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Feynman-Kac penalization problem for additive functionals with jumping functions

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Abstract. Takeda ([**30**]) solved the Feynman-Kac penalization problem for positive continuous additive functionals. We extend his result to additive functionals with jumps. We further give concrete examples of jumping functions.

1. Introduction.

Let $X := (\Omega, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_x, \{X_t\}_{t\geq 0})_{x\in\mathbb{R}^n}$ be a symmetric α-stable process $(0 < \alpha < 2)$ and let A_t be an additive functional of X. We call the next problems the *Feynman-Kac penalization problem.*

(i) Does there exist a probability measure $\tilde{\mathbb{P}}_x$ such that

$$\lim_{t\to\infty} \frac{\mathbb{E}_x[e^{A_t}S]}{\mathbb{E}_x[e^{A_t}]} = \int S d\tilde{\mathbb{P}}_x$$

for every $x \in \mathbb{R}^n$, every $s \geq 0$, and every bounded $S \in \mathcal{M}_s$?

(ii) Does there exist a martingale M by which the limit distribution $\tilde{\mathbb{P}}_x$ is determined:

$$d\tilde{\mathbb{P}}_x = M_s d\mathbb{P}_x?$$

Roynette, Vallois, and Yor considered the Feynman-Kac penalization problem of one or two dimensional Brownian motions ([**22**], [**23**], and [**25**]). K. Yano, Y. Yano, and Yor solved that of one dimensional recurrent symmetric α-stable processes ([**35**]) $(1 < \alpha \leq 2)$. Though the previous results treated the case that Feynman-Kac functionals are killing, we deal with Feynman-Kac functionals with creation.

Takeda solved the Feynman-Kac penalization problem for $e^{A_t^\mu}$ ([**30**]), where A_t^μ is a positive continuous additive functional (PCAF, as an abbreviation) with Revuz measure μ which is Green-tight. We consider this problem in the case that symmetric jumps are added to A_t^μ:

$$A_t^{\mu,F} := A_t^\mu + \sum_{0<u\leq t} F(X_{u-}, X_u),$$

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where F is a bounded measurable positive symmetric function.

The Feynman-Kac multiplicative functional (MF, as abbreviation) $e^{A_t^{\mu,F}}$ is non-local. We decompose this non-local Feynman-Kac MF as the product of an exponential type martingale L_t and the local Feynman-Kac MF $e^{A_t^{\mu+\mu_{F_1}}}$:

$$e^{A_t^{\mu,F}} = L_t e^{A_t^{\mu+\mu_{F_1}}}. \tag{1}$$

Here,

$$L_t := \exp\left(\sum_{0<s\le t} F(X_{s-}, X_s) - c_{\alpha,n}\int_0^t \int F_1(X_s, y)|X_s - y|^{-(\alpha+n)} dy ds\right),$$

$$\mu_{F_1}(dx) := c_{\alpha,n}\left\{\int_{\mathbb{R}^n} F_1(x,y)|x-y|^{-(n+\alpha)} dy\right\} dx,$$

$F_1 := e^F - 1$ and $c_{\alpha,n}$ is a positive constant. We assume that μ_{F_1} is a Green-tight Kato measure. We then transform the symmetric α-stable process X by the martingale MF L_t and denote by Y the transformed process. The Dirichlet form of the transformed process Y is given by

$$\mathcal{E}^Y(u,u) = \frac{c_{\alpha,n}}{2}\int_{d^c} (u(x)-u(y))^2 e^{F(x,y)}|x-y|^{-(\alpha+n)} dx dy,$$

where d is the diagonal set, that is, $d := \{(x,x); x \in \mathbb{R}^n\}$. We then see that the Lévy kernel of the transformed process is equivalent to that of the symmetric stable process: it holds

$$c^{-1}|x-y|^{-(\alpha+n)} \le e^{F(x,y)}|x-y|^{-(\alpha+n)} \le c|x-y|^{-(\alpha+n)}$$

for some $c > 1$. This implies the equivalence of transition probabilities (Bass and Levin ([**3**])). Thus Kato classes are invariant under the transform by L_t.

We define the function $\lambda(\theta)$ for $\theta \ge 0$ by

$$\lambda(\theta) := \inf\left\{\mathcal{E}_\theta^Y(u,u); \int u(x)^2 d(\mu + \mu_{F_1}) = 1\right\}.$$

We see by the definition that $\lambda(\theta)$ is increasing and concave and satisfies $\lim_{\theta\to\infty}\lambda(\theta) = \infty$. We denote the generator of the transformed process Y by $\mathcal{L}^Y$: let

$$\mathcal{L}^Y u(x) := \lim_{t\to 0}\frac{\mathbb{E}_x^L[u(Y_t)] - u(x)}{t}, \tag{2}$$

where $d\mathbb{P}_x^L := L_t d\mathbb{P}_x$. Note that $\mathcal{E}^Y(u,u) = (-\mathcal{L}^Y u, u)$. We divide cases in terms of the value of $\lambda(0)$: if $\lambda(0) > 1$, $\lambda(0) < 1$, and $\lambda(0) = 1$, then $\mathcal{L}^Y + \mu + \mu_{F_1}$ is said to be *subcritical*, *supercritical*, and *critical* respectively.

(a) $\lambda(0) > 1$: We define

$$h(x) := \mathbb{E}_x^L\big[e^{A_\infty^{\mu+\mu_{F_1}}}\big].$$

Z.-Q. Chen proved that the boundedness of $h(x)$ is equivalent to $\lambda(0) > 1$. We change his proof by using the equivalence of β-resolvent kernel of X and Y. The weight process M_s is identified with

$$M_s := \frac{h(X_s)}{h(x)} e^{A_s^{\mu,F}}.$$

We treat this case in Section 4.

(b) $\lambda(0) < 1$: Since there exists $\theta_0 > 0$ such that $\lambda(\theta_0) = 1$ and μ and μ_{F_1} are in the Green-tight Kato class, the embedding of $\mathcal{D}[\mathcal{E}^Y](= \mathcal{D}[\mathcal{E}])$ into $L^2(\mu + \mu_{F_1})$ is compact so that we can take a positive function h in $\mathcal{D}[\mathcal{E}]$ such that $\mathcal{E}_{\theta_0}^Y(h,h) = 1$. We use the limit theorem of Feynman-Kac MFs like [**30**, Theorem 4.1] in the supercritical case. The weight process is then given by

$$M_s := e^{-\theta_0 s}\frac{h(X_s)}{h(x)} e^{A_s^{\mu,F}}.$$

We treat this case in Section 5.

(c) $\lambda(0) = 1$: We use the compact embedding theorem of the extended Dirichlet form $\mathcal{D}_e[\mathcal{E}^Y]$ into $L^2(\mu+\mu_{F_1})$ by Takeda and Tsuchida ([**33**, Theorem 10]). This implies the existence of a positive function h in $\mathcal{D}_e[\mathcal{E}^Y](= \mathcal{D}_e[\mathcal{E}])$ such that $\mathcal{E}^Y(h,h) = 1$. We then obtain a h-transformed process $(\mathbb{P}_x^{L,h}, Y_t, h^2dx)_{x\in\mathbb{R}^n}$ and see that the semigroup of this process becomes recurrent. The function

$$\psi(t) := \mathbb{E}_x^{L,h}\bigg[\int_0^t k(Y_u)du\bigg], \quad k \in C_0^+(\mathbb{R}^n)$$

diverges to infinity as $t \to \infty$ and $\mathbb{E}_x^{L,h}[e^{A_t^{\mu+\mu_{F_1}}}S]/\psi(t)$ and $\mathbb{E}_x^{L,h}[e^{A_t^{\mu+\mu_{F_1}}}]/\psi(t)$ converge only if μ and μ_{F_1} are in the special Kato class. Then the problem is solved for a restricted class of Feynman-Kac MFs. The weight process of the critical case is given by

$$M_s := \frac{h(X_s)}{h(x)} e^{A_s^{\mu,F}}.$$

We treat this case in Section 6.

Let $\mathcal{A}$ (resp. $\mathcal{A}_s$) be the set of jumping functions such that μ_F is in the Green-tight Kato class (resp. the special Kato class). The conditions that $F \in \mathcal{A}$ or $F \in \mathcal{A}_s$ are then analytically characterized. We have used the equivalence of transition probabilities of X and Y instead of the conditional gaugeability to solve the Feynman-Kac penalization problem. Since the condition $F \in \boldsymbol{A_2}$ (see [**7**, Definition 2.3] for the definition of $\boldsymbol{A_2}$) is needed for the conditional gaugeability, we then see $\mathcal{A} \supset \boldsymbol{A_2}$. Furthermore, there exists a jumping function with a full support in $\mathcal{A}$. For example, $F \in \mathcal{A}$ and F has a full support if the jumping function is

$$F(x,y) := (1 \wedge |x-y|^p)\langle x\rangle^{-q}\langle y\rangle^{-q} \quad \text{for } p > \alpha \text{ and } q > n, \tag{3}$$

where $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$. If we further assume $q > 2n-\alpha$ in this example, then $F \in \mathcal{A}_s$.

In Section 2, we prepare for fundamental notations related to Green functions and Kato classes to describe our main results. In Section 3, we show the decomposition (1) of $e^{A_t^{\mu,F}}$, the equivalence of transition probabilities of X and Y, and the invariance of Kato classes under the transform by L_t. We solve our problem in Sections 4, 5, and 6 in the subcritical, supercritical, and critical case respectively. In Section 7, we check that $F \in \mathcal{A}$ or $F \in \mathcal{A}_s$ for the functions F described by (3).

2. Preliminaries.

Let X be a symmetric α-stable process ($0 < \alpha < 2$). The Dirichlet form of X is given by

$$\mathcal{E}(u,u) := \frac{c_{\alpha,n}}{2}\int_{d^c}\frac{|u(x)-u(y)|^2}{|x-y|^{\alpha+n}}dxdy,$$
$$\mathcal{D}[\mathcal{E}] := \left\{u \in L^2(\mathbb{R}^n, dx); \int_{d^c}\frac{|u(x)-u(y)|^2}{|x-y|^{\alpha+n}}dxdy < \infty\right\},$$

where $c_{\alpha,n}$ is given by

$$c_{\alpha,n} := \frac{\alpha 2^{n-1}\Gamma((\alpha+n)/2)}{\pi^{n/2}\Gamma(1-(\alpha/2))}. \tag{4}$$

It is well known that Lévy system of the symmetric stable process X_t is $(c_{\alpha,n}|x-y|^{-(n+\alpha)}, t)$. Note that the Revuz measure of t is the Lebesgue measure (see [**7**, Example 2.1] for further details). Let A_t^μ be a PCAF with the corresponding Revuz measure μ and let F be a bounded measurable positive symmetric function vanishing on diagonal set throughout this paper. We consider following additive functionals (AFs, as an abbreviation) with symmetric jumps

$$A_t^{\mu,F} := A_t^\mu + \sum_{0<s\le t} F(X_{s-}, X_s).$$

We define (β-)resolvent kernels, (β-)potentials, and Kato classes.

Definition 2.1. Let $p(t,x,y)$ be the transition probability of the symmetric α-stable process X. The following function $G_\beta(x,y)$ is said to be β-resolvent kernel.

$$G_\beta(x,y) := \int_0^\infty e^{-\beta t} p(t,x,y)dt \quad \text{for } \beta \geq 0.$$

We write

$$G(x,y) := G_0(x,y)$$

if the process X is transient. Let μ be a positive Radon measure. We denote the β-potential of μ by $G_\beta\mu$, that is,

$$G_\beta\mu(x) := \int_{\mathbb{R}^n} G_\beta(x,y)\mu(dy) \quad \text{for } \beta \geq 0.$$

Definition 2.2. A positive Radon measure μ on $\mathbb{R}^n$ is said to be in the Kato class $\mathcal{K}$ if it satisfies

$$\lim_{\beta\to\infty} \|G_\beta\mu\|_\infty = 0.$$

Given $\beta \geq 0$, a measure $\mu \in \mathcal{K}$ is said to be in β-Green-tight Kato class if

$$\lim_{R\to\infty, r\to 0} \|G_\beta(\mathbf{1}_{B(0,R)^c \cup B(x,r)}\mu)\|_\infty = 0.$$

We denote by $\mathcal{K}_{\infty,\beta}$ the set of β-Green-tight measures. We write $\mathcal{K}_\infty$ for $\mathcal{K}_{\infty,0}$ simply and call this Green-tight Kato class.

It follows from the definition of $\mathcal{K}_{\infty,\beta}$ that $\mathcal{K}_\infty \subset \mathcal{K}_{\infty,1}$ and $\mathcal{K}_{\infty,\beta} = \mathcal{K}_{\infty,1}$ for all $\beta > 0$.

We define a measure

$$\mu_F(dx) := c_{\alpha,n}\left\{\int_{\mathbb{R}^n} F(x,y)|x-y|^{-(\alpha+n)}dy\right\}dx. \tag{5}$$

We define the class $\mathcal{A}$ of jumping functions. This class plays an important role in our results.

Definition 2.3. The function F is said to be in the class $\mathcal{A}$ if $\mu_F \in \mathcal{K}_\infty$ (resp. $\mu_F \in \mathcal{K}_{\infty,1}$) for $n > \alpha$ (resp. $n \leq \alpha$).

Our goal is to obtain the next theorem.

Theorem 2.4. *Assume that the Revuz measure μ is in Kato class $\mathcal{K}_\infty$ (resp. $\mathcal{K}_{\infty,1}$) for $n > \alpha$ (resp. $n \leq \alpha$) and that the function $F_1 := e^F - 1$ belongs to the class $\mathcal{A}$. Then* *there exists a probability measure* $\mathbb{P}_x^M$ *such that it holds*

$$\lim_{t\to\infty} \frac{\mathbb{E}_x[e^{A_t^{\mu,F}} S]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} = \mathbb{E}_x^M[S]$$

for every $s \geq 0$*, every bounded* $\mathcal{M}_s$*-measurable random variable* S*, and every* $x \in \mathbb{R}^n$*. Moreover, the limit distribution* $\mathbb{P}_x^M$ *is characterized as*

$$\mathbb{P}_x^M[A] := \int_A M_s d\mathbb{P}_x \quad \textit{for } A \in \mathcal{M}_s,$$

where M_s *is a martingale MF defined in* (9)*,* (12)*, and* (15) *below.*

Here and in what follows, we let $F_1 = e^F - 1$ without mentioning.

3. Decomposition of non-local Feynman-Kac MF.

In this section, to employ the result for local Feynman-Kac functionals we decompose a non-local Feynman-Kac MF as the product of a local Feynman-Kac MF and an exponential martingale.

We define an exponential martingale L_t by

$$L_t = \exp\left(\sum_{0<u\leq t} F(X_{u-}, X_u) - c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y)|X_u - y|^{-(n+\alpha)} dy du \right).$$

This is the unique solution of Doléans-Dade equation

$$Z_t = 1 + \int_0^t Z_{u-} dK_u,$$

where K_t is a purely discontinuous martingale defined by

$$K_t := \sum_{0<u\leq t} F_1(X_{u-}, X_u) - c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y)|X_u - y|^{-(n+\alpha)} dy du$$

(see [**8**, Remark 3.4]). We thus obtain

$$\begin{aligned} e^{A_t^{\mu,F}} &= e^{A_t^{\mu}} \prod_{0<u\leq t} (1 + F_1(X_{u-}, X_u)) \\ &= L_t \exp\left(A_t^{\mu} + c_{\alpha,n} \int_0^t \int_{\mathbb{R}^n} F_1(X_u, y)|X_u - y|^{-(n+\alpha)} dy du \right) \\ &= L_t e^{A_t^{\mu+\mu_{F_1}}}. \end{aligned}$$

Note that $t \mapsto c_{\alpha,n}\big(\int_0^t \int_{\mathbb{R}^n} F_1(X_u, y)|X_u - y|^{-(n+\alpha)}dydu\big)$ is a PCAF and its Revuz measure is μ_{F_1} as in the formula (5). We transform the symmetric stable process X by the martingale MF L_t and denote its law by $\mathbb{P}_x^L$, that is, $d\mathbb{P}_x^L := L_t d\mathbb{P}_x$. We further denote the associated symmetric strong Markov process by $(\mathbb{P}_x^L, Y_t)_{x\in\mathbb{R}^n}$. The Dirichlet form $\mathcal{E}^Y$ of the process Y is identified as follows (see also [**7**]):

$$\begin{aligned}\mathcal{E}^Y(u,u) &= \mathcal{E}(u,u) + \frac{c_{\alpha,n}}{2}\int_{d^c}(u(x)-u(y))^2F_1(x,y)|x-y|^{-(n+\alpha)}dydx \\ &= \frac{c_{\alpha,n}}{2}\int_{d^c}(u(x)-u(y))^2e^{F(x,y)}|x-y|^{-(n+\alpha)}dydx.\end{aligned}$$

Recall that F is bounded. We then find that the Lévy kernel of Y is equivalent to that of X, that is,

$$c^{-1}|x-y|^{-(n+\alpha)} \le e^{F(x,y)}|x-y|^{-(n+\alpha)} \le c|x-y|^{-(n+\alpha)} \tag{6}$$

for some $c > 1$. We then see from Bass and Levin ([**3**]) that the transition probability of Y is equivalent to that of X.

Theorem 3.1 ([**3**]). *If the Lévy kernel of Y satisfies* (6)*, then the transition probability p^Y is also equivalent to p : it holds*

$$c^{-1}p(t,x,y) \le p^Y(t,x,y) \le cp(t,x,y) \tag{7}$$

for some $c > 1$, every $t \ge 0$, and every $x, y \in \mathbb{R}^n$.

In the sequel, we denote positive constants by c or C. They may be different at each occurrence.

Theorem 3.1 implies the equivalence of the β-resolvent kernel of X and Y.

Corollary 3.2. *Let G^Y (resp. G^Y_β) be the Green function (resp. the β-resolvent kernel) of Y. It then holds*

$$c^{-1}G_\beta(x,y) \le G^Y_\beta(x,y) \le cG_\beta(x,y) \text{ for every } x,y \in \mathbb{R}^n \text{ and some } c > 1.$$

For the rest part of this section, if the process X is transient (resp. recurrent), then we assume that $\mu \in \mathcal{K}_\infty$ (resp. $\mu \in \mathcal{K}_{\infty,1}$) and $F_1 \in \mathcal{A}$.

We define the spectral function $\lambda(\theta)$ by the transformed Dirichlet form $\mathcal{E}^Y$.

$$\lambda(\theta) := \inf\left\{\mathcal{E}^Y_\theta(u,u); \int_{\mathbb{R}^n} u(x)^2 d(\mu+\mu_{F_1}) = 1\right\} \quad \text{for } \theta \ge 0, \tag{8}$$

where $\mathcal{E}^Y_\theta(\cdot,\cdot) := \mathcal{E}^Y(\cdot,\cdot) + \theta(\cdot,\cdot)_{L^2(dx)}$. We here summarize some properties of the spectral function.

Theorem 3.3. (i) $\lambda(\theta)$ *is concave* (*and hence continuous*) *and increasing.*
(ii) $\lim_{\theta\to\infty}\lambda(\theta)=\infty$.

Proof. (i) follows just as [**30**, Lemma 3.1]. Note that it holds

$$\int_{\mathbb{R}^n} u(x)^2 d(\mu+\mu_{F_1}) \le \|G_\theta^Y(\mu+\mu_{F_1})\|_\infty \mathcal{E}_\theta^Y(u,u)$$

for all $u\in\mathcal{D}[\mathcal{E}]$ (see [**29**, Proposition 2.3]). It then follows that

$$\lambda(\theta)\ge\frac{1}{\|G_\theta^Y(\mu+\mu_{F_1})\|_\infty}.$$

Since $G_\theta^Y(\mu+\mu_{F_1})$ is equivalent to $G_\theta(\mu+\mu_{F_1})$ and so $\|G_\theta^Y(\mu+\mu_{F_1})\|_\infty\to 0$ as $\theta\to\infty$, we complete the proof of (ii). □

Since the transformed Dirichlet form of Y is equivalent to that of X, we obtain the compact embedding of the domain of Dirichlet forms into $L^2(\mu+\mu_{F_1})$ (Takeda and Tsuchida [**33**]).

Theorem 3.4. (i) *If* $\mu\in\mathcal{K}_\infty$, $F_1\in\mathcal{A}$ *and* $\mathcal{E}^Y$ *is transient, then the embedding of* $\mathcal{D}_e[\mathcal{E}^Y]$ *into* $L^2(\mu+\mu_{F_1})$ *is compact, where* $(\mathcal{D}_e[\mathcal{E}^Y],\mathcal{E}^Y)$ *is the extended Dirichlet form.*
(ii) *If* $\mu\in\mathcal{K}_{\infty,1}$ *and* $F_1\in\mathcal{A}$, *then the embedding of* $\mathcal{D}[\mathcal{E}^Y]$ *into* $L^2(\mu+\mu_{F_1})$ *is compact.*

Proof. Note that if $\mathcal{E}^Y$ is transient then $(\mathcal{D}_e[\mathcal{E}^Y],\mathcal{E}^Y)$ is a Hilbert space whose norm is $\sqrt{\mathcal{E}^Y(\cdot,\cdot)}$. One can prove this by imitating the proofs of [**33**, Theorem 10] and [**31**, Theorem 2.7]. □

We will divide the following three cases in terms of the value of $\lambda(0)$. If $\lambda(0)>1$, $\lambda(0)<1$, and $\lambda(0)=1$, then we call $\mathcal{L}^Y+\mu+\mu_{F_1}$ subcritical, supercritical, and critical respectively. $\mathcal{L}^Y$ is the generator defined by the formula (2) of the process Y.

Remark 3.5. The formula (8) implies

$$\lambda(0)\int_{\mathbb{R}^n} u(x)^2 d(\mu+\mu_{F_1}) \le \mathcal{E}^Y(u,u)$$

for all $u\in\mathcal{D}[\mathcal{E}]$. The recurrence of the semigroup associated with Y implies the existence of $\{u_n\}_{n\ge1}(\subset\mathcal{D}[\mathcal{E}])$ such that $\lim_{n\to\infty}\mathcal{E}(u_n,u_n)=0$ and $\lim_{n\to\infty}u_n=1$ a.e. by [**13**, Theorem 1.6.3 (i)(ii)], that is, $1\in\mathcal{D}_e[\mathcal{E}]$ from [**13**, Theorem 1.6.3 (iii)]. [**13**, Theorem 2.1.7] then yields the last "a.e." can be replaced by "q.e." If $\lambda(0)>0$ then the last inequality causes contradiction as $n\to\infty$. Therefore, we find that if the process Y is recurrent then $\lambda(0)=0$.

4. Subcritical cases.

We use the gaugeability of $(Y, A^{\mu+\mu_{F_1}})$ in the subcritical case. One can modify the proof of ([**7**, Theorem 3.4]) by using the equivalence of the β-resolvent kernel of X and Y. This modification is needed for the extension of the class of jumping functions.

Theorem 4.1. *Assume that $\mu \in \mathcal{K}_\infty$ and $F_1 \in \mathcal{A}$. The following three conditions are equivalent.*

(i) $\lambda(0) > 1$
(ii) $(X, A^{\mu,F})$ *is gaugeable, that is, the function* $x \mapsto \mathbb{E}_x[e^{A_\infty^{\mu,F}}]$ *is bounded.*
(iii) $(Y, A^{\mu+\mu_{F_1}})$ *is gaugeable, that is, the function* $x \mapsto \mathbb{E}_x^L[e^{A_\infty^{\mu+\mu_{F_1}}}]$ *is bounded.*

In the subcritical case, we define the function h by

$$h(x) := \mathbb{E}_x^L\left[e^{A_\infty^{\mu+\mu_{F_1}}}\right].$$

We now solve the Feynman-Kac penalization problem in the subcritical case. We have only to consider the following ratio.

$$\begin{aligned}
\frac{\mathbb{E}_x[e^{A_t^{\mu,F}}|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} &= \frac{\mathbb{E}_x[e^{A_s^{\mu,F}} \cdot (e^{A_{t-s}^{\mu,F}} \circ \theta_s)|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} \\
&= \frac{e^{A_s^{\mu,F}}\mathbb{E}_x[e^{A_{t-s}^{\mu,F}} \circ \theta_s|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} \\
&= \frac{e^{A_s^{\mu,F}}\mathbb{E}_{X_s}[e^{A_{t-s}^{\mu,F}}]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} \\
&= L_s\frac{e^{A_s^{\mu+\mu_{F_1}}}\mathbb{E}_{X_s}^L[e^{A_{t-s}^{\mu+\mu_{F_1}}}]}{\mathbb{E}_x^L[e^{A_t^{\mu+\mu_{F_1}}}]}.
\end{aligned}$$

Letting $t \to \infty$,

$$\lim_{t\to\infty}\frac{\mathbb{E}_x[e^{A_t^{\mu,F}}|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} = L_s\frac{e^{A_s^{\mu+\mu_{F_1}}}h(X_s)}{h(x)}.$$

The problem is solved in this case by setting M_s as follows:

$$M_s := \frac{e^{A_s^{\mu,F}}h(X_s)}{h(x)}. \tag{9}$$

Remark 4.2. Let $P_t^{\mu+\mu_{F_1}}$ be the Feynman-Kac semigroup.

$$P_t^{\mu+\mu_{F_1}} f(x) = \mathbb{E}_x^L \big[e^{A_t^{\mu+\mu_{F_1}}} f(Y_t) \big]. \tag{10}$$

Let $\mathcal{L}^{\mu+\mu_{F_1}}$ be the generator of $P_t^{\mu+\mu_{F_1}}$. We may regard the above h as a harmonic function such that $\mathcal{L}^{\mu+\mu_{F_1}} h(x) = 0$. Indeed, the Markov property implies

$$\begin{aligned} h(x) &= \mathbb{E}_x^L \big[e^{A_t^{\mu+\mu_{F_1}}} \mathbb{E}_x^L \big[e^{A_\infty^{\mu+\mu_{F_1}}} \circ \theta_t | \mathcal{M}_t \big]\big] \\ &= \mathbb{E}_x^L \big[e^{A_t^{\mu+\mu_{F_1}}} E_{Y_t}^{\mu+\mu_{F_1}} \big[e^{A_\infty^{\mu+\mu_{F_1}}} \big]\big] \\ &= \mathbb{E}_x^L \big[e^{A_t^{\mu+\mu_{F_1}}} h(Y_t) \big] \\ &= P_t^{\mu+\mu_{F_1}} h(x) \end{aligned}$$

so that we obtain

$$\mathcal{L}^{\mu+\mu_{F_1}} h(x) = \lim_{t\to 0} \frac{P_t^{\mu+\mu_{F_1}} h(x) - h(x)}{t} = 0.$$

5. Supercritical cases.

If the process X is transient (resp. recurrent) then we assume $\mu \in \mathcal{K}_\infty$ (resp. $\mu \in \mathcal{K}_{\infty,1}$) and $F_1 \in \mathcal{A}$. Since $\lambda(0) < 1$, there exists $\theta_0 > 0$ such that $\lambda(\theta_0) = 1$. We then see the asymptotic behavior of $\mathbb{E}_x[e^{A_t^{\mu,F}}]$ by using the next theorem.

Theorem 5.1. *Suppose that the process X is transient (resp. recurrent). If $\mu \in \mathcal{K}_\infty$ (resp. $\mu \in \mathcal{K}_{\infty,1}$) and $F_1 \in \mathcal{A}$, then there exists a positive function $h \in L^2(\mu + \mu_{F_1})$ and $\theta_0 > 0$ such that $\lambda(\theta_0) = 1$, $\mathcal{E}_{\theta_0}^Y(h,h) = 1$, and*

$$\lim_{t\to\infty} e^{-\theta_0 t} \mathbb{E}_x \big[e^{A_t^{\mu,F}} \big] = h(x) \int_{\mathbb{R}^n} h(x) dx.$$

Proof. The existence of $\theta_0 > 0$ immediately follows from Theorem 3.3. It is trivial that if both μ and μ_{F_1} are in $\mathcal{K}_\infty$ (resp. $\mathcal{K}_{\infty,1}$) then $\mu + \mu_{F_1}$ is also a member of $\mathcal{K}_\infty$ (resp. $\mathcal{K}_{\infty,1}$).

The compactness of the embedding from $\mathcal{D}[\mathcal{E}]$ into $L^2(\mu + \mu_{F_1})$ (see [**31**, Theorem 2.7]) implies the existence of the function $h \in L^2(\mu+\mu_{F_1})$. In particular, if X is transient then the uniform boundedness principle implies the existence of $h \in L^2(\mu + \mu_{F_1})$ since $(\mathcal{E}_{\theta_0}^Y, \mathcal{D}[\mathcal{E}])$ is a Hilbert space. Applying Lemma 5.2 stated below with $g = 1/h$ completes the proof. Note that h is in $L^1(dx)$. □

Lemma 5.2. (i) *Let $\mu \in \mathcal{K}_{\infty,1}$ and let h be the function given in the proof of Theorem 5.1. Then it holds*

$$\int_{\mathbb{R}^n} p^h(t,x,y)^q h(y)^2 dy < \infty$$

for every $t \geq 0$, every $x \in \mathbb{R}^n$, and all $q > 1$. Here, p^h is the heat kernel given by

$$p^h(t,x,y) = e^{-\theta_0 t} \frac{p^{\mu+\mu_{F_1}}(t,x,y)}{h(x)h(y)} \tag{11}$$

and $p^{\mu+\mu_{F_1}}$ is the heat kernel of the Feynman-Kac semigroup defined in (10).

(ii) *Let $\mu \in \mathcal{K}_{\infty,1}$, let h be the function as in* (i) *and let P_t^h be a semigroup whose heat kernel is p^h. Then it holds*

$$\lim_{t\to\infty} P_t^h g(x) = \int_{\mathbb{R}^n} g(x)h(x)^2 dx$$

for all $x \in \mathbb{R}^n$, all $g \in L^p(h^2dx)$, and all $p > 1$.

Proof. (i) Note that h is a harmonic function of the equation.

$$(\mathcal{L}^Y + \mu + \mu_{F_1})h(x) = \theta_0 h(x).$$

Since $\mu + \mu_{F_1} \in \mathcal{K}_{\infty,1}$, [**31**, Lemma 4.1] implies that it holds

$$c|x|^{-(n+\alpha)} \leq h(x) \leq C|x|^{-(n+\alpha)/q_1}$$

for all $1 < q_1 < 2$ and all $|x| > 1$. We also find the upper bound of the heat kernel $p^{\mu+\mu_{F_1}}$ off the diagonal set:

$$p^{\mu+\mu_{F_1}}(t,x,y) \leq c|x-y|^{-(n+\alpha)/q_2}$$

for all $q_2 > 1$ and every $t > 0$ from [**31**, Lemma 4.3]. Combining the last two results, we find $L^q(h^2dx)$-integrabiilty of the heat kernel p^h if we take q_1 and q_2 close to 1. (ii) Take $g \in L^p(h^2dx)$ arbitrarily. The maximal ergodic theorem follows from [**31**, Lemma 4.5] since $\mu + \mu_{F_1} \in \mathcal{K}_{\infty,1}$.

$$\left\| \sup_{t>0} P_t^h g \right\|_{L^p(h^2dx)} \leq C_p \|g\|_{L^p(h^2dx)}.$$

This implies $\sup_{t>0} P_t^h g(x) < \infty$. Thus we have the desired result

$$\lim_{t\to\infty} P_t^h g(x) = \int_{\mathbb{R}^n} g(x)h(x)^2 dx$$

for every $x \in \mathbb{R}^n$. □

We now solve the Feynman-Kac penalization problem in the supercritical case. Using Theorem 5.1, we find

$$\frac{\mathbb{E}_x[e^{A_t^{\mu,F}}|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} = \frac{\mathbb{E}_x[e^{-\theta_0 t}e^{A_t^{\mu,F}}|\mathcal{M}_s]}{\mathbb{E}_x[e^{-\theta_0 t}e^{A_t^{\mu,F}}]}$$
$$= \frac{e^{-\theta_0 s}e^{A_s^{\mu,F}}\mathbb{E}_x[e^{-\theta_0(t-s)}e^{A_{t-s}^{\mu,F}}\circ\theta_s|\mathcal{M}_s]}{\mathbb{E}_x[e^{-\theta_0 t}e^{A_t^{\mu,F}}]}$$
$$= \frac{e^{-\theta_0 s}e^{A_s^{\mu,F}}\{e^{-\theta_0(t-s)}\mathbb{E}_{X_s}[e^{A_{t-s}^{\mu,F}}]\}}{e^{-\theta_0 t}\mathbb{E}_x[e^{A_t^{\mu,F}}]}.$$

Letting $t\to\infty$, we obtain

$$\lim_{t\to\infty}\frac{\mathbb{E}_x[e^{A_t^{\mu,F}}|\mathcal{M}_s]}{\mathbb{E}_x[e^{A_t^{\mu,F}}]} = \frac{e^{-\theta_0 s}e^{A_s^{\mu,F}}h(X_s)\int_{\mathbb{R}^n}h(x)dx}{h(x)\int_{\mathbb{R}^n}h(x)dx}$$
$$= \frac{e^{-\theta_0 s}e^{A_s^{\mu,F}}h(X_s)}{h(x)}.$$

Scheffe's lemma implies that the above convergence is in $L^1(\mathbb{P}_x)$. The weight process M_s is given by

$$M_s := \frac{e^{-\theta_0 s}e^{A_s^{\mu,F}}h(X_s)}{h(x)}. \tag{12}$$

6. Critical cases.

We use Chacon-Ornstein type ergodic theorem to solve the Feynman-Kac penalization problem in this case. We have to treat a subclass of the Green-tight Kato class to use this theorem.

Note that the extended Dirichlet space $\mathcal{D}_e[\mathcal{E}]$ is a Hilbert space (see [**13**, Lemma 1.5.5]). We further see that the embedding of $\mathcal{D}_e[\mathcal{E}]$ into $L^2(\mu+\mu_{F_1})$ is also compact by [**33**, Theorem 10]. Then we obtain a harmonic function in $\mathcal{D}_e[\mathcal{E}]$. We weight the probability measure by an exponential martingale L_t and the harmonic function h:

$$d\mathbb{P}_x^{L,h} := N_t d\mathbb{P}_x^L, \quad N_t := \frac{h(Y_t)}{h(x)}e^{A_t^{\mu+\mu_{F_1}}}.$$

Hereafter, we consider the Markov process $(\Omega,\mathcal{M},\mathcal{M}_t,\mathbb{P}_x^{L,h},Y_t)$, or $\mathbb{M}^{L,h}$ for short.

We define the special Kato class.

Definition 6.1. (i) Let μ be a measure of Kato class $\mathcal{K}$. μ is said to be in the special Kato class if it holds

$$\sup_{x\in\mathbb{R}^n}|x|^{n-\alpha}\int_{\mathbb{R}^n}|x-y|^{\alpha-n}\mu(dy)<\infty.$$

We denote this class by $\mathcal{K}_s$.

(ii) A PCAF A is said to be special with respect to $\mathbb{M}^{L,h}$, if it holds

$$\sup_{x\in\mathbb{R}^n}\mathbb{E}_x^{L,h}\left[\int_0^\infty \exp\left(-\int_0^t g(X_u)du\right)dA_t\right]<\infty$$

for any positive Borel function g with $\int_{R^n} g(x)dx>0$.

Definition 6.2. A bounded measurable symmetric positive function F vanishing on the diagonal set is said to be in the class $\mathcal{A}_s$ if μ_F is in $\mathcal{K}_s$.

For the rest part of this section, we assume $\mu\in\mathcal{K}_s$ and $F_1\in\mathcal{A}_s$. One can easily check the following properties.

Lemma 6.3. *(i) $\mathcal{K}_s$ is the subset of $\mathcal{K}_\infty$.*
(ii) $\mu+\mu_{F_1}$ is also a member of $\mathcal{K}_s$.

Proof. See [**30**, Section 4] about (i). Noting $G(x,y)=c_{\alpha,n}|x-y|^{\alpha-n}$, (ii) also immediately follows the equivalence of Green function G and the transformed Green function G^Y. □

One can prove the following lemmas just as in [**30**, Section 4].

Lemma 6.4. *For all PCAFs B, it holds*

$$\mathbb{E}_x^L\left[\int_0^t e^{A_u^{\mu+\mu_{F_1}}-B_u}dA_u\right]=h(x)\mathbb{E}_x^{L,h}\left[\int_0^t e^{-B_u}\frac{dA_u^{\mu+\mu_{F_1}}}{h(Y_u)}\right]$$

for every $x\in\mathbb{R}^n$ and $t\geq 0$.

Lemma 6.5. *$\int_0^t(1/h(Y_u))dA_u^{\mu+\mu_{F_1}}$ is special with respect to $\mathbb{M}^{L,h}$ if $F_1\in\mathcal{A}_s$.*

Proof. See the proof of [**30**, Lemma 4.3] and [**30**, Lemma 4.4]. □

By using Lemma 6.4,

$$\mathbb{E}_x^L\left[e^{A_t^{\mu+\mu_{F_1}}}\right]=1+\mathbb{E}_x^L\left[\int_0^t e^{A_u^{\mu+\mu_{F_1}}}dA_u^{\mu+\mu_{F_1}}\right]=1+h(x)\mathbb{E}_x^{L,h}\left[\int_0^t\frac{dA_u^{\mu+\mu_{F_1}}}{h(Y_u)}\right].$$

Integrating by an arbitrary finite positive measure ν, we see

$$\mathbb{E}_\nu^L\left[e^{A_t^{\mu+\mu_{F_1}}}\right]=\nu(\mathbb{R}^n)+\langle\nu,h\rangle\mathbb{E}_{\nu^h}^{L,h}\left[\int_0^t\frac{dA_u^{\mu+\mu_{F_1}}}{h(Y_u)}\right],\tag{13}$$

where $\nu^h:=(h\cdot\nu)/\langle\nu,h\rangle$ and $\langle\nu,h\rangle:=\int_{\mathbb{R}^n}h(x)d\nu$.

We define a function ψ as follows.

$$\psi(t) := \mathbb{E}_x^{L,h}\Big[\int_0^t k(Y_u)du\Big], \tag{14}$$

where k is an arbitrary continuous and positive function with a compact support. We here give some properties of the function ψ.

Lemma 6.6. *Let ψ be as in* (14)*.*

(i) $\lim_{t\to\infty}\psi(t)=\infty$.
(ii) *For every $s>0$ it holds*

$$\lim_{t\to\infty}\frac{\psi(t+s)}{\psi(t)}=1.$$

Proof. Let $G^h k(y) := \int_{\mathbb{R}^n} G^h(y,z)k(z)h^2(z)dz$ and $G^h(y,z) := \int_0^\infty p^h(t,y,z)dt$ for every $y,z\in\mathbb{R}^n$, where p^h is the heat kernel given by the formula (11). The recurrence of $\mathbb{M}^{L,h}$ implies $G^h k(y)=\infty$ h^2dy-a.e. We then obtain (i): The Markov property implies

$$\begin{aligned}\psi(t) &\geq \mathbb{E}_x^{L,h}\Big[\int_1^t k(Y_u)du\Big]\\ &= \mathbb{E}_x^{L,h}\Big[\mathbb{E}_x^{L,h}\Big[\Big(\int_0^{t-1} k(Y_u)du\Big)\circ\theta_1\Big|\mathcal{M}_1\Big]\Big]\\ &= \mathbb{E}_x^{L,h}\Big[\mathbb{E}_{Y_1}^{L,h}\Big[\int_0^{t-1} k(Y_u)du\Big]\Big]\\ &= \int_{\mathbb{R}^n} p^h(1,x,y)\int_{\mathbb{R}^n}\int_0^{t-1} p^h(u,y,z)du k(z)h^2(z)dz h^2(y)dy\\ &\to \int_{\mathbb{R}^n} p^h(1,x,y)G^h k(y)h^2(y)dy\\ &=\infty\end{aligned}$$

as $t\to\infty$. Combining (i) and the boundedness of k, we see that (ii) follows. □

We quote Chacon-Ornstein type ergodic theorem.

Theorem 6.7 ([**4**]). *Let ν_1 and ν_2 be arbitrary probability measures and let B_t and C_t be special PCAFs with respect to $\mathbb{M}^{L,h}$. Suppose $\int_0^t f(Y_u)dB_u$ and $\int_0^t g(Y_u)dC_u$ are special PCAFs with respect to $\mathbb{M}^{L,h}$. It then holds*

$$\lim_{t\to\infty}\frac{\mathbb{E}_{\nu_1}^{L,h}\big[\int_0^t f(Y_u)dB_u\big]}{\mathbb{E}_{\nu_2}^{L,h}\big[\int_0^t g(Y_u)dC_u\big]}=\frac{\langle h^2\mu_B,f\rangle}{\langle h^2\mu_C,g\rangle}$$

for arbitrary bounded positive Borel-measurable functions f and g. Here, μ_B and μ_C are Revuz measures corresponding to B_t and C_t respectively.

Now, we solve the Feynman-Kac penalization problem in the critical case. Using the formula (13), Lemma 6.6, and Theorem 6.7,

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathbb{E}^L_\nu[e^{A^{\mu+\mu_{F_1}}_t}]}{\psi(t)} &= \lim_{t\to\infty}\frac{\nu(\mathbb{R}^n)}{\psi(t)} + \langle\nu,h\rangle\frac{\mathbb{E}^{L,h}_{\nu^h}\big[\int_0^t(1/h(Y_u))dA^{\mu+\mu_{F_1}}_u\big]}{\mathbb{E}^{L,h}_x\big[\int_0^t k(Y_u)du\big]}\\ &= \lim_{t\to\infty}\langle\nu,h\rangle\frac{\mathbb{E}^{L,h}_{\nu^h}\big[\int_0^t(1/h(Y_u))dA^{\mu+\mu_{F_1}}_u\big]}{\mathbb{E}^{L,h}_x\big[\int_0^t k(Y_u)du\big]}\\ &= \langle\nu,h\rangle\frac{\langle\mu+\mu_{F_1},h\rangle}{\langle h^2dx,k\rangle}.\end{aligned}$$

We set a finite positive measure ν for every $B\in\mathcal{B}^n$ as follows:

$$\nu(B) := \mathbb{E}^L_x\big[e^{A^{\mu+\mu_{F_1}}_s}S; Y_s\in B\big].$$

Note that the Markov property implies

$$\begin{aligned}\mathbb{E}^L_x\big[e^{A^{\mu+\mu_{F_1}}_t}S\big] &= \mathbb{E}^L_x\big[e^{A^{\mu+\mu_{F_1}}_s}S\mathbb{E}^L_x\big[e^{A^{\mu+\mu_{F_1}}_{t-s}}\circ\theta_s|\mathcal{M}_s\big]\big]\\ &= \mathbb{E}^L_x\big[e^{A^{\mu+\mu_{F_1}}_s}S\mathbb{E}^L_{Y_s}\big[e^{A^{\mu+\mu_{F_1}}_{t-s}}\big]\big]\\ &= \mathbb{E}^L_\nu\big[e^{A^{\mu+\mu_{F_1}}_{t-s}}\big].\end{aligned}$$

Lemma 6.6 and Theorem 6.7 yield

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathbb{E}_x[e^{A^{\mu,F}_t}S]}{\mathbb{E}_x[e^{A^{\mu,F}_t}]} &= \lim_{t\to\infty}\frac{\mathbb{E}^L_x[e^{A^{\mu+\mu_{F_1}}_t}S]}{\mathbb{E}^L_x[e^{A^{\mu+\mu_{F_1}}_t}]}\\ &= \lim_{t\to\infty}\frac{\mathbb{E}^L_x[e^{A^{\mu+\mu_{F_1}}_t}S]/\psi(t)}{\mathbb{E}^L_x[e^{A^{\mu+\mu_{F_1}}_t}]/\psi(t)}\\ &= \lim_{t\to\infty}\frac{\{\mathbb{E}^L_\nu[e^{A^{\mu+\mu_{F_1}}_{t-s}}]/\psi(t-s)\}\cdot\{\psi(t-s)/\psi(t)\}}{\mathbb{E}^L_x[e^{A^{\mu+\mu_{F_1}}_t}]/\psi(t)}\\ &= \frac{\{\langle\nu,h\rangle\langle\mu+\mu_{F_1},h\rangle\}/\langle h^2dx,k\rangle}{\{\langle\delta_x,h\rangle\langle\mu+\mu_{F_1},h\rangle\}/\langle h^2dx,k\rangle}\\ &= \frac{\langle\nu,h\rangle}{h(x)}.\end{aligned}$$

Rewriting the last limit, we completely solve the problem.

$$\begin{aligned}\frac{\langle \nu, h\rangle}{h(x)} &= \frac{\mathbb{E}_x^L[e^{A_s^{\mu+\mu_{F_1}}} h(Y_s)S]}{h(x)} \\ &= \mathbb{E}_x\left[L_s \frac{e^{A_s^{\mu+\mu_{F_1}}} h(X_s)}{h(x)} S\right] \\ &= \mathbb{E}_x[M_s S] \\ &= \mathbb{E}_x^M[S].\end{aligned}$$

Here, the weight process M_s is as follows:

$$M_s := \frac{e^{A_s^{\mu,F}} h(X_s)}{h(x)}. \tag{15}$$

Remark 6.8. Since $\mathbb{M}^{L,h}$ is an irreducible recurrent h^2dx-symmetric right process, the ergodic theorem yields

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = \begin{cases} \langle h^2dx, k\rangle & \text{if } h \in L^2(\mathbb{R}^n, dx) \\ 0 & \text{if } h \notin L^2(\mathbb{R}^n, dx). \end{cases}$$

We see that $h \in L^2(\mathbb{R}^n, dx)$ (positive critical) if and only if $n > 2\alpha$ and $h \notin L^2(\mathbb{R}^n, dx)$ (null critical) if and only if $\alpha < n \leq 2\alpha$, since $c^{-1}|x|^{\alpha-n} \leq h(x) \leq c|x|^{\alpha-n}$ for all $|x| > 1$. Consequently, we see the asymptotic behavior of the non-local Feynman-Kac semigroup $P_t^{\mu,F} f(x) := \mathbb{E}_x[e^{A_t^{\mu,F}} f(X_t)]$:

$$P_t^{\mu,F}1(x) \begin{cases} \sim \mathbb{E}_x[e^{A_\infty^{\mu,F}}] & \text{if } \lambda(0) > 1 \\ \sim \left(h(x)\displaystyle\int_{\mathbb{R}^n} h(x)dx\right) e^{\theta_0 t} & \text{if } \lambda(0) < 1 \\ \sim \left(h(x)\displaystyle\int_{\mathbb{R}^n} h(x)d(\mu+\mu_{F_1})\right) t & \text{if } \lambda(0) = 1 \text{ and } n \geq 2\alpha \\ = o(t) & \text{if } \lambda(0) = 1 \text{ and } \alpha < n \leq 2\alpha \end{cases}$$

as $t \to \infty$. We further see the growth of L^p-spectral bounds for all $1 \leq p \leq \infty$ (see [**32**, Theorem 5.6]). Let $l_p := -\lim_{t\to\infty}(1/t)\log\|P_t^{\mu,F}\|_{p,p}$. Then

$$l_p = \begin{cases} 0 & \text{if } \lambda(0) \geq 1 \\ -\theta_0 & \text{if } \lambda(0) < 1. \end{cases}$$

This implies that our definition of (sub-, super-)criticality corresponds to Simon's definition (see p. 218 of [**26**]).

7. Examples of jumping functions.

We give some concrete examples of jumping functions which belong to the class $\mathcal{A}$ and the class $\mathcal{A}_s$ (see Definition 2.3 and Definition 6.2 for the definitions of them). We assume $n > \alpha$ in this section. Since the Green function of X is $c_{\alpha,n}|x-y|^{\alpha-n}$ and the Lévy kernel of X is $c_{\alpha,n}|x-y|^{-\alpha-n}$, $F \in \mathcal{A}$ is equivalent to

$$\lim_{R\to\infty,\, r\to 0} \sup_{x\in\mathbb{R}^n} \int_{B(0,R)^c\cup B(x,r)} dy\, |x-y|^{\alpha-n} \int_{\mathbb{R}^n} dz\, |y-z|^{-\alpha-n} F(y,z) = 0 \tag{16}$$

and $F \in \mathcal{A}_s$ is equivalent to

$$\sup_{x\in\mathbb{R}^n} |x|^{n-\alpha} \int_{\mathbb{R}^n} dy\, |x-y|^{\alpha-n} \int_{\mathbb{R}^n} dz\, |y-z|^{-\alpha-n} F(y,z) < \infty. \tag{17}$$

We first give a well-known example.

Example 7.1. Let K_1 and K_2 be two disjoint compact subsets on $\mathbb{R}^n$ and let $F(x,y)$ be as follows:

$$F(x,y) := \mathbf{1}_{K_1}(x)\mathbf{1}_{K_2}(y) + \mathbf{1}_{K_2}(x)\mathbf{1}_{K_1}(y).$$

We here check that this satisfies the conditions (16) and (17). It suffices to estimate the integral

$$I(x) := \int_{B(0,R)^c\cup B(x,r)} dy|x-y|^{\alpha-n} \int_{\mathbb{R}^n} dz|y-z|^{-\alpha-n}(\mathbf{1}_{K_1}(y)\mathbf{1}_{K_2}(z) + \mathbf{1}_{K_2}(y)\mathbf{1}_{K_1}(z)).$$

$\sup\{|y-z|^{-\alpha-n}; y \in K_1 \text{ and } z \in K_2\}$ is bounded so that the integral $I(x)$ can be estimated:

$$\begin{aligned} I(x) &\le c\int_{B(0,R)^c\cup B(x,r)} dy\, |x-y|^{\alpha-n} \int_{\mathbb{R}^n} dz\, (\mathbf{1}_{K_1}(y)\mathbf{1}_{K_2}(z) + \mathbf{1}_{K_2}(y)\mathbf{1}_{K_1}(z)) \\ &\le c\int_{B(0,R)^c\cup B(x,r)} dy\, |x-y|^{\alpha-n} \, (|K_2|\mathbf{1}_{K_1}(y) + |K_1|\mathbf{1}_{K_2}(y)) \\ &\le c\int_{(B(0,R)^c\cup B(x,r))\cap(K_1\cup K_2)} dy\, |x-y|^{\alpha-n} \\ &\le c\int_{B(x,r)} dy\, |y-x|^{\alpha-n}, \end{aligned}$$

where $|K_j|$ is the Lebesgue measure of K_j $(j=1,2)$. We take $R>0$ such that $B(0,R) \supset K_1 \cup K_2$ in the fourth line. We then see that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} I(x) &\le c\int_{B(0,r)} dy\,|y|^{\alpha-n} \\ &\le c\int_0^r \rho^{\alpha-1}d\rho \\ &\le cr^{\alpha}.\end{aligned}$$

We have used the polar coordinates transform in the second line. Since the constant c is independent of x, letting $r\to 0$ completes the check. One can also check the condition (17).

We give another example. Thus far, it has been unknown whether there exists a jumping function of the class $\mathcal{A}$ that has a full support. We provide such a function in the following example.

Example 7.2.

$$F(x,y) = (1\wedge|x-y|^p)\langle x\rangle^{-q}\langle y\rangle^{-q}\quad \text{for } p>\alpha \text{ and } q>n.$$

Here, $\langle x\rangle := \sqrt{1+|x|^2}$. We check the condition (16). Note that $\langle x+y\rangle \le \sqrt{2}\langle x\rangle\langle y\rangle$ for all $x,y\in\mathbb{R}^n$. We take x arbitrarily.

$$\begin{aligned}&\int_{B(0,R)^c\cup B(x,r)} dy\int_{\mathbb{R}^n} dz|x-y|^{\alpha-n}|y-z|^{-\alpha-n}(1\wedge|y-z|^p)\langle y\rangle^{-q}\langle z\rangle^{-q} \\ &\le \int_{B(x,R)^c\cup -B(x,r)+x} dy\int_{\mathbb{R}^n} dz|y|^{\alpha-n}|z|^{-\alpha-n}(1\wedge|z|^p)\langle x-y\rangle^{-q}\langle x-y-z\rangle^{-q} \\ &\le c\int_{B(x,R)^c\cup -B(x,r)+x} dy\int_{\mathbb{R}^n} dz|y|^{\alpha-n}|z|^{-\alpha-n}(1\wedge|z|^p)\langle x-y\rangle^{-q}\langle x\rangle^{\delta q}\langle y\rangle^{\delta q}\langle z\rangle^{-\delta q} \\ &\le c\biggl(\langle x\rangle^{\delta q}\int_{B(x,R)^c\cup -B(x,r)+x} dy|y|^{\alpha-n}\langle x-y\rangle^{-q}\langle y\rangle^{\delta q}\biggr) \\ &\quad\times\biggl(\int_{\mathbb{R}^n} dz|z|^{-\alpha-n}(1\wedge|z|^p)\langle z\rangle^{-\delta q}\biggr).\end{aligned}$$

Here, $-B(x,r)+x := \{x-y; y\in B(x,r)\}$ and $\delta>0$ is so close to 0. We have replaced $x-y$ and $y-z$ by y and z in the second line respectively. We have used two estimates: $\langle y-z\rangle^{-q}\le c\langle y-z\rangle^{-\delta q}$ and $\langle y-z\rangle^{-1}\le\sqrt{2}\langle y\rangle\langle z\rangle^{-1}$ in the third line. It is easy to see that the second factor of the fourth line is dominated by a constant independent of x. Note that the condition $p>\alpha$ is needed for this estimate.

The first factor of the fourth line is estimated as follows.

$$\begin{aligned}&\langle x\rangle^{\delta q}\int_{-B(x,r)+x\cup B(x,R)^c} |y|^{\alpha-n}\langle y-x\rangle^{-q}\langle y\rangle^{\delta q}dy \\ &\quad\le c\langle x\rangle^{\delta q}\int_{B(x,r)\cup B(0,R)^c} |x-w|^{\alpha-n}\langle x-w\rangle^{\delta q}\langle x\rangle^{-\delta q}\langle w\rangle^{-(1-\delta)q}\langle x-w\rangle^{\delta q}dw\end{aligned}$$

$$= c\int_{B(x,r)\cup B(0,R)^c} |x-w|^{\alpha-n}\langle x-w\rangle^{2\delta q}\langle w\rangle^{-(1-\delta)q}dw \ (=: I(x)).$$

Here, $w := x-y$. We consider two cases: one is $x \in B(0,R)$ and the other is $x \in B(0,R)^c$.

$$\begin{aligned}
\sup_{x\in B(0,R)} I(x) &\le \sup_{x\in B(0,R)} \left(\int_{B(x,r)} + \int_{B(0,R)^c}\right)|x-w|^{\alpha-n}\langle x-w\rangle^{2\delta q}\langle w\rangle^{-(1-\delta)q}dw \\
&\le c \sup_{x\in B(0,R)} \int_0^r \rho^{\alpha-n}\cdot\rho^{n-1}d\rho + r^{\alpha-n}\int_R^\infty \rho^{-(1-\delta)q+n-1}d\rho \\
&\le c(r^\alpha + r^{\alpha-n}R^{n-q+\delta q}). \\
\sup_{x\in B(0,R)^c} I(x) &\le \sup_{x\in B(0,R)^c} \left(\int_{B(x,r)\cap B(0,R)^c} + \int_{B(x,r)^c\cap B(0,R)^c}\right)|x-w|^{\alpha-n} \\
&\quad \times \langle x-w\rangle^{2\delta q}\langle w\rangle^{-(1-\delta)q}dw \\
&\le cr^\alpha + cr^{\alpha-n}R^{n-q+\delta q} \\
&\le c(r^\alpha + r^{\alpha-n}R^{n-q+\delta q}).
\end{aligned}$$

Here, $|B(0,r)|$ is the volume of $B(0,r)$. If for an arbitrary $\varepsilon > 0$ we take $r, R > 0$ such that $r^\alpha < \varepsilon$ and $r^{\alpha-n}R^{n-q+\delta q} < \varepsilon$ then $F \in \mathcal{A}$.

We see that some jumping functions of $\mathcal{A}$ have full support and are in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. We also give a concrete example of jumping functions in the class $\mathcal{A}_s$.

Example 7.3.

$$F(x,y) = (1\wedge|x-y|^p)\langle x\rangle^{-q}\langle y\rangle^{-q} \quad \text{for } p > \alpha \text{ and } q > 2n-\alpha.$$

We check that this function satisfies the condition (17).

$$\begin{aligned}
&|x|^{n-\alpha}\int_{\mathbb{R}^n} dy\int_{\mathbb{R}^n} dz|x-y|^{\alpha-n}|y-z|^{-\alpha-n}(1\wedge|y-z|^p)\langle y\rangle^{-q}\langle z\rangle^{-q} \\
&\le |x|^{n-\alpha}\int_{\mathbb{R}^n} dy\int_{\mathbb{R}^n} dz|y|^{\alpha-n}|z|^{-\alpha-n}(1\wedge|z|^p)\langle x-y\rangle^{-q}\langle x-y-z\rangle^{-q} \\
&\le c|x|^{n-\alpha}\int_{\mathbb{R}^n} dy\int_{\mathbb{R}^n} dz|y|^{\alpha-n}|z|^{-\alpha-n}(1\wedge|z|^p)\langle x-y\rangle^{-q}\langle x\rangle^{\delta q}\langle y\rangle^{\delta q}\langle z\rangle^{-\delta q} \\
&\le c\left(|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\mathbb{R}^n} dy|y|^{\alpha-n}\langle x-y\rangle^{-q}\langle y\rangle^{\delta q}\right)\cdot\left(\int_{\mathbb{R}^n} dz|z|^{-\alpha-n}(1\wedge|z|^p)\langle z\rangle^{-\delta q}\right).
\end{aligned}$$

Here, $\delta > 0$ is so close to 0. It is easy to see that the second factor of the fourth line is dominated by a constant independent of x. The condition $p > \alpha$ is then needed.

Consequently, we have only to prove that

$$|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\mathbb{R}^n}|y|^{\alpha-n}\langle y-x\rangle^{-q}\langle y\rangle^{\delta q}dy$$

is uniformly dominated by a constant independent of x. We divide the last integral into three parts:

$$\int_{\mathbb{R}^n}=\int_{\{|y|\le 1\}}+\int_{\{|y|>1 \text{ and } |y-x|\le|x|/2}+\int_{\{|y|>1 \text{ and } |y-x|>|x|/2\}}\quad (=: \mathrm{I}+\mathrm{II}+\mathrm{III}).$$

Noting $\langle y-x\rangle^{-q}\le 2^{q/2}\langle x\rangle^{-q}\langle y\rangle^{q}$, $\langle x\rangle^{(\delta-1)q}\le 1$, and $\langle y\rangle^{(1+\delta)q}\le 2^{2q}$ for all $x\in\mathbb{R}^n$ and $|y|\le 1$,

$$\begin{aligned}\mathrm{I}&\le c|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\{|y|\le 1\}}|y|^{\alpha-n}\langle x\rangle^{-q}\langle y\rangle^{q}\langle y\rangle^{(1+\delta)q}dy\\&\le c|x|^{n-\alpha}\langle x\rangle^{(\delta-1)q}\int_0^1\rho^{\alpha-n}\rho^{n-1}d\rho\\&\le c\big(|x|^{n-\alpha}\wedge|x|^{n-\alpha+(\delta-1)q}\big).\end{aligned}$$

We use the polar coordinates transform in the second line. Thus I is dominated by a constant independent of x. The estimate of II is tricky. Note that if $|x|<2/3$ then II $=0$ since the subset $\{|y|\ge 1$ and $|y-x|\le|x|/2\}$ is empty, that $|x-y|\le|x|/2$ implies $|y|^{\alpha-n}\le 2^{n-\alpha}|x|^{\alpha-n}$ and that $\langle y\rangle^{\delta q}\le 2^{\delta q/2}\langle x\rangle^{\delta q}\langle y-x\rangle^{\delta q}$ holds. We then see

$$\begin{aligned}\mathrm{II}&\le c\langle x\rangle^{\delta q}\int_{\{|y|\ge 1 \text{ and } |y-x|\le|x|/2\}}\langle y-x\rangle^{-q}\langle y\rangle^{\delta q}dy\\&\le c\langle x\rangle^{2\delta q}\int_{\{|y|\ge 1 \text{ and } |y-x|\le|x|/2\}}\langle y-x\rangle^{(\delta-1)q}dy\\&\le c|x|^{2\delta q}\int_0^{|x|/2}\rho^{n-1}(1+\rho^2)^{(\delta-1)q/2}d\rho\\&\le c|x|^{2\delta q}\cdot|x|^{n+(\delta-1)q}\\&\le c|x|^{n+(3\delta-1)q}\end{aligned}$$

for $|x|\ge 2/3$. Since $n+(3\delta-1)q<0$, II is also dominated by a constant independent of x. The estimate of III is also tricky.

$$\begin{aligned}\mathrm{III}&\le c|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\{|y|\ge 1 \text{ and } |y-x|\ge|x|/2\}}|y|^{\alpha-n}\langle y-x\rangle^{(\delta-1)q}\langle y\rangle^{\delta q}dy\\&\le c|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\{|y|\ge 1 \text{ and } |y-x|\ge|x|/2\}}\langle y\rangle^{\alpha-n+\delta q}\langle y-x\rangle^{(\delta-1)q}dy\end{aligned}$$

$$\begin{aligned}
&\leq c|x|^{n-\alpha}\langle x\rangle^{\delta q}\int_{\{|y|\geq 1 \text{ and } |y-x|\geq |x|/2\}}\langle x-y\rangle^{\alpha-n+\delta q}\langle x\rangle^{n-\alpha-\delta q}\langle y-x\rangle^{(\delta-1)q}dy\\
&\leq c|x|^{n-\alpha}\langle x\rangle^{n-\alpha}\int_{\{|y-x|\geq|x|/2\}}\langle x-y\rangle^{\alpha-n+(\delta-1)q}dy\\
&\leq c|x|^{n-\alpha}\langle x\rangle^{n-\alpha}\int_{|x|/2}^{\infty}(1+\rho^2)^{(\alpha-n+(\delta-1)q)/2}\rho^{n-1}d\rho\\
&\leq c|x|^{n-\alpha}\langle x\rangle^{n-\alpha}(|x|^n\wedge|x|^{\alpha+(\delta-1)q})\\
&\leq c(|x|^{2n-\alpha}\wedge|x|^{2n-\alpha+(\delta-1)q}).
\end{aligned}$$

We have used the estimate $|y|^{\alpha-n}\leq 2^{(n-\alpha)/2}\langle y\rangle^{\alpha-n}$ for all $|y|\geq 1$ in the first line. Since $\alpha-n+\delta q<0$, $\langle y\rangle^{\alpha-n+\delta q}\leq 2^{(n-\alpha-\delta q)/2}\langle x-y\rangle^{\alpha-n+\delta q}\langle x\rangle^{n-\alpha-\delta q}$. We use this in the third line. Since $2n-\alpha+(\delta-1)q<0$, III is also uniformly dominated by a constant independent of x.

Remark 7.4. We further see that the jumping function of Example 7.3 does not belong to $\boldsymbol{A_2}$ (see also [**7**, Definition 2.3]), that is, it does not hold

$$\begin{aligned}
&\lim_{R\to\infty,r\to 0}\sup_{(x,w)\in d^c}|x-w|^{n-\alpha}\int_{B(x,r)\cup B(0,R)^c\times B(x,r)\cup B(0,R)^c}|x-y|^{\alpha-n}\\
&\qquad\times(1\wedge|y-z|^{-(\alpha+n)})\langle y\rangle^{-q}\langle z\rangle^{-q}|z-w|^{\alpha-n}|y-z|^{-(n+\alpha)}dydz=0.
\end{aligned}$$

Indeed, we may take a closed ball $B_{x,w}$ with radius 1 in $\{(y,z);|y-x|\leq 1,|z-w|\leq 1, 1\leq|y-z|\leq 5,\ \text{and } |y|,|z|\geq R\}$ for an arbitrary $R>0$. It then follows

$$\begin{aligned}
&|x-w|^{n-\alpha}\int_{B(x,r)\cup B(0,R)^c\times B(x,r)\cup B(0,R)^c}|x-y|^{\alpha-n}(1\wedge|y-z|^{-(\alpha+n)})\\
&\qquad\times\langle y\rangle^{-q}\langle z\rangle^{-q}|z-w|^{\alpha-n}|y-z|^{-(n+\alpha)}dydz\\
&\quad\geq c|B(0,1)||x-w|^{n-\alpha}\langle x\rangle^{-q}\langle w\rangle^{-q}.
\end{aligned}$$

Here, $|B(0,1)|$ is the volume of $B(0,1)$. Therefore we find that $\boldsymbol{A_2}\not\supset\mathcal{A}_s$ and $\boldsymbol{A_2}\subsetneq\mathcal{A}$.

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