$. Let $\mathrm{MW}(S)$ be the set of sections of $\varphi : S \to C$. From our assumption, $\mathrm{MW}(S) \neq \emptyset$. By regarding O as the zero element of $\mathrm{MW}(S)$ and considering fiberwise addition (see [**11**, Section 9] or [**27**, Section 1] for the addition on singular fibers), $\mathrm{MW}(S)$ becomes an abelian group. We denote its addition by $\dot{+}$.

Also for $k \in \mathbb{Z}$ and $s \in \mathrm{MW}(S)$, we write

$$[k]s := \begin{cases} k\text{-times addition of } s \text{ if } k \geq 0 \\ k\text{-times addition of the inverse of } s \text{ if } k < 0. \end{cases}$$

The following two theorems are fundamental:

Theorem 1.2 ([**20**, Theorem 1.2]). *Let $\mathrm{NS}(S)$ be the Néron-Severi group of S. Under our assumption, $\mathrm{NS}(S)$ is torsion free.*



THEOREM 1.3 ([**20**, Theorem 1.3]). *Let T_φ be the subgroup of* $\mathrm{NS}(S)$ *generated by O, F and $\Theta_{v,i}$ ($v \in \mathrm{Red}(\varphi)$, $1 \leq i \leq m_v - 1$). Under our assumption, there is a natural map $\tilde{\psi} : \mathrm{NS}(S) \to \mathrm{MW}(S)$ which induces an isomorphisms of groups*

$$\psi : \mathrm{NS}(S)/T_\varphi \cong \mathrm{MW}(S).$$

In particular, $\mathrm{MW}(S)$ *is a finitely generated abelian group.*

In the following, $\operatorname{rank} \mathrm{MW}(S)$ means that of the free part of $\mathrm{MW}(S)$.

LEMMA 1.4 ([**20**, Lemma 5.1]). *Let D be a divisor on S and put $s(D) = \psi(D)$. Then D is uniquely written in the form*:

$$D \approx s(D) + (d-1)O + nF + \sum_{v \in \mathrm{Red}(\varphi)} \sum_{i=1}^{m_v - 1} b_{v,i} \Theta_{v,i},$$

where $\approx$ denotes the algebraic equivalence of divisors, and d, n and $b_{v,i}$ are integers defined as follows:

$$d = D \cdot F \qquad n = (d-1)\chi(\mathcal{O}_S) + O \cdot D - s(D) \cdot O,$$

and

$$\begin{pmatrix} b_{v,1} \\ \vdots \\ b_{v,m_v-1} \end{pmatrix} = A_v^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} - s_D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,m_v-1} - s_D \cdot \Theta_{v,m_v-1} \end{pmatrix}$$

Here A_v is the intersection matrix $(\Theta_{v,i} \cdot \Theta_{v,j})_{1 \le i,j \le m_v-1}$.

For a proof, see [**20**].

Put $\mathrm{NS}_{\mathbb{Q}} := \mathrm{NS}(S) \otimes \mathbb{Q}$ and $T_{\varphi,\mathbb{Q}} := T_\varphi \otimes \mathbb{Q}$. Since $\mathrm{NS}(S)$ is torsion free under our setting, there is no harm in considering $\mathrm{NS}_{\mathbb{Q}}$. By using the intersection pairing, we have the orthogonal decomposition $\mathrm{NS}_{\mathbb{Q}} = T_{\varphi,\mathbb{Q}} \oplus (T_{\varphi,\mathbb{Q}})^\perp$. In [**20**], the homomorphism $\phi : \mathrm{MW}(S) \to (T_{\varphi,\mathbb{Q}})^\perp \subset \mathrm{NS}_{\mathbb{Q}}$ is defined as follows:

$$\begin{aligned} \phi : \mathrm{MW}(S) \ni s \mapsto s - O - (s \cdot O + \chi(\mathcal{O}_S))F \\ + \sum_{v \in \mathrm{Red}(\varphi)} (\Theta_{v,1}, \dots, \Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix} s \cdot \Theta_{v,1} \\ \vdots \\ s \cdot \Theta_{v,m_v-1} \end{pmatrix} \in (T_{\varphi,\mathbb{Q}})^\perp. \end{aligned}$$

Also, in [**20**], a $\mathbb{Q}$-valued bilinear form $\langle\, ,\, \rangle$ on $\mathrm{MW}(S)$ is defined by $\langle s_1, s_2 \rangle := -\phi(s_1) \cdot \phi(s_2)$, where the right hand side means the intersection pairing in $\mathrm{NS}_{\mathbb{Q}}$. Here are two basic properties of $\langle\, ,\, \rangle$:

- $\langle s, s \rangle \ge 0$ for $\forall s \in \mathrm{MW}(S)$ and the equality holds if and only if s is an element of finite order in $\mathrm{MW}(S)$.
- An explicit formula for $\langle s_1, s_2 \rangle$ ($s_1, s_2 \in \mathrm{MW}(S)$) is given as follows:

$$\langle s_1, s_2 \rangle = \chi(\mathcal{O}_S) + s_1 \cdot O + s_2 \cdot O - s_1 \cdot s_2 - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{Contr}_v(s_1, s_2),$$

  where $\mathrm{Contr}_v(s_1, s_2)$ is given by

$$\mathrm{Contr}_v(s_1, s_2) = (s_1 \cdot \Theta_{v,1}, \dots, s_1 \cdot \Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix} s_2 \cdot \Theta_{v,1} \\ \vdots \\ s_2 \cdot \Theta_{v,m_v-1} \end{pmatrix}.$$

  As for explicit values of $\mathrm{Contr}_v(s_1, s_2)$, we refer to [**20**, (8.16)].

1.2. Double cover construction of an elliptic surface.

For details about this subsection, see [**13**, Lectures III and IV]. Let $\varphi : S \to C$ be an elliptic surface. By our assumption, the generic fiber of φ can be considered as an elliptic curve over $\mathbb{C}(C)$, the rational function field of C. The inverse morphism with respect to the group law induces an involution $[-1]_\varphi$ on S. Let $S/\langle[-1]_\varphi\rangle$ be the quotient by $[-1]_\varphi$. The quotient surface $S/\langle[-1]_\varphi\rangle$ is known to be smooth and $S/\langle[-1]_\varphi\rangle$ can be blown down to its relatively minimal model W over C satisfying the following condition:

Let us denote

- $f : S \to S/\langle[-1]_\varphi\rangle$: the quotient morphism,

- $q : S/\langle[-1]_\varphi\rangle \to W$: the blowing-down, and
- $S \overset{\mu}{\to} S' \overset{f'}{\to} W$: the Stein factorization of $q \circ f$.

Then we have:

1. The branch locus $\Delta_{f'}$ of f' consists of a section Δ_0 and the trisection T such that its singularities are at most simple singularities (see [**8**, Chapter II, Section 8] for simple singularities and their notation) and $\Delta_0 \cap T = \emptyset$.
2. $\Delta_0 + T$ is 2-divisible in $\mathrm{Pic}(W)$.
3. The morphism μ is obtained by contracting all the irreducible components of singular fibers not meeting O.

Conversely, if Δ_0 and T on W satisfy the above condition, we obtain an elliptic surface $\varphi : S \to \mathbb{P}^1$, as the canonical resolution of a double cover $f' : S' \to W$ with $\Delta_{f'} = \Delta_0 + T$, and the diagram (see [**10**] for the canonical resolution):

$$\begin{array}{ccc} S' & \xleftarrow{\ \mu\ } & S \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xleftarrow[\ q\ ]{} & \widehat{W}. \end{array} \tag{1.1}$$

Here q is a composition of blowing-ups so that $\widehat{W} = S/\langle[-1]_\varphi\rangle$. Hence any elliptic surface is obtained as above. In the following, we call the diagram above *the double cover diagram for S*.

In the case of $C = \mathbb{P}^1$, W is the Hirzebruch surface, Σ_d, of degree $d = 2\chi(\mathcal{O}_S) > 0$ and $\Delta_{f'}$ is of the form $\Delta_0 + T$, where Δ_0 is a section with $\Delta_0^2 = -d$ and $T \sim 3(\Delta_0 + d\mathfrak{f})$, $\mathfrak{f}$ being a fiber of the ruling $\Sigma_d \to \mathbb{P}^1$. Moreover, $\dim H^1(S, \mathcal{O}_S) = 0$ by Propositon 1.1.

REMARK 1.5. (i) For each $v \in \mathrm{Sing}(\varphi)$, the type of $\varphi^{-1}(v)$ is determined by the type of singularity of T on $\mathfrak{f}_v$ and the relative position between $\mathfrak{f}_v$ and T (see [**14**, Table 6.2]).

(ii) Note that the covering transformation, σ_f, of f coincides with $[-1]_\varphi$.

2. D_{2n}-covers.

In this section, we summarize some facts on Galois covers. We refer to [**21**] and [**7**, Section 3] for details.

We start with terminology on Galois covers. Let X and Y be normal projective varieties with finite morphism $\pi : X \to Y$. We say that X is a Galois cover of Y if the induced field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ by π^* is Galois, where $\mathbb{C}(\bullet)$ means the rational function field of $\bullet$. Note that the Galois group acts on X such that Y is obtained as the quotient space with respect to this action (*cf.* [**22**, Section 1]). If the Galois group $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$ is isomorphic to a finite group G, we call X a G-cover of Y. The branch locus of $\pi : X \to Y$, which we denote by Δ_π or $\Delta(X/Y)$, is the subset of Y consisting of points y of Y, over which π is not locally isomorphic. It is well-known that Δ_π is an algebraic subset of pure codimension 1 if Y is smooth ([**32**]).

Suppose that Y is smooth. Let B be a reduced divisor on Y whose irreducible decomposition $B = \sum_{i=1}^{r} B_i$. A G-cover $\pi : X \to Y$ is said to be branched at $\sum_{i=1}^{r} e_i B_i$ if (i) $\Delta_\pi = B$ (here we identify B with its support) and (ii) the ramification index along B_i is e_i for each i, where the ramification index means the one along the smooth part of B_i for each i. Note that the study of G-covers is related to that of the fundamental group of the complement of B, since we have the following proposition (see [**7**] for details):

Proposition 2.1 ([**7**, Proposition 3.6]). *Under the notation as above, let γ_i be a meridian around B_i, and $[\gamma_i]$ denote its class in the topological fundamental group $\pi_1(Y \setminus B, p_o)$. If there exists a G-cover $\pi : X \to Y$ branched at $e_1B_1 + \cdots + e_rB_r$, then there exists a normal subgroup H_π of $\pi_1(Y \setminus B, p_o)$ such that*:

(i) $[\gamma_i]^{e_i} \in H_\pi, [\gamma_i]^k \notin H_\pi, (1 \le k \le e_i - 1)$, *and*
(ii) $\pi_1(Y \setminus B, p_o)/H_\pi \cong G$.

Conversely, if there exists a normal subgroup H of $\pi_1(Y \setminus B, p_o)$ satisfying the above two conditions for H_π, then there exists a G-cover $\pi_H : X_H \to Y$ branched at $e_1B_1 + \cdots + e_rB_r$.

Let D_{2n} be the dihedral group of order $2n$. In order to present D_{2n}, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle.$$

By a D_{2n}-cover, we mean a Galois cover whose Galois group is isomorphic to D_{2n}. Given a D_{2n}-cover, we obtain a double cover, $D(X/Y)$, canonically by considering the $\mathbb{C}(X)^\tau$-normalization of Y, where $\mathbb{C}(X)^\tau$ denotes the fixed field of the subgroup generated by τ. The variety X is an n-fold cyclic cover of $D(X/Y)$ and we denote these covering morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively. Here are two propositions for later use.

Proposition 2.2. *Let n be an odd integer ≥ 3. Let Z be a smooth double cover of a smooth projective variety Y. We denote its covering morphism and covering transformation by f and σ_f, respectively. Let D be an effective divisor on Z satisfying the following conditions*:

(i) *D and σ_f^*D have no common component.*
(ii) *If $D = \sum_i a_i D_i$ denotes its irreducible decomposition, then* $\gcd(a_i, n) = 1$ *for every i.*
(iii) *$D - \sigma_f^*D$ is n-divisible in* $\mathrm{Pic}(Z)$.

Then there exists a D_{2n}-cover $\pi : X \to Y$ such that

(a) *$\beta_2(\pi)$ is branched at $n((D + \sigma_f^*D)_{\mathrm{red}})$, and*
(b) *$D(X/Y) = Z$ and $f = \beta_2(\pi)$.*

Proof. By [**21**, Proposition 0.4], our statements except the ramification indices are straightforward. As for the ramification indices, it follows from the last line of the proof of [**21**, Proposition 0.4]. □

PROPOSITION 2.3. *Let n be an odd integer ≥ 3. Let $\pi : X \to Y$ be a D_{2n}-cover such that both Y and $D(X/Y)$ are smooth. Let σ_{β_1} be the covering transform of $\beta_1(\pi)$. If $\beta_2(\pi)$ is branched at $n\mathcal{D}$ for some non-empty reduced divisor $\mathcal{D}$ on $D(X/Y)$, then there exists an effective divisor D, whose irreducible decomposition is $\sum_i a_i D_i$ satisfying the following conditions*:

(i) *D and $\sigma_{\beta_1}^* D$ have no common component.*
(ii) *$D - \sigma_{\beta_1}^* D$ is n-divisible in $\mathrm{Pic}(D(X/Y))$.*
(iii) *For every i, $\gcd(a_i, n) = 1$.*
(iv) *$\mathcal{D} = (D + \sigma_{\beta_1}^* D)_{\mathrm{red}}$.*

PROOF. The statement essentially follows from Proposition 0.5 and its proof in [**21**]. We, however, give another simple proof based on the idea of versal D_{2n}-covers (see [**26**], [**28**] for versal Galois covers). By [**28**], there exists an element $\xi \in \mathbb{C}(X)$ such that the action of D_{2n} on ξ is given in such a way that:

$$\begin{cases} \xi^\sigma = \dfrac{1}{\xi} \\ \xi^\tau = \zeta_n \xi, \quad \zeta_n = \exp\left(\dfrac{2\pi i}{n}\right). \end{cases}$$

By using ξ, we have $\mathbb{C}(D(X/Y)) = \mathbb{C}(Y)(\xi^n)$, $\mathbb{C}(X) = \mathbb{C}(Y)(\xi)$. Put $\theta = \xi^n \in \mathbb{C}(D(X/Y))$. Let $(\theta), (\theta)_0$ and $(\theta)_\infty$ be the divisor of θ, the zero and polar divisors of θ, respectively. Write $(\theta)_0$ in such a way that

$$(\theta)_0 = \sum_i a_i D_i + nD',$$

where D_i's are irreducible divisor on $D(X/Y)$ with $1 \leq a_i < n$ and D' is an effective divisor on $D(X/Y)$. Since σ induces σ_{β_1} on $D(X/Y)$ and $\theta^\sigma (= \theta^{\sigma_{\beta_1}}) = 1/\theta$, we have equalities of divisors:

$$\begin{aligned} (\theta)_\infty &= \sum_i a_i \sigma_{\beta_1}^* D_i + n\sigma_{\beta_1}^* D' \\ (\theta) &= (\varphi)_0 - (\varphi)_\infty \\ &= \sum_i a_i (D_i - \sigma_{\beta_1}^* D_i) + n(D' - \sigma_{\beta_1}^* D'). \end{aligned}$$

Now we put $D = \sum_i a_i D_i$. Since we may assume that $(\theta)_0$ and $(\theta)_\infty$ have no common components, our statements (i) and (ii) follow. Also as $\mathbb{C}(X) = \mathbb{C}(D(X/Y))(\sqrt[n]{\theta})$ and X is the $\mathbb{C}(X)$-normalization of $D(X/Y)$ and the ramification index along D_i is $n/\gcd(a_i, n)$, our statements (iii) and (iv) follow. □

COROLLARY 2.4. *Under the same assumption of Proposition 2.3, if D is an irreducible divisor on Y such that $(\beta_1(\pi))^{-1}(D) \subset \Delta_{\beta_2(\pi)}$, then $\beta_1(\pi)^* D$ consists of two* *irreducible components. In particular, in the case of* $\dim Y = 2$*, the intersection multiplicity at x,* $I_x(D, \Delta_{\beta_1(\pi)})$*, is even for* $\forall x \in D \cap \Delta_{\beta_1(\pi)}$.

Proof. The first statement is immediate from Proposition 2.3. For the second statement, let $\widetilde{D}$ be the normalization of D. If there exists $x \in D \cap \Delta_{\beta_1(\pi)}$ such that $I_x(D, \Delta_{\beta_1(\pi)})$ is odd, $\beta_1(\pi)$ induces a double cover of $\widetilde{D}$ with non-empty branch set. This means $\beta_1(\pi)^*D$ is irreducible. □

In [**25**], we introduce a notion of an elliptic D_{2n}-cover, whose definition is as follows:

Definition 2.5. A D_{2n}-cover $\pi : X \to Y$ is called an elliptic D_{2n}-cover if it satisfies the following condition:

- $D(X/Y)$ has a structure of an elliptic fiber space $\varphi : D(X/Y) \to S$ over a projective variety S with a section $O : S \to D(X/Y)$.
- On the generic fiber $D(X/Y)_\eta$, the group law is given by regarding O as the zero element. The involution on $D(X/Y)_\eta$ induced by the covering transformation $\sigma_{\beta_1(\pi)}$ coincides with the inversion with respect to the group law on $D(X/Y)_\eta$.

In this article, we consider elliptic D_{2n}-covers as follows:

(i) $D(X/Y)$ has an elliptic fibration $\varphi : D(X/Y) \to \mathbb{P}^1$.
(ii) $\beta_1(\pi) : D(X/Y) \to Y$ coincides with $f : D(X/Y) \to \widehat{\Sigma}_d$ in the double cover diagram for $\varphi : D(X/Y) \to \mathbb{P}^1$.

3. Elliptic D_{2p}-covers and p-divisibility of sections.

Let $\varphi : S \to \mathbb{P}^1$ be an elliptic surface over $\mathbb{P}^1$. Let $f : S \to \widehat{\Sigma}_d$ be the double cover appearing in the double cover diagram (1.1) for S.

We first note that, by its definition, any elliptic D_{2p}-cover (p: odd prime) $\pi_p : X_p \to \widehat{\Sigma}_d$ satisfies the following conditions:

- $S = D(X_p/\widehat{\Sigma}_d)$ and $\beta_1(\pi_p) = f$.
- The branch locus of $\beta_2(\pi_p)$ is of the form

$$\mathcal{D} + \sigma_f^*\mathcal{D} + \Xi + \sigma_f^*\Xi$$

where

1. all irreducible components of $\mathcal{D}$ are horizontal with respect to the elliptic fibration and there is no common component between $\mathcal{D}$ and $\sigma_f^*\mathcal{D}$, and
2. all irreducible component of Ξ are vertical and there is no common component between Ξ and $\sigma_f^*\Xi$.

Remark 3.1. (i) By Remark 1.5 (ii) and [**11**, Theorem 9.1], the action of σ_f on irreducible components of singular fibers is described as in the table below. We here use the labeling for irreducible components introduced in Section 1.1. Hence possible irreducible components of Ξ can be determined.

(ii) Under the above notation, the case when $\mathcal{D} = \emptyset$ (resp. = a section) is considered in the author's previous works ([**21**], [**22**], [**23**], [**24**]) (resp. [**25**]).

Type of a singular fiber	The action on irreducible component
I_n	$\Theta_0 \mapsto \Theta_0$ $\Theta_i \mapsto \Theta_{n-i} \quad i = 1, \dots, n-1$
I_n^* (n: even)	$\Theta_i \mapsto \Theta_i \quad \forall i$ $\Theta_{ij} \mapsto \Theta_{ij} \quad \forall i, j$
I_n^* (n: odd)	$\Theta_i \mapsto \Theta_i \quad i \neq 1, 3$ $\Theta_1 \mapsto \Theta_3 \quad \Theta_3 \mapsto \Theta_1$
II, II*, III, III*	$\Theta_i \mapsto \Theta_i \quad \forall i$
IV	$\Theta_0 \mapsto \Theta_0$ $\Theta_1 \mapsto \Theta_2 \quad \Theta_2 \mapsto \Theta_1$
IV*	$\Theta_i \mapsto \Theta_i \quad i = 0, 3, 6$ $\Theta_1 \mapsto \Theta_2 \quad \Theta_2 \mapsto \Theta_1$ $\Theta_4 \mapsto \Theta_5 \quad \Theta_5 \mapsto \Theta_4$

In the following, we always assume that

$$(*) \quad \mathcal{D} \neq \emptyset.$$

The proposition below, which is a generalization of [**25**, Propositions 4.1 and 4.2], plays an important role in this article:

THEOREM 3.2. *Let p be an odd prime. Let $C_1, \dots, C_r$ be irreducible horizontal divisors on S such that $\sum_{i=1}^r C_i$ and $\sum_{i=1}^r \sigma_f^* C_i$ have no common component. Then* (I) *and* (II) *in the below are equivalent*:

(I) *Put $\mathcal{C} = \sum_{i=1}^r C_i$. There exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ such that*
- *$D(X_p/\widehat{\Sigma}_d) = S$ and $\beta_1(\pi_p) = f$.*
- *$\beta_2(\pi_p)$ is branched at*

$$p\big((\mathcal{C} + \sigma_f^*\mathcal{C} + \Xi + \sigma_f^*\Xi)_{\mathrm{red}}\big)$$

for some effective divisor Ξ on S such that irreducible components of Ξ are all vertical and there is no common component between Ξ and $\sigma_f^*\Xi$.

(II) *Let $s(C_i) = \tilde{\psi}(C_i)$ $(i = 1, \dots, r)$. There exist integers a_i $(i = 1, \dots, r)$ such that*
- *$1 \leq a_i < p$ $(i = 1, \dots, r)$ and*
- *$\sum_{i=1}^r [a_i] s(C_i)$ is p-divisible in $\mathrm{MW}(S)$, i.e.,*

$$\sum_{i=1}^r [a_i] s(C_i) \in [p]\,\mathrm{MW}(S) := \{[p]s \mid s \in \mathrm{MW}(S)\}.$$

PROOF. (I) $\Rightarrow$ (II) Let D be the effective divisor in Proposition 2.3. We put $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$, where the irreducible components of D_{hor} are all horizontal, while those of D_{ver} are all in fibers of φ. By Proposition 2.3 (iv), $(D_{\mathrm{hor}} + \sigma_f^* D_{\mathrm{hor}})_{\mathrm{red}} = \sum_{i=1}^r C_i + \sum_{i=1}^r \sigma_f^* C_i$.

Claim. We may assume that D_{hor} is of the form $D_{\mathrm{hor}} = \sum_{i=1}^r a_i C_i$ $(0 \le a_i < p)$.

Proof of Claim. If $\sigma_f^* C_i$ is an irreducible component of D_{hor}, then we consider

$$D'_{\mathrm{hor}} := D_{\mathrm{hor}} + (p - a_i)C_i - a_i \sigma_f^* C_i,$$

and put $D' = D'_{\mathrm{hor}} + D_{\mathrm{ver}}$. Then we infer that D' also satisfies all four conditions in Proposition 2.3. After repeating this process finitely many times, we can choose D_{hor} as in Claim.

We first recall that the irregularity of S is 0 by Proposition 1.1, since we always assume that $\mathrm{Sing}(\varphi) \neq \emptyset$ and the base curve is $\mathbb{P}^1$. Hence linear equivalence coincides with algebraic equivalence on S. By Claim and Proposition 2.3 (iii), there exists $\mathcal{L} \in \mathrm{Pic}(S)$ such that

$$\sum_{i=1}^r a_i (C_i - \sigma_f^* C_i) + D_{\mathrm{ver}} - \sigma_f^* D_{\mathrm{ver}} \sim p\mathcal{L},$$

where $\sim$ means linear equivalence of divisors. This implies

$$\tilde{\psi}\left(\sum_{i=1}^r a_i (C_i - \sigma_f^* C_i)\right) = [p]\tilde{\psi}(\mathcal{L}) \quad \text{in } \mathrm{MW}(S).$$

As $\tilde{\psi}(\sigma_f^* C_i) = [-1]s(C_i)$, we have

$$\tilde{\psi}\left(\sum_{i=1}^r a_i (C_i - \sigma_f^* C_i)\right) = [2]([a_1]s(C_1) \dot{+} \cdots \dot{+} [a_r]s(C_r)).$$

Since p is an odd prime, we infer that $[a_1]s(C_1) \dot{+} \cdots \dot{+} [a_r]s(C_r) \in [p]\,\mathrm{MW}(S)$.

(II) $\Rightarrow$ (I) Our proof is similar to that of [**25**, Proposition 4.2]. By Lemma 1.4, we have

$$C_i \sim s(C_i) + (d_i - 1)O + n_i F + \sum_{v \in \mathrm{Red}(\varphi)} \sum_{j=1}^{m_v - 1} b_{v,j}^{(i)} \Theta_{v,i}.$$

This implies

$$\sum_{i=1}^r a_i C_i \sim \sum_{i=1}^r a_i s(C_i) + \sum_{i=1}^r a_i \left((d_i - 1)O + n_i F + \sum_{v \in \mathrm{Red}(\varphi)} \sum_{j=1}^{m_v - 1} b_{v,j}^{(i)} \Theta_{v,j}\right).$$

By our assumption, there exists s_o such that $\sum_{i=1}^r [a_i]s(C_i) = [p]s_o$ in $\mathrm{MW}(S)$. By Theorem 1.3, this implies that

$$\sum_{i=1}^r a_i s(C_i) \sim ps_o + \Big(-p + \sum_{i=1}^r a_i\Big)O + n_o F + \sum_{v\in\mathrm{Red}(\varphi)} \sum_{j=1}^{m_v-1} c_{v,j}\Theta_{v,j}$$

for some integers $n_o, c_{v,j}$. Hence we have

$$\begin{aligned}\sum_{i=1}^r a_i C_i \sim ps_o &+ \Big(-p + \sum_{i=1}^r a_i d_i\Big)O + \Big(n_o + \sum_i a_i n_i\Big)F \\ &+ \sum_{v\in\mathrm{Red}(\varphi)} \sum_{j=1}^{m_v-1} \Big(c_{v,j} + \sum_{i=1}^r a_i b_{v,j}^{(i)}\Big)\Theta_{v,j},\end{aligned}$$

and put

$$D' := \sum_{i=1}^r a_i C_i + \sum_{v\in\mathrm{Red}(\varphi)} \sum_{j=1}^{m_v-1} \Big(c_{v,j} + \sum_{i=1}^r a_i b_{v,j}^{(i)}\Big)\sigma_f^*\Theta_{v,j}.$$

Then we have

$$D' - \sigma_f^* D' \sim p(s_o - \sigma_f^* s_o).$$

The left hand side of the above equivalence contains some redundancy in the sum for $\Theta_{v,i}$ and $\sigma_f^*\Theta_{v,i}$. By taking the action of σ_f on $\Theta_{v,i}$'s (see Remark 1.5) into account, we can find divisors $D = \sum_{i=1}^r a_i C_i + \sum_j k_j \Xi_j$ and Ξ' on S such that

(i) all Ξ_j and all irreducible components of Ξ' are those in fibers not meeting O,
(ii) D and $\sigma_f^* D$ have no common component,
(iii) $1 \le k_j < p$, and
(iv) $D' - \sigma_f^* D' = D - \sigma_f^* D + p\Xi'$.

Now we easily infer that D satisfies the three conditions in Proposition 2.2 for p. □

THEOREM 3.3. *Let p be an odd prime such that $p \not| \sharp(\mathrm{MW}_{\mathrm{tor}})$. Choose two distinct sections $s_1, s_2 \in \mathrm{MW}(S)$ such that $s_i \notin [p]\,\mathrm{MW}(S)$ $(i = 1, 2)$. There exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ such that the horizontal part of $\Delta_{\beta_2(\pi_p)}$ is*

$$s_1 + s_2 + \sigma_f^*(s_1 + s_2)$$

if and only if the images $\overline{s}_i$ $(i = 1, 2)$ of s_i $(i = 1, 2)$ in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$ are linearly dependent over $\mathbb{Z}/p\mathbb{Z}$.

PROOF. Since $p \not| \sharp(\mathrm{MW}_{\mathrm{tor}})$, we have $\mathrm{MW}(S)/[p]\,\mathrm{MW}(S) \cong \mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus r}$. By our assumption, $\overline{s}_i \neq 0$ $(i = 1, 2)$. If $\overline{s}_1$ and $\overline{s}_2$ are linearly dependent, we have $\overline{s}_1 + c\overline{s}_2 = 0$ for some non-zero $c \in \mathbb{Z}/p\mathbb{Z}$. This means that there exists an integer a $(0 < a < p)$ such that $s_1 \dot{+} [a]s_2 \in [p]\,\mathrm{MW}(S)$. Hence the existence of $\pi_p : X_p \to \widehat{\Sigma}_d$ as above follows from Theorem 3.2. Conversely if $\pi_p : X_p \to \widehat{\Sigma}_d$ exists, then $s_1 \dot{+} [a]s_2 \in [p]\,\mathrm{MW}(S)$ for some integer a $(0 < a < p)$ by Thorem 3.2. This shows that $\overline{s}_1$ and $\overline{s}_2$ are linearly dependent. □

4. Applications.

Let $\varphi : S \to \mathbb{P}^1$ be an elliptic surface and we keep our notation for the double cover diagram for S in Section 1.2. We fix an isomorphism $\mathrm{MW}(S) \cong M_o \oplus \mathrm{MW}_{\mathrm{tor}}$, $M_o \cong \mathbb{Z}^{\oplus r}$, $r = \mathrm{rank}\,\mathrm{MW}(S)$. Let us start with the following proposition:

Proposition 4.1. *Choose $s \in M_o$ such that $M_o/\mathbb{Z}s$ is free. For any finite number of odd prime numbers $p_1, \ldots, p_l$, there exists a section $s_{p_1,\ldots,p_l}$ satisfying the following conditions*:

- (i) $\langle s_{p_1,\ldots,p_l}, s_{p_1,\ldots,p_l}\rangle = (p_1 \cdots p_l)^2\langle s, s\rangle$.
- (ii) *For any odd prime $p \notin \{p_1, \ldots, p_l\}$, there exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ such that*
    - $D(X_p/\widehat{\Sigma}_d) = S$, $\beta_1(\pi_p) = f$, *and*
    - $\beta_2(\pi_p)$ *is branched at* $p(s + s_{p_1,\ldots,p_l} + \sigma_f^*(s + s_{p_1,\ldots,p_l}) + \Xi_o)$, *where all irreducible components of Ξ_o are those of the singular fibers not meeting O.*
- (iii) *For $p \in \{p_1, \ldots, p_l\}$, there exists no elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ as in* (ii)
- (iv) $\{s_{p_1,\ldots,p_l}, [-1]s_{p_1,\ldots,p_l}\}$ *is unique up to torsion elements.*

Proof. Define $s_{p_1,\ldots,p_l} := [\Pi_{i=1}^r p_i]s$. By Theorem 3.2, our statements (i), (ii) and (iii) are immediate. Suppose that $s' \in \mathrm{MW}(S)$ satisfies the statements (i), (ii) and (iii). Put $s' = s'_o + t'_o$, $s'_o \in M_o$, $t'_o \in \mathrm{MW}_{\mathrm{tor}}$. Since $M_o/\mathbb{Z}s$ is free, we can choose a free basis of M_o such that $s_1 = s, \ldots, s_r$, $r = \mathrm{rank}\,\mathrm{MW}(S)$. By Theorem 3.2, for $p \notin \{p_1, \ldots, p_l\}$, there exists an integer a_1 $(1 \le a_1, a_2 < p)$ such that

$$[a_1]s \dot{+} [a_2]s'_o \equiv 0 \bmod pM_o.$$

Hence we infer that $s'_o = [b_1]s_1 \dot{+} p(\sum_{i=2}^r [b_i]s_i)$ for some integers $b_1, \ldots, b_r$. Since p is any odd prime $\notin \{p_1, \ldots, p_l\}$, we infer $b_i = 0$ $(2 \le i \le r)$. Thus

$$\langle s'_o,\, s'_o\rangle = b_1^2\langle s,\, s\rangle = (p_1 \cdots p_l)^2\langle s,\, s\rangle.$$

Since $\langle s, s\rangle \neq 0$ by the basic properties of $\langle\ ,\ \rangle$ (see Section 1), we have $b_1 = \pm p_1 \cdots p_l$. Hence s' is equal to $[\pm 1]s_{p_1,\ldots,p_l}$ up to torsion elements. □

The following theorem is essential to prove Theorem 5.

Theorem 4.2. *Choose $s_1, s_2 \in M_o$ so that s_1 and s_2 are a part of a basis of $\mathbb{Z}^{\oplus r}$, i.e., $M_o/\mathbb{Z}s_1 + \mathbb{Z}s_2$ is free of rank $r - 2$. Put $s_3 := [2]s_1$. For any odd prime p with $p \not| \sharp(\mathrm{MW}_{\mathrm{tor}})$, we have the following*:

- *There exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ such that the horizontal part of the branch locus of $\beta_2(\pi_p)$ is $s_1 + s_3 + \sigma_f^*(s_1 + s_3)$.*
- *There exists no elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_d$ such that the horizontal part of the branch locus of $\beta_2(\pi_p)$ is $s_2 + s_3 + \sigma_f^*(s_2 + s_3)$.*

Proof. We apply Theorem 3.3 to s_1, s_3 and s_2, s_3. □

By Proposition 2.1 and Theorem 4.2, we have:

Corollary 4.3. *Let T be the trisection on Σ_d appearing in the double cover diagram for S. Put $\Delta_i := q \circ f(s_i)$ $(i = 1, 2, 3)$. Then there exists a D_{2p}-cover of Σ_d branched at $2(\Delta_0 + T) + p(\Delta_1 + \Delta_3)$, while there exists no D_{2p}-cover of Σ_d branched at $2(\Delta_0 + T) + p(\Delta_2 + \Delta_3)$. In particular, there exists no homeomorphism $h : (\Sigma_d, \Delta_0 + \Delta_1 + \Delta_3 + T) \to (\Sigma_d, \Delta_0 + \Delta_2 + \Delta_3 + T)$ such that $f(\Delta_0) = \Delta_0$ and $f(T) = T$.*

Proof. Since every vertical component of $\Delta_{\beta_2(\pi_p)}$ is mapped to a singular point of T, our statement for D_{2p}-covers follows. The last statement follows from Proposition 2.1. □

We end this section by considering the case when S is a rational elliptic surface. In this case, as $\chi(\mathcal{O}_S) = 1$, the ruled surface in the the double cover diagram (1.1) for S is Σ_2. Hence we have the following diagram:

$$\begin{array}{ccc} S' & \xleftarrow{\ \mu\ } & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma_2 & \xleftarrow[\ q\ ]{} & \widehat{\Sigma}_2. \end{array}$$

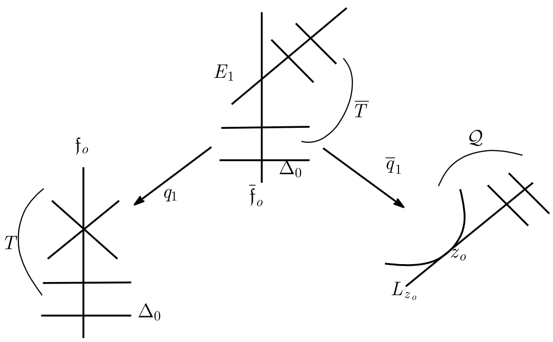
Write $q := q_1 \circ \cdots \circ q_r : \widehat{\Sigma}_2 = \Sigma_2^{(r)} \to \cdots \to \Sigma_2^{(1)} \to \Sigma_2^{(0)} = \Sigma_2$, where q_i is a blowing up at a point at $\Sigma_2^{(i-1)}$. Put $\Delta_{f'} = \Delta_0 + T$. In the following, we assume that

$$T \text{ has a node } x_o.$$

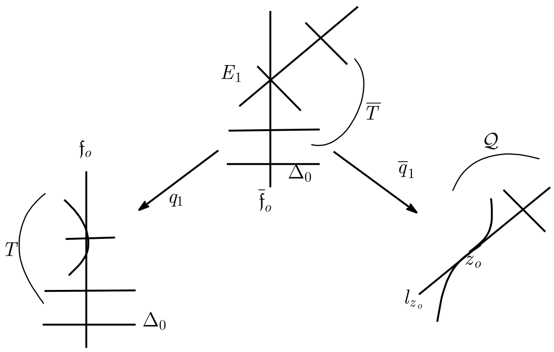
Note that this is equivalent to the fact that S has a singular fiber of type I_2 or III by [**14**, Table 6.2]. We may assume that q_1 is a blowing-up at x_o. Let E_1 be the exceptional divisor of q_1 and let $\bar{\mathfrak{f}}_o$ and $\overline{T}$ be the proper transforms of a fiber, $\mathfrak{f}_o$, through x_o and T, respectively. Then we have the following picture:

Note that if $\mathfrak{f}_o$ meets both of the local branches of T at x_o transversely, we have the case (a), while if $\mathfrak{f}_o$ is tangent to one of the local branches of T at x_o, we have the case (b).

Blow down $\bar{\mathfrak{f}}_o$ and Δ_0 in this order. Then the resulting surface is $\mathbb{P}^2$. We denote this composition of blowing downs by $\overline{q}_1 : \Sigma_2^{(1)} \to \mathbb{P}^2$ and put $\mathcal{Q} := \overline{q}_1(T)$. Then $\mathcal{Q}$ is a reduced quartic with the distinguished point $z_o := \overline{q}_1(\mathfrak{f}_o \cup \Delta_0)$. Note that $\overline{q}_1(E_1)$ is the tangent line L_{z_o} of $\mathcal{Q}$ at z_o. Put $\overline{q} := \overline{q}_1 \circ q_2 \circ \cdots \circ q_r$ and we have the following diagram:



The case (a).



The case (b).

$$\begin{array}{ccc} S'' & \xleftarrow{\overline{\mu}} & S \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^2 & \xleftarrow[\overline{q}]{} & \widehat{\Sigma}_2. \end{array}$$

Here $\overline{q} : S \to S''$ is the Stein factorization of $\overline{q} \circ f$. Note that S'' is a double cover with branch locus $\mathcal{Q}$ and that the pencil of lines through z_o gives rise to the elliptic fibration of S. Now we have the following proposition.

Proposition 4.4. *Let s_1, s_2 and s_3 be sections as in Corollary* 4.2 *and put $\mathcal{C}_i := \overline{q}(s_i)$ $(i = 1, 2, 3)$. There is no homeomorphism $h : (\mathbb{P}^2, \mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_3) \to (\mathbb{P}^2, \mathcal{Q} + \mathcal{C}_2 + \mathcal{C}_3)$ such that $h(\mathcal{Q}) = \mathcal{Q}$. In particular, if* (i) *$\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_3$ and $\mathcal{Q} + \mathcal{C}_2 + \mathcal{C}_3$ have the same combinatorics and* (ii) *the set of irreducible components of $\mathcal{Q}$ is invariant under the induced bijection $\varphi_{\mathrm{Irr}} : \mathrm{Irr}(\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_2) \to \mathrm{Irr}(\mathcal{Q} + \mathcal{C}_2 + \mathcal{C}_3)$ for any equivalence of the combinatorics between $\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_3$ and $\mathcal{Q} + \mathcal{C}_2 + \mathcal{C}_3$, then $(\mathcal{Q} + \mathcal{C}_1 + \mathcal{C}_3, \mathcal{Q} + \mathcal{C}_2 + \mathcal{C}_3)$ is a Zariski pair.*

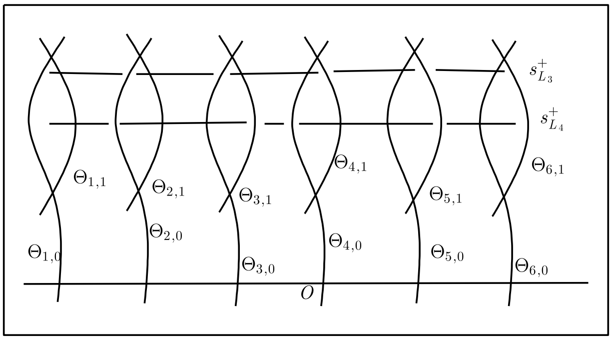
Proof. Our statement is immediate from Proposition 2.1 and Corollary 4.2 and the following lemma. □

LEMMA 4.5. *Let p be an odd prime. For $i = 1, 2$, there exists a D_{2p}-cover $\varpi_p : \mathcal{X}_p \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched at $2\mathcal{Q} + p(\mathcal{C}_i + \mathcal{C}_3)$ if and only if there exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_2$ of $\widehat{\Sigma}_2$ such that the horizontal part of $\Delta_{\beta_2(\pi_p)}$ is $s_i + s_3 + \sigma_f^*(s_i + s_3)$.*

PROOF. Suppose that there exists a D_{2p}-cover $\varpi_p : \mathcal{X}_p \to \mathbb{P}^2$ branched at $2\mathcal{Q} + p(\mathcal{C}_i + \mathcal{C}_3)$. Let $\varpi_p^{(i)} : \mathcal{X}_p^{(i)} \to \Sigma_2^{(i)}$ be the induced D_{2p}-cover, i.e., $\mathcal{X}_p^{(i)}$ is the $\mathbb{C}(\mathcal{X}_p)$-normalization of $\Sigma_2^{(i)}$. Since $D(\mathcal{X}_p/\mathbb{P}^2) = S''$ and $\beta_1(\varpi_p) = f''$, $D(\mathcal{X}_p^{(1)}/\Sigma_2^{(1)})$ is the $\mathbb{C}(S'')$-normalization of $\Sigma_2^{(1)}$. Hence $\Delta_{\beta_1(\varpi_p^{(1)})} = \Delta_0 + \overline{T}$ as $\overline{q}_1^*\mathcal{Q} = \Delta_0 + \overline{T} + 2\bar{\mathfrak{f}}_o$. This implies that $D(\mathcal{X}_p^{(r)}/\widehat{\Sigma}_2) = S$ and $\beta_1(\varpi_p^{(r)}) = f$. As $\mathcal{C}_i = \overline{q} \circ f(s_i) (i = 1, 2, 3)$, $\varpi_p^{(r)} : \mathcal{X}_p^{(r)} \to \widehat{\Sigma}_2$ is an elliptic D_{2p}-cover such that the horizontal part of $\Delta_{\beta_2(\varpi_p^{(r)})}$ is $s_i + s_3 + \sigma_f^*(s_i + s_3)$. Conversely, suppose that there exists an elliptic D_{2p}-cover $\pi_p : X_p \to \widehat{\Sigma}_2$ such that the horizontal part of $\Delta_{\beta_2(\pi_p)}$ is $s_i + s_3 + \sigma_f^*(s_i + s_3)$. Since E_1 gives rise to an irreducible component "Θ_1" of singular fiber of type I_2 or III, the preimage of E_1 in $\widehat{\Sigma}_2$ is not contained in the branch locus of π_p by Corollary 2.4 and Remark 1.5. Now let $\overline{X}_p$ be the Stein factorization of $\overline{q} \circ \pi_p$. Then the induced D_{2p}-cover $\overline{\pi}_p : \overline{X}_p \to \mathbb{P}^2$ is branched at $2\mathcal{Q} + p(\mathcal{C}_i + \mathcal{C}_3)$. □

5. Proof of Theorem 5.

PROOF OF THEOREM 5 (i). Put $\mathcal{Q} = C_1 + L_1 + L_2$ and choose a point $z_o \in C_1 \cap C_2$ as the distinguished point. Let $f''_{\mathcal{Q}} : S''_{\mathcal{Q}} \to \mathbb{P}^2$ be a double cover with branch locus $\mathcal{Q}$ and let $\varphi_{z_o} : S_{(\mathcal{Q}, z_o)} \to \mathbb{P}^1$ be the rational elliptic surface as in Section 4. By our construction of $S_{\mathcal{Q}, z_o}$, both L_3 and L_4 give rise to sections, which we denote by $s_{L_i}^+$ and $s_{L_i}^-$ $(= \sigma_f^* s_{L_i}^+ = [-1]s_{L_i})$ $(i = 3, 4)$, respectively. Reducible singular fibers of φ_{z_o} are of type I_2 or III depending on z_o. As the difference between I_2 and III does not affect computation below, we may assume that all reducible singular fibers are of type I_2. By labeling singular fibers suitably, we may assume that $s_{L_i}^+$ $(i = 3, 4)$ and reducible singular fibers meet as in the following picture:



Here we assume that $\Theta_{1,0}$ and O come from z_o. By the explicit formula of $\langle\, , \,\rangle$, we have

$$\langle s_{L_i}^{\pm}, s_{L_i}^{\pm} \rangle = \frac{1}{2}, \ (i = 3, 4) \quad \langle s_{L_3}^+, s_{L_4}^+ \rangle = 0.$$

By [**16**], $\mathrm{MW}(S_{(\mathcal{Q},z_o)}) \cong (A_1^*)^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and we may assume that

$$(A_1^*)^{\oplus 2} \cong \mathbb{Z}s_{L_3}^+ \oplus \mathbb{Z}s_{L_4}^+,$$

and that the 2-torsions sections arise from C_1, L_1 and L_2.

As for $(\overline{q} \circ f)^*(C_2)$, it also gives rise to two sections $s_{C_2}^{\pm}$. Since C_2 does not pass through any singularities of $\mathcal{Q}$ and $s_{C_2}^{\pm}O = 0$, we have $\langle s_{C_2}^{\pm}, s_{C_2}^{\pm} \rangle = 2$.

On the other hand, any element $s \in \mathrm{MW}(S_{(\mathcal{Q},z_o)})$ with $\langle s, s \rangle = 2$ is of the form

$$[2]s_{L_i}^{\pm} \dot{+} \tau, \quad (i = 3, 4) \quad \tau \in \mathrm{MW}(S_{(\mathcal{Q},z_o)})_{\mathrm{tor}}.$$

If $\tau \neq 0$, then $s_{C_2}^{\pm}$ meets $\Theta_{i,1}$ for some i by considering the addition on singular fibers (see [**11**, Theorem 9.1] or [**27**, Section 1]). Hence, by the explicit formula for $\langle\, ,\, \rangle$, we have $s_{C_2}^{\pm}O \neq 0$. On the other hand, $s_{C_2}^{\pm}O = 0$ by our construction. Thus we infer $\tau = 0$ and we may assume that $s_{C_2}^+ = [2]s_{L_3}^+$ after relabeling $\pm, L_3$ and L_4, if necessary. Therefore

$$s_{C_2}^+ \dot{+} [p-2]s_{L_3}^+ \in [p]\,\mathrm{MW}(S_{(\mathcal{Q},z_o)})$$

for any odd prime p, while

$$s_{C_2}^+ \dot{+} [k]s_{L_4}^+ \notin [p]\,\mathrm{MW}(S_{(\mathcal{Q},z_o)})$$

for any odd prime p and $1 \leq k \leq p-1$. As for any equivalence of the combinatorics between $\mathcal{Q} + C_2 + L_3$ and $\mathcal{Q} + C_2 + L_4$, $\{C_1, L_1, L_2\}$ is invariant under the induced bijection $\varphi_{\mathrm{Irr}} : \mathrm{Irr}(\mathcal{Q} + C_2 + L_3) \to \mathrm{Irr}(\mathcal{Q} + C_2 + L_4)$, by Proposition 4.4, we infer that $(\mathcal{Q} + C_2 + L_3, \mathcal{Q} + C_2 + L_4)$ is a Zariski pair. □

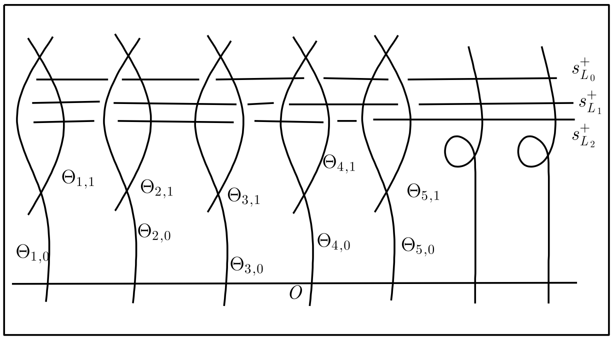
Proof for Theorem 5 (ii). Put $\mathcal{Q} = C_1 + C_2$ and choose a point $z_o \in C_1 \cap C_3$ as the distinguished point. Let $f''_{\mathcal{Q}} : S''_{\mathcal{Q}} \to \mathbb{P}^2$ be a double cover with branch locus $\mathcal{Q}$ and let $\varphi_{z_o} : S_{(\mathcal{Q},z_o)} \to \mathbb{P}^1$ be the rational elliptic surface as in Section 4. By our construction of $S_{\mathcal{Q},z_o}$, L_0, L_1 and L_2 give rise to sections, which we denote by $s_{L_i}^+$ and $s_{L_i}^-$ $(= \sigma_f^* s_{L_i}^+ = [-1]s_{L_i})$ $(i = 0, 1, 2)$, respectively. Likewise our proof for Theorem 5 (i), we may also assume that all reducible singular fibers are of type I_2. By labeling singular fibers suitably, we may assume that $s_{L_i}^+$ $(i = 0, 1, 2)$ and reducible singular fibers meet as in the following picture:

Here we assume that $\Theta_{1,0}$ and O come from z_o. By the explicit formula of $\langle\, ,\, \rangle$, we have

$$\langle s_{L_i}^{\pm}, s_{L_i}^{\pm} \rangle = \frac{1}{2}, \ (i = 0, 1, 2) \quad \langle s_{L_i}^+, s_{L_j}^+ \rangle = 0. \ (i, j = 0, 1, 2, i \neq j)$$

By [**16**], $\mathrm{MW}(S_{(\mathcal{Q},z_o)}) \cong (A_1^*)^{\oplus 3} \oplus (\mathbb{Z}/2\mathbb{Z})$ and we may assume that

$$(A_1^*)^{\oplus 3} \cong \mathbb{Z}s_{L_0}^+ \oplus \mathbb{Z}s_{L_1}^+ \oplus \mathbb{Z}s_{L_2}^+,$$



and that the unique 2-torsion section arises from C_1.

By [**11**, Theorem 9.1], $[2]s^{\pm}_{L_i}$ $(i = 0, 1, 2)$ meet the identity component at each singular fiber. Hence by the explicit formula for $\langle\,,\,\rangle$, we have $[2]s^{\pm}_{L_i}O = 0$ for each i. This implies that, for each i, $C_{L_i} := \overline{q} \circ f([2]s^{\pm}_{L_i})$ is a conic not passing through P_j $(j = 1, 2, 3, 4)$. If C_{L_i} and $\mathcal{Q}$ has an intersection point at which intersection multiplicity is odd, then we easily see that the closure of $(\overline{q} \circ f)^{-1}(C_{L_i} \setminus z_o)$ is irreducible. This is impossible, as C_{L_i} is the image of $[2]s^{\pm}_{L_i}$. Hence we have three conic satisfying the first two conditions.

Conversely, suppose that there exists a conic C_o satisfying the first two conditions. We infer that C_o gives rise to two sections $s^{\pm}_{C_o}$. Since C_o does not pass through any singularities of $\mathcal{Q}$ and $s^{\pm}_{C_o}O = 0$, we have $\langle s^{\pm}_{C_o}, s^{\pm}_{C_o}\rangle = 2$. On the other hand, any element $s \in \mathrm{MW}(S_{(\mathcal{Q},z_o)})$ with $\langle s, s\rangle = 2$ is of the form

$$[2]s^{\pm}_{L_i} \dot{+} \tau, \quad (i = 0, 1, 2) \quad \tau \in \mathrm{MW}(S_{(\mathcal{Q},z_o)})_{\mathrm{tor}}.$$

By a similar argument to that in the case of Conic-line arrangement 1, we infer that $\tau = 0$. Hence C_{L_i} $(i = 0, 1, 2)$ are only conics satisfying the first two conditions and no other such conics. Now we may assume that $C_3^{(i)} := C_{L_i}$ and $s^+_{C_3^{(i)}} := [2]s^+_{L_i}$ $(i = 0, 1, 2)$. For $i, j = 0, 1, 2$ $(i \neq j)$, we have

$$s^+_{C_3^{(i)}} \dot{+} [p-2]s^+_{L_i} \in [p]\,\mathrm{MW}(S_{(\mathcal{Q},z_o)})$$

for any odd prime p, while

$$s^+_{C_3^{(i)}} \dot{+} [k]s^+_{L_j} \notin [p]\,\mathrm{MW}(S_{(\mathcal{Q},z_o)})$$

for any odd prime p and $1 \leq k \leq p-1$.

Now suppose that both of $\mathcal{Q} + C_3^{(i)} + L_i$ and $\mathcal{Q} + C_3^{(i)} + L_j$ have the combinatorics for Conic-line arrangement 2 of the same type. Then, as for any equivalence of the combinatorics between $\mathcal{Q} + C_3^{(i)} + L_i$ and $\mathcal{Q} + C_3^{(i)} + L_j$ $(i, j = 0, 1, 2, i \neq j)$, $\{C_1, C_2\}$ is invariant under the induced bijection $\varphi_{\mathrm{Irr}} : \mathrm{Irr}(\mathcal{Q} + C_3^{(i)} + L_i) \to \mathrm{Irr}(\mathcal{Q} + C_3^{(i)} + L_j)$, $(\mathcal{Q} + C_3^{(i)} + L_i, \mathcal{Q} + C_3^{(i)} + L_j)$ $(i, j = 0, 1, 2, i \neq j)$ are Zariski pairs by Proposition 4.4.

□

REMARK 5.1. Let B be one of the conic-line arrangements as in Theorem 5. By Corollary 2.4, if there exists a D_{2p}-cover $\pi : X \to \mathbb{P}^2$ with branch locus B, then $\Delta_{\beta_1(\pi)} = L_1 + L_2 + C_1$ (resp. $C_1 + C_2$) for Conic-line arrangement 1 (resp. 2). This means that the D_{2p}-covers in our proof of Theorem 5 are the only possible ones. Therefore for the fundamental group $\pi_1(\mathbb{P}^2 \setminus B, *)$, we infer that

$$\pi_1(\mathbb{P}^2 \setminus (C_1 + C_2 + L_1 + L_2 + L_3), *) \not\cong \pi_1(\mathbb{P}^2 \setminus (C_1 + C_2 + L_1 + L_2 + L_4), *)$$

for Conic-line arrangement 1, and

$$\pi_1(\mathbb{P}^2 \setminus (C_1 + C_2 + C_3^{(i)} + L_i), *) \not\cong \pi_1(\mathbb{P}^2 \setminus (C_1 + C_2 + C_3^{(i)} + L_j), *) \quad (i \neq j)$$

for Conic-line arrangement 2. In particular, the complements are not homeomorphic for both of Conic-line arrangements 1 and 2.

EXAMPLE 5.2. Let $[T, X, Z]$ be homogeneous coordinates of $\mathbb{P}^2$ and let $(t, x) := (T/Z, X/Z)$ be affine coordinates for $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$ and consider a conic and four lines as follows:

$$C_1 : x - t^2 = 0, \qquad L_1 : x - 3t + 2 = 0, \quad L_2 : x + 3t + 2 = 0,$$
$$L_3 : x - t - 2 = 0, \quad L_4 : x - 1 = 0.$$

Note that $C_1 \cap (L_1 \cup L_2) = \{[\pm 1, 1, 1], [\pm 2, 4, 1]\}$. Put $\mathcal{Q} = C_1 + L_1 + L_2$ and choose $[0, 1, 0]$ as the distinguished point z_o. Let $S_{(\mathcal{Q}, z_o)}$ be the rational elliptic surface obtained as in Section 4. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation:

$$y^2 = (x - t^2)(x - 3t + 2)(x + 3t + 2).$$

Under this setting, we may assume that the sections $s_{L_i}^{\pm}\,(i = 3, 4)$ are as follows:

$$s_{L_3}^{\pm} = \left(t + 2, \pm 2\sqrt{2}(t - 2)(t + 1)\right), \quad s_{L_4}^{\pm} = (1, \pm 3(t + 1)(t - 1)).$$

Hence we have

$$[2]s_{L_3}^{+} = \left(\frac{9}{8}t^2, \frac{1}{32}\sqrt{2}t(9t^2 - 16)\right), \quad [2]s_{L_4}^{+} = \left(t^2 + \frac{1}{4}, \frac{1}{2}t^2 - \frac{9}{8}\right).$$

Now put

$$C_2 : x - \frac{9}{8}t^2 = 0, \quad C_2' : x - t^2 - \frac{1}{4} = 0.$$

Then $(\mathcal{Q} + C_2 + L_3, \mathcal{Q} + C_2 + L_4)$ is a Zariski pair for Conic-line arrangement 1 of type (a), and $(\mathcal{Q} + C_2' + L_3, \mathcal{Q} + C_2' + L_4)$ is a Zariski pair for Conic-line arrangement 1 of type (b).

EXAMPLE 5.3. We keep the same coordinates as Example 5.2.

Conic-line arrangement 2 *of type* (a). Consider two conics and two lines:

$$C_1 : x - t^2 + 2 = 0, \quad C_2 : x^2 - 2x + t^2 - 4 = 0,$$
$$L_1 : x - t = 0, \qquad L_2 : x - 3t + 4 = 0.$$

Note that $C_1 \cap C_2 = \{[\pm 2, 2, 1], [\pm 1, -1, 1]\}$. Put $\mathcal{Q} = C_1 + C_2$ and choose $[0, 1, 0]$ as the distinguished point z_o. Let $S_{(\mathcal{Q}, z_o)}$ be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation:

$$y^2 = (x - t^2 + 2)(x^2 - 2x + t^2 - 4).$$

Then we may assume that the sections $s^{\pm}_{L_2} (i = 1, 2)$ are as follows:

$$s^{\pm}_{L_1} = \big(t, \pm\sqrt{-2}(t + 1)(t - 2)\big), \quad s^{\pm}_{L_2} = \big(3t - 4, \pm\sqrt{-10}(t - 1)(t - 2)\big).$$

Thus we have

$$[2]s^{+}_{L_1} = \left(\frac{1}{2}t^2 - 2, -\frac{1}{4}\sqrt{-2}t(t^2 - 4)\right), \quad [2]s^{+}_{L_2} = \left(\frac{1}{10}t^2 - 2, -\frac{3}{100}\sqrt{-10}t(t^2 + 20)\right).$$

Now we put

$$C_3 : x - \frac{1}{2}t^2 + 2 = 0, \quad C_3' : x - \frac{1}{10}t^2 + 2 = 0.$$

Then both $(\mathcal{Q} + C_3 + L_1, \mathcal{Q} + C_3 + L_2)$ and $(\mathcal{Q} + C_3' + L_1, \mathcal{Q} + C_3' + L_2)$ are Zariski pairs for Conic-line arrangement 2 of type (a).

Conic-line arrangement 2 *of type* (b). Consider two conics and two lines:

$$C_1 : x - t^2 + 2 = 0, \quad C_2 : x^2 - 2x + t^2 - 4 = 0,$$
$$L_1 : x - t = 0, \qquad L_2 : x + 1 = 0.$$

Put $\mathcal{Q} = C_1 + C_2$ and choose $[0, 1, 0]$ as the distinguished point z_o. Let $S_{(\mathcal{Q}, z_o)}$ be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation:

$$y^2 = (x - t^2 + 2)(x^2 - 2x + t^2 - 4).$$

Then we may assume that the sections $s^{\pm}_{L_2}$ $(i = 1, 2)$ is as follows:

$$s^{\pm}_{L_2} = \big(-1, \pm\sqrt{-1}(t - 1)(t + 1)\big).$$

Thus we have

$$[2]s_{L_2}^{+} = \left(t^2 - \frac{17}{4}, \frac{3}{8}\sqrt{-1}(4t^2 - 19)\right).$$

Now we put

$$C_3 : x - t^2 + \frac{17}{4} = 0.$$

As C_3 is tangent to C_1 (resp. C_2) at one point (resp. two distinct points), we infer that $(\mathcal{Q} + C_3 + L_1, \mathcal{Q} + C_3 + L_2)$ is a Zariski pair for Conic-line arrangement 2 of type (b).

Conic-line arrangement 2 *of type* (c)*.* Consider two conics and two lines:

$$C_1 : x - t^2 + \frac{1}{2} = 0, \quad C_2 : x^2 - x + t^2 = 0,$$

$$L_1 : x = \frac{1}{\sqrt{2}}, \qquad L_2 : \frac{\sqrt{2}}{4}\big(\sqrt{-1}c_1 - c_2\big)x + t - \frac{1}{4}\big(\sqrt{-1}c_1 + c_2\big) = 0,$$

where $c_1 = \sqrt{2 + 2\sqrt{2}}$, $c_2 = \sqrt{-2 + 2\sqrt{2}}$. Note that

$$C_1 \cap C_2 = \left\{\left[\pm\sqrt{-1/2 + 1/\sqrt{2}}, 1/\sqrt{2}, 1\right], \left[\pm\sqrt{-1/2 - 1/\sqrt{2}}, -1/\sqrt{2}, 1\right]\right\}.$$

Put $\mathcal{Q} = C_1 + C_2$ and choose $[0, 1, 0]$ as the distinguished point z_o. Let $S_{(\mathcal{Q}, z_o)}$ be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation:

$$y^2 = \left(x - t^2 - \frac{1}{2}\right)(x^2 - x + t^2).$$

Then we may assume that the sections $s_{L_1}^{\pm}$ are as follows:

$$s_{L_1}^{\pm} = \left(\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{-1}}{2}\big(-2t^2 - 1 + \sqrt{2}\big)\right).$$

Thus we have

$$[2]s_{L_1}^{+} = \left(t^2, \sqrt{-\frac{1}{2}}t^2\right).$$

Now we put

$$C_3 : x - t^2 = 0$$

Then $(\mathcal{Q}+C_3+L_1, \mathcal{Q}+C_3+L_2)$ is a Zariski pair for Conic-line arrangement 2 of type (c).

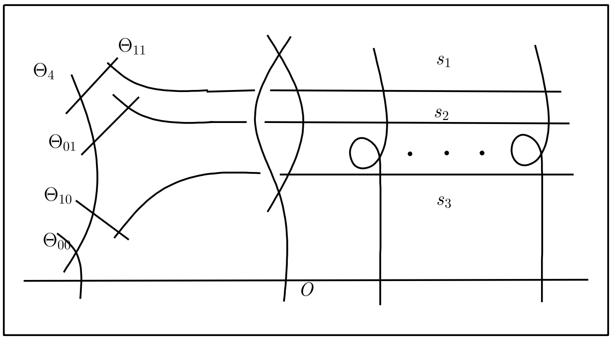
REMARK 5.4. Note that we have examples with real equations in Examples 5.2 and 5.3 except the case of Conic-line arrangement 2 of type (c).

We end this section by giving another example of a Zariski pair whose irreducible components are all rational curves:

PROPOSITION 5.5. *Let $\mathcal{Q}$ be an irreducible quartic with a $\mathbb{D}_4$ singularity, P. Let z_o be a point on $\mathcal{Q}$ such that the tangent line L_{z_o} at z_o meets $\mathcal{Q}$ with two other distinct points. Let L_1, L_2 and L_3 be the three tangent lines which meet $\mathcal{Q}$ at P with multiplicity* 4 *(i.e., the tangent lines to the smooth branches). Then there exist three conics C_i $(i = 1, 2, 3)$ satisfying the following properties*:

- (i) *(a) $z_o \in C_i$, (b) $P \notin C_i$ and (c) for $\forall x \in C_i \cap \mathcal{Q}$, $I_x(C_i, \mathcal{Q})$ is even.*
- (ii) *For any odd prime p, there exists a D_{2p}-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q}+p(C_i+L_i)$ for each $i = 1, 2, 3$, while there exists no D_{2p}-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q}+p(C_i+L_j)$ for any i, j $(i \neq j)$.*

PROOF. (i) Let $f''_{\mathcal{Q}} \to \mathbb{P}^2$ be a double cover with branch locus $\mathcal{Q}$ and let φ_{z_o} : $S_{(\mathcal{Q},z_o)} \to \mathbb{P}^1$ be the rational elliptic surface obtained as in Section 4. By our assumption on $\mathcal{Q}$ and z_o, the configuration of reducible singular fiber of φ_{z_o} is $\mathrm{I}_0^*, \mathrm{I}_2$ and three lines L_i $(i = 1, 2, 3)$ give rise to sections $s^{\pm}_{L_i}$ $(i = 1, 2, 3)$, respectively. By labeling irreducible components of singular fibers suitably, we have the following picture for $s^{+}_{L_i}$ $(i = 1, 2, 3)$:



By the explicit formula for $\langle\, ,\, \rangle$, we have

$$\langle s^{+}_{L_i}, s^{+}_{L_i} \rangle = \frac{1}{2}\ (i = 1, 2, 3), \quad \langle s^{+}_{L_i}, s^{+}_{L_j} \rangle = 0\ (i \neq j).$$

By [**16**], we have $\mathrm{MW}(S_{(\mathcal{Q},z_o)}) \cong (A_1^*)^{\oplus 3}$. Hence we may assume that

$$\mathrm{MW}(S_{\mathcal{Q},z_o}) \cong \mathbb{Z}s^{+}_{L_1} \oplus \mathbb{Z}s^{+}_{L_2} \oplus \mathbb{Z}s^{+}_{L_3}.$$

By the lattice structure of $\mathrm{MW}(S_{\mathcal{Q},z_o})$, all elements $s \in \mathrm{MW}(S_{\mathcal{Q},z_o})$ with $\langle s, s \rangle = 2$

given by $[2]s^{\pm}_{L_i}$ $(i = 1, 2, 3)$. By [**11**, Theorem 9.1], $[2]s^{\pm}_{L_i}$ $(i = 1, 2, 3)$ meet the identity component at each singular fiber. Hence, $[2]s^{\pm}_{L_i}O = 0$ $(i = 1, 2, 3)$ by the explicit formula for $\langle\, ,\, \rangle$. By our construction of $S_{(\mathcal{Q},z_o)}$, $\Delta_i := q \circ f([2]s^{\pm}_{L_i}) \sim \Delta_0 + 2\mathfrak{f}$ $(i = 1, 2, 3)$. Hence $C_i := \overline{q} \circ f([2]s^{\pm}_{L_i})$ $(i = 1, 2, 3)$ are all conic and $z_o \in C_i, P \notin C_i$. Moreover as $[2]s^{+}_{L_i} \neq [2]s^{-}_{L_I}$ $(i = 1, 2, 3)$, our assertion for the intersection multiplicities follows.

(ii) By Corollary 4.2 and Lemma 4.5, our statement follows. □

Corollary 5.6. *If L_i and L_j $(i \neq j)$ meet C_i transversely, then $(\mathcal{Q}+L_i+C_i, \mathcal{Q}+L_j + C_i)$ is a Zariski pair.*

Proof. Since the combinatorics of $\mathcal{Q} + L_i + C_i$ and $\mathcal{Q} + L_j + C_i$ are the same, our assertion follows from Proposition 5.5. □

Remark 5.7. First examples of Zariski pairs whose are all rational curves appeared in [**4**].

Example 5.8. We keep the same coordinates as in Examples 5.2 and 5.3. Consider Q, L_1 and L_2 as follows:

$$\mathcal{Q} : f_{\mathcal{Q}}(t,x) := x^3 + \frac{343}{64}\left(\frac{121}{49}t^2+\frac{768}{2401}t\right)x^2 + \frac{343}{64}\left(\frac{384}{2401}t^2+\frac{92}{49}t^3\right)x + \frac{35}{16}t^4 + \frac{1}{7}t^3 = 0$$

$$L_1 : x + t = 0, \qquad L_2 : x - \frac{\zeta_3 - 2}{7}t = 0,\ \zeta_3 = \exp(2\pi i/3).$$

$\mathcal{Q}$ is irreducible and has a $\mathbb{D}_4$ singularity at $(0, 0)$. Both L_1 and L_2 meet $\mathcal{Q}$ at $(0, 0)$ with multiplicity 4. Choose $[0, 1, 0]$ as the distinguished point z_o. Let $S_{(\mathcal{Q},z_o)}$ be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation $y^2 = f_{\mathcal{Q}}(t, x)$. Under these circumstances, we have

$$s^{\pm}_{L_1} = \left(-t, \pm\frac{\sqrt{343}}{8}t^2\right), \qquad s^{\pm}_{L_2} = \left(\frac{\zeta_3 - 2}{7}t, \pm\frac{\sqrt{71 + 39\sqrt{-3}}}{8\sqrt{14}}t^2\right).$$

Then we have

$$[2]s^{+}_{L_1} = \left(\frac{144}{16807} - \frac{127}{343}t - \frac{19}{28}t^2, -\frac{\sqrt{7}(55296 + 1947456t + 1450204t^2 + 167649825t^3)}{184473632}\right).$$

Now put

$$C : x - \frac{144}{16807} + \frac{127}{343}t + \frac{19}{28}t^2 = 0.$$

Since one can see that both of L_1 and L_2 meet C with two distinct points, $\mathcal{Q} + C + L_1$ and $\mathcal{Q}+C+L_2$ have the same combinatorics. By Corollary 5.6, $(\mathcal{Q}+C+L_1, \mathcal{Q}+C+L_2)$ is a Zariski pair.

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Note added in proof. After this paper was accepted, the author was informed that examples of Zariski pairs of degree 6 for conic arrangements had been already known. They are given explicitly in [**17**, Section 6] or [**29**, Section 5.1].