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Trivializing number of knots

By Ryo Hanaki

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Abstract. We introduce a numerical invariant, called trivializing number, of knots and investigate it. The trivializing number gives an upper bound of unknotting number and canonical genus for knots. We present a table of trivializing numbers for up to 10 crossings knots. We conjecture that twice of the unknotting number of any positive knot is equal to the trivializing number of it and give a partial answer.

1. Introduction.

We consider oriented knots in $\boldsymbol{R}^3$ and do not distinguish between a knot and its knot type so long as no confusion occurs. For the standard definitions and results of knots and links, we refer to [**1**]. Let p be a natural projection from $\boldsymbol{R}^3$ to $\boldsymbol{R}^2$. We say that p is a *projection* of a knot K if the multiple points of $p|_K$ are only finitely many transversal double points. Then we call $p(K)$ a *(knot) projection* and denote it by $P = p(K)$. (See Figure 1.) A *diagram* D is a projection P with over/under information at every double point. Then we say that P is the *projection of* D. A diagram D uniquely represents a knot up to ambient isotopy.

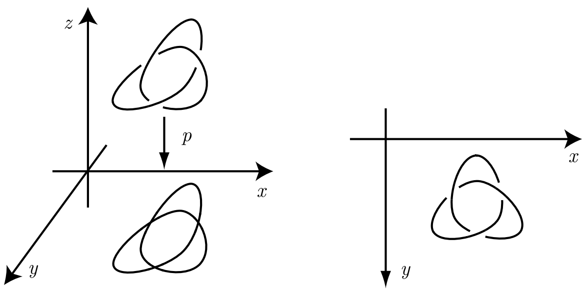

Figure 1. Knot projection and diagram.

We have the following question on knot projections. *Which double points of a projection and which over/under informations at them should we know in order to determine that the original knot is trivial or knotted?* Then, the author introduced a notion of the pseudo diagram in [**3**]. In this paper, a double point with over/under information is called a *crossing*, in contrast a double point without over/under information is called a *pre-crossing*. We say that a *pseudo diagram* Q is a projection P with over/under infor-

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mation at some pre-crossings of P. Here, we allow the possibility that a pseudo diagram is a projection or a diagram. We defined the pseudo diagram for links and spatial graphs and investigated them [**3**]. Let Q and Q' be pseudo diagrams of a projection. Then we say that *a pseudo diagram Q' is obtained from a pseudo diagram Q* if each crossing of Q has the same over/under information with Q'. A pseudo diagram Q is said to be *trivial* if every diagram obtained from Q represents the trivial knot. For example, in Figure 2, (a) is trivial but (b) and (c) are not trivial.

(a) (b) (c)

Figure 2. Pseudo diagrams.

We define that the *trivializing number of P*, denoted by $\mathrm{tr}(P)$, is the minimal number of the crossings of Q where Q varies over all trivial pseudo diagrams obtained from P. For example, let P be the projection as illustrated in Figure 1, then $\mathrm{tr}(P) = 2$. Recently, A. Henrich etc. expanded pseudo diagrams for virtual knots in [**4**]. They discuss relation between trivializing number and unknotting number (resp. genus) in the paper.

Our purpose in this paper is to define the trivializing number for knots and investigate them. Note that the definition is defined in [**4**] independently. First, we define it for diagrams. Let D be a diagram of a knot and P the projection of D. Then we define that $\mathrm{tr}(D) = \mathrm{tr}(P)$. For a knot K, we define the following:

$$\mathrm{tr}(K) = \min\{\mathrm{tr}(D) \mid A \text{ diagram } D \text{ represents } K\}.$$

Then we call $\mathrm{tr}(K)$ the *trivializing number of K*. This is a numerical invariant of knots. We see from Theorem 13 shown in [**3**] that $\mathrm{tr}(K)$ is even for any knot K. Similarly, we can define the trivializing number for links. The following proposition holds. The proposition is shown in [**4**] independently.

Proposition 1. *Let K be a knot. Then $u(K) \leq \mathrm{tr}(K)/2$ holds where $u(K)$ is the unknotting number of K.*

Proof. Let D be a diagram of K which realizes the trivializing number of K. Let P be the projection of D. Let Q be a trivial pseudo diagram of P which realizes the trivializing number of P. Let $\bar{Q}$ be the pseudo diagram of P obtained from Q by reversing over/under information of the crossings in Q. Then, $\bar{Q}$ is also trivial. We can deform D into the diagram D' such that over/under information of the crossings in Q and that of the corresponding crossings in D' agree by n ($\leq \mathrm{tr}(K)$) crossing changes. Then, we can deform D into the diagram D'' such that over/under information of the crossings in $\bar{Q}$ and that of the corresponding crossings in D'' agree by $\mathrm{tr}(K) - n$ crossing changes. Each of D' and D'' represents the trivial knot. Here, either n or $\mathrm{tr}(K) - n$ is less than or equal to the half of $\mathrm{tr}(K)$. This implies that the unknotting number of K is less than or equal to the half of $\mathrm{tr}(K)$. □

We search knots which satisfy the equality in Proposition 1. We see from the definition of the trivializing number that the following holds.

PROPOSITION 2. *Let K be a knot with* $\mathrm{tr}(K) = 2u(K)$ *and D a diagram which realizes the trivializing number of K. Then D realizes the unknotting number of K, namely* $u(D) = u(K)$.

We present a table of trivializing numbers for up to 10 crossings knots in the bottom of the paper. We refer to KnotInfo [**5**] to develop the table and determine the trivializing number by applying Proposition 1 and Theorems 13 and 11 for diagrams in KnotInfo. In Table 1, + means the positive knot. A *flype* is a transformation of a projection as illustrated in Figure 3. Then, the following holds and we give a proof in Section 2.

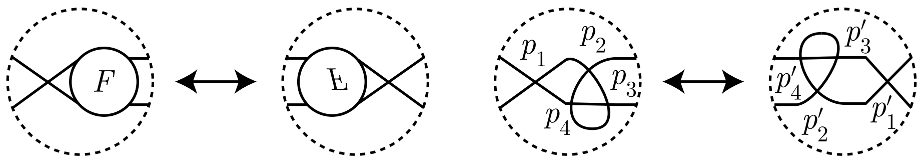


Figure 3. Flype.

PROPOSITION 3. *Let P be a projection of a knot and P' a projection obtained from P by a flype. Then* $\mathrm{tr}(P) = \mathrm{tr}(P')$.

By Main Theorem of [**6**], that is, a positive solution of Tait flyping conjecture, minimal crossing diagrams of an alternating knot are related by a sequence of flypes. Therefore, we have the following.

PROPOSITION 4. *The trivializing number of any alternating knot stays constant in its minimal crossing diagrams.*

It is known that there exist exactly 42 positive knots in up to 10 crossing knots. For example, see [**8**]. Then, we have the following from Table 1.

PROPOSITION 5. *Let K be a positive knot with up to* 10 *crossings. Then* $\mathrm{tr}(K) = 2u(K)$. *Moreover, a positive diagram of K realizes the trivializing number of K.*

Then we have the following conjecture.

CONJECTURE 1. *For any positive knot K,* $\mathrm{tr}(K) = 2u(K)$ *holds. Moreover, for any positive diagram D of K,* $\mathrm{tr}(D) = \mathrm{tr}(K)$ *holds.*

The following is a question associated with the conjecture above. If we give a positive answer to Conjecture 1 then we can give a positive answer to Question 1.

QUESTION 1 ([**12**, Question 9.6]). Does every positive knot realize its unknotting number in a positive diagram?

We note that there exist knots K such that $\mathrm{tr}(K) \neq 2u(K)$, see Table 1. Then, we give a partial answer and give its proof in Section 2.

Theorem 6. *Let K be a positive braid knot. Then* $\mathrm{tr}(K) = 2u(K)$. *Moreover, let D be a positive braid diagram of K. Then* $\mathrm{tr}(D) = 2u(K)$.

Next, we provide an answer to a question of whether the trivializing number of every knot K is realized in minimal crossing diagrams of K in [**4**].

Proposition 7. *The knot* 11_{550} *does not realize the trivializing number in minimal crossing diagrams. The positive* 12 *crossing diagram as in Figure* 4 (b) *realizes the trivializing number of* 11_{550}.

It is known in [**11**] that 11_{550} has only one 11 crossing diagram D as in Figure 4 (a) which is not positive but has a positive 12 crossing diagram D' as in Figure 4 (b). Their chord diagram is defined in Section 2. Then, we have $\mathrm{tr}(D) = 8$ and $\mathrm{tr}(D') = 6$ by Theorem 13. We see that $\mathrm{tr}(11_{550}) = 6$ from Propositions 1 and 15 and Theorems 13 and 14.

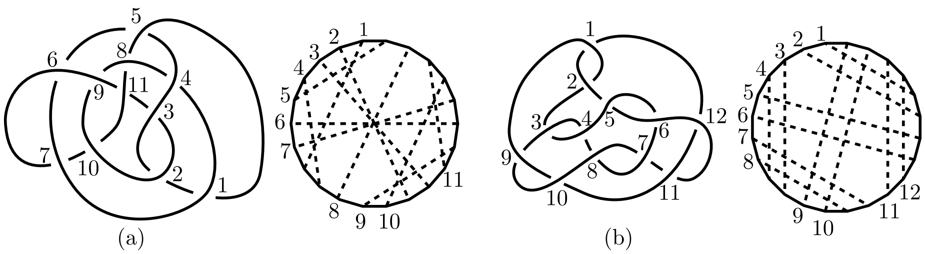


Figure 4. Diagrams of 11_{550}.

There exists a knot whose minimal crossing diagrams have different trivializing numbers. For example, we see that Perko's pair as in Figure 5 which represent 10_{161} have different trivializing numbers by Theorem 13. The diagram (a) in Figure 5 is not a positive diagram, another is a positive diagram. Note that the trivializing number of the diagram (a) is eight and that of the diagram (b) is six. Again, we see that the positive diagram realizes the trivializing number of the knot.

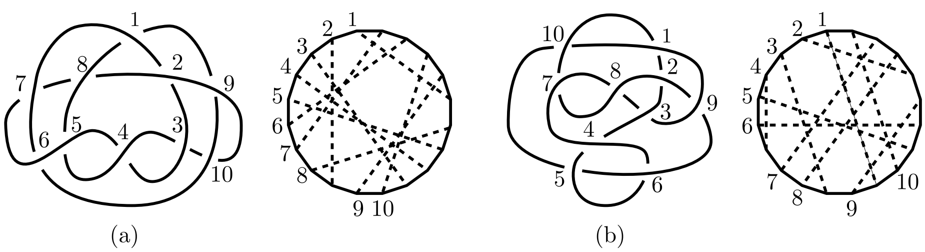


Figure 5. Perko's pair

We have the following by applying results of projections and give their proofs in Section 2.

Theorem 8. *Let K be a nontrivial knot. Then $2 \leq \mathrm{tr}(K) \leq c(K) - 1$ where $c(K)$ is the crossing number of K. The second equality holds if and only if K is a $(2, p)$-torus knot where p is some odd number, namely the braid index of K is equal to two.*

Theorem 9. *Let K be a knot. Then $\mathrm{tr}(K) = 2$ if and only if K is a twist knot where a twist knot is represented by a diagram obtained from some projection in Figure 6 (a).*

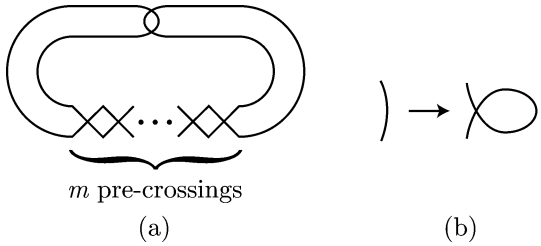


Figure 6.

Proposition 10. *Let K_1 and K_2 be knots and $K_1 \sharp K_2$ the connected sum of K_1 and K_2. Then,*

$$\mathrm{tr}(K_1 \sharp K_2) \leq \mathrm{tr}(K_1) + \mathrm{tr}(K_2).$$

It is open whether the above inequality holds.

Theorem 7.11 in [**4**] implies that the following relation between the canonical genus and the trivializing number holds.

Theorem 11 ([**4**]). *Let K be a knot. Then $g_c(K) \leq \mathrm{tr}(K)/2$ holds where $g_c(K)$ is the canonical genus of K, namely the minimal genus taken over all orientable surfaces constructed by applying Seifert's algorithm.*

Then, we have the following proposition and question and give its proof in Section 2.

Proposition 12. *For any non-negative integer n, there exists an alternating knot K such that $(\mathrm{tr}(K)/2) - u(K) = n$.*

Question 2. For any non-negative integer n, does there exist a knot K such that $(\mathrm{tr}(K)/2) - g_c(K) = n$?

We note that $(\mathrm{tr}(7_4)/2) - g_c(7_4) = 1$ and $(\mathrm{tr}(9_{35})/2) - g_c(9_{35}) = 2$ and they are positive pretzel knots. We consider positive pretzel knots as candidates.

In the Section 2, we give proofs. In the Section 3, we introduce an application.

2. Proofs.

First of all, we give a proof of Proposition 3.

Proof of Proposition 3. Note that each pre-crossings of P is corresponding to the pre-crossing of P' as illustrated in Figure 3. Let Q be a trivial pseudo diagram obtained from P with $\mathrm{tr}(P) = m$ crossings. Let $p_1, \ldots, p_m$ be the pre-crossings of Q with over/under information. Let Q' be the pseudo diagram obtained from P' such that the pre-crossing of P' corresponding to p_i has the same sign as p_i in Q ($i = 1, \ldots, m$). We show that Q' is trivial. Let D' be a diagram obtained from Q'. Then, let D be the diagram obtained from Q such that each crossing of D is the same sign of the crossing of D' corresponding to the crossing of D. Since D represents the trivial knot and D and D' represent same knot, D' also represents the trivial knot. Thus Q' is trivial, hence $\mathrm{tr}(P) \geq \mathrm{tr}(P')$. Similarly, we see that $\mathrm{tr}(P) \leq \mathrm{tr}(P')$. Therefore, $\mathrm{tr}(P) = \mathrm{tr}(P')$. □

We recall some results on pseudo diagrams and a method for calculating the trivializing number of a projection.

First, we recall a chord diagram. Let P be a projection with n pre-crossings. A *chord diagram* of P, denote by CD_P, is a circle with n chords marked on it by dashed line segment where the preimage of each pre-crossing is connected by a chord. For example, let P be a projection (a) in Figure 7. Then a chord diagram (b) in Figure 7 is CD_P. We have the following.

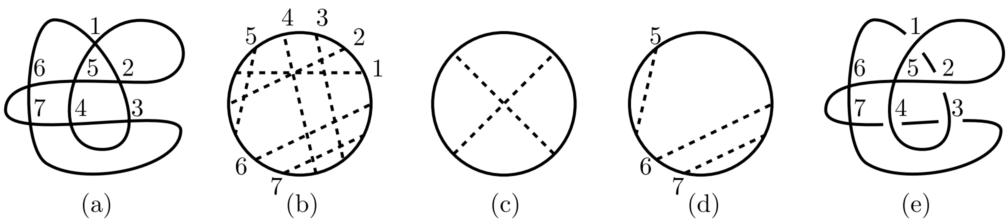


Figure 7.

Theorem 13 ([**3**]). *Let P be a knot projection. Then,* $\mathrm{tr}(P) = \min\{n|$ *Deleting some n chords from CD_P yields a chord diagram which does not contain a sub-chord diagram as* (c) *in Figure* 7$\}$ *and* $\mathrm{tr}(P)$ *is even.*

For a projection P, by Theorem 13, we can calculate $\mathrm{tr}(P)$ from CD_P. For example, we consider the projection as (a) in Figure 7. Any chord diagram obtained from CD_P by deleting at most three chords contains a sub-chord diagram as Figure 7 (c). A chord diagram as (d) in Figure 7 obtained from CD_P by deleting four chords does not contain a sub-chord diagram as Figure 7 (c). Therefore, we get $\mathrm{tr}(P) = 4$ and a pseudo diagram (e) in Figure 7 is a trivial pseudo diagram which realizes the trivializing number of P.

We recall the theorem and the proposition to estimate the unknotting number before proving Theorem 6.

Theorem 14 ([**8**], [**10**]). *Let D be a positive diagram and K the knot represented by D. Then $2g_4(K) = 2g(K) = c(D) - O(D) + 1$ holds where $c(D)$ is the number of the crossings of D, $O(D)$ is the number of the Seifert circles of D and $g_4(K)$ is the minimal genus of a compact orientable surface properly and locally flatly embedded in the upper-half* 4*-space with boundary K.*

We note that $s(K) = c(D) - O(D) + 1$ is called the Rasmussen invariant for a positive knot K and a positive diagram D of K. The following is well-known.

PROPOSITION 15. *Let K be a knot. Then $u(K) \geq g_4(K)$.*

PROOF OF THEOREM 6. Let D be a positive m-braid diagram of K. Let P be the projection of D. By Propositions 1 and 15 and Theorem 14,

$$\mathrm{tr}(P) \geq \mathrm{tr}(K) \geq 2u(K) \geq 2g_4(K) = c(D) - O(D) + 1.$$

Next, we prove that $\mathrm{tr}(P) \leq c(D) - O(D) + 1$. An m-component link projection is obtained from P by smoothing some $m-1$ pre-crossings. Therefore, there exist $m-1(= O(D)-1)$ chords each of whose two chords is not as Figure 7 (c) in CD_P. This implies that $\mathrm{tr}(P) \leq c(D) - O(D) + 1$ by Theorem 13. Hence, $\mathrm{tr}(K) = 2u(K)$. □

We recall the following theorems by applying Theorem 13 and give proofs of Theorems 8 and 9.

THEOREM 16 ([**3**]). *Let P be a projection with at least one pre-crossing. Then it holds that* $\mathrm{tr}(P) \leq p(P) - 1$. *The equality holds if and only if P is one of the projections as illustrated in Figure* 8 *where m is some positive odd integer.*

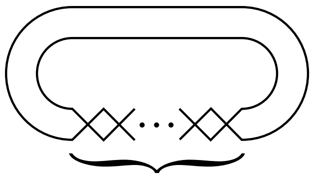


Figure 8.

THEOREM 17 ([**3**]). *Let P be a projection. Then* $\mathrm{tr}(P) = 2$ *if and only if P is obtained from the projection as illustrated in Figure* 6 (a) *where m is a positive integer by a series of replacing a sub-arc of P as illustrated in Figure* 6 (b).

PROOF OF THEOREM 8. First, we show that $2 \leq \mathrm{tr}(K) \leq c(K) - 1$. The condition that the trivializing number of K is equal to zero implies that K is trivial and hence $2 \leq \mathrm{tr}(K)$ by Theorem 13. Let D be a minimal crossing diagram of K and P the projection of D. Then we have

$$\mathrm{tr}(K) \leq \mathrm{tr}(P) \leq p(P) - 1 = c(K) - 1$$

by Theorem 16.

Next, we prove the case where the equality holds. The 'if' part holds since the projection of a minimal diagram of a $(2,p)$-torus knot is one of the projections in Figure 8. The 'only if' part holds since K is represented by some diagram obtained from some projection in Figure 8 by Theorem 16 and any diagram obtained from projections as

Figure 8 represents a $(2,p)$-torus knot for some odd integer p or the trivial knot. □

Proof of Theorem 9. The 'if' part holds since a twist knot has one of the projections in Figure 6 (a).

The 'only if' part holds since K is represented by some diagram obtained from some projection in Figure 6 (a) by Theorem 17 and any diagram obtained from projections as Figure 6 (a) represents some twist knot or the trivial knot. □

Then the following proposition holds and we give a proof of Proposition 10.

Proposition 18 ([**3**]). *Let P_1 and P_2 be a knot projection. Let P be the connected sum of P_1 and P_2 as illustrated in Figure* 9. *Then* $\mathrm{tr}(P) = \mathrm{tr}(P_1) + \mathrm{tr}(P_2)$.

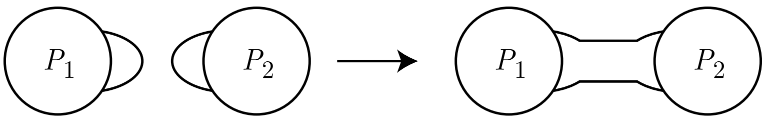


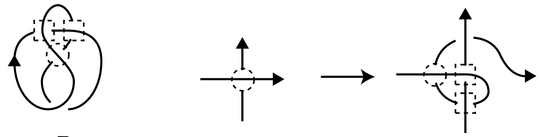
Figure 9.

Proof of Proposition 10. Let D_1 (resp. D_2) be a diagram which realizes the trivializing number of K_1 (resp. K_2). Let D be a connected sum of D_1 and D_2. Then D represents $K_1 \sharp K_2$ and $\mathrm{tr}(K_1 \sharp K_2) \leq \mathrm{tr}(D) = \mathrm{tr}(K_1) + \mathrm{tr}(K_2)$ by Proposition 18. □

Finally, we give a proof of Proposition 12.

Proof of Proposition 12. Let D_0 be the diagram as illustrated in Figure 10 (a). In the case $n = 0$, let D'_0 be the alternating diagram obtained from D_0 by changing the crossing at the crossing framed by a dash circle in Figure 10 (a). Let K_0 be the knot represented by D'_0. Since K_0 is an alternating knot and it is known in [**7**], [**2**] that an orientable surface obtained by Seifert's algorithm in an alternating diagram realizes the minimal genus, we get $g(K_0) = 1$. We see from Theorem 11 and a chord diagram of the projection of D_0 that $\mathrm{tr}(K_0) = 2$. Note that deleting the two chords corresponding to the crossings framed by dash squares in Figure 10 (a) yields the chord diagram which does not contain a sub-chord diagram as (c) in Figure 7. It is obvious that $u(K_0) = 1$. Therefore, $(\mathrm{tr}(K_0)/2) - u(K_0) = 0$.

In the case $n \geq 1$, let D_n be the almost alternating diagram obtained from D_0 by the deformation as illustrated in Figure 10 (b) n times. Then, let D'_n be the alternating



D_0

(a) (b)

Figure 10.

diagram obtained from D_n by changing the crossing at the crossing framed by a dash circle. Let K_n be the knot represented by D'_n. Since deleting the chords corresponding to the crossings framed by dash squares in Figure 10 (a) and (b) yields the chord diagram which does not contain a sub-chord diagram as (c) in Figure 7 and $g(K_n) = n+1$, $\mathrm{tr}(K_n) = 2(n+1)$. Similarly, we see that $u(K_n) = 1$. Therefore, $(\mathrm{tr}(K_n)/2) - u(K_n) = n$. □

3. Application to the partial order of knots.

In this section, we consider unoriented links. We denote the set of all projections of L by $\mathrm{PROJ}(L)$. Taniyama define that L_1 is a *minor* of L_2, denoted by $L_1 \leq L_2$ ($L_2 \geq L_1$) if $\mathrm{PROJ}(L_1) \supset \mathrm{PROJ}(L_2)$ in [**13**], [**14**]. We denote the set of all μ-component links by $\mathfrak{L}^\mu$. In particular, $\mathfrak{L}^1$ is the set of all knots. The following holds.

Proposition 19 ([**13**]). *The pair $(\mathfrak{L}^\mu, \geq)$ is a pre-ordered set for each natural number μ. Namely the following* (1) *and* (2) *hold for any L_1, L_2 and L_3 in $\mathfrak{L}^\mu$.*

(1) $L_1 \geq L_1$ *(reflexive law).*
(2) *If $L_1 \geq L_2$ and $L_2 \geq L_3$ then $L_1 \geq L_3$ (transitive law).*

Proposition 20 ([**13**]). *Let L_1, L_2 be μ-component links. If $L_1 \leq L_2$ then it holds that $c(L_1) \leq c(L_2)$, $br(L_1) \leq br(L_2)$ and $b(L_1) \leq b(L_2)$ where $br(L)$ and $b(L)$ are the minimal number of the bridge index and the braid index of L respectively.*

As a summary of many results, Taniyama has the Hasse diagram of knots (resp. links) in [**13**] (resp. [**14**]). Then Przytycki and Taniyama show a characterization of 2-almost positive diagram which represent a trivial link using the signature of links and an argument of a partial order of knots in [**9**]. We have the following.

Proposition 21. *Let L_1, L_2 be μ-component links. If $L_1 \leq L_2$ then it holds that* $\mathrm{tr}(L_1) \leq \mathrm{tr}(L_2)$.

Proof. Let D be a diagram of L_2 which realizes the trivializing number of L_2 and P the projection of D. There exists a diagram D' obtained from P which represents L_1 by $L_1 \leq L_2$. Therefore, we have $\mathrm{tr}(L_1) \leq \mathrm{tr}(L_2)$. □

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	unknotting	genus	trivializing	+		unknotting	genus	trivializing	+
3_1	1	1	2	+	9_9	3	3	6	+
4_1	1	1	2		9_{10}	3	2	6	+
5_1	2	2	4	+	9_{11}	2	3	6	
5_2	1	1	2	+	9_{12}	1	2	4	
6_1	1	1	2		9_{13}	3	2	6	+
6_2	1	2	4		9_{14}	1	2	4	
6_3	1	2	4		9_{15}	2	2	4	
7_1	3	3	6	+	9_{16}	3	3	6	+
7_2	1	1	2	+	9_{17}	2	3	6	
7_3	2	2	4	+	9_{18}	2	2	4	+
7_4	2	1	4	+	9_{19}	1	2	4	
7_5	2	2	4	+	9_{20}	2	3	6	
7_6	1	2	4		9_{21}	1	2	4	
7_7	1	2	4		9_{22}	1	3	6	
8_1	1	1	2		9_{23}	2	2	4	+
8_2	2	3	6		9_{24}	1	3	6	
8_3	2	1	4		9_{25}	2	2	4	
8_4	2	2	4		9_{26}	1	3	6	
8_5	2	3	6		9_{27}	1	3	6	
8_6	2	2	4		9_{28}	1	3	6	
8_7	1	3	6		9_{29}	2	3	6	
8_8	2	2	4		9_{30}	1	3	6	
8_9	1	3	6		9_{31}	2	3	6	
8_{10}	2	3	6		9_{32}	2	3	6	
8_{11}	1	2	4		9_{33}	1	3	6	
8_{12}	2	2	4		9_{34}	1	3	6	
8_{13}	1	2	4		9_{35}	3	1	6	+
8_{14}	1	2	4		9_{36}	2	3	6	
8_{15}	2	2	4	+	9_{37}	2	2	4	
8_{16}	2	3	6		9_{38}	3	2	6	+
8_{17}	1	3	6		9_{39}	1	2	4,6	
8_{18}	2	3	6		9_{40}	2	3	6	
8_{19}	3	3	6	+	9_{41}	2	2	4,6	
8_{20}	1	2	4		9_{42}	1	2	4,6	
8_{21}	1	2	4		9_{43}	2	3	6	
9_1	4	4	8	+	9_{44}	1	2	4,6	
9_2	1	1	2	+	9_{45}	1	2	4,6	
9_3	3	3	6	+	9_{46}	2	1	4	
9_4	2	2	4	+	9_{47}	2	3	6	
9_5	2	1	4	+	9_{48}	2	2	4	
9_6	3	3	6	+	9_{49}	3	2	6	+
9_7	2	2	4	+	10_1	1	1	2	
9_8	2	2	4		10_2	3	4	8	

	unknotting	genus	trivializing	+		unknotting	genus	trivializing	+
10_{3}	2	1	4		10_{46}	3	4	8	
10_{4}	2	2	4		10_{47}	2,3	4	8	
10_{5}	2	4	8		10_{48}	2	4	8	
10_{6}	3	3	6		10_{49}	3	3	6	+
10_{7}	1	2	4		10_{50}	2	3	6	
10_{8}	2	3	6		10_{51}	2,3	3	6	
10_{9}	1	4	8		10_{52}	2	3	6	
10_{10}	1	2	4		10_{53}	3	2	6	+
10_{11}	2,3	2	4,6		10_{54}	2,3	3	6	
10_{12}	2	3	6		10_{55}	2	2	4	+
10_{13}	2	2	4		10_{56}	2	3	6	
10_{14}	2	3	6		10_{57}	2	3	6	
10_{15}	2	3	6		10_{58}	2	2	4	
10_{16}	2	2	4,6		10_{59}	1	3	6	
10_{17}	1	4	8		10_{60}	1	3	6	
10_{18}	1	2	4		10_{61}	2,3	3	6	
10_{19}	2	3	6		10_{62}	2	4	8	
10_{20}	2	2	4		10_{63}	2	2	4	+
10_{21}	2	3	6		10_{64}	2	4	8	
10_{22}	2	3	6		10_{65}	2	3	6	
10_{23}	1	3	6		10_{66}	3	3	6	+
10_{24}	2	2	4		10_{67}	2	2	4	
10_{25}	2	3	6		10_{68}	2	2	4,6	
10_{26}	1	3	6		10_{69}	2	3	6	
10_{27}	1	3	6		10_{70}	2	3	6	
10_{28}	2	2	4,6		10_{71}	1	3	6	
10_{29}	2	3	6		10_{72}	2	3	6	
10_{30}	1	2	4,6		10_{73}	1	3	6	
10_{31}	1	2	4		10_{74}	2	2	4,6	
10_{32}	1	3	6		10_{75}	2	3	6	
10_{33}	1	2	4,6		10_{76}	2,3	3	6	
10_{34}	2	2	4		10_{77}	2,3	3	6	
10_{35}	2	2	4		10_{78}	2	3	6	
10_{36}	2	2	4		10_{79}	2,3	4	8	
10_{37}	2	2	4		10_{80}	3	3	6	+
10_{38}	2	2	4		10_{81}	2	3	6	
10_{39}	2	3	6		10_{82}	1	4	8	
10_{40}	2	3	6		10_{83}	2	3	6	
10_{41}	2	3	6		10_{84}	1	3	6	
10_{42}	1	3	6		10_{85}	2	4	8	
10_{43}	2	3	6		10_{86}	2	3	6	
10_{44}	1	3	6		10_{87}	2	3	6	
10_{45}	2	3	6		10_{88}	1	3	6	

	unknotting	genus	trivializing	+		unknotting	genus	trivializing	+
10_{89}	2	3	6		10_{128}	3	3	6	+
10_{90}	2	3	6		10_{129}	1	2	4,6	
10_{91}	1	4	8		10_{130}	2	2	4,6	
10_{92}	2	3	6		10_{131}	1	2	4,6	
10_{93}	2	3	6		10_{132}	1	2	4	
10_{94}	2	4	8		10_{133}	1	2	4,6	
10_{95}	1	3	6		10_{134}	3	3	6	+
10_{96}	2	3	6		10_{135}	2	2	4,6	
10_{97}	2	2	4,6		10_{136}	1	2	4,6	
10_{98}	2	3	6		10_{137}	1	2	4,6	
10_{99}	2	4	8		10_{138}	2	3	6	
10_{100}	2,3	4	8		10_{139}	4	4	8	+
10_{101}	3	2	6	+	10_{140}	2	2	4,6	
10_{102}	1	3	6		10_{141}	1	3	6	
10_{103}	3	3	6		10_{142}	3	3	6	+
10_{104}	1	4	8		10_{143}	1	3	6	
10_{105}	2	3	6		10_{144}	2	2	4,6	
10_{106}	2	4	8		10_{145}	2	2	4,6	
10_{107}	1	3	6		10_{146}	1	2	4,6	
10_{108}	2	3	6		10_{147}	1	2	4,6	
10_{109}	2	4	8		10_{148}	2	3	6	
10_{110}	2	3	6		10_{149}	2	3	6	
10_{111}	2	3	6		10_{150}	2	3	6	
10_{112}	2	4	8		10_{151}	2	3	6	
10_{113}	1	3	6		10_{152}	4	4	8	+
10_{114}	1	3	6		10_{153}	2	3	6	
10_{115}	2	3	6		10_{154}	3	3	6	+
10_{116}	2	4	8		10_{155}	2	3	6,8	
10_{117}	2	3	6		10_{156}	1	3	6	
10_{118}	1	4	8		10_{157}	2	3	6,8	
10_{119}	1	3	6		10_{158}	2	3	6	
10_{120}	3	2	6	+	10_{159}	1	3	6,8	
10_{121}	2	3	6		10_{160}	2	3	6	
10_{122}	2	3	6		10_{161}	3	3	6	+
10_{123}	2	4	8		10_{162}	2	2	4,6	
10_{124}	4	4	8	+	10_{163}	2	3	6	
10_{125}	2	3	6		10_{164}	1	2	4,6	
10_{126}	2	3	6		10_{165}	2	2	4,6	
10_{127}	2	3	6						

Table 1. The unknotting number, genus and trivializing number of knots and whether a knot is positive or not.

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Ryo Hanaki

Department of Mathematics
Nara University of Education
Takabatake
Nara 630-8528, Japan
E-mail: hanaki@nara-edu.ac.jp