

Central exact sequences of tensor categories, equivariantization and applications

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Abstract. We define equivariantization of tensor categories under tensor group scheme actions and give necessary and sufficient conditions for an exact sequence of tensor categories to be an equivariantization under a finite group or finite group scheme action. We introduce the notion of central exact sequence of tensor categories and use it in order to present an alternative formulation of some known characterizations of equivariantizations for fusion categories, and to extend these characterizations to equivariantizations of finite tensor categories under finite group scheme actions. In particular, we obtain a simple characterization of equivariantizations under actions of finite abelian groups. As an application, we show that if \mathcal{C} is a fusion category and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor of Frobenius-Perron index p , then F is an equivariantization if $p = 2$, or if \mathcal{C} is weakly integral and p is the smallest prime factor of $\text{FPdim } \mathcal{C}$.

1. Introduction.

In this paper we pursue the study of exact sequences of tensor categories initiated in [3]. Exact sequences of tensor categories generalize (strict) exact sequences of Hopf algebras, due to Schneider, and in particular, exact sequences of groups.

By a tensor category over a field \mathbf{k} , we mean a monoidal rigid category $(\mathcal{C}, \otimes, \mathbf{1})$ endowed with a \mathbf{k} -linear abelian structure such that

- Hom spaces are finite dimensional and all objects have finite length,
- the tensor product \otimes is \mathbf{k} -linear in each variable and the unit object $\mathbf{1}$ is scalar, that is, $\text{End}(\mathbf{1}) = \mathbf{k}$.

A tensor category is *finite* if it is \mathbf{k} -linearly equivalent to the category of finite dimensional right modules over a finite dimensional \mathbf{k} -algebra.

We will mostly work with finite tensor categories, with special attention to *fusion categories*. A fusion category is a split semisimple finite tensor category (*split* semisimple means semisimple with scalar simple objects).

A *tensor functor* is a strong monoidal, \mathbf{k} -linear exact functor between tensor categories over \mathbf{k} ; it is faithful. A tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *dominant*¹ if any object Y of \mathcal{D} is a subobject of $F(X)$ for some object X of \mathcal{C} .

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¹Dominant functors between finite tensor categories are called *surjective* in [6].

A tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *normal* if any object X of \mathcal{C} admits a subobject $X' \subset X$ such that $F(X')$ is the largest trivial subobject of $F(X)$. An object is *trivial* if it is isomorphic to $\mathbf{1}^n$ for some natural integer n .

We denote by $\mathfrak{Ker}_F \subset \mathcal{C}$ the full tensor subcategory of objects X of \mathcal{C} such that $F(X)$ is trivial.

Let $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ be tensor categories over \mathbf{k} . A sequence of tensor functors

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}'' \tag{1.1}$$

is called an *exact sequence of tensor categories* if

- F is dominant and normal,
- i is a full embedding whose essential image is \mathfrak{Ker}_F .

An exact sequence of finite tensor categories $\mathcal{C}' \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$ is *perfect* if the left (or equivalently, the right) adjoint of F is exact. Such is always the case if \mathcal{C}'' is a fusion category.

The monadic approach.

Let $F : \mathcal{C} \rightarrow \mathcal{C}''$ be a tensor functor between finite tensor categories. Then F is monadic, that is, it admits a left adjoint L . The endofunctor $T = FL$ of \mathcal{C}'' is a \mathbf{k} -linear Hopf monad on \mathcal{C}'' , and \mathcal{C} is tensor equivalent to the category \mathcal{C}''^{T} of T -modules in \mathcal{C}'' . Moreover, F is dominant if and only if T is faithful, and F is normal if and only if $T(\mathbf{1})$ is trivial, in which case T is said to be normal.

Via this construction, exact sequences of finite tensor categories

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

are classified by \mathbf{k} -linear right exact faithful normal Hopf monads on \mathcal{C}'' [3, Theorem 5.8].

Examples.

Any (strictly) exact sequence of Hopf algebras $H' \rightarrow H \rightarrow H''$ over a field \mathbf{k} in the sense of Schneider [16] gives rise to an exact sequence of tensor categories of finite dimensional comodules:

$$(\mathcal{E}_S) \quad \text{comod-}H' \rightarrow \text{comod-}H \rightarrow \text{comod-}H'',$$

and if H is finite-dimensional we also have an exact sequence of tensor categories of finite dimensional modules:

$$\text{mod-}H'' \rightarrow \text{mod-}H \rightarrow \text{mod-}H'.$$

Equivariantization is another source of examples. Let G be a finite group acting on a tensor category \mathcal{D} by tensor autoequivalences. Then the *equivariantization* \mathcal{D}^G is a tensor category and the forgetful functor $\mathcal{D}^G \rightarrow \mathcal{D}$ gives rise to a (perfect) exact sequence

of tensor categories

$$\text{rep } G \rightarrow \mathcal{D}^G \rightarrow \mathcal{D}, \tag{1.2}$$

see [3, Section 5.3]. This is extended in Section 3.2 to the case where G is a finite group scheme. If \mathcal{D} is a fusion category, \mathbf{k} is algebraically closed and the order of G is not a multiple of $\text{char}(\mathbf{k})$, then (1.2) is an exact sequence of fusion categories.

A tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivariantization* if there is an action of a finite group scheme G on \mathcal{D} and a tensor equivalence $\mathcal{C} \simeq \mathcal{D}^G$ such that the triangle of tensor functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\simeq} & \mathcal{D}^G \\ & \searrow & \swarrow \\ & \mathcal{D} & \end{array}$$

commutes up to a \mathbf{k} -linear monoidal isomorphism.

An exact sequence of tensor categories $\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ is called an *equivariantization exact sequence* if it is equivalent to an exact sequence defined by an equivariantization, or equivalently, if F is an equivariantization.

A *braided exact sequence* is an exact sequence of tensor categories where all categories and functors are braided. If $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is a braided exact sequence, then \mathcal{C}' is a subcategory of the category $\mathcal{T} \subset \mathcal{C}$ of transparent objects of \mathcal{C} (see [1]). We say that $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is a *modularization exact sequence* if \mathcal{C}'' is modular, that is, if all transparent objects of \mathcal{C}'' are trivial. In that case $\mathcal{C}' = \mathcal{T}$. Examples of modularization exact sequences of fusion categories arise through the modularization procedures introduced in [1], [10].

Equivariantization criteria.

Generalizing a result of [3], we show that an exact sequence of finite tensor categories $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is an equivariantization exact sequence if and only if the associated normal Hopf monad is exact and cocommutative in the sense of [3]. If \mathcal{C}' is finite and \mathbf{k} is an algebraically closed field such that $\text{char}(\mathbf{k})$ does not divide $\dim \mathcal{C}'$, then the corresponding group scheme is discrete so that we have an equivariantization in the usual sense.

In particular, any perfect braided exact sequence of finite tensor categories is an equivariantization exact sequence.

On the other hand, [7, Proposition 2.10](i) affirms that a fusion category \mathcal{C} is an equivariantization under the action of a finite group G if there is a full braided embedding j of the category $\text{rep } G$ into the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} , such that $Uj : \text{rep } G \rightarrow \mathcal{C}$ is full, where U denotes the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. See also [5, Theorem 4.18].

In order to unify those two result, we introduce the notion of central exact sequence of tensor categories. An exact sequence of finite tensor categories

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

is *central* if, denoting by (A, σ) its central commutative algebra, the tensor functor $i :$

$\mathcal{C}' \rightarrow \mathcal{C}$ lifts to a tensor functor $\tilde{i} : \mathcal{C}' \rightarrow \mathcal{Z}(\mathcal{C})$ such that $\tilde{i}(A) = (A, \sigma)$. Such a lift, if it exists, is essentially unique.

We show that an exact sequence of finite tensor categories is central if and only if its normal Hopf monad is cocommutative. We give two proofs of this result.

The first one works in the fusion case. In that situation our characterization is a reformulation in terms of exact sequences of the characterization of equivariantizations given in [7, Proposition 2.10], [5, Theorem 4.18]. However our proof is organized differently, and boils down to showing that a central exact sequence of fusion categories is ‘dominated’ in a canonical way by a modularization exact sequence, thus:

$$\begin{array}{ccccc}
 \mathcal{C}' & \longrightarrow & C_{\mathcal{Z}(\mathcal{C})}(\mathcal{C}') & \longrightarrow & \mathcal{Z}(\mathcal{C}'') \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'',
 \end{array}$$

where $C_{\mathcal{Z}(\mathcal{C})}(\mathcal{C}')$ denotes the centralizer of \mathcal{C}' viewed as a fusion subcategory of $\mathcal{Z}(\mathcal{C})$, and that an exact sequence dominated by an equivariantization exact sequence is itself an equivariantization exact sequence.

The second one works for finite tensor (not necessarily semisimple) categories and actions of finite group schemes, and relies on the construction of the double of a Hopf monad in [4]. It is based on the existence of a commutative diagram of tensor categories

$$\begin{array}{ccccc}
 \mathcal{C}' & \longrightarrow & \mathcal{Z}(\mathcal{C}) & \longrightarrow & \mathcal{Z}_F(\mathcal{C}'') \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}' & \longrightarrow & \mathcal{C} & \xrightarrow{F} & \mathcal{C}'',
 \end{array}$$

whose first line is an exact sequence of tensor categories, $\mathcal{Z}_F(\mathcal{C}'')$ denoting the center of \mathcal{C}'' relative to the functor F .

Application.

The *Frobenius-Perron index* of a dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between fusion categories is defined in [3] to be the ratio

$$\text{FPind}(F) = \text{FPind}(\mathcal{C} : \mathcal{D}) = \frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{D}}.$$

It is an algebraic integer by [6, Corollary 8.11]. According to [3, Proposition 4.13], a dominant tensor functor F of Frobenius-Perron index 2 is normal. In this paper we prove the following refinement of this result:

THEOREM 6.1. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories over a field of characteristic 0. If $\text{FPind}(\mathcal{C} : \mathcal{D}) = 2$, then F is an equivariantization.*

This generalizes the fact that subgroups of index 2 are normal. An analogue in the context of finite dimensional semisimple Hopf algebras was proved in [9, Proposition 2], [13, Corollary 1.4.3]. We also generalize the fact that subgroups of a finite group whose index is the smallest prime factor of the order of the larger group are normal. Recall that a fusion category is *weakly integral* if its Frobenius-Perron dimension is a natural integer.

THEOREM 6.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories over a field of characteristic 0. Assume that $\text{FPdim } \mathcal{C}$ is a natural integer, and that $\text{FPind}(\mathcal{C} : \mathcal{D})$ is the smallest prime number dividing $\text{FPdim } \mathcal{C}$. Then F is an equivariantization.*

In particular under the hypotheses of Theorem 6.2 the functor F is normal. An analogue in the context of semisimple Hopf algebras was proved in [9, Proposition 2], [13, Corollary 1.4.3]. Observe that Theorem 6.2 gives some positive evidence in favor of the conjecture that every weakly integral fusion category is weakly group-theoretical [7].

Regarding the ‘dual’ situation, namely, when \mathcal{C} is a weakly integral fusion category and $\mathcal{D} \subseteq \mathcal{C}$ is a full fusion subcategory such that the quotient $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is the smallest prime factor of $\text{FPdim } \mathcal{C}$, it may be the case that \mathcal{D} is *not* normal in \mathcal{C} ; we give an example of this where $p = 2$ and \mathcal{C} is a Tambara-Yamagami category (see Proposition 6.5). This actually provides examples of simple fusion categories of Frobenius-Perron dimension $2q$, where q is an odd prime number.

Organization of the text.

In Section 2 we introduce central exact sequences of tensor categories. We also discuss dominant tensor functors on weakly integral fusion categories in terms of induced central algebras and show that this class of fusion categories is closed under extensions; see Corollary 2.13. We give a general criterion for an exact sequence to be an equivariantization exact sequence in Section 3. In order to do so, we generalize the notion of equivariantization to tensor actions of group schemes on tensor categories in Section 3.2. The main results, Theorems 3.5 and 3.6, assert that equivariantization exact sequences coincide with central exact sequences, and also with exact sequences whose Hopf monad is exact cocommutative. It is proved in Section 5 using the notion of double of a Hopf monad. We apply this characterization in Section 4 to several special cases: braided exact sequences of tensor categories, exact sequences of fusion categories, equivariantizations under the action of abelian groups. Lastly we prove Theorems 6.1 and 6.2 in Section 6.

Conventions and notation.

We retain the conventions and notation of [3].

If \mathcal{C} is a monoidal category and A is an algebra in \mathcal{C} , we denote by \mathcal{C}_A the category of right A -modules in \mathcal{C} , and by $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ the free A -module functor, defined by $X \mapsto X \otimes A$. If \mathcal{C} is additive, so is \mathcal{C}_A . In that case, we say that A is *self-trivializing* if $F_A(A) \simeq F_A(\mathbf{1})^n$ for some natural integer n .

Let \mathcal{C} be a tensor category over a field \mathbf{k} , and let \mathcal{X} be an object, or a set of objects of \mathcal{C} . We denote by $\langle \mathcal{X} \rangle$ the smallest full replete tensor subcategory of \mathcal{C} containing \mathcal{X} . Its objects are the subquotients of finite direct sums of tensor products of elements of \mathcal{X} .

and their duals. We denote by $\mathbf{1}$ the unit object of \mathcal{C} . The tensor subcategory $\langle \mathbf{1} \rangle$ is the category of trivial objects of \mathcal{C} and it is tensor equivalent to $\text{vect}_{\mathbf{k}}$.

Given an object X of \mathcal{C} , we denote by ${}^\vee X$ and X^\vee the left dual and the right dual of X respectively. An object X of \mathcal{C} is called *invertible* if there exists an object Y of \mathcal{C} such that $X \otimes Y \simeq \mathbf{1} \simeq Y \otimes X$. In that case $Y \simeq {}^\vee X \simeq X^\vee$. Invertible objects of \mathcal{C} are both simple and scalar. We denote by $\text{Pic}(\mathcal{C})$ the set of isomorphism classes of invertible objects of \mathcal{C} ; it is a group for the tensor product, called the *Picard group of \mathcal{C}* . We set $\mathcal{C}_{\text{pt}} = \langle \text{Pic}(\mathcal{C}) \rangle \subset \mathcal{C}$.

Now assume \mathcal{C} is a fusion category. The multiplicity of a simple object X in an object Y of \mathcal{C} is defined as $m_X(Y) = \dim \text{Hom}_{\mathcal{C}}(X, Y)$. We have

$$Y \simeq \bigoplus_{X \in \text{Irr}(\mathcal{C})} X^{m_X(Y)},$$

where $\text{Irr}(\mathcal{C})$ denotes the set of simple objects of \mathcal{C} up to isomorphism. An object X of \mathcal{C} is invertible if and only if its Frobenius-Perron dimension is 1.

2. Central exact sequences of tensor categories.

2.1. Tensor functors.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between tensor categories over \mathbf{k} . We denote by $\mathfrak{ker}_F \subset \mathcal{C}$ the full subcategory of \mathcal{C} of objects c such that $F(c)$ is trivial in \mathcal{D} . The category \mathfrak{ker}_F is a tensor category over \mathbf{k} , and it is endowed with a fibre functor

$$\omega_F \begin{cases} \mathfrak{ker}_F \rightarrow \text{vect}_{\mathbf{k}} \\ x \mapsto \text{Hom}(\mathbf{1}, F(x)). \end{cases}$$

By Tannaka reconstruction, this defines a Hopf algebra

$$H = \text{coend}(\omega_F) = \int^{x \in \mathfrak{ker}_F} \omega(x)^\vee \otimes \omega(x)$$

such that $\mathfrak{ker}_F \simeq \text{comod-}H$.

The tensor functor F admits a left adjoint L if and only if it admits a right adjoint R ; if they exist, the adjoints of F are related by $R(X) = L({}^\vee X)^\vee$. If \mathcal{C} is finite, then F admits adjoints.

If the tensor functor F admits adjoints, we say that F is *perfect* if its left, or equivalently, its right adjoint is exact. Such is always the case if \mathcal{D} is a fusion category.

The tensor functor F is *dominant* if for any object d of \mathcal{D} , there exists an object c of \mathcal{C} such that d is a subobject of $F(c)$. It is *normal* if for any object c of \mathcal{C} , there exists a subobject $c_0 \subset c$ such that $F(c_0)$ is the largest trivial subobject of $F(c)$.

If F admits adjoints L and R , then F is dominant if and only if L , or equivalently R , is faithful, and F is normal if and only if $L(\mathbf{1})$, or equivalently $R(\mathbf{1})$, belongs to \mathfrak{ker}_F .

If F is a normal tensor functor, the Hopf algebra $H = \text{coend}(\omega_F)$ is called the *induced Hopf algebra of F* .

2.2. Central induced (co)algebras.

Let \mathcal{C} and \mathcal{D} be finite tensor categories over \mathbf{k} and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor. Then F admits a left adjoint $L : \mathcal{D} \rightarrow \mathcal{C}$ which is faithful and comonoidal. Consequently $\hat{C} = L(\mathbf{1})$ is a coalgebra in \mathcal{C} , called *the induced coalgebra of F* , with coproduct $L_2(\mathbf{1}, \mathbf{1})$ and unit L_0 , where (L_2, L_0) denotes the comonoidal structure of L . We have $\text{Hom}_{\mathcal{C}}(\hat{C}, \mathbf{1}) \simeq \mathbf{k}$. In addition, \hat{C} is endowed with a canonical half-braiding $\hat{\sigma} : \hat{C} \otimes \text{id}_{\mathcal{C}} \rightarrow \text{id}_{\mathcal{C}} \otimes \hat{C}$, which makes it a cocommutative coalgebra in the center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} ; see [2] for details of this construction. The cocommutative coalgebra $(\hat{C}, \hat{\sigma})$ is called *the induced central coalgebra of F* .

Dually, under the same hypotheses the functor F also admits a right adjoint R , related to L by $R(X) = L(\vee X)^\vee$. The functor R is faithful and monoidal. As a result, $A = R(\mathbf{1}) = \hat{C}^\vee$ is an algebra in \mathcal{C} , called *the induced algebra of F* , and it is endowed a canonical half-braiding $\sigma : A \otimes \text{id}_{\mathcal{C}} \rightarrow \text{id}_{\mathcal{C}} \otimes A$, making it a commutative algebra in $\mathcal{Z}(\mathcal{C})$. The commutative algebra (A, σ) , which is the right dual of $(\hat{C}, \hat{\sigma})$, is called *the induced central algebra of F* . See [3, Section 6].

The category $\mathcal{C}_A = \mathcal{C}_{(A, \sigma)}$ of right A -modules in \mathcal{C} is an abelian \mathbf{k} -linear monoidal category over \mathbf{k} with tensor product induced by \otimes_A and the half-braiding σ , and the functor $F_{(A, \sigma)} : \mathcal{C} \rightarrow \mathcal{C}_A$, $F_{(A, \sigma)}(X) = X \otimes A$, is strong monoidal and \mathbf{k} -linear.

If F is dominant and perfect (that is, R is faithful exact), then by [3, Proposition 6.1] \mathcal{C}_A is a tensor category, and there is a tensor equivalence $\kappa : \mathcal{D} \rightarrow \mathcal{C}_A$ such that the following diagram of tensor functors commutes up to tensor isomorphisms:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow F_A & \downarrow \kappa \\ & & \mathcal{C}_A. \end{array}$$

Note that if F is dominant and \mathcal{D} is a fusion category, then R is exact, so $\mathcal{C}_A \simeq \mathcal{D}$ is a fusion category and in that case, A is semisimple.

LEMMA 2.1. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between finite tensor categories, with induced algebra A . The following assertions are equivalent:*

- (i) F is normal;
- (ii) A belongs to \mathfrak{Ker}_F ;
- (iii) A is a self-trivializing algebra, that is, $A \otimes A \simeq A^n$ in \mathcal{C}_A for some integer n .

If these hold, then $\mathfrak{Ker}_F = \langle A \rangle$. Moreover, the integer n of assertion (iii) is the dimension of the induced Hopf algebra of F .

PROOF. Since $A = R(\mathbf{1})$, we have (i) \iff (ii). If A is in \mathfrak{Ker}_F , then $F_A(A) \simeq \kappa F(A) \simeq \kappa F(\mathbf{1})^n \simeq F_A(\mathbf{1})^n$, i.e. A is self-trivializing, so (ii) \implies (iii). Conversely, assume A is self-trivializing, that is, $A \otimes A \simeq A^n$ as right A -modules. By adjunction, $\text{Hom}_{\mathcal{D}}(F(A), \mathbf{1}) \simeq \text{Hom}_{\mathcal{C}}(A, R(\mathbf{1})) = \text{Hom}_{\mathcal{C}}(A, A) \simeq \text{Hom}_{\mathcal{C}_A}(F_A(A), F_A(\mathbf{1})) \simeq \text{Hom}_{\mathcal{C}_A} \cdot (F_A(\mathbf{1})^n, F_A(\mathbf{1})) \simeq \text{Hom}_{\mathcal{C}}(\mathbf{1}, A)^n \simeq \mathbf{k}^n$. Thus, there exists an epimorphism $s : F(A) \rightarrow \mathbf{1}^n$ in \mathcal{D} . Now $s \otimes F(A)$ is an epimorphism $F(A) \otimes F(A) \rightarrow F(A)^n$, and since those

two objects are isomorphic and of finite length, $s \otimes F(A)$ is an isomorphism. Thus $\ker(s) \otimes F(A) = 0$, so $\ker(s) = 0$ and s is an isomorphism $F(A) \xrightarrow{\sim} \mathbf{1}^n$. This shows (iii) \implies (ii).

If the assertions of the lemma hold, then $\mathfrak{ker}_F = \langle A \rangle$, in view of [3, Proposition 6.7]. Moreover, $F(A) = FR(\mathbf{1}) \simeq F(L(\mathbf{1})^\vee) \simeq F(L(\mathbf{1}))^\vee$. Hence ${}^\vee F(A) \simeq H \otimes \mathbf{1}$, and we get $n = \dim H$. □

REMARK 2.2. A terminological summary might help: if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a tensor functor between finite tensor categories, the induced central algebra $\mathbf{A} = (A, \sigma)$ of F is a commutative algebra in $\mathcal{Z}(\mathcal{C})$; the induced algebra A of F is an algebra in \mathcal{C} . If F is normal, with induced Hopf algebra $H = \text{coend}(\omega_F)$, then $F(A) \simeq H \otimes \mathbf{1}$ and, denoting by $T = FL$ the Hopf monad of F , we have $T(\mathbf{1}) \simeq K \otimes \mathbf{1}$ with $K = H^*$.

2.3. Exact sequences of tensor categories.

In this section we recall some basic facts about exact sequences of tensor categories which we will use throughout this paper, see [3] for details. Let \mathbf{k} be a field.

An *exact sequence of tensor categories* over \mathbf{k} is a diagram of tensor functors

$$(\mathcal{E}) \quad \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

between tensor categories $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ over \mathbf{k} , such that F is normal and dominant, $i(\mathcal{C}') \subset \mathfrak{ker}_F$ and i induces a tensor equivalence $\mathcal{C}' \rightarrow \mathfrak{ker}_F$.

The induced Hopf algebra $H = \text{coend}(\omega_F)$ of F is also called the *induced Hopf algebra of* (\mathcal{E}) , and we have an equivalence of tensor categories $\mathcal{C}'' \simeq \text{comod-}H$. By [3, Proposition 3.15], the induced Hopf algebra of (\mathcal{E}) is finite dimensional if and only if the tensor functor F has a left adjoint, or equivalently, a right adjoint. In that case, we say that (\mathcal{E}) is *perfect* if F is perfect, that is, R (or F) is exact.

If $(\mathcal{E}_1) = (\mathcal{C}'_1 \rightarrow \mathcal{C}_1 \rightarrow \mathcal{C}''_1)$ and $(\mathcal{E}_2) = (\mathcal{C}'_2 \rightarrow \mathcal{C}_2 \rightarrow \mathcal{C}''_2)$ are two exact sequences of tensor categories over \mathbf{k} , a *morphism of exact sequences of tensor categories* from (\mathcal{E}_1) to (\mathcal{E}_2) is a diagram of tensor functors:

$$\begin{array}{ccccc} \mathcal{C}'_1 & \longrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{C}''_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}'_2 & \longrightarrow & \mathcal{C}_2 & \longrightarrow & \mathcal{C}''_2 \end{array}$$

which commutes up to tensor isomorphisms. Such a morphism induces a morphism of Hopf algebras $w : H_1 \rightarrow H_2$, where H_1 and H_2 denote the induced Hopf coalgebras of (\mathcal{E}_1) and of (\mathcal{E}_2) , respectively.

A morphism of exact sequences of tensor categories is an *equivalence of exact sequences of tensor categories* if the vertical arrows are equivalences.

LEMMA 2.3. *Consider a morphism of exact sequences of finite tensor categories*

$$\begin{array}{ccccc}
 (\mathcal{E}_1) & & \mathcal{C}'_1 & \xrightarrow{i_1} & \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}''_1 \\
 & & \downarrow U' & & \downarrow U & & \downarrow U'' \\
 & & \mathcal{C}'_2 & \xrightarrow{i_2} & \mathcal{C}_2 & \xrightarrow{F_2} & \mathcal{C}''_2.
 \end{array}$$

Denote by H_1, H_2 the induced Hopf algebras of F_1 and F_2 respectively.

- (1) The morphism of exact sequences induces a Hopf algebra morphism $\phi : H_1 \rightarrow H_2$, and U' is dominant (respectively, an equivalence) if and only if ϕ is a monomorphism (respectively, and isomorphism);
- (2) If U' and U'' are dominant, then so is U .

PROOF. 1) We may assume $\mathcal{C}'_1 = \text{comod-}H_1, \mathcal{C}'_2 = \text{comod-}H_2, U'$ being compatible with the forgetful functors. Then U' is of the form ϕ_* , for some Hopf algebra morphism $\phi : H_1 \rightarrow H_2, U'$ is an equivalence if and only if ϕ is an isomorphism, and by [3, Remark 3.12], U' is dominant if and only if ϕ is surjective.

2) Let X be an object of \mathcal{C}_2 . We are to show that X is a subobject of $U(Y)$ for some $Y \in \text{Ob}(\mathcal{C}_1)$. Now $F_2U = U''F_1$ is dominant, so $F_2(X) \subset F_2U(Z)$ for some $Z \in \text{Ob}(\mathcal{C}_1)$. Denote by R_2 the right adjoint of F_2 ; being a right adjoint, it preserves monomorphisms so $R_2F_2(X) \subset R_2F_2U(Z)$. We have $A_2 = R_2(\mathbf{1})$, and we have an isomorphism $RF_2 \simeq A_2 \otimes ?$ coming from the fact that the adjunction (F_2, R) is a Hopf monoidal adjunction (see [3, Proof of Proposition 6.1], and [2] for the original statement in terms of Hopf monads).

Thus $A_2 \otimes X \subset A_2 \otimes U(Z)$. Since we have $\mathbf{1} \subset A_2$, we obtain $X \subset A_2 \otimes U(Z)$. On the other hand, A_2 belongs to the essential image of i_2 and U' is dominant, so there exists $Z' \in \text{Ob}(\mathcal{C}'_1)$ such that $A_2 \subset i_2U'(Z') \simeq Ui_1(Z')$. Consequently $X \subset U(i_1(Z') \otimes Z)$, which shows that U is dominant, as claimed. \square

2.4. Central exact sequences of finite tensor categories.

If $i : \mathcal{C}' \rightarrow \mathcal{C}$ is a strong monoidal functor between monoidal categories, a *central lifting of i* is a strong monoidal functor $\tilde{i} : \mathcal{C}' \rightarrow \mathcal{Z}(\mathcal{C})$ such that $\mathcal{U}\tilde{i} = i$, where \mathcal{U} denotes the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. Note that if i is full, given a central lifting \tilde{i} there exists a unique braiding on \mathcal{C}' such that \tilde{i} is braided.

An exact sequence of finite tensor categories $\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$, with induced central algebra $\mathbf{A} = (A, \sigma)$, is called *central* if the restriction of the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ induces an equivalence of categories $\langle \mathbf{A} \rangle \rightarrow \langle A \rangle$.

THEOREM 2.4. Consider an exact sequence of finite tensor categories

$$(\mathcal{E}) \quad \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

with induced central algebra (A, σ) and induced central coalgebra $(\hat{C}, \hat{\sigma})$. The following assertions are equivalent:

- (i) The exact sequence (\mathcal{E}) is central;
- (ii) There exists a central lifting \tilde{i} of i such that $\tilde{i}(A) = (A, \sigma)$;
- (iii) There exists a central lifting \tilde{i} of i such that $\tilde{i}(\hat{C}) = (\hat{C}, \hat{\sigma})$.

Moreover, if these assertions hold, the central liftings \tilde{i} of i appearing in assertions (ii) and (iii) are essentially unique and they coincide.

The central lifting \tilde{i} is called the *canonical central lifting of the central exact sequence* (\mathcal{E}) .

PROOF. Assertions (ii) and (iii) are equivalent because (A, σ) is the right dual of $(\hat{C}, \hat{\sigma})$ and strong monoidal functors preserve duals.

Denote by j the tensor functor $\langle\langle A, \sigma \rangle\rangle \rightarrow \langle A \rangle$ induced by the forgetful functor \mathcal{U} . In particular $j(\mathbf{A}) = A$.

It follows from exactness of the sequence (\mathcal{E}) that $i : \mathcal{C}' \rightarrow \mathcal{C}$ induces an equivalence $\mathcal{C}' \simeq \langle A \rangle$ (see Lemma 2.1).

We have (ii) \implies (i) because if \tilde{i} is a central lifting of i such that $\tilde{i}(A) = (A, \sigma)$, then $\tilde{i}(\langle A \rangle) \subset \langle\langle A, \sigma \rangle\rangle$ and by definition of a central lifting, $j\tilde{i} = \text{id}_{\langle A \rangle}$. This shows that j is full and essentially surjective; since on the other hand j is faithful, it is therefore an equivalence, with quasi-inverse \tilde{i} .

We have (i) \implies (ii) because if j is an equivalence, then it admits a quasi-inverse k , which is also a tensor functor. One defines k by picking for each object X of $\langle A \rangle$ an object $k(X)$ in $\mathcal{Z}(\mathcal{C})$ such that $\mathcal{U}k(X) \simeq X$. One may further impose that $\mathcal{U}k(X) = X$, and $k(A) = (A, \sigma)$. Then $\tilde{i} : \langle A \rangle \rightarrow \mathcal{Z}(\mathcal{C}), X \mapsto k(X)$ is a central lifting of i sending A to (A, σ) , which proves assertion (ii).

If \tilde{i} exists, it is a quasi-inverse of j and as such, it is essentially unique. □

EXAMPLE 2.5. If G is a finite group acting on a tensor category \mathcal{C} by tensor autoequivalences, then the corresponding exact sequence of tensor categories

$$\text{rep } G \xrightarrow{i} \mathcal{C}^G \xrightarrow{F} \mathcal{C}$$

is central. Indeed, if (V, r) is a representation of G and (X, ρ) an object of \mathcal{C}^G , one verifies that the trivial isomorphism $V \otimes X \xrightarrow{\sim} X \otimes V$ lifts to an isomorphism $\sigma_{(V,r),(X,\rho)} = i(V, r) \otimes (X, \rho) \simeq (X, \rho) \otimes i(V, r)$ in \mathcal{C}^G , and this defines a central lifting $\tilde{i} : \text{rep } G \rightarrow \mathcal{Z}(\mathcal{C}^G)$ of i , $(V, r) \mapsto ((V, r), \sigma_{(V,r),-})$. Moreover, the induced central algebra (A, σ) of F is defined by $A = \mathbf{k}^G$, with G -action defined by right translations, and $\sigma = \sigma_{A,-}$, hence centrality of the exact sequence.

If $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is a central exact sequence of tensor categories, then \mathcal{C}' is symmetric. More precisely, we have the following lemma.

PROPOSITION 2.6. *Consider a central exact sequence*

$$(\mathcal{E}) \quad \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}'',$$

with canonical central lifting \tilde{i} . Then the induced Hopf algebra H of (\mathcal{E}) is commutative, so that $\mathcal{C}' \simeq \text{comod } H$ is endowed with a symmetry, and with this symmetry on \mathcal{C}' , the tensor functor $\tilde{i} : \mathcal{C}' \rightarrow \mathcal{Z}(\mathcal{C})$ is braided. If in addition H is split semisimple, then $G = \text{Spec } H$ is a discrete finite group and $\mathcal{C} \simeq \text{rep } G$.

REMARK 2.7. As a special case of Proposition 2.6, for any finite dimensional Hopf algebra H the corresponding exact sequence

$$\text{comod-}H \rightarrow \text{comod-}H \rightarrow \text{vect}_{\mathbf{k}}$$

is central if and only if H is commutative.

REMARK 2.8. Note that if $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is a central exact sequence of fusion categories over an algebraically closed field of characteristic 0, $G = \text{Spec } H$ is a discrete finite group and we have $\mathcal{C}' \simeq \text{rep } G$.

PROOF OF PROPOSITION 2.6. Let (A, σ) be the induced central algebra of (\mathcal{E}) . We may replace (\mathcal{E}) with the equivalent exact sequence

$$\mathfrak{Ker}_F = \langle A \rangle \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}''.$$

Consider the morphism of exact sequences of tensor categories:

$$\begin{array}{ccccc} \langle A \rangle & \longrightarrow & \langle A \rangle & \longrightarrow & \langle \mathbf{1} \rangle \\ \downarrow = & & \downarrow \text{incl.} & & \downarrow \text{incl.} \\ \langle A \rangle & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' \end{array}$$

Denote by (\mathcal{E}_0) the top exact sequence in this diagram. Its induced central algebra is $(A, \sigma|_{\langle A \rangle})$. Moreover it is central, with canonical lifting \tilde{i}_0 defined by $\tilde{i}_0(X) = (X, s_{X|_{\langle A \rangle}})$, where $\tilde{i}(X) = (X, s_X)$ denotes the canonical lifting of (\mathcal{E}) .

Thus, it is enough to prove the theorem for (\mathcal{E}_0) . We may again replace (\mathcal{E}_0) by the equivalent exact sequence

$$\text{comod-}H \rightarrow \text{comod-}H \rightarrow \text{vect}_{\mathbf{k}},$$

where H is the induced Hopf algebra of (\mathcal{E}) , which is the situation of Remark 2.7.

In that situation, we have $A = (H, \Delta)$ and the half-braiding σ is defined by

$$\begin{aligned} \sigma_V : H \otimes V &\rightarrow V \otimes H \\ h \otimes v &\mapsto v_{(0)} \otimes S(v_{(1)})hv_{(2)} \end{aligned} \tag{2.1}$$

in Sweedler’s notation, for any finite dimensional right H -comodule V (see [3, Example 6.3]).

Centrality of the exact sequence means that we have a central lifting of the identity of $\text{comod-}H$, which is nothing but a braiding c on $\text{comod-}H$, and in addition this braiding is required to be such that $c_{A,?} = \sigma$. What we have to prove is that H is commutative and c is the standard symmetry.

Let $r : H \otimes H \rightarrow \mathbf{k}$ be the coquasitriangular structure corresponding to this braiding,

so that for any pair of right comodules V and W , we have

$$c_{V,W} : V \otimes W \rightarrow W \otimes V, \quad c_{V,W}(v \otimes w) = r(v_{(1)}, w_{(1)})w_{(0)} \otimes v_{(0)}. \tag{2.2}$$

Comparing 2.1 and 2.2, we obtain by a straightforward computation that the condition $c_{A,A} = \sigma_A$ implies $r = \varepsilon \otimes \varepsilon$. That means that the forgetful functor $\text{comod-}H \rightarrow \text{vect}_{\mathbf{k}}$ is braided, so H is commutative and c is the standard symmetry. This concludes the proof of the proposition. \square

2.5. Normality and centrality criteria.

The following theorem gives a sufficient condition for a dominant tensor functor between finite tensor categories to be normal in terms of the induced central algebra.

THEOREM 2.9. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between finite tensor categories \mathcal{C}, \mathcal{D} , and let (A, σ) be its induced central algebra. Assume that A decomposes as a direct sum of invertible objects of \mathcal{C} . Then:*

- (i) *The functor F is normal, the isomorphism classes of simple direct summands of A form a group Γ , and we have an exact sequence of tensor categories*

$$(\mathcal{E}) \quad \Gamma\text{-vect} \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D},$$

where $\Gamma\text{-vect}$ denotes the tensor category of finite dimensional Γ -graded vector spaces.

- (ii) *If in addition (A, σ) decomposes as a direct sum of invertible objects in $\mathcal{Z}(\mathcal{C})$, then the exact sequence (\mathcal{E}) is central.*

PROOF. An invertible object in a tensor category is both simple and scalar. Let R denote the right adjoint of F . For any invertible object g of \mathcal{C} , we have by adjunction $\dim \text{Hom}_{\mathcal{C}}(g, A) = \dim \text{Hom}_{\mathcal{D}}(F(g), \mathbf{1})$, because $A = R(\mathbf{1})$. Now $F(g)$ is invertible in \mathcal{D} , so $\dim \text{Hom}_{\mathcal{C}}(g, A) = 1$ if $F(g) \simeq \mathbf{1}$, and $\dim \text{Hom}_{\mathcal{C}}(g, A) = 0$ otherwise.

In other words: (1) the invertible factors of A are exactly the invertible objects of \mathcal{C} which are trivialized by F ; therefore, they form a group for the tensor product; and (2) their multiplicity in A is exactly one.

In particular if A is the direct sum of its invertible factors, then A itself is trivialized by F , that is, F is normal. In that case, we have an exact sequence $\mathfrak{Ker}_F \rightarrow \mathcal{C} \rightarrow \mathcal{D}$. Now \mathfrak{Ker}_F is the tensor subcategory of \mathcal{C} generated by A ; it is a pointed tensor category whose invertible objects are the invertible factors of A , whose isomorphism classes form a group Γ . So \mathfrak{Ker}_F is a pointed tensor category; in addition, \mathfrak{Ker}_F admits a fiber functor, hence it is tensor equivalent to the category $\Gamma\text{-vect}$. This proves assertion (i).

Now assume that (A, σ) decomposes as a direct sum of invertible objects of $\mathcal{Z}(\mathcal{C})$, that is, $(A, \sigma) = \bigoplus_{i=1}^n (g_i, \sigma_i)$. Then $A = \bigoplus_{i=1}^n g_i$, where the g_i 's are invertible in A , so that the first part of the theorem applies. The category $\langle (A, \sigma) \rangle$ is generated as a tensor category by the (g_i, σ_i) . Let us show that $\langle (A, \sigma) \rangle$ is *additively* generated by the (g_i, σ_i) . For this it is enough to show that $(g_i, \sigma_i) \otimes (g_j, \sigma_j)$ is a direct factor of (A, σ) for all $i, j \in \{1, \dots, n\}$. Now the product $\mu : A \otimes A \rightarrow A$ embeds $g_i \otimes g_j$ into A , and lifts to a

morphism in $\mathcal{Z}(\mathcal{C})$, namely the product of (A, σ) ; consequently it embeds $(g_i, \sigma_i) \otimes (g_j, \sigma_j)$ into (A, σ) . Thus $\langle (A, \sigma) \rangle \rightarrow \langle A \rangle$ is full, which proves that (\mathcal{E}) is central. \square

2.6. Example: Tambara-Yamagami categories.

A Tambara-Yamagami category is a fusion category having exactly one non-invertible simple object X , with the additional condition that X is not a factor of $X \otimes X$. These categories, which are in a sense the simplest non-pointed categories, have been classified in [17].

Let $\mathcal{T}\mathcal{Y}$ be a Tambara-Yamagami category. Denote by Γ the Picard group of $\mathcal{T}\mathcal{Y}$. It is a finite abelian group. Denote by X a non-invertible simple object. The maximal pointed fusion subcategory of \mathcal{C} , denoted by \mathcal{C}_{pt} , is tensor equivalent to the category Γ -vect of finite dimensional Γ -graded vector spaces. The following proposition characterizes normal tensor functors on \mathcal{C} .

PROPOSITION 2.10. *Let $\mathcal{T}\mathcal{Y}$ be a Tambara-Yamagami category, with Picard group Γ , and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor, with induced central algebra (A, σ) . Then F is normal if and only if F is a fiber functor, or A belongs to \mathcal{C}_{pt} .*

In the latter case, we have an exact sequence of tensor categories:

$$G\text{-vect} \longrightarrow \mathcal{T}\mathcal{Y} \xrightarrow{F} \mathcal{D},$$

where G is a subgroup of Γ .

PROOF. The only proper fusion subcategories of \mathcal{C} are those contained in \mathcal{C}_{pt} . This shows the ‘only if’ direction. Conversely, any fiber functor F on \mathcal{C} is normal, with $\mathfrak{Ker}_F = \mathcal{C}$. Suppose on the other hand that F is not a fiber functor. Then F is normal by Theorem 2.9. This finishes the proof of the proposition. \square

It is known that if a Tambara-Yamagami category $\mathcal{T}\mathcal{Y}$ admits a fiber functor, so that $\mathcal{T}\mathcal{Y} \simeq \text{rep } H$ for some semisimple Hopf algebra H , then H fits into an abelian exact sequence of Hopf algebras $\mathbf{k}^{\mathbf{Z}_2} \rightarrow H \rightarrow \mathbf{k}\Gamma$ [12]. Hence in this case $\mathcal{T}\mathcal{Y}$ fits into an exact sequence of fusion categories $\text{rep } \Gamma \rightarrow \mathcal{T}\mathcal{Y} \rightarrow \text{rep } \mathbf{Z}_2$. In particular, $\mathcal{T}\mathcal{Y}$ is not simple.

2.7. Extensions of weakly integral fusion categories.

Recall that a fusion category is weakly integral if its Frobenius-Perron dimension is a natural integer. In this section we discuss dominant tensor functors on weakly integral fusion categories.

LEMMA 2.11. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories, with induced central algebra (A, σ) . Then we have:*

- (i) $\text{FPind}(\mathcal{C} : \mathcal{D}) = \text{FPdim } A$.
- (ii) *If $X \in \text{Irr}(\mathcal{C})$, then $m_X(A) \leq \text{FPdim } X$.*
- (iii) *F is normal if and only if for all $X \in \text{Irr}(\mathcal{C})$ we have $m_X(A) = 0$ or $m_X(A) = \text{FPdim } X$.*

PROOF. By [3, Proposition 4.3], we have $\text{FPind}(\mathcal{C} : \mathcal{D}) = \text{FPdim } R(\mathbf{1})$. This

proves (i) since $A = R(\mathbf{1})$.

By adjunction, we have $\text{Hom}_{\mathcal{D}}(F(X), \mathbf{1}) \simeq \text{Hom}_{\mathcal{C}}(X, A)$, so that $m_X(A) = m_{\mathbf{1}}(F(X))$. This implies (ii) since $m_{\mathbf{1}}(F(X)) \leq \text{FPdim } F(X) = \text{FPdim } X$.

We next show (iii). The only if part follows from [3, Proposition 6.9]. Conversely, suppose that for all $X \in \text{Irr}(\mathcal{C})$ we have $m_{\mathbf{1}}(F(X)) = m_X(A) \in \{0, \text{FPdim } X\}$. Let $X \in \text{Irr}(\mathcal{C})$ and assume $m_{\mathbf{1}}(F(X)) \neq 0$. Then $m_{\mathbf{1}}(F(X)) = \text{FPdim } X = \text{FPdim}(F(X))$, so $F(X)$ is trivial. Thus F is normal, which completes the proof of (iii) and of the lemma. \square

PROPOSITION 2.12. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories \mathcal{C} and \mathcal{D} and let (A, σ) be the induced central algebra of F . Then the following assertions are equivalent:*

- (i) \mathcal{C} is weakly integral.
- (ii) \mathcal{D} is weakly integral and $\text{FPdim } A \in \mathbf{Z}$.

PROOF. (ii) \Rightarrow (i) results immediately from Lemma 2.11 (i).

(i) \Rightarrow (ii). Notice first that since $\text{FPdim } A \text{FPdim } \mathcal{D} = \text{FPdim } \mathcal{C}$ and $\text{FPdim } A$ is an algebraic integer, it is enough to verify that $\text{FPdim } \mathcal{D}$ is a natural integer, that is, \mathcal{D} is weakly integral.

Recall that \mathcal{D} is tensor equivalent to the fusion category \mathcal{C}_A of right A -modules in \mathcal{C} . Since A is an indecomposable algebra in \mathcal{C} , the category ${}_A\mathcal{C}_A$ of A -bimodules in \mathcal{C} is a fusion category and it satisfies $\text{FPdim } {}_A\mathcal{C}_A = \text{FPdim } \mathcal{C}$ [6, Corollary 8.14]. Therefore ${}_A\mathcal{C}_A$ is weakly integral. We have a full tensor embedding $\mathcal{C}_A \subset {}_A\mathcal{C}_A$.

Now, in a weakly integral fusion category the Frobenius-Perron dimensions of simple objects are square roots of natural integers [6, Proposition 8.27], and as a result, a full fusion subcategory of a weakly integral fusion category is weakly integral. So $\mathcal{D} \simeq \mathcal{C}_A$ is weakly integral, and we are done. \square

In the case where the functor F is normal, we have $\text{FPdim } A = \text{FPdim } \mathfrak{Ker}_F$, and since \mathfrak{Ker}_F admits a fibre functor it is weakly integral. Thus we have:

COROLLARY 2.13. *Let $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ be an exact sequence of fusion categories. Then \mathcal{C} is weakly integral if and only if \mathcal{C}'' is weakly integral. In particular, the class of weakly integral fusion categories is closed under extensions.* \square

3. Equivariantization revisited.

The aim of this section is to state and discuss equivariantization criteria. In order arrive at a synthetic statement, we have to extend the notion of equivariantization to actions of finite group schemes. Thanks to this generalization, we can state that an exact sequence of finite tensor categories is central if and only if its Hopf monad is normal cocommutative, and that it is an equivariantization exact sequence if and only if it is perfect and central, which extends a result of [3] concerning discrete groups, and also reformulates and extends a result of [7] concerning fusion categories.

3.1. Cocommutative normal Hopf monads.

Let \mathcal{C} be a tensor category. A k -linear right exact normal Hopf monad T on \mathcal{C} is *cocommutative* (see [3]) if for any morphism $x : T(\mathbf{1}) \rightarrow \mathbf{1}$ and any object X of \mathcal{C}

$$(x \otimes TX)T_2(\mathbf{1}, X) = (TX \otimes x)T_2(X, \mathbf{1}).$$

Note that, if V is a trivial object, and X is an *arbitrary* object of \mathcal{C} , there is a canonical isomorphism $\tau_{V,X} : V \otimes X \xrightarrow{\sim} X \otimes V$, which is characterized by the fact that for all $x : V \rightarrow \mathbf{1}$, we have $(X \otimes x)\tau_{V,X} = x \otimes X$. The natural isomorphism $\tau_{V,-}$ is a half-braiding, called the *trivial half-braiding* of V .

We have the following characterizations of normal cocommutative Hopf monads.

LEMMA 3.1. *Let \mathcal{C} be a tensor category and let T be a normal Hopf monad on \mathcal{C} , with induced central coalgebra $(\hat{C}, \hat{\sigma})$. The following assertions are equivalent:*

- (i) T is cocommutative;
- (ii) $T_2(X, \mathbf{1}) = \tau_{T\mathbf{1}, TX}T_2(\mathbf{1}, X)$ for X in \mathcal{C} ;
- (iii) $\hat{\sigma}_{(M,r)} = \tau_{T\mathbf{1}, M}$ for (M, r) in \mathcal{C}^T , or in short: $\hat{\sigma}$ ‘is the trivial half-braiding’.

PROOF. Assertion (ii) is just a reformulation of the definition of cocommutativity in terms of trivial half-braidings, so (i) \iff (ii). Let (M, r) be a T -module. Since T is a Hopf monad, we have fusion isomorphisms

$$\begin{aligned} \Phi^r_{(M,r)} &= (T\mathbf{1} \otimes r)T_2(\mathbf{1}, M) : TM \xrightarrow{\sim} T\mathbf{1} \otimes M, \\ \Phi^l_{(M,r)} &= (r \otimes T\mathbf{1})T_2(M, \mathbf{1}) : TM \xrightarrow{\sim} M \otimes T\mathbf{1}, \end{aligned}$$

and by definition $\hat{\sigma} = \Phi^l\Phi^{r-1}$. If (ii) holds, we have $\Phi^l_{(M,r)} = \tau_{T\mathbf{1}, M}\Phi^r_{(M,r)}$ by functoriality of τ , so $\hat{\sigma}_{(M,r)} = \tau_{T\mathbf{1}, M}$, which shows (ii) \implies (iii). Conversely, applying (iii) to $(M, r) = (TX, \mu_X)$ and composing on the right by $T(\eta_X)$ gives $T_2(X, \mathbf{1}) = \tau_{T\mathbf{1}, TX}T_2(\mathbf{1}, X)$, so (iii) \implies (ii). \square

From [3, Theorem 5.21 and Theorem 5.24], one deduces immediately

PROPOSITION 3.2. *A dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between finite tensor categories is an equivariantization under the action of a finite group G if and only if the following two conditions are met:*

- (1) the Hopf monad T of F is normal and cocommutative;
- (2) the induced Hopf algebra H of F is split semisimple.

If these conditions hold, then F is perfect, that is T is exact, and $G = \text{Spec}(H)$.

This suggests that a (perfect) dominant tensor functor between finite tensor categories is an equivariantization under the action of a finite group scheme if and only if its Hopf monad is normal cocommutative; the group scheme being the spectrum of the induced Hopf algebra of T - which, in this case, is a finite dimensional commutative Hopf algebra.

We will now define group scheme actions in order to give a mathematical meaning to this claim, and then prove it.

3.2. Group scheme actions.

Let \mathcal{A} be a monoidal category and let \mathcal{M} be a category. An action of \mathcal{A} on \mathcal{M} is a strong monoidal functor $\rho : \mathcal{A} \rightarrow \underline{\text{End}}(\mathcal{M})$, where $\underline{\text{End}}(\mathcal{M})$ denotes the strict monoidal category of endofunctors of \mathcal{M} . Given such an action ρ , we say that \mathcal{M} is an \mathcal{A} -module category, and we usually write $\rho(a, m) = a \odot m$. Let $\mathcal{M}, \mathcal{M}'$ be two \mathcal{A} -module categories. A functor of \mathcal{A} -module categories $\mathcal{M} \rightarrow \mathcal{M}'$ is a pair (F, F_2) , where $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor and F_2 is a natural isomorphism $F_2(a, m) : a \odot F(m) \xrightarrow{\simeq} F(a \odot m)$, $a \in \mathcal{A}, m \in \mathcal{M}$, such that the following diagrams commute:

$$\begin{array}{ccc}
 F((a \otimes b) \odot m) \xrightarrow{\simeq} F(a \odot (b \odot m)) & & F(\mathbf{1} \odot m) \\
 \downarrow F_2(a \otimes b, m) & \downarrow F_2(a, b \odot m) & \searrow \simeq \\
 (a \otimes b) \odot F(m) & a \odot F(b \odot m) & F(m) \\
 \downarrow & \downarrow a \odot F_2(b, m) & \nearrow \simeq \\
 (a \otimes b) \odot F(m) \xrightarrow{\simeq} a \odot (b \odot F(m)) & & \mathbf{1} \odot F(m) \\
 & & \downarrow F_2(\mathbf{1}, m)
 \end{array}$$

where the unlabeled isomorphisms come from the monoidal structure of the action.

Now assume \mathcal{A} is endowed with a strong monoidal functor $\omega : \mathcal{A} \rightarrow \text{vect}_{\mathbf{k}}$, \mathcal{M} is a \mathbf{k} -category, and ρ is an action of \mathcal{A} on \mathcal{M} by \mathbf{k} -linear endofunctors. The *equivariantization of \mathcal{M} under the action ρ* is the category \mathcal{M}^ρ , also denoted by $\mathcal{M}^\mathcal{A}$, defined as follows. Objects of \mathcal{M}^ρ are data (m, α) where m is an object of \mathcal{M} and $\alpha = (\alpha_c^\lambda)_{c \in \text{Ob}(\mathcal{A}), \lambda \in \omega(c)^*}$ is a family of morphisms $\alpha_c^\lambda : c \odot m \rightarrow m$ satisfying the following conditions:

- (1) *functoriality*: α_c^λ is linear in λ , and if $f : c \rightarrow c'$ is a morphism in \mathcal{A} and $\lambda \in \omega(c')^*$, then $\alpha_{c'}^\lambda(f \odot m) = \alpha_c^{f^* \lambda}$, where $f^* \lambda = \lambda \omega(f)$;
- (2) *ρ -compatibility*: we have commutative diagrams

$$\begin{array}{ccc}
 (a \otimes b) \odot m \xrightarrow{\alpha_{a \otimes b}^{\lambda \otimes \mu}} m & & \mathbf{1} \otimes m \xrightarrow{\alpha_{\mathbf{1}}^{\omega_0}} m \\
 \downarrow \simeq & \uparrow \alpha_a^\lambda & \searrow \simeq \\
 a \odot (b \odot m) \xrightarrow{a \odot \alpha_b^\mu} a \odot m & & m
 \end{array}$$

with a, b objects of \mathcal{A} and $\lambda \in \omega(a)^*, \mu \in \omega(b)^*$.

Morphisms in $\mathcal{M}^\mathcal{A}$ from (m, α) to (n, β) are morphisms $f : m \rightarrow n$ in \mathcal{M} satisfying $f\alpha = \beta(a \odot f)$.

Note that if G is a discrete group, viewed as a monoidal category \underline{G} whose objects are the elements of the group, and equipped with the trivial strict monoidal functor $\omega : \underline{G} \rightarrow \mathbf{k}, g \mapsto \mathbf{k}$, then a \underline{G} -action is the same thing as a G -action in the usual sense, and in the case of a \mathbf{k} -linear action on a \mathbf{k} -category \mathcal{M} , $\mathcal{M}^{\underline{G}}$ is isomorphic to the usual equivariantization \mathcal{M}^G .

PROPOSITION 3.3. *Let T be a \mathbf{k} -linear faithful exact normal Hopf monad on a tensor category \mathcal{C} , with induced Hopf algebra H . Let $K = H^*$ and $\mathcal{L} = \text{comod-}K$. Then there is a natural action of \mathcal{L} on \mathcal{C} by \mathbf{k} -linear endofunctors, defined by*

$$V \odot X = V \square^K T(X),$$

and $\mathcal{C}^{\mathcal{L}}$ is canonically isomorphic to \mathcal{C}^T as a \mathbf{k} -linear category.

PROOF. For simplicity, we identify a finite dimensional vector space E with the trivial object $E \otimes \mathbf{1}$ in \mathcal{C} . Then $T\mathbf{1} = H$ and the comonoidal structure of T defines a structure of H -bicomodule on $T(X)$. This enables us to define $V \odot X = V \square^K T(X)$ for V a finite-dimensional right K -comodule and X in \mathcal{C} . From the fact that T is a faithful exact Hopf monad, one deduces natural isomorphisms $(V \otimes W) \odot X \simeq V \odot (W \odot X)$ and $(\mathbf{k}, \varepsilon) \odot X \simeq X$ which make \odot an action of \mathcal{L} on \mathcal{C} . Let $A = (H, \Delta)$ be the trivializing algebra of \mathcal{L} . Then A generates \mathcal{L} . An object (m, α) of $\mathcal{C}^{\mathcal{L}}$ is entirely determined by $\alpha_A^\varepsilon : A \odot X \simeq T(X) \rightarrow X$, which can be interpreted as a T -action on X because we have a canonical isomorphism $A \odot X \simeq T(X)$. This defines a \mathbf{k} -linear isomorphism $\mathcal{C}^{\mathcal{L}} \xrightarrow{\sim} \mathcal{C}^T$. □

Note that the action of Proposition 3.3 is not compatible in any clear way with the tensor product of \mathcal{C} . In order to take care of the monoidal structure of \mathcal{C} , we now introduce the notion of \mathcal{L} -module tensor category.

Denote by $\mathbf{ab}_{\mathbf{k}}$ the 2-category of abelian \mathbf{k} -linear categories having finite dimensional Hom spaces and objects of finite length, 1-morphisms being \mathbf{k} -linear left exact functors, and 2-morphisms being natural transformations. We equip $\mathbf{ab}_{\mathbf{k}}$ with a tensor product *à la* Deligne, denoted by \boxtimes , and characterized by the fact that given three objects $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ in $\mathbf{ab}_{\mathbf{k}}$, the category of \mathbf{k} -linear left exact functors $\mathcal{M} \boxtimes \mathcal{M}' \rightarrow \mathcal{M}''$ is equivalent to the category of functors $\mathcal{M} \times \mathcal{M}' \rightarrow \mathcal{M}''$ which are \mathbf{k} -linear left exact in each variable. This tensor product makes $\mathbf{ab}_{\mathbf{k}}$ a monoidal 2-category with unit object $\text{vect}_{\mathbf{k}}$, with a symmetry $\tau_{\mathcal{M}, \mathcal{M}'} : \mathcal{M} \boxtimes \mathcal{M}' \xrightarrow{\sim} \mathcal{M}' \boxtimes \mathcal{M}$ defined by $m \boxtimes m' \mapsto m' \boxtimes m$.

Now if \mathcal{A} is a tensor category over \mathbf{k} , define an \mathcal{A} -module category to be an object \mathcal{C} of $\mathbf{ab}_{\mathbf{k}}$ endowed with a \mathbf{k} -linear action of \mathcal{A} such that the functor $\odot : \mathcal{A} \times \mathcal{C} \rightarrow \mathcal{C}$ is \mathbf{k} -linear right exact in the first variable (it is automatically exact in the second variable because \mathcal{A} is autonomous). Thus we may view \odot as a \mathbf{k} -linear right exact functor $\mathcal{A} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$.

Let L be a finite dimensional cocommutative Hopf algebra. The tensor category $\mathcal{L} = \text{comod-}L$ is tannakian; it is endowed with a strong monoidal symmetric functor $\Delta_* : \mathcal{L} \rightarrow \mathcal{L} \boxtimes \mathcal{L}$, which is coassociative, and the symmetric fiber functor $\varepsilon_* : \mathcal{L} \rightarrow \text{vect}_{\mathbf{k}}$ is a counit for Δ_* . Thus \mathcal{L} is a bialgebra in the monoidal 2-category $\mathbf{ab}_{\mathbf{k}}$.

If (\mathcal{M}, ρ) and (\mathcal{M}', ρ') are two \mathcal{L} -module categories then one defines a new \mathcal{L} -module category $(\mathcal{M}, \rho) \boxtimes (\mathcal{M}', \rho') = (\mathcal{M} \boxtimes \mathcal{M}', \rho'')$, where

$$\rho'' = (\rho \boxtimes \rho')(\mathcal{L} \boxtimes \tau_{\mathcal{L}, \mathcal{M}} \boxtimes \mathcal{M}')(\Delta_* \boxtimes \mathcal{M} \boxtimes \mathcal{M}'), \tag{3.1}$$

and this tensor product defines a monoidal structure on the 2-category of \mathcal{L} -module categories.

An \mathcal{L} -module tensor category is a tensor category \mathcal{C} over \mathbf{k} , endowed with

- a structure of \mathcal{L} -module $\odot : \mathcal{L} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ of \mathcal{L} on \mathcal{C} ;
- natural isomorphisms

$$\alpha_{V,X,Y} : V \odot (X \otimes Y) \xrightarrow{\sim} \otimes(V \odot (X \boxtimes Y))$$

$$\beta : V \odot \mathbf{1} \xrightarrow{\sim} \epsilon_*(V) \otimes \mathbf{1}$$

making the tensor product $\otimes = \otimes_{\mathcal{C}} : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ and the unit functor $u : \text{vect}_{\mathbf{k}} \rightarrow \mathcal{C}$, $\mathbf{k} \mapsto \mathbf{1}$, morphisms of \mathcal{L} -module categories, where the \mathcal{L} -module category structure $\odot : \mathcal{L} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}$ is the one defined by (3.1).

If \mathcal{C} is an \mathcal{L} -module tensor category, then $\mathcal{C}^{\mathcal{L}}$ is monoidal.

PROPOSITION 3.4. *Let \mathcal{C} be a tensor category over \mathbf{k} .*

- (1) *Let L be a finite dimensional cocommutative Hopf algebra, $\mathcal{L} = \text{comod-}L$. Then a structure of \mathcal{L} -module tensor category on \mathcal{C} defines a \mathbf{k} -linear faithful exact Hopf monad $T = A \odot ?$ on \mathcal{C} , where $A = (H, \Delta)$;*
- (2) *If T is a normal cocommutative \mathbf{k} -linear faithful exact Hopf monad on \mathcal{C} , the action of $\mathcal{L} = \text{comod-}L$ on \mathcal{C} defined in Proposition 3.3 makes \mathcal{C} a \mathcal{L} -module tensor category.*

Moreover, these construction are essentially mutually inverse. Given a Hopf monad as in Assertion (2) and the corresponding structure of \mathcal{L} -module tensor category on \mathcal{C} , the canonical isomorphism $\mathcal{C}^{\mathcal{L}} \simeq \mathcal{C}^T$ is a tensor isomorphism.

PROOF. Assume we have a structure of \mathcal{L} -module tensor category on \mathcal{C} , with action $\rho : \mathcal{L} \rightarrow \underline{\text{End}}(\mathcal{C})$, and set $T = \rho(A) = A \odot ?$. Then T is a monad on \mathcal{C} , that is, an algebra in $\underline{\text{End}}(\mathcal{C})$, because $A = (H, \Delta)$ is an algebra in \mathcal{C} and ρ is strong monoidal. Moreover T is \mathbf{k} -linear exact, and it is faithful because ρ is right exact and $\mathbf{1}$ is a subobject of A , so $X = \mathbf{1} \odot X$ is a subobject of $A \odot X = T(X)$. Moreover the \mathcal{L} -module tensor category structure defines isomorphisms $T(X \otimes Y) \simeq T(X) \square^K T(Y)$ and $T(\mathbf{1}) = A \odot \mathbf{1} \simeq K \otimes \mathbf{1}$, which define a Hopf monad structure on T , which is normal and clearly cocommutative.

Conversely, if T is a normal cocommutative \mathbf{k} -linear faithful exact Hopf monad on \mathcal{C} , consider the action ρ of \mathcal{L} on \mathcal{C} defined by $V \odot X = V \square^K T(X)$. Then $X \mapsto \rho(X)$ is \mathbf{k} -linear left exact, so the action may be viewed as a \mathbf{k} -linear exact functor $\mathcal{L} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$. The structure of Hopf monad, cocommutativity and normality define isomorphisms $A \odot (X \otimes Y) \simeq (A \odot X) \square^K (A \odot Y)$ and $A \odot \mathbf{1} \simeq L$, which give rise to structures of morphisms of \mathcal{L} -module morphisms on the tensor product $\mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ and the unit functor $\text{vect}_{\mathbf{k}} \rightarrow \mathcal{C}$, making ρ a structure of \mathcal{L} -module tensor category on \mathcal{C} . □

3.3. Equivariantization and centrality criteria.

Let G be a finite group scheme over \mathbf{k} . A tensor action of G on a tensor category \mathcal{C} is a structure of \mathcal{L} -module tensor category on \mathcal{C} , where $\mathcal{L} = \text{comod-}\mathbf{k}[G] = \mathcal{O}(G)\text{-mod}$, where $\mathcal{O}(G)$ is the Hopf algebra of regular functions on G . The equivariantization of \mathcal{C} under a tensor action of a finite group scheme G is the tensor category $\mathcal{C}^G = \mathcal{C}^{\mathcal{L}}$.

From the results of the previous section, we deduce:

THEOREM 3.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a normal dominant tensor functor between finite tensor categories, and let T be its Hopf monad and H its induced Hopf algebra. The following assertions are equivalent:*

- (i) *The tensor functor F is an equivariantization under the tensor action of a finite group scheme on \mathcal{D} ;*
- (ii) *The normal Hopf monad T is exact and cocommutative.*

If these assertions hold then the induced Hopf algebra H of F is commutative and the group scheme of assertion (i) is $G = \text{Spec } H$.

THEOREM 3.6. *Let $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ be an exact sequence of finite tensor categories, and let T be the associated normal Hopf monad on \mathcal{C}'' . Then the following assertions are equivalent:*

- (i) *The exact sequence $\mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$ is central;*
- (ii) *The normal Hopf monad T is cocommutative.*

Theorem 3.6 will be proved in Section 5.3.

COROLLARY 3.7. *Consider a morphism of exact sequences of finite tensor categories*

$$\begin{array}{ccccc}
 (\mathcal{E}_0) & & \mathcal{C}'_0 & \longrightarrow & \mathcal{C}_0 & \longrightarrow & \mathcal{C}''_0 \\
 & & \downarrow W & & \downarrow U & & \downarrow V \\
 & & \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' \\
 (\mathcal{E}) & & & & & &
 \end{array}$$

such that the vertical arrows are dominant tensor functors. Then

- (1) *if (\mathcal{E}_0) is central, so is (\mathcal{E}) .*
- (2) *If (\mathcal{E}_0) is an equivariantization exact sequence for a finite group scheme G , with G discrete or (\mathcal{E}) perfect, then (\mathcal{E}) is an equivariantization exact sequence for a subgroup $G' \subset G$ acting on \mathcal{C}'' in a manner compatible with V . Moreover if W is an equivalence, then $G' = G$.*

PROOF. Let T_0, T be the normal Hopf monads, and H_0, H the induced Hopf algebras of the exact sequences (\mathcal{E}_0) and (\mathcal{E}) respectively. Then we may assume that (\mathcal{E}_0) and (\mathcal{E}) are of the form $\text{comod-}H_0 \rightarrow \mathcal{C}''_0{}^{T_0} \rightarrow \mathcal{C}''_0$ and $\text{comod-}H \rightarrow \mathcal{C}''^T \rightarrow \mathcal{C}''$ respectively, and we have a diagram of tensor functors:

$$\begin{array}{ccccc}
 \text{comod-}H_0 & \longrightarrow & \mathcal{C}''_0{}^{T_0} & \xrightarrow{U_{T_0}} & \mathcal{C}''_0 \\
 \downarrow W & & \downarrow U & & \downarrow V \\
 \text{comod-}H & \longrightarrow & \mathcal{C}''^T & \xrightarrow{U_T} & \mathcal{C}''
 \end{array}$$

with U', U, U'' dominant, which commutes up to tensor isomorphisms. By transport of structure, we may assume that $U_T U = V U_{T_0}$ as tensor functors.

Let us assume (\mathcal{E}_0) is central, and let us show that (\mathcal{E}) is central. By Theorem 3.6, T_0 is cocommutative, and we are to prove that T is cocommutative too.

If (X, r) is an object of $\mathcal{C}_0''^{T_0}$ then $U(X, r) = (V(X), \lambda(X, r))$, so that we have a natural transformation $\lambda : TVU_{T_0} \rightarrow VU_{T_0}$, which by adjunction can be encoded as a natural transformation $\Lambda : TV \rightarrow VT_0$ such that

$$U(X, r) = (VX, Vr \Lambda_X), \quad \text{for any } (X, r) \text{ in } \mathcal{C}_0''^{T_0}.$$

The transformation Λ is compatible with the monad structures of T_0 and T , and it is comonoidal because $U_T U = V U_{T_0}$ as tensor functors.

The tensor functor W is induced by a morphism of Hopf algebras $\phi : H_0 \rightarrow H$, which is surjective because W is dominant (see Lemma 2.3). In particular H is commutative, and the group scheme $G = \text{Spec } H$ is a subgroup of the group scheme $G_0 = \text{Spec}(H_0)$ associated with the central exact sequence (\mathcal{E}_0) .

On the other hand, we have $TV(\mathbf{1}) = H^* \otimes \mathbf{1}$ and $VT_0(\mathbf{1}) = H_0^* \otimes \mathbf{1}$, and via these isomorphisms Λ_1 is the transpose of ϕ ; therefore Λ_1 is a monomorphism (in fact, one can show that Λ is monomorphism, a fact we will not use). Denote by $(\hat{C}_0, \hat{\sigma}_0)$ and $(\hat{C}, \hat{\sigma})$ the induced central coalgebras of T_0 and T respectively. Let (M, r) be a T_0 -module. One deduces easily from the comonoidality of Λ that the following diagram commutes:

$$\begin{array}{ccccc}
 T\mathbf{1} \otimes VM & \xrightarrow{\Lambda_1 \otimes VM} & VT_0\mathbf{1} \otimes VM & \xrightarrow{\cong} & V(T_0\mathbf{1} \otimes M) \\
 \downarrow \hat{\sigma}_{U(M,r)} & \swarrow \cong & \downarrow \Lambda_M & \nearrow \cong & \downarrow V((\hat{\sigma}_0)_{(M,r)}) \\
 & TV(M) & \xrightarrow{\Lambda_M} & VT_0(M) & \\
 & \swarrow \cong & & \searrow \cong & \\
 VM \otimes T\mathbf{1} & \xrightarrow{VM \otimes \Lambda_1} & VM \otimes VT_0\mathbf{1} & \xrightarrow{\cong} & V(M \otimes T_0\mathbf{1})
 \end{array}$$

where the slanted arrows are the fusion isomorphisms. Since Λ_1 is a monomorphism and, by assumption, $\hat{\sigma}_0$ is the trivial half-braiding, we see that $\hat{\sigma}_{U(M,r)}$ is the trivial half-braiding. The tensor functor U being dominant, $\hat{\sigma}$ is the trivial half-braiding, that is, T is cocommutative.

In particular if (\mathcal{E}_0) is an equivariantization under the action of a group scheme G , it is central so T is cocommutative. If G is discrete or (\mathcal{E}) is perfect (hence T is exact), then by Proposition 3.2 or Theorem 3.5 (\mathcal{E}) is an equivariantization exact sequence, corresponding with a tensor action of $G \subset G_0$ on \mathcal{C}'' , which by construction is compatible with the tensor action of G_0 on \mathcal{C}_0'' via V . If W is an equivalence, ϕ is an isomorphism so $G = G_0$. □

4. Equivariantization: special cases.

4.1. The braided case.

A *braided exact sequence of tensor categories* is an exact sequence of tensor categories

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

such that the tensor categories \mathcal{C} , \mathcal{C}'' and the tensor functor F are braided. This implies that \mathcal{C}' admits a unique braiding such that i is braided, too.

PROPOSITION 4.1. *A braided exact sequence of finite tensor categories is central. In particular, if it is perfect, it is an equivariantization exact sequence.*

PROOF. Let $\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ be a braided exact sequence of tensor categories. Then the Hopf monad of F is braided by [3, Proposition 5.29], and it is \mathbf{k} -linear, right exact, and normal, so it is cocommutative by [3, Proposition 5.30]. Therefore, the exact sequence is central by Theorem 3.6 and, if it is perfect, it is equivariantization exact sequence by Theorem 3.5. □

4.2. The fusion case.

Recall that if \mathcal{B} is a braided category with braiding c , and $\mathcal{A} \subset \mathcal{B}$ is a set of objects, or a full subcategory of \mathcal{B} , then the *centralizer of \mathcal{A} in \mathcal{B}* , denoted by $C_{\mathcal{B}}(\mathcal{A})$, is the full monoidal subcategory of \mathcal{B} of objects b satisfying $c_{b,a}c_{a,b} = \text{id}_{a \otimes b}$ for any object a in \mathcal{A} . If \mathcal{B} is a fusion braided category, then $C_{\mathcal{B}}(\mathcal{A})$ is a full fusion subcategory of \mathcal{B} (see [11]).

PROPOSITION 4.2. *Let \mathbf{k} be an algebraically closed field of characteristic 0. Consider a central exact sequence of fusion categories*

$$(\mathcal{E}) \quad \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}''$$

with canonical lifting $\tilde{i} : \mathcal{C}' \rightarrow \mathcal{Z}(\mathcal{C})$, and set $\mathcal{A} = \tilde{i}(\mathcal{C}')$. Let $C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})$ denote the centralizer of \mathcal{A} in $\mathcal{Z}(\mathcal{C})$. Then the following holds:

- (1) *we have a braided exact sequence of fusion categories (in fact a modularization exact sequence)*

$$(\mathcal{E}_0) \quad \mathcal{A} \longrightarrow C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) \longrightarrow \mathcal{Z}(\mathcal{C}'');$$

- (2) *we have a morphism $(\mathcal{E}_0) \rightarrow (\mathcal{E})$ of exact sequences of fusion categories:*

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) & \longrightarrow & \mathcal{Z}(\mathcal{C}'') \\ \cong \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' \end{array},$$

where the vertical arrows are dominant forgetful functors;

- (3) *there is a finite group G acting on $\mathcal{Z}(\mathcal{C}'')$ and on \mathcal{C}'' by tensor autoequivalences in a compatible way, in such a way that \mathcal{E}_0 and \mathcal{E} are the equivariantization exact sequences relative to these actions.*

REMARK 4.3. This proposition says essentially the same thing as [7, Proposition 2.10](i), with a different viewpoint. Indeed [7, Proposition 2.10](i) asserts that if \mathcal{C} is a fusion category over \mathbf{C} and $\mathcal{A} \subset \mathcal{Z}(\mathcal{C})$ is a full tannakian subcategory such that the forgetful functor $\mathcal{U} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ induces an equivalence of \mathcal{A} with a tensor subcategory of \mathcal{C} , then \mathcal{A} encodes a de-equivariantization of \mathcal{C} , that is, a tensor action of a finite group G (such that $\mathcal{A} \simeq \text{rep } G$) on a category \mathcal{D} such that $\mathcal{C} \simeq \mathcal{D}^G$. This can be deduced from Proposition 4.2, as follows. Assume $\mathcal{A} \subset \mathcal{Z}(\mathcal{C})$ is as above. Since \mathcal{A} is tannakian, it contains a self-trivializing semisimple commutative algebra $\mathbf{A} = (A, \sigma)$ such that $\text{Hom}(\mathbf{1}, \mathbf{A}) = \mathbf{k}$. The forgetful functor \mathcal{U} induces by assumption a full tensor embedding $i : \mathcal{A} \rightarrow \mathcal{C}$. Moreover, we have a tensor functor $F_{\mathbf{A}} : \mathcal{C} \rightarrow \mathcal{C}_{\mathbf{A}}$ and an exact sequence of fusion categories

$$(\mathcal{E}) \quad \mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{F_{\mathbf{A}}} \mathcal{C}_{\mathbf{A}}.$$

The inclusion $\mathcal{A} \subset \mathcal{Z}(\mathcal{C})$ is a central lifting of i which makes (\mathcal{E}) a central exact sequence, hence the tensor functor $\mathcal{C} \rightarrow \mathcal{C}_{\mathbf{A}}$ is an equivariantization.

PROOF. Notice first that assertion (3) derives immediately from assertions (1) and (2) and previous results: if (1) and (2) hold, then by Proposition 4.1 (\mathcal{E}_0) is an equivariantization exact sequence for the action of a finite group G on $\mathcal{Z}(\mathcal{C}'')$. By Corollary 3.7, (\mathcal{E}) is also an equivariantization exact sequence for an action of the same group G .

So the whole point is to construct the exact sequence (\mathcal{E}_0) of assertion (1) and the morphism of exact sequences of assertion (2). This is based on the following lemma. Let us say that an object X of a braided category \mathcal{B} with braiding c is *symmetric* if $c_{X,X}^2 = \text{id}_{X \otimes X}$.

LEMMA 4.4. *Let \mathcal{C} be a fusion category, let $\mathbf{A} = (A, \sigma)$ be a semisimple, symmetric, self-trivializing commutative algebra in $\mathcal{Z}(\mathcal{C})$ such that $\text{Hom}(\mathbf{1}, \mathbf{A}) = \mathbf{k}$, and let $\mathcal{A} = \langle \mathbf{A} \rangle$ be the fusion subcategory of $\mathcal{Z}(\mathcal{C})$ generated by \mathbf{A} . Then*

$$C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}} = \text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}} \simeq \mathcal{Z}(\mathcal{C}_{\mathbf{A}}),$$

where $\text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}}$ denotes the category of dyslectic \mathbf{A} -modules in $\mathcal{Z}(\mathcal{C})$.

PROOF. Recall that if \mathcal{B} is a braided category, with braiding c , and A is a commutative algebra in \mathcal{B} , then a *dyslectic A -module* ([14, Definition 2.1]) is a right A -module $(M, r : M \otimes A \rightarrow M)$ in \mathcal{B} satisfying $rc_{\mathbf{A}, M} c_{M, \mathbf{A}} = r$. The category $\text{dys } \mathcal{B}_{\mathbf{A}}$ of dyslectic A -modules is a full monoidal subcategory of $\mathcal{B}_{\mathbf{A}}$ and it is braided with braiding induced by c .

In the situation of the lemma, $\mathcal{Z}(\mathcal{C})_{\mathbf{A}}$ is a fusion category (because \mathbf{A} is semisimple and $\text{Hom}(\mathbf{1}, \mathbf{A}) = \mathbf{k}$), and $\text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}}$ is a full fusion category of $\mathcal{Z}(\mathcal{C})_{\mathbf{A}}$.

On the other hand, the fact that \mathbf{A} is symmetric means that it belongs to $C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})$, and the category $C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}}$ is also a full fusion subcategory of $\mathcal{Z}(\mathcal{C})_{\mathbf{A}}$. Moreover, we have

$$C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}} \subset \text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}}$$

because a A -module (M, r) such that M belongs to the centralizer of \mathcal{A} is dyslectic.

Now it follows from [15, Corollary 4.5] that there is a natural equivalence of braided tensor categories $\text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}} \simeq \mathcal{Z}(\mathcal{C}_A)$ which is compatible with the forgetful functors to \mathcal{C} .

All that remains to do is to show that the full, replete inclusion of $C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}}$ in $\text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}}$ is an equality, which we do by showing that those two fusion categories have the same Frobenius-Perron dimension. Now \mathcal{C}_A is a fusion category and $\text{FPdim } \mathcal{C}_A = \text{FPdim } \mathcal{C} / \text{FPdim } A$ by Lemma 2.11(i). Since $\text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}} \simeq \mathcal{Z}(\mathcal{C}_A)$, we have

$$\text{FPdim } \text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}} = \text{FPdim } \mathcal{Z}(\mathcal{C}_A) = \left(\frac{\text{FPdim } \mathcal{C}}{\text{FPdim } A} \right)^2$$

by [6, Proposition 8.12]. On the other hand,

$$\text{FPdim } C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) = \frac{\text{FPdim } \mathcal{Z}(\mathcal{C})}{\text{FPdim } \mathbf{A}}$$

by [5, Theorem 3.14], since $\mathcal{Z}(\mathcal{C})$ is a nondegenerate fusion category. We have $\text{FPdim } \mathcal{Z}(\mathcal{C}) = \text{FPdim }(\mathcal{C})^2$ again by [6, Proposition 8.12], and since \mathbf{A} is self-trivializing, $\text{FPdim }(\mathcal{A}) = \text{FPdim }(\mathbf{A}) = \text{FPdim } A$, because the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ preserves Frobenius-Perron dimensions. So

$$\text{FPdim } C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}} = \frac{\text{FPdim }(\mathcal{C})^2}{\text{FPdim } A^2} = \text{FPdim } \text{dys } \mathcal{Z}(\mathcal{C})_{\mathbf{A}},$$

and we are done. This finishes the proof of the lemma. □

Now we apply the lemma. Let $\tilde{F} = F_{\mathbf{A}} : C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) \rightarrow C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A})_{\mathbf{A}} \simeq \mathcal{Z}(\mathcal{C}_A)$ be the functor ‘free \mathbf{A} -module’ $X \mapsto X \otimes \mathbf{A}$. Then \tilde{F} is a braided dominant normal fusion functor because \mathbf{A} is semisimple and self-trivializing, so we have a braided exact sequence of fusion categories

$$(\mathcal{E}_0) \quad \mathcal{A} \rightarrow C_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) \rightarrow \mathcal{Z}(\mathcal{C}_A),$$

which is an equivariantization exact sequence by Proposition 4.1, for an action of a certain finite group G on $\mathcal{Z}(\mathcal{C})$. All our constructions are compatible with the forgetful functors, hence we get a morphism of exact sequences of fusion categories $(\mathcal{E}_0) \rightarrow (\mathcal{E})$:

$$\begin{array}{ccccc}
 \mathcal{A} & \longrightarrow & \mathcal{C}_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}) & \longrightarrow & \mathcal{Z}(\mathcal{C}'') \\
 \downarrow \simeq & & \downarrow & & \downarrow \\
 \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' .
 \end{array}$$

The vertical arrow on the right is an equivalence because (\mathcal{E}) is central and $\mathcal{C}' \simeq \langle A \rangle$, and the left vertical arrow is dominant because it is the forgetful functor of the center. The middle vertical arrow is therefore dominant by virtue of Lemma 2.3, so assertion (2) holds. By Corollary 3.7, (\mathcal{E}) is an equivariantization exact sequence, for an action of G on \mathcal{C} which is compatible with the action of G on $\mathcal{Z}(\mathcal{C})$ and the forgetful functor $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$. □

4.3. The abelian case.

Let \mathbf{k} be a field. We say that a finite abelian group G has the Kummer property (w.r.t. \mathbf{k}) if \mathbf{k} contains e distinct e -th roots of 1, where e is the exponent of G . If such is the case, the group of characters $\hat{\Gamma}_{\mathbf{k}}$ of G is isomorphic to G . If \mathbf{k} is algebraically closed of characteristic 0, all finite abelian groups have the Kummer property.

PROPOSITION 4.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between finite tensor categories over field \mathbf{k} , and denote by $\mathbf{A} = (A, \sigma)$ its induced central algebra. The following assertions are equivalent:*

- (i) *The functor F is an equivariantization associated with an action of a finite abelian group G having the Kummer property;*
- (ii) *The induced central algebra \mathbf{A} of F is a direct sum of invertible objects of $\mathcal{Z}(\mathcal{C})$, and the finite abelian group Γ formed by the isomorphism classes of these invertible objects has the Kummer property;*

If these equivalent assertion hold, the groups G of assertion (i) and Γ of assertion (ii) are in duality, that is $G = \hat{\Gamma}_{\mathbf{k}}$.

PROOF. (i) \implies (ii). Assume that F is an equivariantization under a finite abelian group G having the Kummer property. We have a tensor action of G on \mathcal{D} , and we may assume that $\mathcal{C} = \mathcal{D}^G$. The exact sequence

$$\text{rep } G \xrightarrow{i} \mathcal{C} \longrightarrow \mathcal{D}$$

is central, that is, i admits a central lifting $\tilde{i} : \text{rep } G \rightarrow \mathcal{Z}(\mathcal{C})$ such that $\tilde{i}(A) = (A, \sigma) = \mathbf{A}$ (see Example 2.5). Now, since G is abelian and has the Kummer property, $\text{mod-}G \simeq \Gamma\text{-vect}$ is pointed, so A splits as a sum of invertible objects, and so does $\mathbf{A} = \tilde{i}(A)$, so (ii) holds.

(ii) \implies (i). Assume that \mathbf{A} splits as a direct sum of invertible objects of $\mathcal{Z}(\mathcal{C})$. Then by Theorem 2.9, F is normal and fits into a central exact sequence

$$\mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}.$$

Denote by H the induced Hopf algebra of this exact sequence, which is commutative by Proposition 2.6. The tensor category \mathcal{C}' is pointed with Picard group Γ because it is tensor equivalent to $\langle \mathbf{A} \rangle$ via the canonical lifting \tilde{i} , and since $\mathcal{C}' = \text{comod-}H$, we see that H is cocommutative and split cosemisimple. Thus $G = \text{Spec } H = \hat{\Gamma}_{\mathbf{k}}$ is a discrete abelian group, so H is split semisimple. We conclude by Proposition 3.2 that F is an equivariantization under the group G . \square

5. Equivariantization and the double of a Hopf monad.

5.1. Relative centers and centralizers.

Let \mathcal{C}, \mathcal{D} be monoidal categories, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a comonoidal functor. Define a half-braiding relative to F to be a pair (d, σ) , where d is an object of \mathcal{D} and σ is a natural transformation $d \otimes F \rightarrow F \otimes d$ satisfying:

$$(F_2(c, c') \otimes d)\sigma_{c \otimes c'} = (F(c) \otimes \sigma_{c'}) (\sigma_c \otimes F(c')) (d \otimes F_2(c, c')),$$

$$(F_0 \otimes d)\sigma_{\mathbf{1}} = d \otimes F_0.$$

Half-braidings relative to F form a category called the *center of \mathcal{D} relative to F* and denoted by $\mathcal{Z}_F(\mathcal{D})$, or $\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ if the functor F is clear from the context. It is monoidal, with the tensor product defined by

$$(d, \sigma) \otimes (d', \sigma') = (d \otimes d', (\sigma \otimes d')(d \otimes \sigma')),$$

and the forgetful functor $\mathcal{U} : \mathcal{Z}_F(\mathcal{D}) \rightarrow \mathcal{D}$ is monoidal strict.

Now assume F is strong monoidal (in particular, it can be viewed as a comonoidal functor). Then we have a strong monoidal functor $\tilde{F} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}_F(\mathcal{D})$, defined by $\tilde{F}(c, \sigma) = (F(c), \tilde{\sigma})$, where $\tilde{\sigma} = F(\sigma)$ up to the structure isomorphisms of F .

If F is strong monoidal and has a left adjoint L , then $T = FL$ is a bimonad on \mathcal{D} and by adjunction, $\mathcal{Z}_F(\mathcal{D})$ is isomorphic as a monoidal category to the center $\mathcal{Z}_T(\mathcal{D})$ of \mathcal{D} relative to the bimonad T defined in [4, Section 5.5].

Let $(\mathcal{E}) = (\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}'')$ be an exact sequence of finite tensor categories over a field \mathbf{k} . We will show that, if (\mathcal{E}) is central, with canonical central lifting \tilde{i} , then we have an exact sequence of tensor categories

$$\mathcal{C}' \xrightarrow{\tilde{i}} \mathcal{Z}(\mathcal{C}) \xrightarrow{\tilde{F}} \mathcal{Z}_F(\mathcal{C}'')$$

and a morphism of exact sequences of tensor categories

$$\begin{array}{ccccc} \mathcal{C}' & \longrightarrow & \mathcal{Z}(\mathcal{C}) & \longrightarrow & \mathcal{Z}_F(\mathcal{C}'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}'' \end{array}$$

5.2. Quantum double of a Hopf monad.

Let T be a Hopf monad on a rigid monoidal category \mathcal{C} . We say that T is *centralizable* ([4]) if for all objects X in \mathcal{C} the coend

$$Z_T(X) = \int^{Y \in \mathcal{C}} \vee TY \otimes X \otimes Y$$

exists. In that case the assignment $X \rightarrow Z_T(X)$ defines a Hopf monad Z_T on \mathcal{C} , called the *centralizer* of T . Denoting by $j_{X,Y} : \vee TY \otimes X \otimes Y \rightarrow Z_T(X)$ the universal dinatural transformation (in Y) associated with the coend $Z_T(X)$, set

$$\partial_{X,Y} = (TX \otimes j_{X,Y})(\text{coev}_{TY} \otimes X \otimes Y) : X \otimes Y \rightarrow TY \otimes Z_T(X).$$

If T is centralizable, we have an isomorphism of tensor categories $\mathcal{Z}_T(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{Z_T}$, and so, an isomorphism of tensor categories $K : \mathcal{Z}_{U_T}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{Z_T}$.

Moreover, if T is centralizable there also exists a canonical comonoidal distributive law $\Omega : TZ_T \rightarrow Z_T T$, which is an isomorphism. It is characterized by the following equation

$$(\mu_X \otimes \Omega_Y)T_2(TY, Z_T(X))T(\partial_{X,Y}) = (\mu_X \otimes Z_T T(Y))\partial_{TX, TY}T_2(X, Y). \tag{5.1}$$

This invertible distributive law serves two purposes: it defines (via its inverse) a lift \tilde{T} of the Hopf monad T to \mathcal{C}^{Z_T} , and it also defines a structure of a Hopf monad $D_T = Z_T \circ_\Omega T$ on the endofunctor $Z_T T$ of \mathcal{C} . The Hopf monad D_T is called the double of T ; it is quasi-triangular, so that \mathcal{C}^{D_T} is braided, and we have a canonical braided isomorphism $K' : \mathcal{C}^{D_T} \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}^T)$.

Lastly, we have a commutative diagram of tensor functors

$$\begin{array}{ccccc} \mathcal{Z}(\mathcal{C}^T) & \xrightarrow[\sim]{K'} & \mathcal{C}^{D_T} & \xrightarrow{U_{D_T}} & \mathcal{C}^T \\ \downarrow \tilde{U}_T & & \downarrow \tilde{U} & & \downarrow U_T \\ \mathcal{Z}_{\tilde{U}_T}(\mathcal{C}) & \xrightarrow[\sim]{K} & \mathcal{C}^{Z_T} & \xrightarrow{U_{Z_T}} & \mathcal{C}. \end{array}$$

If \mathcal{C} is a finite tensor category over a field \mathbf{k} , and T a \mathbf{k} -linear right exact Hopf monad on \mathcal{C} , then \mathcal{C}^T is a finite tensor category and the forgetful functor $U_T : \mathcal{C}^T \rightarrow \mathcal{C}$ is a tensor functor. Moreover, T is centralizable and Z_T is a \mathbf{k} -linear Hopf monad on \mathcal{C} , which is right exact (being an inductive limit of right exact functors) so $\mathcal{Z}_T(\mathcal{C}) \simeq \mathcal{C}^{Z_T}$ is a finite tensor category and the forgetful functor $\mathcal{Z}_T(\mathcal{C}) \rightarrow \mathcal{C}$ is a tensor functor.

5.3. Proof of Theorem 3.6.

Let (\mathcal{E}) be an exact sequence of finite tensor categories over a field \mathbf{k} . Up to equivalence, we may assume that (\mathcal{E}) is of the form

$$\langle A \rangle \longrightarrow \mathcal{C}^T \xrightarrow{U_T} \mathcal{C},$$

where \mathcal{C} is a finite tensor category, T is a k -linear right exact normal Hopf monad on \mathcal{C} , and A is the induced algebra of U_T .

We are to show that the following assertions are equivalent:

- (i) T is cocommutative;
- (ii) (\mathcal{E}) is a central exact sequence.

Our proof will rely on the following

LEMMA 5.1. *Let T be a centralizable Hopf monad on an rigid category \mathcal{C} . Then the induced central algebra (resp. coalgebra) of U_T is the induced algebra (resp. coalgebra) of \tilde{U}_T .*

Before we prove this lemma, let us show how it enables us to conclude. Consider the tensor functor $\tilde{U}_T : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}_T(\mathcal{C})$. It is dominant. Indeed U_T is dominant by assumption, which means that the unit of T is a monomorphism, and so is the unit of \tilde{T} because it is a lift of T .

Moreover, \tilde{U}_T is normal if and only if T is cocommutative. This can be seen as follows. Denote by $(\hat{\mathcal{C}}, \hat{\sigma})$ the induced central coalgebra of $U_T : \mathcal{C}_T \rightarrow \mathcal{C}$, which is also the induced coalgebra of \tilde{U}_T by Lemma 5.1. We have $\tilde{T}(\mathbf{1}) = \tilde{U}_T(\hat{\mathcal{C}}, \hat{\sigma})$, and also $T(\mathbf{1}) = \hat{\mathcal{C}}$. In particular, \tilde{U}_T is normal if and only if $(\hat{\mathcal{C}}, \hat{\sigma})$ is trivial in $\mathcal{Z}_T(\mathcal{C})$, that is $\hat{\mathcal{C}}$ is trivial in \mathcal{C} (which is true because we have assumed T is normal) and $\hat{\sigma}$ coincides with the trivial half-braiding. The latter condition means that T is cocommutative by Lemma 3.1.

Denote by $\mathbf{A} = (A, \sigma)$ the induced central algebra of U_T , which is the right dual of $(\hat{\mathcal{C}}, \hat{\sigma})$. It is also the induced algebra of \tilde{U}_T .

Now assume T is cocommutative. As we have just seen this means that \tilde{U}_T is normal and dominant, so we have an exact sequence of tensor categories

$$(\mathcal{E}_0) \quad \langle \mathbf{A} \rangle \longrightarrow \mathcal{Z}(\mathcal{C}) \xrightarrow{\tilde{U}_T} \mathcal{Z}_T(\mathcal{C}).$$

Moreover, we have a morphism of exact sequences of tensor categories

$$\begin{array}{ccccc} \langle \mathbf{A} \rangle & \longrightarrow & \mathcal{Z}(\mathcal{C}^T) & \longrightarrow & \mathcal{Z}_T(\mathcal{C}) \\ v \downarrow & & U \downarrow & & w \downarrow \\ \langle A \rangle & \longrightarrow & \mathcal{C}^T & \longrightarrow & \mathcal{C} \end{array}$$

where U, V, W denote the forgetful functors. We have $U(\mathbf{A}) = A$, so V is an equivalence of categories, that is, (\mathcal{E}) is central.

Conversely, assume (\mathcal{E}) is central. That means that the forgetful functor induces a tensor equivalence $\langle \mathbf{A} \rangle \rightarrow \langle A \rangle$. Since A is self-trivializing, so is \mathbf{A} . But by Lemma 5.1, \mathbf{A} is also the induced algebra of \tilde{U}_T , so this tensor functor is normal by Lemma 2.1; and as we have seen above, this implies that T is cocommutative. Thus, we have shown the equivalence of (i) and (ii).

PROOF OF LEMMA 5.1. The induced (central) algebra being the dual of the induced (central) coalgebra, it is enough to prove the assertion for coalgebras. Let \tilde{L} denote the left adjoint of \tilde{U} . The induced coalgebra \tilde{C} of \tilde{U}_T is $K^{-1}\tilde{L}(\mathbf{1})$.

The functor \tilde{U} is monadic, and its monad \tilde{T} is the lift of T defined by the distributive law Ω^{-1} . This means that we have $\tilde{L}(c, r) = (T(c), T(r)\Omega_c^{-1})$ for (c, r) in \mathcal{C}^{Z_T} . We also have $K^{-1}(c, \rho) = ((c, r), s)$, where $r = \rho\eta_{Z_T T_c}$ and s is the half-braiding defined by $s_{(x,r)} = (r \otimes \rho Z_T(\eta_c))\partial_{c,x}$ for (x, r) in \mathcal{C}^T . As a result, $\tilde{C} = ((T\mathbf{1}, \mu_{\mathbf{1}}), \Sigma)$, where

$$\Sigma_{(c,r)} = (r \otimes T((Z_T)_0)\Omega_{\mathbf{1}}^{-1})\partial_{T\mathbf{1},c}.$$

On the other hand, we have $\hat{C} = (T(\mathbf{1}), \mu_{\mathbf{1}})$, and the half-braiding $\tilde{\sigma}$ is characterized by the fact that the following diagram commutes:

$$\begin{array}{ccc} T\mathbf{1} \otimes c & \xrightarrow{\tilde{\sigma}_{(c,r)}} & c \otimes T\mathbf{1} \\ T\mathbf{1} \otimes r \uparrow & \swarrow \simeq & \searrow \simeq \uparrow r \otimes T\mathbf{1} \\ T\mathbf{1} \otimes T_c & \xleftarrow{T_2(\mathbf{1},c)} Tc & \xrightarrow{T_2(c,\mathbf{1})} Tc \otimes T\mathbf{1} \end{array}$$

so it is enough to verify: $\Sigma_{(c,r)}(T\mathbf{1} \otimes r)T_2(\mathbf{1}, c) = (r \otimes T\mathbf{1})T_2(\mathbf{1}, c)$. Now we have

$$\begin{aligned} & \Sigma_{(c,r)}(T\mathbf{1} \otimes r)T_2(\mathbf{1}, c) \\ &= (r \otimes T((Z_T)_0)\Omega_{\mathbf{1}}^{-1})\partial_{T\mathbf{1},c}(T\mathbf{1} \otimes r)T_2(\mathbf{1}, c) \\ &= (rT(r) \otimes T((Z_T)_0)\Omega_{\mathbf{1}}^{-1})\partial_{T\mathbf{1},Tc}T_2(\mathbf{1}, c) && \text{by functoriality of } \partial \\ &= (r\mu_c \otimes T((Z_T)_0)\Omega_{\mathbf{1}}^{-1})\partial_{T\mathbf{1},Tc}T_2(\mathbf{1}, c) \\ &= (r\mu_c \otimes T((Z_T)_0))T_2(Tc, Z_T\mathbf{1})T(\partial_{\mathbf{1},c}) && \text{by (5.1)} \\ &= (rT(r) \otimes T((Z_T)_0))T_2(Tc, Z_T\mathbf{1})T(\partial_{\mathbf{1},c}) \\ &= (r \otimes T\mathbf{1})T_2(c, \mathbf{1})T((r \otimes (Z_T)_0)\partial_{\mathbf{1},c}) && \text{by functoriality of } T_2 \\ &= (r \otimes T\mathbf{1})T_2(c, \mathbf{1})T(r\eta_c) && \text{by construction of } (Z_T)_0 \\ &= (r \otimes T\mathbf{1})T_2(c, \mathbf{1}). \end{aligned}$$

This concludes the proof of the lemma. □

6. Tensor functors of small Frobenius-Perron index.

In this section we prove Theorems 6.1 and 6.2.

6.1. Tensor functors of Frobenius-Perron index 2.

THEOREM 6.1. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories over a field of characteristic 0. If $\text{FPind}(\mathcal{C} : \mathcal{D}) = 2$, then F is an equivariantization associated with an action of Z_2 on \mathcal{D} .*

PROOF. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor of Frobenius-Perron index 2 between fusion categories. By [3, Proposition 4.13], F is normal and so it induces an exact sequence of fusion categories

$$\text{rep } \mathbb{Z}_2 \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}. \tag{6.1}$$

Now let $\mathbf{A} = (A, \sigma)$ be the induced central algebra of F . We have $\text{FPdim } \mathbf{A} = \text{FPdim } A = 2$ by Lemma 2.11. Since \mathbf{A} contains the unit object, we have $\mathbf{A} = \mathbf{1} \oplus S$, with S invertible. By Proposition 4.5, F is an equivariantization relative to an action of \mathbb{Z}_2 . This concludes the proof of the theorem. \square

6.2. Tensor functors of small prime Frobenius-Perron index.

THEOREM 6.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dominant tensor functor between fusion categories over a field of characteristic 0. Assume that $\text{FPdim } \mathcal{C}$ is a natural integer, and that $\text{FPind}(\mathcal{C} : \mathcal{D})$ is the smallest prime number dividing $\text{FPdim } \mathcal{C}$. Then F is an equivariantization associated with an action of \mathbb{Z}_p on \mathcal{D} .*

PROOF. Assume \mathcal{C} is a weakly integral fusion category and let p be the smallest prime factor of $\text{FPdim } \mathcal{C}$. Consider a dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{D} is a fusion category, such that $\text{FPind}(\mathcal{C} : \mathcal{D}) = p$.

Recall that a fusion category is *integral* if its objects all have integral Frobenius-Perron dimension. If \mathcal{C} is not integral, then by [8, Theorem 3.10] it is \mathbb{Z}_2 -graded, and so $\text{FPdim } \mathcal{C}$ is even. Thus $p = 2$, and so Theorem 6.1 applies and we are done.

From now on we assume that \mathcal{C} is integral. Then $\mathcal{Z}(\mathcal{C})$ is also an integral fusion category. Let $\mathbf{A} = (A, \sigma)$ be the induced central algebra of F . We have $\text{FPdim } \mathbf{A} = \text{FPdim } A = \text{FPind}(\mathcal{C} : \mathcal{D}) = p$. Let us decompose \mathbf{A} as a direct sum of simple objects of $\mathcal{Z}(\mathcal{C})$:

$$\mathbf{A} = W_1 \oplus \cdots \oplus W_r. \tag{6.2}$$

We have $r \geq 2$ because \mathbf{A} is not simple (it contains the unit object), so the Frobenius-Perron dimension of W_i is an integer $< p$ for all i .

The center $\mathcal{Z}(\mathcal{C})$ is a non-degenerate braided fusion category. According to [7, Theorem 2.11] (i), $(\text{FPdim } W_i)^2$ divides $\text{FPdim } \mathcal{Z}(\mathcal{C}) = (\text{FPdim } \mathcal{C})^2$, and so $\text{FPdim } W_i$ divides $\text{FPdim } \mathcal{C}$. We have $\text{FPdim } W_i = 1$, because p is by assumption the smallest prime divisor of $\text{FPdim } \mathcal{C}$, and so W_i is invertible in $\mathcal{Z}(\mathcal{C})$.

This implies that \mathbf{A} belongs to $\mathcal{Z}(\mathcal{C})_{\text{pt}}$. By Theorem 2.9, we have an exact sequence

$$\text{rep } \mathbb{Z}_p \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}. \tag{6.3}$$

which is central, and by Theorem 3.5, it is an equivariantization exact sequence. This concludes the proof of the theorem. \square

6.3. Fusion subcategories of index 2 are not always normal.

Let \mathcal{C} be a fusion category. A full fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is *normal in \mathcal{C}* (see [3]) if the inclusion $\mathcal{D} \subset \mathcal{C}$ extends to an exact sequence of fusion categories $\mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$. In that case we have $\text{FPdim } \mathcal{C}'' = \text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$.

If \mathcal{D} is a full fusion subcategory of \mathcal{C} then the ratio $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is an algebraic integer (see [6, Proposition 8.15]). If \mathcal{C} is weakly integral, so is \mathcal{D} , and therefore $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is a natural integer.

Is it true that if \mathcal{C} is weakly integral and $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is the smallest prime number dividing $\text{FPdim } \mathcal{C}$, then \mathcal{D} is normal in \mathcal{C} ? We show that even for $p = 2$ such is not the case, by exhibiting counterexamples in Tambara-Yamagami categories (see Section 2.6).

PROPOSITION 6.3. *Let \mathcal{C} be a Tambara-Yamagami category. Then we have*

$$\frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{C}_{\text{pt}}} = 2,$$

but \mathcal{C}_{pt} is not normal in \mathcal{C} if the order of the Picard group of \mathcal{C} is not a square.

PROOF. Let $\Gamma = \text{Pic}(\mathcal{C})$ and denote by X the simple non-invertible object of \mathcal{C} . We have $X \otimes X \simeq \sum_{g \in \Gamma} g$, so $\text{FPdim } X = \sqrt{|\Gamma|}$. We have $\mathcal{C}_{\text{pt}} = \langle \Gamma \rangle$, and

$$\frac{\text{FPdim } \mathcal{C}}{\text{FPdim } \mathcal{C}_{\text{pt}}} = \frac{2|\Gamma|}{|\Gamma|} = 2.$$

Now assume assume that $|\Gamma|$ is not a square. Then \mathcal{C} is not integral because $\text{FPdim } X$ is not an integer. We conclude by the following lemma.

LEMMA 6.4. *Let \mathcal{C} be a fusion category and let $\mathcal{D} \subset \mathcal{C}$ be a normal fusion subcategory such that $\text{FPdim } \mathcal{C} / \text{FPdim } \mathcal{D}$ is prime. Then \mathcal{C} is integral.*

PROOF. Consider the exact sequence of fusion categories $\mathcal{D} \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}''$ coming from the fact that \mathcal{D} is normal in \mathcal{C} . We have $\text{FPdim } \mathcal{C}'' = p$. By [6, Corollary 8.30], \mathcal{C}'' admits a quasi-fiber functor $\omega : \mathcal{C}'' \rightarrow \text{vect}_{\mathbf{k}}$. Then \mathcal{C} admits a quasi-fiber functor ωK , and therefore \mathcal{C} is integral. □

A contrario the lemma shows that \mathcal{C}_{pt} is not normal in \mathcal{C} . □

PROPOSITION 6.5. *Let \mathcal{C} be a Tambara-Yamagami category with Picard group of prime order. Then \mathcal{C} is a simple fusion category.*

PROOF. Assume we have an exact sequence of fusion categories $\mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{C}''$. Since \mathcal{C} contains a simple object of Frobenius-Perron dimension \sqrt{p} , it is not integral. Consequently \mathcal{C}'' admit no quasi-fiber functor. In particular $\text{FPdim}(\mathcal{C}'')$ is neither 1 nor a prime number. Therefore $\text{FPdim}(\mathcal{C}'') = 2p$, and \mathcal{D} is trivial. Hence \mathcal{C} is simple. □

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