J. Math. Soc. Japan
Vol. 66, No. 1 (2014) pp. 161–203
doi: 10.2969/jmsj/06610161

On the topology of stable maps

By Nicolas Dutertre and Toshizumi Fukui

(Received Sep. 14, 2010)
(Revised Apr. 13, 2012)

Abstract. We investigate how Viro's integral calculus applies for the study of the topology of stable maps. We also discuss several applications to Morin maps and complex maps.

1. Introduction.

It is well known that there is a deep relation between the topology of a manifold and the topology of the critical locus of maps. The best example of this fact is Morse Theory which gives the homotopy type of a compact manifold in terms of the Morse indices of the critical points of a Morse function. Let us mention other examples.

R. Thom [**23**] proved that the Euler characteristic $\chi(M)$ of a compact manifold M of dimension at least 2 has the same parity as the number of cusps of a generic map $f : M \to \mathbb{R}^2$. Later H. I. Levine [**16**] improved this result giving an equality relating $\chi(M)$ and the critical set of f. In [**10**], T. Fukuda generalized R. Thom's result to Morin maps $f : M \to \mathbb{R}^p$ when $\dim M \geq p$. He proved that

$$\chi(M) + \sum_{k=1}^{p} \chi(\overline{A_k(f)}) \equiv 0 \bmod 2, \tag{1.1}$$

where $A_k(f)$ is the set of points x in M such that f has a singularity of type A_k at x (see Section 4 for the definition of A_k). Furthermore if f has only fold points (i.e., singularities of type A_1), then T. Fukuda gave an equality relating $\chi(M)$ to the critical set of f. T. Fukuda's formulas were extended to the case of a Morin mapping $f : M \to N$, where $\dim M \geq \dim N$, by O. Saeki [**21**] and I. Nakai [**18**]. When $\dim M = \dim N$, similar formulas were obtained by J. M. Eliashberg [**7**], J. R. Quine [**20**] and I. Nakai [**18**]. On the other hand, Y. Yomdin [**27**] showed the equality among Euler characteristics of singular sets of holomorphic maps. As Y. Yomdin and I. Nakai showed in this context, the integral calculus due to O. Viro [**25**] is useful to find relations like (1.1) for stable maps. In this paper, we investigate how O. Viro's integral calculus applies in sufficiently wide setup. To do this we introduce the notion of local triviality at infinity and give some examples to illustrate this notion in Section 3. T. Ohmoto showed that Yomdin-Nakai's

---

2010 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R20, 57R70, 58C25.
Key Words and Phrases. singularities of maps, stable maps, Euler integration, Euler characteristic, Morin maps.
The first author was supported by Agence Nationale de la Recherche (reference ANR-08-JCJC-0118-01).

formula is generalized to a statement in terms of characteristic classes and discussed a relation with Thom polynomial in his lecture of the conference on the occasion of 70th birthday of T. Fukuda held on 20 July 2010.

We consider a stable map $f : M \to N$ between two smooth manifolds M and N. We assume that $\dim M \geq \dim N$, that N is connected and that M and N have finite homological type, i.e. that their homology groups are finitely generated. We also assume that f is locally trivial at infinity (see Definition 3.1) and has finitely many singularity types. Then the singular set $\Sigma(f)$ of f is decomposed into a finite union $\bigsqcup_\nu \nu(f)$, where $\nu(f)$ is the set of singular points of f of type ν. In Theorems 5.1, 5.6, 5.7 and 5.11, we establish several formulas between the Euler characteristics with closed support of M, N and the $\nu(f)$'s. We apply them to maps having singularities of type A_k or D_k in Corollaries 5.4, 5.5, 5.9, 5.10, 5.12 and 5.13.

In Section 6 of this paper, we apply the results of Section 5 to Morin maps and we use the link between the Euler characteristic with closed support and the topological Euler characteristic to recover and improve several results of J. M. Eliashberg, T. Fukuda, T. Fukuda and G. Ishikawa, I. Nakai, J. R. Quine, O. Saeki. We end the paper with some remarks in the complex case in Section 7.

The authors thank the anonymous referee for very careful reading and pointing out several errors in a previous version.

2. Viro's integral calculus.

In this section, we recall the method of integration with respect to a finitely-additive measure due to O. Viro [**25**].

Let X be a topological space and $\mathcal{S}(X)$ denote a collection of subsets of X which satisfies the following properties:

- If A, $B \in \mathcal{S}(X)$, then $A \cup B \in \mathcal{S}(X)$, $A \cap B \in \mathcal{S}(X)$.
- If $A \in \mathcal{S}(X)$, then $X \setminus A \in \mathcal{S}(X)$.

Let R be a commutative ring. Let $\mu_X : \mathcal{S}(X) \to R$ be a map which satisfies the following properties:

- If A and B are homeomorphic then $\mu_X(A) = \mu_X(B)$.
- For $A, B \in \mathcal{S}(X)$, $\mu_X(A \cup B) = \mu_X(A) + \mu_X(B) - \mu_X(A \cap B)$.

Example 2.1. When the elements of $\mathcal{S}(X)$ have finite homological type and are locally compact, the Euler characteristic of Borel-Moore homology (the homology with closed support, see [**2**]), denoted by χ_c, satisfies these conditions for μ_X with $R = \mathbb{Z}$. The mod 2 Euler characteristic of Borel-Moore homology also satisfies these conditions for μ_X with $R = \mathbb{Z}/2\mathbb{Z}$.

Let $\mathrm{Cons}(X, \mathcal{S}(X), R)$ (or $\mathrm{Cons}(X)$, for short) denote the set of finite R-linear combinations of characteristic functions $\mathbf{1}_A$ of elements A of $\mathcal{S}(X)$. For $B \in \mathcal{S}(X)$ and $\varphi \in \mathrm{Cons}(X, \mathcal{S}(X), R)$, we define the integral of φ over B with respect to μ_X, denoted by $\int_B \varphi d\mu_X$, by

$$\int_B \varphi(x) d\mu_X(x) = \sum_A \lambda_A \, \mu_X(A \cap B) \quad \text{where} \quad \varphi = \sum_A \lambda_A \mathbf{1}_A.$$

We remark that $\mu_X(B) = \int_B d\mu_X$.

Now we are going to state a Fubini type theorem for this integration. We need to introduce some notations.

We say that $(\mathcal{S}(X), \mathcal{S}(Y))$ *fits to* the map $f : X \to Y$ if the following conditions hold:

- If $A \in \mathcal{S}(X)$, then $f(A) \in \mathcal{S}(Y)$.
- $f^{-1}(y) \in \mathcal{S}(X)$ for $y \in Y$.
- For $A \in \mathcal{S}(X)$, $B \in \mathcal{S}(Y)$ with $f(A) = B$, if $f|_A : A \to B$ is a locally trivial fibration with fiber F, $F \in \mathcal{S}(X)$, then

$$\mu_X(A) = \mu_X(F)\mu_Y(B).$$

- For $A \in \mathcal{S}(X)$, there is a filtration $\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_l = Y$ with $B_i \in \mathcal{S}(Y)$ such that

$$f|_{f^{-1}(B_i \setminus B_{i-1}) \cap A} : f^{-1}(B_i \setminus B_{i-1}) \cap A \to B_i \setminus B_{i-1} \quad (i = 0, 1, \ldots, l)$$

  is a locally trivial fibration with fiber F_i, $F_i \in \mathcal{S}(X)$.

Lemma 2.2 (Fubini's theorem). *For $\varphi \in \mathrm{Cons}(X)$ and $f : X \to Y$ such that $(\mathcal{S}(X), \mathcal{S}(Y))$ fits to f, we have that $f_*\varphi(y) = \int_{f^{-1}(y)} \varphi(x) d\mu_X$ is a constructible function, and*

$$\int_X \varphi(x) d\mu_X = \int_Y f_*\varphi(y) d\mu_Y.$$

Proof. It is enough to show the case when $\varphi = \mathbf{1}_A$ for $A \in \mathcal{S}(X)$. So let us show that

$$\mu_X(A) = \int_Y \mu_X(A \cap f^{-1}(y)) d\mu_Y.$$

We take a filtration $\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_l = Y$ $(B_i \in \mathcal{S}(Y))$ so that

$$f|_{f^{-1}(B_i \setminus B_{i-1}) \cap A} : f^{-1}(B_i \setminus B_{i-1}) \cap A \to B_i \setminus B_{i-1} \quad (i = 0, 1, 2, \ldots, l)$$

is a locally trivial fibration with a fiber F_i, $F_i \in \mathcal{S}(X)$. Then we have

$$
\begin{aligned}
\mu_X(A) &= \sum_{i=0}^{l} \mu_X(f^{-1}(B_i \setminus B_{i-1}) \cap A) && \text{(additivity of } \mu_X) \\
&= \sum_{i=0}^{l} \mu_X(F_i)\mu_Y(B_i \setminus B_{i-1}) && \text{(local triviality of } f|_A \text{ on } B_i \setminus B_{i-1}) \\
&= \sum_{i=0}^{l} \mu_X(F_i) \int_{B_i \setminus B_{i-1}} d\mu_Y && \text{(definition of } \textstyle\int) \\
&= \sum_{i=0}^{l} \int_{B_i \setminus B_{i-1}} \mu_X(F_i) d\mu_Y \\
&= \sum_{i=0}^{l} \int_{B_i \setminus B_{i-1}} \mu_X(A \cap f^{-1}(y)) d\mu_Y && (F_i = A \cap f^{-1}(y) \ \text{ for } y \in B_i \setminus B_{i-1}) \\
&= \int_Y \mu_X(A \cap f^{-1}(y)) d\mu_Y && \text{(additivity of } \textstyle\int). \qquad \square
\end{aligned}
$$

Corollary 2.3. *Set $X_i = \{x \in X \mid \varphi(x) = i\}$, and $Y_j = \{y \in Y \mid f_*\varphi(y) = j\}$. Then $X_i \in \mathcal{S}(X)$, $Y_j \in \mathcal{S}(Y)$ and we have*

$$\sum_i i\, \mu_X(X_i) = \sum_j j\, \mu_Y(Y_j).$$

Proof. This is clear, since:

$$
\begin{gathered}
\int_X \varphi(x) d\mu_X = \sum_i \int_{X_i} \varphi(x) d\mu_X = \sum_i \int_{X_i} i\, d\mu_X = \sum_i i\, \mu_X(X_i), \\
\int_Y f_*\varphi(y) d\mu_Y = \sum_j \int_{Y_j} f_*\varphi(y) d\mu_Y = \sum_j \int_{Y_j} j\, d\mu_Y = \sum_j j\, \mu_Y(Y_j). \qquad \square
\end{gathered}
$$

Corollary 2.4. *If $f_*\varphi$ is a constant d on $y \in Y$, we have*

$$\sum_i i\, \mu_X(X_i) = d\, \mu_Y(Y).$$

In the sequel, we will apply O. Viro's integral calculus to investigate the topology of stable maps (see [**18**] and [**19**] for a similar strategy).

3. Local triviality at infinity.

In this section, we define the notion of local triviality at infinity for a smooth map. We assume that all the manifolds that appear have finite homological type.

Definition 3.1. Let $f : M \to N$ be a smooth map between two smooth manifolds. We say f is *locally trivial at infinity at $y \in N$* if there are a compact set K in M and an open neighborhood D of y such that $f : (M \setminus K) \cap f^{-1}(D) \to D$ is a trivial fibration. We say f is *locally trivial at infinity* if it is locally trivial at infinity at any $y \in N$.

Note that this definition implies that $\Sigma(f) \cap ((M \setminus K) \cap f^{-1}(D)) = \emptyset$, where $\Sigma(f)$ denotes the critical set of f. Let us give other consequences of this definition.

LEMMA 3.2. *Assume that*

- $\dim M - \dim N$ *is odd and* $\dim M - \dim N > 0$,
- $f : M \to N$ *is locally trivial at infinity at* $y \in N$.

Then $\chi_c(f^{-1}(t))$ *and* $\chi(f^{-1}(t))$ *are constant for any regular value* t *of* f *in a neighborhood of* y.

PROOF. Let us treat first the case $N = \mathbb{R}$. Let t_0 and t_1 be two regular values of f in a small neighborhood of y. Let us assume that $t_0 < t_1$ so that $y \in]t_0, t_1[$. By Definition 3.1, $\Sigma(f) \cap f^{-1}(]t_0, t_1[)$ is a compact subset of M. Making a small perturbation of f if necessary, we can assume that f is a Morse function in $f^{-1}([t_0, t_1])$. Let us denote by $\{p_1, \ldots, p_l\}$ the set of its critical points and by $\{\lambda_1, \ldots, \lambda_l\}$ the set of their respective indices. By Morse theory and since f is locally trivial at infinity over $[t_0, t_1]$, we have

$$\chi(f^{-1}[t_0, t_1]) - \chi(f^{-1}(t_0)) = \sum_{i=1}^{l} (-1)^{\lambda_i},$$

and

$$\chi(f^{-1}[t_0, t_1]) - \chi(f^{-1}(t_1)) = (-1)^{\dim M} \sum_{i=1}^{l} (-1)^{\lambda_i} = \sum_{i=1}^{l} (-1)^{\lambda_i}.$$

Therefore $\chi(f^{-1}(t_0)) = \chi(f^{-1}(t_1))$.

Let us treat now the general case. Let t_0 and t_1 be two regular values of f in a small neighborhood of y. Let $\gamma : [0, 1] \to N$ be a smooth embedded arc transverse to f (see [**5**, Section 4.3] for details) such that $\gamma(0) = t_0$, $\gamma(1) = t_1$ and $\gamma([0, 1])$ is included in a neighborhood of y. Then $W = f^{-1}(\gamma([0, 1]))$ is a manifold with boundary of dimension dim $M-$ dim $N + 1$ with boundary $f^{-1}(t_0) \cup f^{-1}(t_1)$. Applying the previous case to $\gamma^{-1} \circ f : W \to [0, 1]$, we get the result since $\gamma^{-1} \circ f$ is clearly trivial at infinity. We conclude with the fact that $\chi_c(Z) = -\chi(Z)$ for any odd-dimensional smooth manifold Z. □

Similarly, we can prove:

LEMMA 3.3. *Assume that*

- $\dim M - \dim N$ *is odd and* $\dim M - \dim N > 0$,
- N *is connected,*
- $f : M \to N$ *is locally trivial at infinity.*

Then $\chi_c(f^{-1}(t))$ *and* $\chi(f^{-1}(t))$ *are constant for any regular value* t *of* f.

Using the same arguments, we can prove the following two results.

LEMMA 3.4. *Assume that*

- $\dim M - \dim N$ *is even and* $\dim M - \dim N > 0$,
- $f : M \to N$ *is locally trivial at infinity at* $y \in N$.

Then $\chi_c(f^{-1}(t)) \mod 2$ *and* $\chi(f^{-1}(t)) \mod 2$ *are constant for any regular value* t *of* f *in a neighborhood of* y.

LEMMA 3.5. *Assume that*

- $\dim M - \dim N$ *is even and* $\dim M - \dim N > 0$,
- N *is connected,*
- $f : M \to N$ *is locally trivial at infinity.*

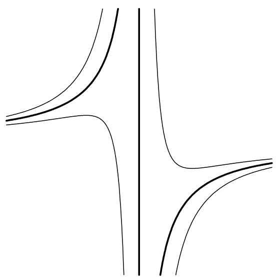
Then $\chi_c(f^{-1}(t)) \mod 2$ *and* $\chi(f^{-1}(t)) \mod 2$ *are constant for any regular value* t *of* f.

Here are some examples of functions not locally trivial at infinity.

EXAMPLE 3.6 (Broughton [**3**]). Consider $f(x, y) = x(xy+1)$. The critical set $\Sigma(f)$ of f is empty. For $t \neq 0$,

$$f^{-1}(t) = \{y = (t - x)/x^2\}.$$

We have $f^{-1}(t) = \mathbb{R}^*$, $f^{-1}(0) = \mathbb{R} \cup \mathbb{R}^*$ and $\chi_c(f^{-1}(t)) = -2$, $\chi_c(f^{-1}(0)) = -3$. So this example is not locally trivial at infinity at $t = 0$. The level curves of f with level $-1/2, 0, 1/2$ are shown in the figure. The thick line shows the level 0.

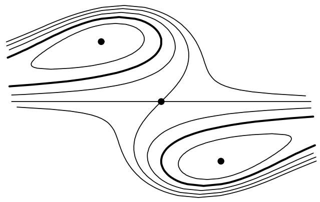


A map $f : \mathbb{R}^2 \to \mathbb{R}$ with $\Sigma(f) = \emptyset$ may not be surjective. M. Shiota remarked that the map $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto (x(xy + 1) + 1)^2 + x^2$, has empty critical set, and is not surjective.

EXAMPLE 3.7 (Tibăr-Zaharia [**24**, Example 3.2]). Consider $f(x, y) = x^2y^2 + 2xy + (y^2 - 1)^2$. Then $\Sigma(f) = \{(0, 0), (1, -1), (-1, 1)\}$ and $f(0, 0) = 1$, $f(1, -1) = f(-1, 1) = -1$. Since $f^{-1}(t)$ consists of two lines (resp. circles) if $0 \leq t < 1$ (resp. $-1 < t < 0$), we have

$$\chi_c(f^{-1}(t)) = \begin{cases} -2 & (0 \leq t < 1), \\ 0 & (-1 < t < 0). \end{cases}$$

So this example is not locally trivial at infinity at $t = 0$. The level curves of f with level $-1, -1/2, 0, 1/2, 1, 3/2$ are shown in the figure. The thick line shows the level 0.



Let us examine now the case $\dim M = \dim N$. We assume first that M and N are both oriented.

LEMMA 3.8. *Assume that*

- $\dim M = \dim N$,
- *f is finite-to-one,*
- *N is connected,*
- *$f : M \to N$ is locally trivial at infinity.*

Then

$$\deg(f, t) = \sum_{x \in f^{-1}(t)} \deg(f : (M, x) \to (N, f(x))),$$

is constant for any regular value t of f.

PROOF. Let t_0 and t_1 be two regular values of f and let $\gamma : [0, 1] \to N$ be a smooth embedded arc transverse to f such that $\gamma(0) = t_0$ and $\gamma(1) = t_1$. We set $I = \gamma([0, 1])$. The set $f^{-1}(I)$ is a smooth curve with boundary. It is enough to prove that $f^{-1}(I)$ is compact (see [**17**], for a similar argument). Let $t \in I$. Since f is locally trivial at infinity, there exist a neighborhood I_t of t and a compact subset K_t of M such that $\# f^{-1}(t') \cap (M \setminus K_t)$ is constant for $t' \in I_t$. Hence we can find another neighborhood $\tilde{I}_t$ of t and another compact subset $\tilde{K}_t$ such that $f^{-1}(t') \cap (M \setminus \tilde{K}_t)$ is empty for $t' \in \tilde{I}_t$. Since I is compact, we see that there is a compact set K' such that $f^{-1}(I) \cap (M \setminus K') = \emptyset$. Therefore $f^{-1}(I)$ is compact. $\square$

DEFINITION 3.9. With the same assumptions, we define $\deg f$ by $\deg f = \deg(f, t)$ where t is any regular value of f.

In the non-oriented case, all these results are still valid replacing $\deg(f, t)$ by $\# f^{-1}(t)$ mod 2. Hence, in this situation, we can define the degree of f modulo 2.

4. Euler characteristics of local generic fibers.

In this section, we present a general method for the computation of the Euler characteristic of the Milnor fibers of a stable map-germ. We start with a lemma.

Lemma 4.1. *Let Y be a manifold and let X be a set defined by*

$$X = \{(x,y) \in \mathbb{R}^p \times Y : {x_1}^2 + \cdots + {x_p}^2 = g(y)\}$$

where $g(y)$ is a smooth positive function. Then X is a smooth manifold and

$$\chi_c(X) = \chi(S^{p-1})\chi_c(Y) = (1-(-1)^p)\chi_c(Y).$$

Proof. It is easy to check that X is a manifold. To obtain the equality, consider the map:

$$X \to Y, \ (x,y) \mapsto y.$$

This is a locally trivial fibration whose fiber is S^{p-1}. □

Example 4.2. Let X be the set defined by

$$X = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q : {x_1}^2 + \cdots + {x_p}^2 = {y_1}^2 + \cdots + {y_q}^2 + t\}$$

where t is a positive constant. Since $X \to \mathbb{R}^q$, $(x,y) \mapsto y$, is a locally trivial fibration whose fiber is S^{p-1}, we have

$$\chi_c(X) = \chi_c(S^{p-1})\chi_c(\mathbb{R}^q) = (1-(-1)^p)(-1)^q = (-1)^q - (-1)^{p+q}.$$

Example 4.3. Let X be the set defined by

$$X = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q : {x_1}^2 + \cdots + {x_p}^2 = {y_1}^2 + \cdots + {y_q}^2\}.$$

Since $X \setminus \{0\} \to \mathbb{R}^q \setminus \{0\}$, $(x,y) \mapsto y$, is a locally trivial fibration whose fiber is S^{p-1}, we have

$$\begin{aligned}\chi_c(X) &= \chi_c(\{0\}) + \chi_c(S^{p-1})\chi_c(\mathbb{R}^q \setminus \{0\}) \\ &= 1 + (1-(-1)^p)((-1)^q - 1) \\ &= (-1)^p + (-1)^q - (-1)^{p+q}.\end{aligned}$$

Next we will apply this lemma and these examples to the computation of Euler characteristics of local nearby fibers of some particular stable map-germs.

Remember that stable-germs are $\mathcal{K}$-versal unfoldings, deleting constant terms, of a map-germ $x \mapsto g(x;0)$, called the genotype (see [**1**, Part I, 9]). Here we consider the unfolding of a function germ with an isolated critical point. Let B be a small open ball in $\mathbb{R}^n$ centered at 0 of radius R and let B' be a small open ball in $\mathbb{R}^{a+b}$ centered at 0 of radius R'. We consider a map f defined by

$$f : B \times B' \times \mathbb{R}^h \to \mathbb{R} \times \mathbb{R}^h, \qquad (x, z, c) \mapsto (g(x; c) + Q(z), c) \tag{4.1}$$

where $Q(z) = {z_1}^2 + \cdots + {z_a}^2 - {z_{a+1}}^2 - \cdots - {z_{a+b}}^2$. We assume that $(x, c) \mapsto g(x; c)$ is a $\mathcal{K}$-versal unfolding of a function-germ $g_0 : x \mapsto g(x; 0)$ and that g_0 has an isolated critical point at the origin.

We want to compute the Euler characteristic of a local Milnor fiber around the point $(0, 0)$, namely the intersection of the fiber $f^{-1}(\varepsilon, c)$, for a regular value (ε, c) of f near 0, with a small Euclidian open ball centered at $(0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^{a+b}$. Here we notice that the regular value c is fixed.

Since $g(x; c)$ is an unfolding of a function-germ, we can suppose that it is a polynomial. Since $g(0; 0) = 0$, we can write

$$g(x; c) = \sum_{i=1}^{d} g_i(x; c),$$

where d is the degree of g and g_i is its homogeneous component of degree i. Therefore for $(x, c) \neq (0, 0)$ and for any ε, we have

$$g(x; c) - \varepsilon = \sum_{i=1}^{d} |(x, c)|^i g_i\left(\frac{x}{|(x, c)|}; \frac{c}{|(x, c)|}\right) - \varepsilon.$$

Hence there exists $C > 0$ such that

$$|g(x; c) - \varepsilon| \leq C|(x, c)| + |\varepsilon|,$$

for (x, c) in a small neighborhood of $(0, 0)$ and for any ε. We conclude easily that there exists $D > 0$ such that

$$|g(x; c) - \varepsilon| \leq D(|x| + |(c, \varepsilon)|),$$

for (x, c, ε) in a small neighborhood of the origin.

For $p \in \mathbb{N}$, let us define $h_p(x, z) = \mathrm{Max}(4D|x|^{1/p}, |z|)$. Remark that $h_p^{-1}(0) = \{0\}$ and that h_p is continuous and semi-algebraic. Let us describe the "ball" of radius R' defined with this distance function, i.e. the set of points $(x, z) \in \mathbb{R}^n \times \mathbb{R}^{a+b}$ such that $h_p(x, z) \leq R'$. We have

$$\begin{aligned} h_p(x, z) \leq R' \Leftrightarrow |z| \leq R' \text{ and } 4D|x|^{1/p} \leq R' \Leftrightarrow |z| \leq R' \text{ and } |x| \leq \left(\frac{R'}{4D}\right)^p \\ \Leftrightarrow (x, z) \in B_{(R'/4D)^p} \times B'_{R'}. \end{aligned}$$

Hence there exist a small neighborhood U of the origin in $\mathbb{R}^h \times \mathbb{R}$ and a constant $D' > 0$ such that for any $x \in B_{(R'/4D)^p}$ and any $(c, \varepsilon) \in U$,

$$|g(x; c) - \varepsilon| \leq D'(R')^p.$$

So if we choose p sufficiently big then $g(x;c)-\varepsilon$ is very small compared with R' as R' tends to 0.

Let us explain why the Milnor fibre of f defined with the Euclidian distance $\omega = \sqrt{|x|^2+|z|^2}$ and the Milnor fiber of f defined with the "distance" function h_p have the same Euler characteristic. Using the results of A. H. Durfee [**6**, Section 3] , we have

$$\chi_c(f_0^{-1}(0)\cap\{\omega\le\varepsilon'\}) = \chi_c(f_0^{-1}(0)\cap\{h_p\le R'\}),$$

and

$$\chi_c(f_0^{-1}(0)\cap\{\omega=\varepsilon'\}) = \chi_c(f_0^{-1}(0)\cap\{h_p=R'\}),$$

where ε' and R' are chosen sufficiently small so that $\{h_p\le R'\}\subset\{\omega<\varepsilon'\}$. Here $f_0: B\times B'\to\mathbb{R}$ is the polynomial function $f_0(x,z)=g_0(x)+Q(z)$. Note that it has an isolated critical point at $(0,0)$. Therefore

$$\chi_c(f_0^{-1}(0)\cap\{\omega<\varepsilon'\}) = \chi_c(f_0^{-1}(0)\cap\{h_p<R'\}),$$

and

$$\chi_c\left(f_0^{-1}(0)\cap(\{\omega<\varepsilon'\}\setminus\{h_p<R'\})\right)=0.$$

In order to prove these last equalities are still valid replacing $f_0^{-1}(0)$ by $\{(x,z)\mid g(x;c)+Q(z)=\varepsilon\}$ where c and ε are small enough, it is enough to prove that $f_0^{-1}(0)$ intersects the spheres $S_{\varepsilon'}$ and $h_p^{-1}(R')$ transversally. It is well-known that $f_0^{-1}(0)$ intersects the sphere $S_{\varepsilon'}$ transversally. The sphere $h_p^{-1}(R')$ is a manifold with corners, which is clearly Whitney stratified. It is straightforward to see that $f_0^{-1}(0)$ intersects the strata $\{|z|=R', |x|<(R'/4D)^p\}$ and $\{|z|<R', |x|=(R'/4D)^p\}$ transversally. It remains to prove that $f_0^{-1}(0)$ intersects the stratum $\{|z|=R', |x|=(R'/4D)^p\}$ transversally. This is achieved using the following lemma.

Lemma 4.4. *If p is big enough then $f_0^{-1}(0)$ intersects the stratum $\{|z|=R', |x|=(R'/4D)^p\}$ transversally.*

Proof. This argument is due to Z. Szafraniec [**22**]. Let

$$\begin{aligned}\Sigma = \big\{&(x,z,\rho,\rho')\in\mathbb{R}^n\times\mathbb{R}^{a+b}\times\mathbb{R}\times\mathbb{R} : f_0(x,z)=0,\\ &\operatorname{rank}(Df_0(x,z),(x,0),(0,z))<3,\ |x|=\rho' \text{ and } |z|=\rho\big\},\end{aligned}$$

and let

$$\begin{aligned}\pi:\mathbb{R}^n\times\mathbb{R}^{a+b}\times\mathbb{R}\times\mathbb{R}&\to\mathbb{R}\times\mathbb{R}\\ (x,z,\rho,\rho')&\mapsto(\rho,\rho'),\end{aligned}$$

be the projection on the last two components. Since $\pi_{|\Sigma}$ is proper, $\Gamma = \pi(\Sigma)$ is a closed semi-algebraic set. Let $\Gamma_1 = \mathbb{R} \times \{0\}$ in $\mathbb{R} \times \mathbb{R}$ and let Γ_2 be the closure of $\Gamma \setminus \Gamma_1$. Since for $R' > 0$ small, the semi-algebraic function $|x|^2$ restricted to the semi-algebraic manifold $f_0^{-1}(0) \cap \{|z| = R'\}$ has a finite number of critical values, $\Gamma_2 \cap \{\rho = R'\}$ is a finite number of points and therefore Γ_2 is a curve and 0 is isolated in $\Gamma_1 \cap \Gamma_2$. By Lojaziewicz's inequality, there exists an integer $p' > 0$ and a constant $D'' > 0$ such that

$$|\rho|^{p'} \leq D''|\rho'|,$$

for $(\rho, \rho') \in \Gamma_2$ in a neighborhood of $(0,0)$. This means that if $f_0^{-1}(0)$ does not intersect the strata $\{|x| = \rho', |z| = \rho\}$ transversally at (x, z) then

$$|z|^{p'} \leq D''|x|,$$

i.e.

$$|x| \geq \frac{|z|^{p'}}{D''}.$$

Hence if we choose p much bigger than p', then $f_0^{-1}(0)$ will intersect $\{|x| = (R'/4D)^p, |z| = R'\}$ transversally, when R' is small enough. $\square$

Hence for the computation of the Euler characteristic of the Milnor fibers, we can replace the open Euclidian ball with the product $B \times B'$, where the radius R of B is much smaller than the radius R' of B'. Let

$$F = \{(x, z) \in B \times B' : g(x; c) + Q(z) = \varepsilon\}.$$

Note that $\dim F = n + a + b - 1$.

Lemma 4.5. *We have*

$$\chi_c(F) = \begin{cases} \chi_c(B_0) & a \text{ even, } b \text{ even,} \\ (-1)^n + \chi_c(B_+) - \chi_c(B_-) & a \text{ even, } b \text{ odd,} \\ (-1)^n - \chi_c(B_+) + \chi_c(B_-) & a \text{ odd, } b \text{ even,} \\ -2(-1)^n - \chi_c(B_0) & a \text{ odd, } b \text{ odd,} \end{cases}$$

where

$$B_+ = \{x \in B : g(x; c) > \varepsilon\},$$
$$B_0 = \{x \in B : g(x; c) = \varepsilon\},$$
$$B_- = \{x \in B : g(x; c) < \varepsilon\}.$$

Remark that B_+, B_- and B_0 depend on ε, c and it would be better to denote them by $B_+(\varepsilon, c)$, $B_-(\varepsilon, c)$ and $B_0(\varepsilon, c)$ respectively. But we keep the notation in the lemma for shortness.

PROOF. Consider the map: $\varphi : F \to B$, $(x, z) \mapsto x$. Since Q is homogeneous of degree 2 and $g(x;c)-\varepsilon$ is very small compared with R', we see that $\varphi^{-1}(x)$ is homeomorphic to one of the sets of Examples 4.2 and 4.3. Namely, we have the following:

$$\chi_c(\varphi^{-1}(x)) = \begin{cases} (-1)^a - (-1)^{a+b} & x \in B_+, \\ (-1)^b - (-1)^{a+b} & x \in B_-, \\ (-1)^a + (-1)^b - (-1)^{a+b} & x \in B_0. \end{cases}$$

In other words, $\chi_c(\varphi^{-1}(x))$ is given by the following table:

	$x \in B_+$	$x \in B_-$	$x \in B_0$
a even, b even	0	0	1
a even, b odd	2	0	1
a odd, b even	0	2	1
a odd, b odd	-2	-2	-3

Now, since $g(x;c)$ is an unfolding of a function-germ, we can suppose that it is a polynomial. Then the map $\varphi : F \to B$ is semi-algebraic and thanks to Hardt's theorem, we can apply Lemma 2.2 to get

$$\chi_c(F) = \begin{cases} \chi_c(B_0) & a \text{ even}, b \text{ even}, \\ 2\chi_c(B_+) + \chi_c(B_0) = (-1)^n + \chi_c(B_+) - \chi_c(B_-) & a \text{ even}, b \text{ odd}, \\ 2\chi_c(B_-) + \chi_c(B_0) = (-1)^n - \chi_c(B_+) + \chi_c(B_-) & a \text{ odd}, b \text{ even}, \\ -2\chi_c(B_+) - 2\chi_c(B_-) - 3\chi_c(B_0) = -2(-1)^n - \chi_c(B_0) & a \text{ odd}, b \text{ odd}. \end{cases}$$

Here we use the fact that $\chi_c(B_+) + \chi_c(B_-) + \chi_c(B_0) = \chi_c(B) = (-1)^n$. $\square$

We conclude that

$$1 + (-1)^{n+a+b}\chi_c(F) = \begin{cases} (-1)^a + (-1)^{a+n}\chi_c(B_0) & a+b \text{ even}, \\ (-1)^{a+n+1}[\chi_c(B_+) - \chi_c(B_-)] & a+b \text{ odd}. \end{cases}$$

Remember that F, B_0, B_+ and B_- depend on ε and c. When $n+a+b$ is even, we have

$$1 + \chi_c(F) = \begin{cases} (-1)^b(1 + \chi_c(B_0)) & \text{if } n \text{ is even and } a+b \text{ is even}, \\ (-1)^b(\chi_c(B_-) - \chi_c(B_+)) & \text{if } n \text{ is odd and } a+b \text{ is odd}. \end{cases}$$

Note that if n is even, then

$$\chi(\overline{B_0}) = \frac{1}{2}\chi(\overline{B_0} \cap \partial\overline{B}) = \frac{1}{2}\chi(g_0^{-1}(0) \cap \partial\overline{B}),$$

because g_0 is polynomial, hence conical, with isolated singularity. Therefore, we have

$$\chi_c(B_0) = \chi(\overline{B_0}) - \chi(\overline{B_0} \cap \partial\overline{B}) = -\frac{1}{2}\chi(g_0^{-1}(0) \cap \partial\overline{B}),$$

and we see that $\chi_c(B_0)$ does not depend on c nor on ε. If n is odd, then $\overline{B_-}$ and $\overline{B_+}$ are odd-dimensional compact manifolds with corners and

$$\begin{aligned}
\chi(\overline{B_-}) &= \frac{1}{2}\chi\big(\{x \in \partial\overline{B} : g(x;c) \leq \varepsilon\}\big) + \frac{1}{2}\chi\big(\{x \in \overline{B} : g(x;c) = \varepsilon\}\big), \\
\chi(\overline{B_+}) &= \frac{1}{2}\chi\big(\{x \in \partial\overline{B} : g(x;c) \geq \varepsilon\}\big) + \frac{1}{2}\chi\big(\{x \in \overline{B} : g(x;c) = \varepsilon\}\big).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\chi_c(B_-) - \chi_c(B_+) &= -\chi(\overline{B_-}) + \chi(\overline{B_+}) \\
&= -\frac{1}{2}\chi\big(\{x \in \partial\overline{B} : g(x;c) \leq \varepsilon\}\big) + \frac{1}{2}\chi\big(\{x \in \partial\overline{B} : g(x;c) \geq \varepsilon\}\big) \\
&= -\frac{1}{2}\chi\big(\{g_0 \leq 0\} \cap \partial\overline{B}\big) + \frac{1}{2}\chi\big(\{g_0 \geq 0\} \cap \partial\overline{B}\big).
\end{aligned}$$

We see that $\chi_c(B_-) - \chi_c(B_+)$ does not depend on c nor on ε.

When $n + a + b$ is odd, we have

$$1 - \chi_c(F) = \begin{cases} (-1)^b[1 - \chi_c(B_0)] & \text{if } n \text{ is odd and } a+b \text{ is even,} \\ (-1)^b[\chi_c(B_+) - \chi_c(B_-)] & \text{if } n \text{ is even and } a+b \text{ is odd.} \end{cases}$$

In this case, $1 - \chi_c(F)$ may depend on c and on ε, but its parity does not. Indeed we have $\chi_c(B_0) \equiv \chi_c(\overline{B_0}) \bmod 2$ and, since $\overline{B_0}$ is stably parallelizable, $\chi_c(\overline{B_0}) \equiv \psi(\overline{B_0} \cap \partial\overline{B}) \bmod 2$ for n odd, where ψ denotes the semi-characteristic, i.e., half the sum of the mod 2 Betti numbers (see [**28**]). Therefore $\chi_c(B_0) \equiv \psi(g_0^{-1}(0) \cap \partial\overline{B}) \bmod 2$. This proves that $1 - \chi_c(F) \bmod 2$ does not depend on c nor on ε if n is odd and $a + b$ is even. If n is even and $a + b$ is odd, it is enough to use the congruence

$$\chi_c(B_+) - \chi_c(B_-) \equiv \chi_c(B) + \chi_c(B_0) \bmod 2.$$

Definition 4.6. Let σ denote the singularity type of the map $g_0 : x \mapsto g(x;0)$. When $n + a + b$ is even, define s_σ by

$$s_\sigma = \begin{cases} 1 + \chi_c(B_0) & \text{if } n \text{ is even,} \\ \chi_c(B_-) - \chi_c(B_+) & \text{if } n \text{ is odd.} \end{cases}$$

When $n+a+b$ is odd, define s_σ by

$$s_\sigma \equiv \begin{cases} 1-\chi_c(B_0) \mod 2 & \text{if } n \text{ is odd,} \\ \chi_c(B_+)-\chi_c(B_-) \mod 2 & \text{if } n \text{ is even,} \end{cases}$$

and define $s_\sigma^{\max}$, $s_\sigma^{\min}$ by

$$s_\sigma^{\max} = \begin{cases} \min\{1-\chi_c(B_0)\} & \text{if } n \text{ is odd,} \\ \min\{\chi_c(B_+)-\chi_c(B_-)\} & \text{if } n \text{ is even,} \end{cases}$$

$$s_\sigma^{\min} = \begin{cases} \max\{1-\chi_c(B_0)\} & \text{if } n \text{ is odd,} \\ \max\{\chi_c(B_+)-\chi_c(B_-)\} & \text{if } n \text{ is even,} \end{cases}$$

where we take the maximal (or minimal) over all the regular values (ε, c) near 0.

Now let us apply this machinery to A_k and D_k singularities.

4.1. A_k singularities.
We set $n=1$, and

$$g_c(x) = g(x;c) = x^{k+1} + c_1 x^{k-1} + \cdots + c_{k-2}x^2 + c_{k-1}x.$$

Then we have $\chi_c(B_0) = \#\{x \in B : g_c(x) = \varepsilon\}$ and

$$\chi_c(B_+)-\chi_c(B_-) = \begin{cases} 0 & k \text{ even,} \\ -1 & k \text{ odd.} \end{cases}$$

Then we obtain

$$1-(-1)^{a+b}\chi_c(F) = \begin{cases} (-1)^b[1-\#\{x\in B : g_c(x)=\varepsilon\}] & a+b \text{ even,} \\ 0 & a+b \text{ odd, } k \text{ even,} \\ (-1)^b & a+b \text{ odd, } k \text{ odd.} \end{cases}$$

4.2. Unfoldings of functions $(x_1, x_2, z) \mapsto g(x;0)+Q(z)$.
We set $n=2$. We consider the map defined by

$$\begin{aligned} &(\mathbb{R}^{2+a+b+h}, 0) \to (\mathbb{R}^{1+h}, 0), \\ &(x_1, x_2, z_1, \ldots, z_{a+b}, c_1, \ldots, c_h) \\ &\quad \mapsto (g(x_1, x_2; c_1, \ldots, c_h) + {z_1}^2 + \cdots + {z_a}^2 - {z_{a+1}}^2 - \cdots - {z_{a+b}}^2, c_1, \ldots, c_h). \end{aligned} \tag{4.2}$$

Let r denote the number of branches of the curve defined by $g(x;0)=0$. Since $\chi_c(B_0) = -r$, we obtain that

$$1+(-1)^{a+b}\chi_c(F)=\begin{cases}(-1)^b(1-r) & a+b \text{ even},\\ (-1)^b[\chi_c(B_+)-\chi_c(B_-)] & a+b \text{ odd}.\end{cases}$$

O. Viro [**26**] described the list of possible smoothings of D_k ($k \geq 4$), E_6, E_7, E_8, J_{10} and non-degenerate r-fold points. In next subsection, we use this list to compute $\chi_c(B_+) - \chi_c(B_-)$ for D_k singularities. We leave to the reader the computation in the other cases.

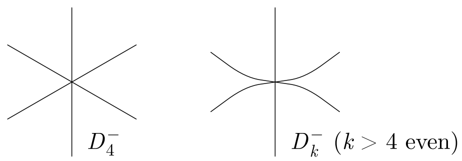
4.3. D_k singularities.

We denote by $D_k^{\pm}$ the singularity which is defined by (4.2) with

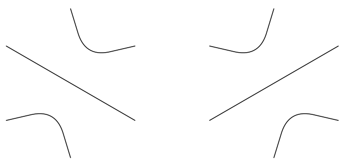
$$g(x;c) = x_1({x_1}^{k-2} \pm {x_2}^2) + c_1x_1 + \cdots + c_{k-2}{x_1}^{k-2} + c_{k-1}x_2.$$

We denote by D_k the singularities defined by this formula.

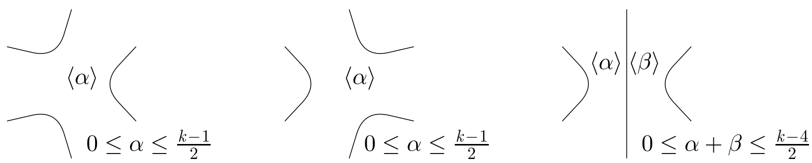
First case: k is even and $\{x \in \mathbb{R}^2 : g(x;0) = 0\}$ has 3 branches.
The zero set of $g(x;0)$ looks like the following:



First consider the smoothing described by the following picture:



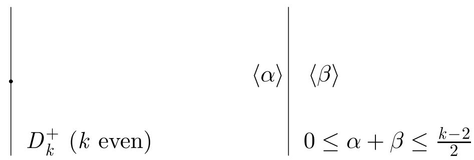
For such a smoothing, it is easy to see that $\chi_c(B_+) - \chi_c(B_-) = 0$.
Next we consider the smoothings described by the following pictures:



Here $\langle\alpha\rangle$ represents a group of α ovals without nests. For such smoothings, we see that $\chi_c(B_+) - \chi_c(B_-) = 2(1+\alpha)$, $-2(1+\alpha)$, $-2(\alpha-\beta)$ respectively. Then we obtain:

$$\chi_c(B_+) - \chi_c(B_-) = -k, -k+2, \ldots, k-2, k.$$

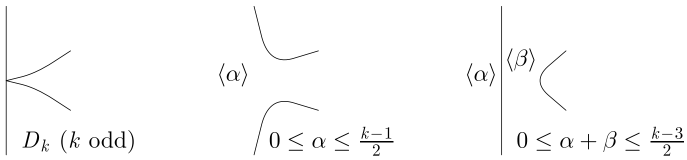
Second case: k is even and $\{x \in \mathbb{R}^2 : g(x;0) = 0\}$ has 1 branch.
The smoothings are described by the figure on the right-hand side.



For such smoothings, we see that $\chi_c(B_+) - \chi_c(B_-) = 2(\alpha - \beta)$. Thus we have

$$\chi_c(B_+) - \chi_c(B_-) = 2 - k, 4 - k, \ldots, k - 4, k - 2.$$

Third case: k is odd.



For such smoothings, we see that $\chi_c(B_+) - \chi_c(B_-) = -1 - 2\alpha$, $1 - 2(\alpha - \beta)$, respectively. Thus we have

$$\chi_c(B_+) - \chi_c(B_-) = -k, 2 - k, \ldots, k - 4, k - 2.$$

Remark 4.7. We remark that the signs in the notations B_+ and B_- have ad hoc meaning, since we are talking about map-germs not unfoldings of functions. The map-germ obtained by changing the sign of the first component in (4.2) has the same $\mathcal{A}$-type as the map-germ defined by (4.2). In the case of a D_k singularity with k odd, after this change of sign, the quantity $\chi_c(B_+) - \chi_c(B_-)$ takes the following values: $2 - k, 4 - k, \ldots, k - 2, k$. But, since the values of a and b are exchanged, we see that this change does not affect the possible values of the Euler characteristic $\chi_c(F)$ of the local Milnor fibers of the singularity.

5. Study of stable maps $f : M \to N$ with $\dim M \geq \dim N$.

Let $f : M \to N$ be a stable map between two smooth manifolds M and N. We assume that $\dim M \geq \dim N$, that N is connected and that M and N have finite homological type. Let σ denote the singularity type given by the genotype: $x \mapsto g(x; 0)$ in the notation of (4.1). We set

$$\sigma(f) = \{x \in M : \text{the genotype of } f_x \text{ is } \sigma\},$$

where $f_x : (M, x) \to (N, f(x))$ is the germ of f at x.

If $\dim M - \dim N$ is odd and $s_\sigma \neq 0$ then the genotype σ gives rise to two kinds of singularity types of f: we say that f is of type σ_+ (resp. σ_-) if, with the notations of Section 4, $1 + \chi_c(F) = s_\sigma$ (resp. $1 + \chi_c(F) = -s_\sigma$).

Similarly, if $\dim M - \dim N$ is even and $s_\sigma^{\max} + s_\sigma^{\min} \neq 0$, we say that f is of type σ_+ (resp. σ_-) if $1 - \max\{\chi_c(F)\} = s_\sigma^{\max}$ and $1 - \min\{\chi_c(F)\} = s_\sigma^{\min}$ (resp. $1 - \max\{\chi_c(F)\} = -s_\sigma^{\min}$ and $1 - \min\{\chi_c(F)\} = -s_\sigma^{\max}$). We set

$$\sigma_\pm(f) = \{x \in M : f_x \text{ has singularity of type } \sigma_\pm\}.$$

Let $\Sigma(f)$ denote the critical set of f.

Since f is stable, $\Sigma(f) \cap f^{-1}(y)$ is a finite set for each $y \in N$. Then f defines a multi-germ:

$$f_y : (M, \Sigma(f) \cap f^{-1}(y)) \to (N, y).$$

5.1. Case $\dim M - \dim N$ is odd.

If $\dim M - \dim N$ is odd, then $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$ does not depend on the choice of the regular value y' near $f(x)$, where $B_\varepsilon(x)$ denotes the open ball of small radius ε centered at x in M. Indeed, $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is a compact odd-dimensional manifold with boundary and so

$$\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \frac{1}{2}\chi(f^{-1}(y') \cap \partial\overline{B_\varepsilon(x)}).$$

But the last Euler characteristic is equal to $\chi(f^{-1}(f(x)) \cap \partial\overline{B_\varepsilon(x)})$. If x is of type ν then we denote by c_ν the Euler characteristic $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$.

If we assume that f is locally trivial at infinity, then we know by Lemma 3.3 that $\chi_c(f^{-1}(y))$ does not depend on the choice of the regular value y of f. We denote this Euler characteristic by χ_f.

THEOREM 5.1. *Assume that* $\dim M - \dim N$ *is odd. Assume that a stable map* $f : M \to N$ *is locally trivial at infinity and has finitely many singularity types* (*this is the case when* $(\dim M, \dim N)$ *is a pair of nice dimensions in Mather's sense*). *Then we have*

$$\sum_\nu c_\nu \chi_c(\nu(f)) = \chi_f \chi_c(N), \tag{5.1}$$

provided that the $\chi_c(\nu(f))$*'s and* χ_f *are finite, where the* ν*'s denote the possible singularity types of* f. *Moreover, if all singularities of* f *are unfoldings of function-germs as in* (4.1) *then we have*

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_{\sigma : s_\sigma \neq 0} s_\sigma \big[\chi_c(\sigma_+(f)) - \chi_c(\sigma_-(f))\big], \tag{5.2}$$

where σ *denotes the singularity type of the genotype.*

PROOF. We consider the stratification of f defined by the types of singularities (see Nakai's paper [**19**, Section 1]) and we define $\mathcal{S}(M)$, $\mathcal{S}(N)$ as the subset algebras generated by the strata and fibers of f. Then $(\mathcal{S}(M), \mathcal{S}(N))$ fits to the map f. Set

$\mu_M = \chi_c$, $\mu_N = \chi_c$ and

$$\varphi(x) = \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}),$$

where y' is a regular value near $f(x)$. Applying Corollary 2.3 for φ and Lemma 5.2 below, we obtain that

$$\sum_\nu c_\nu \chi_c(\nu(f)) = \chi_f \chi_c(N). \tag{5.3}$$

By the additivity of the Euler characteristic with closed support, we get

$$\chi_c(M) - \chi_f \chi_c(N) = \sum_\nu (1 - c_\nu)\chi_c(\nu(f)).$$

If all the singularities are unfoldings of function-germs then each genotype gives two kinds of singularity types $\sigma_+(f)$ and $\sigma_-(f)$ whenever $s_\sigma \neq 0$. We complete the proof by Definition 4.6, Remark 5.3 below and the computations made in Section 4. $\square$

Lemma 5.2. *Let $f : M \to N$ be a stable map such that*

- $\dim M - \dim N$ *is odd,*
- f *is locally trivial at infinity.*

Then for each $y \in N$, we have

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi(x) d\chi_c = \chi_c(f^{-1}(y')),$$

where y' is a regular value of f close to y.

Proof. Set $\{x_1, \dots, x_s\} = f^{-1}(y) \cap \Sigma(f)$. Take a regular value y' of f near y. Then,

$$\begin{aligned}
\chi_c(f^{-1}(y')) &= \chi_c\Big(f^{-1}(y') \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&= \chi_c\Big(f^{-1}(y) \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&= \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi(x_i) \\
&= \int_{f^{-1}(y) \setminus \{x_1, \dots, x_s\}} \varphi(x) d\chi_c + \int_{\{x_1, \dots, x_s\}} \varphi(x) d\chi_c \\
&= \int_{f^{-1}(y)} \varphi(x) d\chi_c.
\end{aligned}$$

$\square$

REMARK 5.3. Set $\phi(x) = \chi_c(f^{-1}(y') \cap B_\varepsilon(x))$ where y' is a regular value near $f(x)$. Then

$$\varphi(x) = \phi(x) + \chi_c(f^{-1}(y') \cap S_\varepsilon(x)),$$

where $S_\varepsilon(x)$ is the sphere of radius ε centered at x. If $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is an odd dimensional manifold with boundary $f^{-1}(y') \cap S_\varepsilon(x)$, then we obtain $\phi(x) = -\varphi(x)$, since

$$2\varphi(x) = \chi_c(f^{-1}(y') \cap S_\varepsilon(x)) = -2\phi(x).$$

Similarly if $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is an even dimensional manifold with boundary $f^{-1}(y') \cap S_\varepsilon(x)$, we obtain that $\phi(x) = \varphi(x)$.

COROLLARY 5.4. *Assume that the map f satisfies the assumptions of Theorem* 5.1 *and has at worst A_n singularities. Then, we have*

$$\chi_c(M) - \chi_f\, \chi_c(N) = \sum_{k:\mathrm{odd}} \big[\chi_c((A_k)_+(f)) - \chi_c((A_k)_-(f))\big].$$

PROOF. Using the computations in Section 4, we see that $s_{A_k} = 0$ if k is even and $s_{A_k} = 1$ if k is odd. $\square$

COROLLARY 5.5. *Assume that the map f satisfies the assumptions of Theorem* 5.1 *and has only stable singularities locally defined by* (4.2). *We denote by σ_r the union of singularities types such that the number of branches of $g(x_1, x_2; 0) = 0$ near* 0 *is r. For $r \neq 1$, σ_r splits into two subsets $\sigma_{r,+}$ and $\sigma_{r,-}$. Then, we have*

$$\chi_c(M) - \chi_f\, \chi_c(N) = \sum_{r:r\neq 1} (1-r)\big[\chi_c(\sigma_{r,+}(f)) - \chi_c(\sigma_{r,-}(f))\big].$$

PROOF. Using the computations in Section 4, we see that $s_{\sigma_r} = 1 - r$. $\square$

5.2. Case $\dim M - \dim N$ is even and $\dim M - \dim N > 0$.

If $\dim M - \dim N$ is even and non-zero then $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$ depends on the choice of the regular value y' near $f(x)$ in general. But its parity does not depend on y'. Indeed, $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is a compact even-dimensional manifold with boundary and so

$$\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) \equiv \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) \mod 2.$$

Using local coordinates at x and $f(x)$, we can assume that f is a map from $\mathbb{R}^{\dim M}$ to $\mathbb{R}^{\dim N}$. Hence $f^{-1}(y') \cap \overline{B_\varepsilon(x)}$ is stably parallelizable, because its normal bundle is clearly trivial. This implies that

$$\begin{aligned}\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) &\equiv \psi(f^{-1}(y') \cap S_\varepsilon(x)) \\ &\equiv \psi(f^{-1}(f(x)) \cap S_\varepsilon(x)) \mod 2.\end{aligned}$$

If a point x in M is of singularity type ν then we denote by c_ν the mod 2 Euler characteristic $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)})$. We will denote by χ_f the mod 2 Euler characteristic $\chi_c(f^{-1}(y))$ where y is a regular value of f (see Lemma 3.5). The following theorem is proved in the same way as Theorem 5.1.

Theorem 5.6. *Assume that* $\dim M - \dim N$ *is even and positive. Assume that a stable map* $f : M \to N$ *is locally trivial at infinity and has finitely many singularity types* (*this is the case when* $(\dim M, \dim N)$ *is a pair of nice dimensions in Mather's sense*). *Then we have*

$$\sum_\nu c_\nu \chi_c(\nu(f)) \equiv \chi_f \chi_c(N) \mod 2, \tag{5.4}$$

provided that the $\chi_c(\nu(f))$*'s and* χ_f *are finite, where the* ν*'s denote the possible singularity types of* f. *Moreover, if all singularities of* f *are unfoldings of function-germs as in* (4.1) *then we have*

$$\chi_c(M) - \chi_f \chi_c(N) \equiv \sum_\sigma s_\sigma \chi_c(\sigma(f)) \mod 2, \tag{5.5}$$

where σ *denotes the singularity type of the genotype.*

This theorem gives a mod 2 congruence. Nevertheless, it is still possible to find integral relations between the topology of the source, the target and the singular set.

Let ν denote a singularity type of a map-germ. Let $c_\nu^{\max}$ (resp. $c_\nu^{\min}$) denote the maximal (resp. minimum) of all possible Euler characteristics of local regular fibers near the singular fiber. Set also

$$N_j^{\max} = \big\{ y \in N : j = \max\{\chi_c(f^{-1}(y')) : y' \text{ is a regular value near } y\} \big\},$$
$$N_j^{\min} = \big\{ y \in N : j = \min\{\chi_c(f^{-1}(y')) : y' \text{ is a regular value near } y\} \big\}.$$

Theorem 5.7. *Assume that* $\dim M - \dim N$ *is even and positive. Assume that a stable map* $f : M \to N$ *is locally trivial at infinity and has finitely many singularity types* (*this is the case when* $(\dim M, \dim N)$ *is a pair of nice dimensions in Mather's sense*). *Then we have*

$$\sum_\nu c_\nu^{\max} \cdot \chi_c(\nu(f)) = \sum_j j \chi_c(N_j^{\max}),$$
$$\sum_\nu c_\nu^{\min} \cdot \chi_c(\nu(f)) = \sum_j j \chi_c(N_j^{\min}),$$

provided the $\chi_c(\nu(f))$*'s, the* $\chi_c(N_j^{\max})$*'s and the* $\chi_c(N_j^{\min})$*'s are finite, where the* ν*'s denote the possible singularity types of* f. *Moreover, if all singularities are unfoldings of function-germs as in* (4.1) *then*

$$\chi_c(M) - \sum_j j\chi_c(N_j^{\max})$$
$$= \sum_{\sigma: s_\sigma^{\max}+s_\sigma^{\min}=0} s_\sigma^{\max}\chi_c(\sigma(f)) + \sum_{\sigma: s_\sigma^{\max}+s_\sigma^{\min}\neq 0} [s_\sigma^{\max}\chi_c(\sigma_+(f)) - s_\sigma^{\min}\chi_c(\sigma_-(f))],$$
$$\chi_c(M) - \sum_j j\chi_c(N_j^{\min})$$
$$= \sum_{\sigma: s_\sigma^{\max}+s_\sigma^{\min}=0} s_\sigma^{\min}\chi_c(\sigma(f)) + \sum_{\sigma: s_\sigma^{\max}+s_\sigma^{\min}\neq 0} [s_\sigma^{\min}\chi_c(\sigma_+(f)) - s_\sigma^{\max}\chi_c(\sigma_-(f))],$$

and

$$\chi_c(M) - \sum_j \frac{j}{2}(\chi_c(N_j^{\max}) + \chi_c(N_j^{\min}))$$
$$= \sum_{\sigma: s_\sigma^{\max}+s_\sigma^{\min}\neq 0} \frac{s_\sigma^{\max}+s_\sigma^{\min}}{2}[\chi_c(\sigma_+(f)) - \chi_c(\sigma_-(f))],$$
$$\sum_j j[\chi_c(N_j^{\min}) - \chi_c(N_j^{\max})] = \sum_\sigma [s_\sigma^{\max} - s_\sigma^{\min}]\chi_c(\sigma(f)),$$

where σ denotes the singularity type of the genotype and $s_\sigma^{\max}$ and $s_\sigma^{\min}$ are defined in Definition 4.6.

Proof. To get the first equalities, we apply the same method as we did in the proof of Theorem 5.1 with the following two constructible functions $\varphi_{\max}$ and $\varphi_{\min}$:

$$\varphi_{\max}(x) = \max\{\chi_c(f^{-1}(y') \cap B_\varepsilon(x)) : y' \text{ is a regular value near } f(x)\},$$
$$\varphi_{\min}(x) = \min\{\chi_c(f^{-1}(y') \cap B_\varepsilon(x)) : y' \text{ is a regular value near } f(x)\}.$$

We also use Lemma 5.8 below.

By the additivity of the Euler characteristic with closed support, we get

$$\chi_c(M) - \sum_j j\chi_c(N_j^{\max}) = \sum_\nu (1 - c_\nu^{\max})\chi_c(\nu(f)),$$
$$\chi_c(M) - \sum_j j\chi_c(N_j^{\min}) = \sum_\nu (1 - c_\nu^{\min})\chi_c(\nu(f)).$$

If all the singularities are unfoldings of function-germs as in (4.1), then each genotype σ with $s_\sigma^{\max} + s_\sigma^{\min} \neq 0$ gives two kinds of singularity types σ_+ and σ_-. Using the computations done in Section 4, we see that

$$(1 - c_{\sigma_-}^{\max}) = -(1 - c_{\sigma_+}^{\min}) = -s_\sigma^{\min} \text{ and } (1 - c_{\sigma_-}^{\min}) = -(1 - c_{\sigma_+}^{\max}) = -s_\sigma^{\max},$$

and we complete the proof. □

Lemma 5.8. *Let $f : M \to N$ be a stable map such that*

- $\dim M - \dim N$ *is even and positive,*
- f *is locally trivial at infinity.*

Then we have

$$f_*\varphi_{\max}(y) = \max\{\chi_c(f^{-1}(y')) : y' \textit{ is a regular value near } y\},$$
$$f_*\varphi_{\min}(y) = \min\{\chi_c(f^{-1}(y')) : y' \textit{ is a regular value near } y\}.$$

Proof. Set $\{x_1, \dots, x_s\} = f^{-1}(y) \cap \Sigma(f)$. Take a regular value y' of f near y. Then

$$\begin{aligned}
\chi_c(f^{-1}(y')) &= \chi_c\Big(f^{-1}(y') \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&= \chi_c\Big(f^{-1}(y) \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&\le \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi_{\max}(x_i) \\
&= \int_{f^{-1}(y)\setminus\{x_1,\dots,x_s\}} \varphi_{\max}(x) d\chi_c + \int_{\{x_1,\dots,x_s\}} \varphi_{\max}(x) d\chi_c \\
&= \int_{f^{-1}(y)} \varphi_{\max}(x) d\chi_c = f_*\varphi_{\max}(y).
\end{aligned}$$

But, since f is stable, we see that the equality is attained by some y' using the fact (i)$\Longleftrightarrow$(iii) of [**29**, Lemma 1.5].

Similarly we obtain

$$\begin{aligned}
\chi_c(f^{-1}(y')) &= \chi_c\Big(f^{-1}(y') \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&= \chi_c\Big(f^{-1}(y) \setminus \bigcup_i \overline{B_\varepsilon(x_i)}\Big) + \sum_i \chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x_i)}) \\
&\ge \chi_c(f^{-1}(y) \setminus \{x_1, \dots, x_s\}) + \sum_i \varphi_{\min}(x_i) \\
&= \int_{f^{-1}(y)\setminus\{x_1,\dots,x_s\}} \varphi_{\min}(x) d\chi_c + \int_{\{x_1,\dots,x_s\}} \varphi_{\min}(x) d\chi_c \\
&= \int_{f^{-1}(y)} \varphi_{\min}(x) d\chi_c = f_*\varphi_{\min}(y).
\end{aligned}$$

But, since f is stable, we see that the equality is attained by some y' using the fact (i)$\Longleftrightarrow$(iii) of [**29**, Lemma 1.5]. □

Now let us apply this theorem to the case of a map having at worst D_n singularities. Using the computations in Section 4, we see that

$$s_\sigma^{\max} = \begin{cases} -k & \text{if } \sigma = A_k, \\ -k & \text{if } \sigma = D_k^-,\ k \text{ even}, \\ 2-k & \text{if } \sigma = D_k^+,\ k \text{ even}, \\ -k & \text{if } \sigma = D_k,\ k \text{ odd}, \end{cases} \qquad s_\sigma^{\min} = \begin{cases} 1 & \text{if } \sigma = A_k,\ k \text{ odd}, \\ 0 & \text{if } \sigma = A_k,\ k \text{ even}, \\ k & \text{if } \sigma = D_k^-,\ k \text{ even}, \\ k-2 & \text{if } \sigma = D_k^+,\ k \text{ even}, \\ k-2 & \text{if } \sigma = D_k,\ k \text{ odd}. \end{cases}$$

Corollary 5.9. *If the map f satisfies the assumptions of Theorem 5.7 and has at worst D_n singularities then*

$$\begin{aligned} \chi_c(M) - \sum_j j\chi_c(N_j^{\max}) = {} & -\chi_c(A_1(f)) - \sum_{k>1} k\chi_c((A_k)_+(f)) - \sum_{k>1:\text{odd}} \chi_c((A_k)_-(f)) \\ & - \sum_k k\chi_c(D_k(f)) + 2\sum_{k:\text{even}} \chi_c(D_k^+(f)) + 2\sum_{k:\text{odd}} \chi_c((D_k)_-(f)), \\ \chi_c(M) - \sum_j j\chi_c(N_j^{\min}) = {} & \chi_c(A_1(f)) + \sum_{k>1} k\chi_c((A_k)_-(f)) + \sum_{k>1:\text{odd}} \chi_c((A_k)_+(f)) \\ & + \sum_k k\chi_c(D_k(f)) - 2\sum_{k:\text{even}} \chi_c(D_k^+(f)) - 2\sum_{k:\text{odd}} \chi_c((D_k)_+(f)). \end{aligned}$$

Proof. Combine the previous theorem with the above expressions of $s_\sigma^{\max}$ and $s_\sigma^{\min}$. □

Corollary 5.10. *Assume that a map f satisfies the assumptions of Theorem 5.7 and has at worst A_n singularities.*

When $\dim N = 1$, *we have*

$$\sum_j j\chi_c(N_j^{\max}) = \chi_c(M) + \chi_c(A_1(f)), \qquad \sum_j j\chi_c(N_j^{\min}) = \chi_c(M) - \chi_c(A_1(f)),$$

and thus

$$\begin{aligned} \sum_j \frac{j}{2}[\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})] &= \chi_c(M), \\ \sum_j \frac{j}{2}[\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})] &= \chi_c(A_1(f)) = \chi_c(\Sigma(f)). \end{aligned}$$

When $\dim N = 2$, *we have*

$$\sum_j j\chi_c(N_j^{\max}) = \chi_c(M) + \chi_c(A_1(f)) + 2\#((A_2)_+(f)),$$
$$\sum_j j\chi_c(N_j^{\min}) = \chi_c(M) - \chi_c(A_1(f)) - 2\#((A_2)_-(f)),$$

and thus

$$\sum_j \frac{j}{2}[\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})] = \chi_c(M) + \#((A_2)_+(f)) - \#((A_2)_-(f)),$$
$$\sum_j \frac{j}{2}[\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})] = \chi_c(A_1(f)) + \#(A_2(f)) = \chi_c(\Sigma(f)).$$

When $\dim N = 3$, *we have*

$$\sum_j j\chi_c(N_j^{\max}) = \chi_c(M) + \chi_c(A_1(f)) + 2\chi_c((A_2)_+(f)) + \#(A_3(f)) + 2\#((A_3)_+(f)),$$
$$\sum_j j\chi_c(N_j^{\min}) = \chi_c(M) - \chi_c(A_1(f)) - 2\chi_c((A_2)_-(f)) - \#(A_3(f)) - 2\#((A_3)_-(f)),$$

and thus

$$\begin{aligned}&\sum_j \frac{j}{2}[\chi_c(N_j^{\max}) + \chi_c(N_j^{\min})]\\ &\quad = \chi_c(M) + \chi_c((A_2)_+(f)) - \chi_c((A_2)_-(f)) + \#((A_3)_+(f)) - \#((A_3)_-(f)),\\ &\sum_j \frac{j}{2}[\chi_c(N_j^{\max}) - \chi_c(N_j^{\min})]\\ &\quad = \chi_c(A_1(f)) + \chi_c(A_2(f)) + 2\#(A_3(f)) = \chi_c(\Sigma(f)) + \#(A_3(f)).\end{aligned}$$

5.3. Case $\dim M - \dim N = 0$.

Here we assume that M and N are oriented and have the same dimension. If a point x in M is of type ν, we denote by d_ν the local topological degree of the map-germ $f : (M, x) \to (N, f(x))$. We notice that the singularity type of x depends on the orientations of the source and the target, unlike the case with $\dim M > \dim N$. We assume that f is finite-to-one and that f is locally trivial at infinity. In this situation, we know that it is possible to define the mapping degree of f as follows:

$$\deg f = \sum_{x \in f^{-1}(y)} \deg(f : (M, x) \to (N, f(x))),$$

where y is a regular value of f (see Definition 3.9).

Theorem 5.11. *Assume that* $\dim M - \dim N = 0$. *Assume that a stable map*

$f : M \to N$ is finite-to-one, locally trivial at infinity and has finitely many singularity types. We also assume that M and N are oriented and that N is connected. Then

$$\sum_{\nu} d_\nu \chi_c(\nu(f)) = (\deg f)\chi_c(N),$$

provided that the $\chi_c(\nu(f))$'s are finite.

Proof. We consider the stratification of f defined by the types of singularities (see Nakai's paper [**19**, Section 1]) and we define $\mathcal{S}(M)$, $\mathcal{S}(N)$ as the subset algebras generated by the strata and fibers of f. Then $(\mathcal{S}(M), \mathcal{S}(N))$ fits to the map f. Set $\mu_M = \chi_c$, $\mu_N = \chi_c$ and

$$\varphi(x) = \deg(f : (M, x) \to (N, f(x))).$$

Applying Corollary 2.3 for φ and remarking that $f_*\varphi(y) = \deg f$, we obtain the result. $\square$

If x is a point of type A_k with k even, we say that x belongs to $A_k^+(f)$ (resp. $A_k^-(f)$) if $\deg\{f : (M, x) \to (N, f(x))\} = 1$ (resp. -1).

Corollary 5.12. *Assume that f satisfies the assumptions of Theorem* 5.11 *and has at worst A_n singularities. Then we have*

$$\sum_{k:\text{even}} \left[\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))\right] = (\deg f)\chi_c(N).$$

Proof. Apply the previous theorem and the fact that $\deg\{f : (M, x) \to (N, f(x))\} = 0$ if $x \in A_k(f)$, k odd. $\square$

The map $f : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ is an $I_{2,2}^{\pm}$ singularity if f is defined by

$$(x, y, a, b) \mapsto (x^2 \pm y^2 + ax + by, xy, a, b).$$

We assume that the source and the target are oriented. Then we see that the mapping degree of $I_{2,2}^+$ singularity is zero. We also see that the mapping degree is not zero for $I_{2,2}^-$ singularity. We say it is $(I_{2.2}^-)^+$ (resp. $(I_{2,2}^-)^-$), if its mapping degree is positive (resp. negative) at 0. These are the only singularities of stable-germs which are not Morin singularities from $\mathbb{R}^4$ to $\mathbb{R}^4$. We can state

Corollary 5.13. *Assume that f satisfies the assumptions of Theorem* 5.11 *and that* $\dim M = \dim N = 4$. *Then we have*

$$\sum_{k:\text{even}} \left[\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))\right] + 2\#((I_{2,2}^-)^+(f)) - 2\#((I_{2,2}^-)^-(f)) = (\deg f)\chi_c(N).$$

Proof. Remark that the mapping degree of f_x is ± 2 (resp. 0) when x is an $I^-_{2,2}$ (resp. $I^+_{2,2}$) point. $\square$

A similar discussion shows the following:

Theorem 5.14. *Assume that* $\dim M - \dim N = 0$. *Assume that a stable map* $f : M \to N$ *is finite-to-one, locally trivial at infinity and has finitely many singularity types. We assume that* M *or* N *may not be orientable and that* N *is connected. Then* d_σ *and* $\deg f$ *are well-defined modulo* 2, *and we have*

$$\sum_\sigma d_\sigma \chi_c(\sigma(f)) \equiv (\deg f)\chi_c(N) \mod 2.$$

6. Applications to Morin maps.

In this section, we apply the results of the previous section to Morin maps. We recall that a Morin map is a map which admits only A_k singularities (see Subsection 4.1). We will consider three different settings: Morin maps from a compact manifold M to a connected manifold N such that $\dim M - \dim N$ is odd, Morin maps from a compact manifold M to a connected manifold N with $\dim M = \dim N$, and Morin perturbations of smooth map-germs.

6.1. Morin maps from M^m to N^n, $m - n$ odd and $m - n > 0$.

Let $f : M^m \to N^n$ be a Morin map from a compact m-dimensional manifold M to a connected n-dimensional manifold N.

Let us recall that a point p in M is of type A_k if the genotype of f_p is x^{k+1}. This means that there exist a local coordinate system $(x_1, \dots, x_m)$ centered at p and a local coordinate system $(y_1, \dots, y_n)$ centered at $f(p)$ such that f has the following normal form:

$$y_i \circ f = x_i \quad \text{for } i \le n-1,$$
$$y_n \circ f = x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i} + x_{n+1}^2 + \dots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \dots - x_m^2.$$

If k is odd then we remark that $x \in (A_k)_+(f)$ (resp. $(A_k)_-(f)$) if and only if $\chi_c(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(x)}) = 0$ (resp. 2) where y' is a regular value of f close to $f(x)$ or equivalently, that $x \in (A_k)_+(f)$ (resp. $(A_k)_-(f)$) if and only if $m-n-\lambda+1$ is even (resp. odd) (see the computations in Section 4). It is well known that for $k \ge 1$, the $A_k(f)$'s and the $\overline{A_k(f)}$'s are smooth manifolds of dimension $n-k$ and that $\overline{A_k(f)} = \bigcup_{i \ge k} A_i(f)$. We will describe more precisely the structure of the $(A_k)_\pm(f)$'s.

Proposition 6.1. *If* k *is odd then* $\overline{(A_k)_+(f)}$ *and* $\overline{(A_k)_-(f)}$ *are compact manifolds with boundary of dimension* $n-k$. *Furthermore* $\partial\overline{(A_k)_+(f)} = \partial\overline{(A_k)_-(f)} = \overline{A_{k+1}(f)}$.

Proof. Let p be a point in $A_k(f)$, k odd. There exist local coordinates around p and $f(p)$ such that f has the form

$$y_i \circ f = x_i \quad \text{for } i \le n-1,$$

$$y_n \circ f = x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i} + x_{n+1}^2 + \cdots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \cdots - x_m^2.$$

Let us write $\gamma = y_n \circ f$. Around p, $A_k(f)$ is defined by $\partial\gamma/\partial x_n = \cdots = \partial^k\gamma/\partial x_n^k = 0$ and $x_{n+1} = \cdots = x_m = 0$. It is easy to see that this is equivalent to $x_1 = \cdots = x_{k-1} = 0$ and $x_n = \cdots = x_m = 0$. This proves that $A_k(f)$ is a manifold of dimension $n-k$. Let $q = (q_1, \ldots, q_m) \in A_k(f)$ be a point close to p. We have $q_1 = \ldots = q_{k-1} = 0$ and $q_n = \ldots = q_m = 0$. For $i \in \{k, \ldots, n-1\}$, let us put $z_i = x_i - q_i$ and $w_i = y_i - q_i$. For $i \notin \{k, \ldots, n-1\}$, let us put $z_i = x_i$ and $w_i = y_i$. Then $(z_1, \ldots, z_m)$ and $(w_1, \ldots, w_n)$ are local coordinate systems centered at q and $f(q)$. In these systems, f has the form

$$w_i \circ f = z_i \quad \text{for } i \le n-1,$$

$$w_n \circ f = z_n^{k+1} + \sum_{i=1}^{k-1} z_i z_n^{k-i} + z_{n+1}^2 + \cdots + z_{n+\lambda-1}^2 - z_{n+\lambda}^2 - \cdots - z_m^2.$$

We conclude that q belongs to $(A_k)_+(f)$ (resp. $(A_k)_-(f)$) if and only if p belongs to $(A_k)_+(f)$ (resp. $(A_k)_-(f)$). This proves that the sets $(A_k)_+(f)$ and $(A_k)_-(f)$ are open subsets of $A_k(f)$, hence manifolds of dimension $n-k$.

We know that $\overline{A_k(f)} = \bigcup_{l \ge k} A_l(f)$. Let $l > k$ and let $p \in A_l(f)$. There are local coordinates systems around p and $f(p)$ such that f has the form:

$$y_i \circ f = x_i \quad \text{for } i \le n-1,$$

$$y_n \circ f = x_n^{l+1} + \sum_{i=1}^{l-1} x_i x_n^{l-i} + x_{n+1}^2 + \cdots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \cdots - x_m^2.$$

Let us denote by γ the function $y_n \circ f$. We have

$$A_k(f) = \left\{ \frac{\partial\gamma}{\partial x_n} = \cdots = \frac{\partial^k\gamma}{\partial x_n^k} = 0,\ x_{n+1} = \cdots = x_m = 0,\ \frac{\partial^{k+1}\gamma}{\partial x_n^{k+1}} \neq 0 \right\},$$

and

$$\overline{A_{k+1}(f)} = \left\{ \frac{\partial\gamma}{\partial x_n} = \cdots = \frac{\partial^{k+1}\gamma}{\partial x_n^{k+1}} = 0,\ x_{n+1} = \cdots = x_m = 0 \right\}.$$

Let $q = (q_1, \ldots, q_n, 0, \ldots, 0)$ be a point in $A_k(f)$ close to p. Let us find when $q \in (A_k)_+(f)$ or $q \in (A_k)_-(f)$. For this we have to compute $\varphi(q) = \chi(f^{-1}(y') \cap \overline{B_\varepsilon(q)})$ where y' is a regular value of f close to $f(q)$. Since it does not depend on the choice of the regular value because $m-n$ is odd, let us compute $\chi(f^{-1}(\tilde{y}) \cap \overline{B_\varepsilon(q)})$ where $\tilde{y} = (q_1, \ldots, q_{n-1}, \gamma(q) + \xi)$ and ξ is a small real number. So we have to look for the solutions lying close to q of the following system:

$$\begin{cases} y_i \circ f(x) = q_i & \text{for } i \leq n-1, \\ \gamma(x) = \gamma(q) + \xi. \end{cases}$$

This system is equivalent to

$$\begin{cases} x_i = q_i \quad \text{for } i \leq n-1, \\ \gamma(q_1, \dots, q_{n-1}, q_n + x'_n, x_{n+1}, \dots, x_m) = \gamma(q) + \xi. \end{cases}$$

But we have

$$\begin{aligned} &\gamma(q_1, \dots, q_{n-1}, q_n + x'_n, x_{n+1}, \dots, x_m) \\ &\quad = \gamma(q_1, \dots, q_{n-1}, q_n + x'_n, 0, \dots, 0) + x_{n+1}^2 + \cdots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \cdots - x_m^2 \\ &\quad = \gamma(q) + \sum_{i \geq k+1} \frac{1}{i!} \frac{\partial^i \gamma}{\partial x_n^i}(q) {x'_n}^i + x_{n+1}^2 + \cdots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \cdots - x_m^2 \\ &\quad = \gamma(q) + \gamma'(x'_n, x_{n+1}, \dots, x_m), \end{aligned}$$

where $\gamma'(x'_n, x_{n+1}, \dots, x_m) = \sum_{i \geq k+1} (1/i!)(\partial^i \gamma / \partial x_n^i)(q) {x'_n}^i + x_{n+1}^2 + \cdots + x_{n+\lambda-1}^2 - x_{n+\lambda}^2 - \cdots - x_m^2$. Hence by Khimshiashvili's formula [**15**], we have: $\varphi(q) = 1 - \deg_0 \nabla \gamma'$, where $\deg_0 \nabla \gamma'$ is the topological degree of the map $\nabla \gamma' / \|\nabla \gamma'\| : S_\varepsilon^{m-n} \to S^{m-n}$. Two cases are possible. If λ is even then

$$q \in (A_k)_+(f) \Leftrightarrow \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}}(q) > 0 \text{ and } q \in (A_k)_-(f) \Leftrightarrow \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}}(q) < 0.$$

If λ is odd then

$$q \in (A_k)_+(f) \Leftrightarrow \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}}(q) < 0 \text{ and } q \in (A_k)_-(f) \Leftrightarrow \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}}(q) > 0.$$

Finally we see that the sets $(A_k)_+(f)$ and $(A_k)_-(f)$ are in correspondence with the sets $A_k(f) \cap \{(\partial^{k+1}\gamma)/(\partial x_n^{k+1}) > 0\}$ and $A_k(f) \cap \{(\partial^{k+1}\gamma)/(\partial x_n^{k+1}) < 0\}$, which enables us to conclude. $\square$

We can state our main theorem which is a slight improvement of a result of T. Fukuda [**10**] for $N = \mathbb{R}^n$ and O. Saeki [**21**] for a general N.

Theorem 6.2. *Let $f : M^m \to N^n$ be a Morin map. Assume that M is compact, N is connected and $m - n$ is odd. Then we have*

$$\chi(M) = \sum_{k:\text{odd}} \left[\chi(\overline{(A_k)_+(f)}) - \chi(\overline{(A_k)_-(f)}) \right].$$

Proof. Applying Corollary 5.4, we get

$$\chi_c(M) - \chi_f\, \chi_c(N) = \sum_{k:\text{odd}} \big[\chi_c((A_k)_+(f)) - \chi_c((A_k)_-(f))\big],$$

where χ_f is the Euler characteristic of a regular fiber of f. In this situation, $\chi_f = 0$ because the regular fiber of f is a compact odd-dimensional manifold. Then we remark that $\chi_c(M) = \chi(M)$ because M is compact. Moreover by the additivity of the Euler-Poincaré characteristic with closed support, we have

$$\begin{aligned}
\chi(\overline{(A_k)_+(f)}) = \chi_c(\overline{(A_k)_+(f)}) &= \chi_c((A_k)_+(f)) + \chi_c(\partial(\overline{(A_k)_+(f)}))\\
&= \chi_c((A_k)_+(f)) + \chi_c(\overline{A_{k+1}(f)}),\\
\chi(\overline{(A_k)_-(f)}) = \chi_c(\overline{(A_k)_-(f)}) &= \chi_c((A_k)_-(f)) + \chi_c(\partial(\overline{(A_k)_-(f)}))\\
&= \chi_c((A_k)_-(f)) + \chi_c(\overline{A_{k+1}(f)}).
\end{aligned}$$

This implies that $\chi(\overline{(A_k)_+(f)}) - \chi(\overline{(A_k)_-(f)}) = \chi_c((A_k)_+(f)) - \chi_c((A_k)_-(f))$. $\square$

We end this subsection with two remarks:

(1) If m is odd then n is even and $\chi(M) = 0$. If k is odd, the dimensions of $\overline{(A_k)_+(f)}$ and $\overline{(A_k)_-(f)}$ are odd. Furthermore, we have

$$\chi(\overline{(A_k)_+(f)}) = \frac{1}{2}\chi(\partial\overline{(A_k)_+(f)}) = \frac{1}{2}\chi(\overline{A_{k+1}(f)}) = \frac{1}{2}\chi(\partial\overline{(A_k)_-(f)}) = \chi(\overline{(A_k)_-(f)}),$$

and $\chi(\overline{(A_k)_+(f)}) - \chi(\overline{(A_k)_-(f)}) = 0$. In this case, our theorem is trivial.

(2) If m is even and $n = 1$, then we can apply our theorem. In this situation, there is only a finite number of singular points, which are the elements of $(A_1)_+(f)$ and of $(A_1)_-(f)$. Theorem 6.2 gives that $\chi(M) = \#(A_1)_+(f) - \#(A_1)_-(f)$. We recover the well-known Morse equality.

6.2. Morin maps from M^n to N^n.

Let $f : M^n \to N^n$ be a Morin map from a compact oriented manifold M of dimension n to a connected oriented manifold N of the same dimension. For any $p \in M$, let $\varphi(p)$ be the local topological degree of the map-germ $f : (M, p) \to (N, f(p))$. Recall that $\varphi(p) = 0$ if $p \in A_k(f)$ and k odd and that $|\varphi(p)| = 1$ if $p \in A_k(f)$ and k even. Hence, if k is even, $A_k(f)$ splits into two subsets $A_k^+(f)$ and $A_k^-(f)$ where $A_k^+(f)$ (resp. $A_k^-(f)$) consists of the points p such that $\varphi(p) = 1$ (resp. $\varphi(p) = -1$). It is well known that the $A_k(f)$'s and the $\overline{A_k(f)}$'s are smooth manifolds of dimension $n - k$ and that $\overline{A_k(f)} = \bigcup_{i \geq k} A_i(f)$. Remark that $A_0(f)$ is the set of regular points of f. Let us describe more precisely the structure of the sets $A_k^{\pm}(f)$.

Proposition 6.3. *If k is even, then $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ are manifolds with boundary of dimension $n - k$ and $\partial\overline{A_k^+(f)} = \partial\overline{A_k^-(f)} = \overline{A_{k+1}(f)}$.*

PROOF. Let p be a point in $A_k(f)$, k even. In local coordinates, f is given by

$$y_i \circ f = x_i \quad \text{for } i \leq n-1,$$
$$y_n \circ f = x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i}.$$

Depending on the orientations of the source and the target in the above expression, f has four possible forms:

$$\begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = x_n^{k+1} + \sum_{i=1}^{k-1} x_i x_n^{k-i}, \end{cases} \qquad \begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = -x_n^{k+1} + \sum_{i=1}^{k-1} (-1)^{k-i} x_i x_n^{k-i}, \end{cases}$$
$$\begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = -x_n^{k+1} - \sum_{i=1}^{k-1} x_i x_n^{k-i}, \end{cases} \quad \text{or} \quad \begin{cases} y_i \circ f = x_i & \text{for } i \leq n-1, \\ y_n \circ f = x_n^{k+1} - \sum_{i=1}^{k-1} (-1)^{k-i} x_i x_n^{k-i}, \end{cases}$$

where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are coordinates in positive basis. In the first and fourth cases, $\varphi(p) = 1$ and in the second and third cases $\varphi(p) = -1$.

We can prove the fact that $A_k^+(f)$ and $A_k^-(f)$ are manifolds of dimension $n-k$ with the same method as in Proposition 6.1. Now let $l > k$ and let $p \in A_l(f)$. Locally f is given by

$$y_i \circ f = x_i \quad \text{for } i \leq n-1,$$
$$y_n \circ f = \pm x_n^{l+1} + \sum_{i=1}^{l-1} \pm x_i x_n^{l-i}.$$

Let us denote by γ the function $y_n \circ f$. We have

$$A_k(f) = \left\{ \frac{\partial \gamma}{\partial x_n} = \cdots = \frac{\partial^k \gamma}{\partial x_n^k} = 0,\ \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}} \neq 0 \right\},$$

and

$$\overline{A_{k+1}(f)} = \left\{ \frac{\partial \gamma}{\partial x_n} = \cdots = \frac{\partial^{k+1} \gamma}{\partial x_n^{k+1}} = 0 \right\}.$$

Let $q = (q_1, \ldots, q_n)$ be a point in $A_k(f)$ close to p. Let us find when $q \in A_k^+(f)$ or $q \in A_k^-(f)$. For this we have to compute $\varphi(q)$. Let ξ be a small real number and let us look for the solutions lying close to q of the following system:

$$\begin{cases} y_i \circ f(x) = q_i & \text{for } i \leq n-1, \\ \gamma(x) = \gamma(q) + \xi. \end{cases}$$

This system is equivalent to

$$\begin{cases} x_i = q_i \quad \text{for } i \leq n-1, \\ \gamma(q_1, \ldots, q_{n-1}, q_n + x'_n) = \gamma(q) + \xi. \end{cases}$$

But

$$\gamma(q_1, \ldots, q_{n-1}, q_n + x'_n) = \gamma(q) + \sum_{i \geq k+1} \frac{\partial^i \gamma}{\partial x_n^i}(q) {x'_n}^i.$$

Then we see that $\varphi(q) = \text{sign}((\partial^{k+1}\gamma)/(\partial x_n^{k+1}))(q)$. We conclude as in Proposition 6.1. $\square$

THEOREM 6.4. *Let $f : M^n \to N^n$ be a Morin map. Assume that M is compact and oriented and that N is connected and oriented. We have*

$$\sum_{k:\text{even}} \left[\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)})\right] = (\deg f)\chi(N).$$

This was proved by J. M. Eliashberg [**7**] and J. R. Quine [**20**] when $n = 2$. It appeared in a preprint of I. Nakai [**18**] for any n.

PROOF. By Corollary 5.12, we know that

$$\sum_{k:\text{even}} \left[\chi_c(A_k^+(f)) - \chi_c(A_k^-(f))\right] = (\deg f)\chi_c(N).$$

If N is compact then $\chi_c(N) = \chi(N)$ and if N is not compact then $\deg f = 0$. In both cases the equality $(\deg f)\chi_c(N) = (\deg f)\chi(N)$ is true. With the same arguments as in Theorem 6.2, it is easy to prove that $\chi(\overline{A_k^+(f)}) - \chi(\overline{A_k^-(f)}) = \chi_c(A_k^+(f)) - \chi_c(A_k^-(f))$. $\square$

REMARK 6.5. When n is odd, $\overline{A_k^+(f)}$ and $\overline{A_k^-(f)}$ are odd-dimensional manifolds with the same boundary and so the left hand-side of the equality vanishes. But the right-hand side is also zero because $\chi(N) = 0$ if N is compact and $\deg f = 0$ if N is not compact. Hence our theorem is trivial in this case.

6.3. Local versions.

We give local versions of the global formulas of the previous subsections.

We work first with map-germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p$, which are generic in the sense of Theorem $1'$ in [**9**]. There are two cases:

Case I) If the origin 0 is not isolated in $f^{-1}(0)$, i.e., $0 \in \overline{f^{-1}(0) \setminus \{0\}}$, then there exist a positive number ε_0 and a strictly increasing function $\delta : [0, \varepsilon_0] \to [0, +\infty)$ with $\delta(0) = 0$ such that for every ε and δ with $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta < \delta(\varepsilon)$ the following properties hold:

(1) $f^{-1}(0) \cap S_\varepsilon^{n-1}$ is an $(n-p-1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S_{\varepsilon_0}^{n-1}$.
(2) $\overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1})$ is a smooth manifold with boundary and it is diffeomorphic to $\overline{B_{\varepsilon_0}^n} \cap f^{-1}(S_{\delta(\varepsilon_0)}^{p-1})$.
(3) $\partial(\overline{B_\varepsilon^n} \cap f^{-1}(\overline{B_\delta^p}))$ is homeomorphic to S_ε^{n-1}.
(4) The restricted mapping $f : \overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is topologically stable (C^∞ stable if (n,p) is a nice pair) and its topological type is independent of ε and δ.

Here B_ε^n denotes the open ball of radius ε centered at 0 and S_ε^{n-1} the sphere of radius ε centered at 0 in $\mathbb{R}^n$.

Case II) If the origin 0 is isolated in $f^{-1}(0)$, i.e., $0 \notin \overline{f^{-1}(0) \setminus \{0\}}$, then there exists a positive number ε_0 such that for every ε with $0 < \varepsilon \leq \varepsilon_0$ the following properties hold:

(1) $f^{-1}(S_\varepsilon^{p-1})$ is diffeomorphic to S_ε^{n-1}.
(2) The restricted mapping $f : f^{-1}(S_\varepsilon^{p-1}) \to S_\varepsilon^{p-1}$ is topologically stable (C^∞ stable if (n,p) is a nice pair) and its topological type is independent of ε.

We will focus first on Case I). Note that in this case, $\overline{B_\varepsilon^n} \cap f^{-1}(\overline{B_\delta^p})$ is a manifold with corners whose topological boundary is the manifold with corners $(\overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1})) \cup (S_\varepsilon^{n-1} \cap f^{-1}(\overline{B_\delta^p}))$. We will use the following notations:

$$\begin{aligned} B_{\varepsilon,\delta} &= \overline{B_\varepsilon^n} \cap f^{-1}(\overline{B_\delta^p}), \\ \partial B_{\varepsilon,\delta} &= \left(\overline{B_\varepsilon^n} \cap f^{-1}(S_\delta^{p-1})\right) \cup \left(S_\varepsilon^{n-1} \cap f^{-1}(\overline{B_\delta^p})\right), \\ C_{\varepsilon,\delta} &= B_\varepsilon^n \cap f^{-1}(S_\delta^{p-1}) \end{aligned}$$

and $I_{\varepsilon,\delta}$ is the topological interior of $B_{\varepsilon,\delta}$.

Let us denote by ∂f the restricted mapping $f_{|C_{\varepsilon,\delta}} : C_{\varepsilon,\delta} \to S_\delta^{p-1}$ and let us assume that it is a Morin mapping.

Let us consider a perturbation $\tilde{f}$ of f such that $\tilde{f}_{|I_{\varepsilon,\delta}} : I_{\varepsilon,\delta} \to B_\delta^p$ is a stable Morin map and $\tilde{f} = f$ in a neighborhood of $C_{\varepsilon,\delta}$. Here we assume the existence of such a perturbation. It always exists only when $p = 1, 2$ or 3.

Our aim is to generalize Theorem 2 of [**9**] which deals with map-germs from $\mathbb{R}^n$ to $\mathbb{R}^2$, i.e., to relate the topology of $\mathrm{Lk}(f) = f^{-1}(0) \cap S_\varepsilon^{n-1}$ to the topology of the singular set of $\tilde{f}$ and to the topology of the singular set of ∂f. As in the previous sections, we will denote by $A_k(\tilde{f})$ (resp. $A_k(\partial f)$), the set of singular points of $\tilde{f}$ (resp. ∂f) of type A_k. The first result is a local version of Saeki's formula (Theorem 2.3 in [**21**]).

Theorem 6.6. *We have*

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \sum_{k=1}^{p} \chi(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) \bmod 2,$$

where ψ denotes the semi-characteristic. Furthermore if for $k \in \{1, \ldots, p-1\}$, $\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}$ is stably parallelizable then we have

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \sum_{k=1}^{p-1} \psi(\overline{A_k(\partial f)}) + \# A_p(\tilde{f}) \bmod 2.$$

Proof. Note that for $\tilde{\delta}$ a sufficiently small regular value of $\tilde{f}$ ($|\tilde{\delta}| \leq \delta$), we have

$$\begin{aligned}\chi_c(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) &\equiv \chi(\tilde{f}^{-1}(\tilde{\delta}) \cap I_{\varepsilon,\delta}) \equiv \chi(\tilde{f}^{-1}(\tilde{\delta}) \cap B_{\varepsilon,\delta}) \\ &\equiv \psi(\tilde{f}^{-1}(\tilde{\delta}) \cap S_\varepsilon^{n-1}) \equiv \psi(\mathrm{Lk}(f)) \mod 2.\end{aligned}$$

The last equality comes from the fact that f has an isolated singularity, that $\tilde{f}^{-1}(\tilde{\delta})$ intersects S_ε^{n-1} transversally and that $\tilde{f}$ is close to f.

On the one hand, applying Theorem 5.1, Theorem 5.6 and their corollaries to the restriction of $\tilde{f}$ to $I_{\varepsilon,\delta}$, we obtain:

$$\sum_{k:\text{even}} \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) \equiv \psi(\mathrm{Lk}(f)) \bmod 2.$$

On the other hand, by additivity, we have

$$1 \equiv \chi_c(I_{\varepsilon,\delta}) \equiv \sum_k \chi_c(I_{\varepsilon,\delta} \cap A_k(\tilde{f})) \bmod 2.$$

For each $k \geq 1$, we have

$$\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}} = (A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) \sqcup (\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) \sqcup (\overline{A_k(\tilde{f})} \cap C_{\varepsilon,\delta}),$$

because if ε and δ are small enough the singular set of $\tilde{f}$ does not intersect $f^{-1}(\overline{B_\delta^p}) \cap S_\varepsilon^{n-1}$. Before carrying on with our computations, let us observe that for $k \in \{1, \ldots, p-1\}$, $\overline{A_k(\tilde{f})} \cap C_{\varepsilon,\delta} = \overline{A_k(\partial f)}$. It is not difficult to see this with the characterization of the A_k sets by the ranks of the iterated jacobians. Hence

$$\begin{aligned}\chi(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) &\equiv \chi_c(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) \\ &\equiv \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_k(\tilde{f})} \cap C_{\varepsilon,\delta}) \\ &\equiv \chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) \mod 2,\end{aligned}$$

because $\overline{A_k(\tilde{f})} \cap C_{\varepsilon,\delta}$ is a compact boundary. Furthermore, we have

$$\begin{aligned}\chi(\overline{A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}}) &\equiv \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap C_{\varepsilon,\delta}) \\ &\equiv \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) \mod 2.\end{aligned}$$

Finally, for each k, $\chi_c(A_k(\tilde{f}) \cap I_{\varepsilon,\delta}) \equiv \chi(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) + \chi(\overline{A_{k+1}(\tilde{f}) \cap I_{\varepsilon,\delta}}) \mod 2$, and

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \sum_{k=1}^{p} \chi(\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}) \bmod 2.$$

If for $k \in \{1, \dots, p-1\}$, $\overline{A_k(\tilde{f}) \cap I_{\varepsilon,\delta}}$ is stably parallelizable then we have

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \sum_{k=1}^{p-1} \psi(\overline{A_k(\partial f)}) + \# A_p(\tilde{f}) \bmod 2. \qquad \square$$

Let us examine some special cases. When $p = 1$, we find

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \# A_1(\tilde{f}) \equiv 1 + \deg_0 \nabla f \quad \bmod 2,$$

where $\deg_0 \nabla f$ is the topological degree of the map $\nabla f / \|\nabla f\| : S_\varepsilon^{n-1} \to S^{n-1}$. This is due to the fact that $\tilde{f}$ is a Morse function and the points in $A_1(\tilde{f})$ are exactly its critical points.

When $p = 2$, we find

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \chi(\overline{A_1(\tilde{f}) \cap I_{\varepsilon,\delta}}) + \# A_2(\tilde{f}) \bmod 2.$$

If $\tilde{f}$ is close to f then $\chi(\overline{A_1(\tilde{f}) \cap I_{\varepsilon,\delta}})$ is equal to $(1/2)b(C(f))$ where $C(f)$ denotes the critical locus of f and $b(C(f))$ the number of branches of $C(f)$. Hence

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \frac{1}{2} b(C(f)) + \# A_2(\tilde{f}) \bmod 2.$$

Since $b(C(f))$ is a topological invariant of f, we deduce that $\# A_2(\tilde{f})$ mod 2 is a topological invariant of f. This last result was also obtained in [**14**].

Similarly if $p = 3$, this gives

$$\psi(\mathrm{Lk}(f)) \equiv 1 + \chi(\overline{A_1(\tilde{f}) \cap I_{\varepsilon,\delta}}) + \frac{1}{2} \# A_2(\partial f) + \# A_3(\tilde{f}) \bmod 2.$$

In the sequel, we will improve Theorem 6.6 in some situations. Let us assume that $n - p$ is odd.

Theorem 6.7. *If $n - p$ is odd, then we have*

$$\chi(\mathrm{Lk}(f)) = 2 - 2 \sum_{k:\mathrm{odd}} \left[\chi(\overline{(A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}}) - \chi(\overline{(A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta}}) \right].$$

Furthermore, when n is odd and p is even, we have

$$\chi(\mathrm{Lk}(f)) = 2 - \sum_{k:\mathrm{odd}} \left[\chi(\overline{(A_k)_+(\partial f)}) - \chi(\overline{(A_k)_-(\partial f)}) \right].$$

PROOF. With the same notations as in Theorem 6.6, we can write

$$\chi_c(\tilde f^{-1}(\tilde\delta)\cap B_{\varepsilon,\delta})=\chi_c(\tilde f^{-1}(\tilde\delta)\cap I_{\varepsilon,\delta})+\chi_c(\tilde f^{-1}(\tilde\delta)\cap \partial B_{\varepsilon,\delta}),$$

thus

$$\frac{1}{2}\chi(\mathrm{Lk}(f))=\chi_c(\tilde f^{-1}(\tilde\delta)\cap I_{\varepsilon,\delta})+\chi(\mathrm{Lk}(f)).$$

Therefore, we get

$$\chi_c(\tilde f^{-1}(\tilde\delta)\cap I_{\varepsilon,\delta})=-\frac{1}{2}\chi(\mathrm{Lk}(f)).$$

Applying Corollary 5.4, we obtain

$$\chi_c(I_{\varepsilon,\delta})+\frac{1}{2}\chi(\mathrm{Lk}(f))\chi_c(B^p_\delta)=\sum_{k:\mathrm{odd}}\big[\chi_c((A_k)_+(\tilde f)\cap I_{\varepsilon,\delta})-\chi_c((A_k)_-(\tilde f)\cap I_{\varepsilon,\delta})\big].$$

Note that $\chi_c(I_{\varepsilon,\delta})=(-1)^n$ since $I_{\varepsilon,\delta}$ is homeomorphic to an open unit ball of dimension n. Thus if n is odd and p is even, we have

$$\frac{1}{2}\chi(\mathrm{Lk}(f))=1+\sum_{k:\mathrm{odd}}\big[\chi_c((A_k)_+(\tilde f)\cap I_{\varepsilon,\delta})-\chi_c((A_k)_-(\tilde f)\cap I_{\varepsilon,\delta})\big],$$

which means

$$\chi(\mathrm{Lk}(f))=2+2\sum_{k:\mathrm{odd}}\big[\chi_c((A_k)_+(\tilde f)\cap I_{\varepsilon,\delta})-\chi_c((A_k)_-(\tilde f)\cap I_{\varepsilon,\delta})\big].$$

It remains to relate $\chi_c((A_k)_\pm(\tilde f)\cap I_{\varepsilon,\delta})$ to $\chi(\overline{(A_k)_\pm(\tilde f)\cap I_{\varepsilon,\delta}})$. The argument is the same as the one used in Remark 5.3 except that $\overline{(A_k)_\pm(\tilde f)\cap I_{\varepsilon,\delta}}$ are manifolds with corners. But since every manifold with corners is homeomorphic to a manifold with boundary with a homeomorphism mapping the interior to the interior and the boundary to the boundary, we see that:

$$\begin{aligned}\chi_c((A_k)_+(\tilde f)\cap I_{\varepsilon,\delta})&=-\chi(\overline{(A_k)_+(\tilde f)\cap I_{\varepsilon,\delta}})\\&=-\frac{1}{2}\chi(\overline{(A_k)_+(\tilde f)}\cap C_{\varepsilon,\delta})-\frac{1}{2}\chi(\overline{A_{k+1}(\tilde f)\cap I_{\varepsilon,\delta}}),\\ \chi_c((A_k)_-(\tilde f)\cap I_{\varepsilon,\delta})&=-\chi(\overline{(A_k)_-(\tilde f)\cap I_{\varepsilon,\delta}})\\&=-\frac{1}{2}\chi(\overline{(A_k)_-(\tilde f)}\cap C_{\varepsilon,\delta})-\frac{1}{2}\chi(\overline{A_{k+1}(\tilde f)\cap I_{\varepsilon,\delta}}).\end{aligned}$$

Finally, we obtain

$$\chi(\mathrm{Lk}(f)) = 2 - 2\sum_{k:\mathrm{odd}} \left[\chi(\overline{(A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}}) - \chi(\overline{(A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta}})\right],$$

and so

$$\chi(\mathrm{Lk}(f)) = 2 - \sum_{k:\mathrm{odd}} \left[\chi(\overline{(A_k)_+(\tilde{f})} \cap C_{\varepsilon,\delta}) - \chi(\overline{(A_k)_-(\tilde{f})} \cap C_{\varepsilon,\delta})\right].$$

But using the characterization of the $(A_k)_+$ and $(A_k)_-$ sets by the Euler characteristic of the nearby fiber, we can say that $\overline{(A_k)_+(\tilde{f})}\cap C_{\varepsilon,\delta} = \overline{(A_k)_+(\partial f)}$ and $\overline{(A_k)_-(\tilde{f})}\cap C_{\varepsilon,\delta} = \overline{(A_k)_-(\partial f)}$ for k odd.

If n is even and p is odd, then $\chi_c(I_{\varepsilon,\delta}) = 1$ and

$$1 - \sum_{k:\mathrm{odd}} \chi_c((A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}) + \sum_{k:\mathrm{odd}} \chi_c((A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta}) = \frac{1}{2}\chi(\mathrm{Lk}(f)),$$

and then

$$\chi(\mathrm{Lk}(f)) = 2 - 2\bigg(\sum_{k:\mathrm{odd}} \chi_c((A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}) - \sum_{k:\mathrm{odd}} \chi_c((A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta})\bigg).$$

Here $\dim(A_k)_+(\tilde{f}) = \dim(A_k)_-(\tilde{f}) = p-k$ is even when k is odd. We see that

$$\chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{(A_k)_+(\tilde{f})} \cap C_{\varepsilon,\delta}) = 0,$$

since $(\overline{A_{k+1}(\tilde{f})}\cap I_{\varepsilon,\delta})\cup(\overline{(A_k)_+(\tilde{f})}\cap C_{\varepsilon,\delta})$ is homeomorphic to an odd dimensional compact manifold. Consequently, we have

$$\chi(\overline{(A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}}) = \chi_c((A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}),$$

because

$$\chi(\overline{(A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}}) = \chi_c((A_k)_+(\tilde{f}) \cap I_{\varepsilon,\delta}) + \chi_c(\overline{A_{k+1}(\tilde{f})} \cap I_{\varepsilon,\delta}) + \chi_c(\overline{(A_k)_+(\tilde{f})} \cap C_{\varepsilon,\delta}).$$

Similarly, we have

$$\chi(\overline{(A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta}}) = \chi_c((A_k)_-(\tilde{f}) \cap I_{\varepsilon,\delta}). \qquad \square$$

The same results hold in Case II) replacing $B_\varepsilon^n \cap f^{-1}(S_\varepsilon^{p-1})$ with $f^{-1}(S_\varepsilon^{p-1})$, which is diffeomorphic to S_ε^{n-1}, $B_{\varepsilon,\delta}$ with $f^{-1}(\overline{B_\varepsilon^p})$, $I_{\varepsilon,\delta}$ with the topological interior of $f^{-1}(\overline{B_\varepsilon^p})$ and $\chi(\mathrm{Lk}(f))$ with 0.

Now we work with map-germs from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^n, 0)$. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a map-germ such that 0 is isolated in $f^{-1}(0)$. We assume that f is generic in the sense of Theorem 3 in [**8**]: there exists a positive number ε_0 such that for any number ε with $0 < \varepsilon \leq \varepsilon_0$, we have

(1) $\tilde{S}_\varepsilon^{n-1} = f^{-1}(S_\varepsilon^{n-1})$ is a homotopy $(n-1)$-sphere which, if $n \neq 4, 5$, is diffeomorphic to the standard $(n-1)$-sphere S^{n-1},
(2) the restricted mapping $f|_{\tilde{S}_\varepsilon^{n-1}} : \tilde{S}_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair),
(3) letting $\tilde{B}_\varepsilon^n = f^{-1}(\overline{B_\varepsilon^n})$, the restricted mapping $f|_{\tilde{B}_\varepsilon^n} : \tilde{B}_\varepsilon^n \setminus \{0\} \to \overline{B_\varepsilon^n} \setminus \{0\}$ is proper, topologically stable (C^∞ stable if (n, p) is nice) and topologically equivalent (C^∞ equivalent if (n, p) is nice) to the product mapping:

$$\big(f|_{\tilde{S}_\varepsilon^{n-1}}\big) \times \mathrm{Id}_{(0,\varepsilon]} : \tilde{S}_\varepsilon^{n-1} \times (0, \varepsilon] \to S_\varepsilon^{n-1} \times (0, \varepsilon],$$

defined by $(x, t) \mapsto (f(x), t)$,
(4) consequently, $f|_{\tilde{B}_\varepsilon^n} : \tilde{B}_\varepsilon^n \to \overline{B_\varepsilon^n}$ is topologically equivalent to the cone:

$$C\big(f|_{\tilde{S}_\varepsilon^{n-1}}\big) : \tilde{S}_\varepsilon^{n-1} \times [0, \varepsilon]/\tilde{S}_\varepsilon^{n-1} \times \{0\} \to S_\varepsilon^{n-1} \times [0, \varepsilon]/S_\varepsilon^{n-1} \times \{0\},$$

of the stable mapping $f|_{\tilde{S}_\varepsilon^{n-1}} : \tilde{S}_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$ defined by

$$C(f|_{\tilde{S}_\varepsilon^{n-1}})(x, t) = (f(x), t).$$

Note that in this case $\tilde{B}_\varepsilon = f^{-1}(\overline{B_\varepsilon^n})$ is a smooth manifold with boundary $f^{-1}(S_\varepsilon^{n-1})$. This last manifold has the homotopy type of S^{n-1}.

We will keep the notations of the previous sections. We denote by $\tilde{B}_\varepsilon$ the set $f^{-1}(\overline{B_\varepsilon^n})$, by $\tilde{I}_\varepsilon$ its topological interior and by $\partial \tilde{B}_\varepsilon$ its boundary. We denote by ∂f the restricted mapping $f|_{\partial \tilde{B}_\varepsilon} : \partial \tilde{B}_\varepsilon \to S_\varepsilon^{n-1}$ and we assume that it is a Morin mapping.

Let us consider a perturbation $\tilde{f}$ of f such that $\tilde{f}|_{\tilde{I}_\varepsilon} : \tilde{I}_\varepsilon \to B_\varepsilon^n$ is a Morin mapping and $\tilde{f} = f$ in a neighborhood of $\partial \tilde{B}_\varepsilon$. As above, such a perturbation does not always exist.

The main result is a local version of Corollary 5.12.

Theorem 6.8. *We have*

$$\deg_0 f = \sum_{k:\text{even}} \big[\chi(\overline{A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon}) - \chi(\overline{A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon})\big],$$

where $\deg_0 f$ *is the local topological degree of* f *at* 0.

Proof. Using Corollary 5.12, we obtain

$$(\deg_0 f)(-1)^n = \sum_{k:\text{even}} \big[\chi_c(A_k^+(\tilde{f}) \cap \tilde{I}_\varepsilon) - \chi_c(A_k^-(\tilde{f}) \cap \tilde{I}_\varepsilon)\big].$$

It remains to relate the Euler characteristics with closed support to the topological Euler characteristics. But, as in Theorem 6.7, we have

$$\chi_c(A_k^+(\tilde f)\cap \tilde I_\varepsilon)-\chi_c(A_k^-(\tilde f)\cap \tilde I_\varepsilon)=(-1)^{n-k}\big(\chi(\overline{A_k^+(\tilde f)\cap \tilde I_\varepsilon})-\chi(\overline{A_k^-(\tilde f)\cap \tilde I_\varepsilon})\big).\qquad \square$$

Corollary 6.9. *If n is odd, we have*

$$2\deg_0 f=\sum_{k:\text{even}}\big[\chi(\overline{A_k^+(\partial f)})-\chi(\overline{A_k^-(\partial f)})\big].$$

Corollary 6.10. *We have*

$$\deg_0 f\equiv 1+\sum_{k=1}^{n-1}\chi(\overline{A_k(\tilde f)\cap \tilde I_\varepsilon})+\# A_n(\tilde f) \bmod 2.$$

Furthermore if for $k\in\{1,\dots,n-1\}$, $\overline{A_k(\tilde f)\cap \tilde I_\varepsilon}$ is stably parallelizable, then we have

$$\deg_0 f\equiv 1+\sum_{k=1}^{n-1}\psi(\overline{A_k(\partial f)})+\# A_n(\tilde f) \bmod 2.$$

Proof. We have

$$1=\chi(\tilde B_\varepsilon)=\chi(\overline{A_0^+(\tilde f)\cap \tilde I_\varepsilon})+\chi(\overline{A_0^-(\tilde f)\cap \tilde I_\varepsilon})-\chi(\overline{A_1(\tilde f)\cap \tilde I_\varepsilon}),$$

hence

$$\chi(\overline{A_0^+(\tilde f)\cap \tilde I_\varepsilon})-\chi(\overline{A_0^-(\tilde f)\cap \tilde I_\varepsilon})\equiv 1+\chi(\overline{A_1(\tilde f)\cap \tilde I_\varepsilon}) \bmod 2.$$

Similarly, if k is even and $\dim A_k>0$, then

$$\chi(\overline{A_k^+(\tilde f)\cap \tilde I_\varepsilon})-\chi(\overline{A_k^-(\tilde f)\cap \tilde I_\varepsilon})\equiv \chi(\overline{A_k(\tilde f)\cap \tilde I_\varepsilon})+\chi(\overline{A_{k+1}(\tilde f)\cap \tilde I_\varepsilon}) \bmod 2.$$

Thus we obtain that

$$\chi(\overline{A_k^+(\tilde f)\cap \tilde I_\varepsilon})-\chi(\overline{A_k^-(\tilde f)\cap \tilde I_\varepsilon})\equiv\begin{cases}\chi(\overline{A_k(\tilde f)\cap \tilde I_\varepsilon})+\chi(\overline{A_{k+1}(\tilde f)\cap \tilde I_\varepsilon}) & \text{if } \dim A_k(\tilde f)>1,\\ \chi(\overline{A_k(\tilde f)\cap \tilde I_\varepsilon})+\# A_{k+1}(\tilde f) & \text{if } \dim A_k(\tilde f)=1,\\ \# A_k(\tilde f) & \text{if } \dim A_k(\tilde f)=0,\end{cases}$$

modulo 2.

The second congruence is proved as in Theorem 6.6. $\square$

If $n=2$, this gives

$$\deg_0 f \equiv 1 + \frac{1}{2} b(C(f)) + \# A_2(\tilde{f}) \bmod 2,$$

and we recover Theorem 2.1 of T. Fukuda and G. Ishikawa [**11**].

If $n = 3$, this gives

$$\deg_0 f \equiv 1 + \chi(\overline{A_1(\tilde{f}) \cap \tilde{I}_\varepsilon}) + \frac{1}{2} \# A_2(\partial f) + \# A_3(\tilde{f}) \bmod 2.$$

7. Complex maps.

We end with some remarks in the complex case. Let $f : M \to N$ be a holomorphic map between complex manifolds M and N with $\dim M \geq \dim N$. We assume that N is connected. We assume that f is locally infinitesimally stable in J. Mather's sense.

Let c_σ denote the Euler characteristic of the local generic fiber of the map-germ of singularity type σ. Let χ_f denote the Euler characteristic of the generic fibers of f.

THEOREM 7.1. *If a locally infinitesimally stable map $f : M \to N$ is locally trivial at infinity, then*

$$\sum_\sigma c_\sigma \, \chi_c(\sigma(f)) = \chi_f \, \chi_c(N).$$

PROOF. Apply Corollary 2.4, setting $\varphi(x)$ the Euler characteristic of closed supported homology of the local Milnor fiber of f near x. □

COROLLARY 7.2. *If a Morin map $f : M \to N$ is locally trivial at infinity, then*

$$\chi_c(M) + (-1)^{m-n} \sum_{k=1}^{n} \chi_c(\overline{A_k(f)}) = \chi_f \, \chi_c(N)$$

where m denotes the complex dimension of M and n denotes the complex dimension of N.

We should remark that this formula was firstly formulated by Y. Yomdin (see [**27**]). Note also that when $m = n$, then χ_f is also the topological degree of f.

Let $f = (f_1, f_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic map-germ with $c(f) < \infty$ where

$$c(f) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \Big/ \begin{matrix} \text{the ideal generated} \\ \text{by } 2 \times 2 \text{ minors of} \end{matrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial J}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \frac{\partial J}{\partial x_2} \end{pmatrix}, \qquad J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}.$$

COROLLARY 7.3 ([**13**, (1.8)]). *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be holomorphic map-germs with $c(f) < \infty$, $c(g) < \infty$. Let f_t and g_t denote stable perturbations of f and g, respectively. If f and g are topologically right-left equivalent, then $\# A_2(f_t) = \# A_2(g_t)$.*

PROOF. Since the critical set can be characterized topologically, $(\mathbb{C}^2, \Sigma(f), 0)$ and $(\mathbb{C}^2, \Sigma(g), 0)$ are topologically equivalent, and they have the same Milnor number. Thus their smoothings have the same Euler characteristic and $\chi(\overline{A_1(f_t)}) = \chi(\overline{A_1(g_t)})$. By Corollary 7.2, we have

$$1 + \chi_c(\overline{A_1(f_t)}) + \#A_2(f_t) = \deg_0 f,$$
$$1 + \chi_c(\overline{A_1(g_t)}) + \#A_2(g_t) = \deg_0 g,$$

and, since $\deg_0 f = \deg_0 g$, we conclude the result. □

REMARK 7.4. Consider the map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$, $n > 2$. Take a stable perturbation f_t of f. We have

$$\chi_c(\overline{A_1(f_t)}) + \#A_2(f_t) = (-1)^n(\chi_f - 1). \tag{7.1}$$

Consider the map $F : (\mathbb{C}^n, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ defined by $F(x, t) = (f_t(x), t)$. Since $\overline{A_1(F)}$ is determinantal, it is Cohen-Macaulay. So the map $\overline{A_1(F)} \to (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat. So $\overline{A_1(f_t)}$ is a smoothing of $\overline{A_1(f)}$ and its Euler characteristic is described by the Milnor number $\mu(\overline{A_1(f)})$ of $\overline{A_1(f)}$: $\chi_c(\overline{A_1(f_t)}) = 1 - \mu(\overline{A_1(f)})$, and we conclude that $\mu(\overline{A_1(f)})$ and χ_f determine $\#(A_2(f_t))$. Now we assume that f is $\mathcal{A}$-finite. Then, we have

$$\begin{aligned} 1 - \mu(\overline{A_1(f)}) = \chi_c(\overline{A_1(f_t)}) &= \chi_c(f_t(\overline{A_1(f_t)})) + d(f_t) \\ &= 1 - \mu(f(\overline{A_1(f)})) + 2\#(A_2(f_t)) + 2d(f_t), \end{aligned}$$

where $d(f_t)$ denotes the number of double fold $(A_{1,1})$ points of f_t near 0. Combining this with (7.1), we obtain

$$3\#A_2(f_t) + 2d(f_t) = \mu(f(\overline{A_1(f)})) - 1 + (-1)^n(\chi_f - 1).$$

We conclude that $3\#A_2(f_t) + 2d(f_t)$ (and thus $\#A_2(f_t)$ mod 2) is a topological invariant of f.

REMARK 7.5. Consider a map germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$. Take a stable perturbation f_t of f. Then we obtain

$$1 + \chi_c(\overline{A_1(f_t)}) + \chi_c(\overline{A_2(f_t)}) + \#A_3(f_t) = \deg_0 f.$$

Consider the map $F : (\mathbb{C}^3, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}^3, 0) \times (\mathbb{C}, 0)$ defined by $F(x, t) = (f_t(x), t)$. Since $\overline{A_2(F)}$ is defined by the rank condition of the iterated jacobian, it is determinantal, and thus Cohen-Macaulay. We obtain that the map $\overline{A_2(F)} \to (\mathbb{C}, 0)$, $(x, t) \mapsto t$, is flat. So $\overline{A_2(f_t)}$ is a smoothing of $\overline{A_2(f)}$ and its Euler characteristic $\chi_c(\overline{A_2(f_t)})$ is described by the Milnor number of $\overline{A_2(f)}$ when $\overline{A_2(f)}$ has an isolated singularity at 0. Since $\overline{A_1(F)}$

is determinantal and thus Cohen-Macaulay, the map $\overline{A_1(F)} \to (\mathbb{C},0)$, $(x,t) \mapsto t$, is flat. So $\overline{A_1(f_t)}$ is a smoothing of $\overline{A_1(f)}$ and its Euler characteristic $\chi_c(\overline{A_1(f_t)})$ is determined by the Milnor number of $\overline{A_1(f)}$ when $\overline{A_1(f)}$ has an isolated singularity at 0. So when $\overline{A_1(f)}$ and $\overline{A_2(f)}$ have isolated singularities at 0, $\#(A_3(f_t))$ is determined by $\mu(\overline{A_1(f)})$, $\mu(\overline{A_2(f)})$ and $\deg_0 f$, that is:

$$\#(A_3(f_t)) = \deg_0 f - \mu(\overline{A_1(f)}) + \mu(\overline{A_2(f)}) - 3.$$

Remark 7.6. Consider a map germ $f : (\mathbb{C}^n,0) \to (\mathbb{C}^3,0)$, $n > 3$. Take a stable perturbation f_t of f. Then we obtain

$$\chi_c(\overline{A_1(f_t)}) + \chi_c(\overline{A_2(f_t)}) + \#A_3(f_t) = (-1)^n(1-\chi_f).$$

Consider the map $F : (\mathbb{C}^n,0) \times (\mathbb{C},0) \to (\mathbb{C}^3,0) \times (\mathbb{C},0)$ defined by $F(x,t) = (f_t(x),t)$. Since $\overline{A_1(F)}$ is determinantal, it is Cohen-Macaulay. We obtain that the map $\overline{A_1(F)} \to (\mathbb{C},0)$, $(x,t) \mapsto t$, is flat, and $\overline{A_1(f_t)}$ is a smoothing. So the topology of $\overline{A_1(f_t)}$ is determined by $\overline{A_1(f)}$ when $\overline{A_1(f)}$ has an isolated singularity at 0. By Theorem 2.8 in [**12**], $\overline{A_2(F)}$ is Cohen-Macaulay if and only if $n = 4, 5$. We thus obtain that the map $\overline{A_2(F)} \to (\mathbb{C},0)$, $(x,t) \mapsto t$, is flat, if $n = 4, 5$. Assume that $n = 4, 5$. Then $\overline{A_2(f_t)}$ is a smoothing of $\overline{A_2(f)}$ and its Euler characteristic $\chi_c(\overline{A_2(f_t)})$ is described by the Milnor number of $\overline{A_2(f)}$: $\chi_c(\overline{A_2(f_t)}) = 1 - \mu(\overline{A_2(f)})$. This means $\#(A_3(f_t))$ is determined by $\mu(\overline{A_1(f)})$, $\mu(\overline{A_2(f)})$ and χ_f. When $n \geq 6$, we do not know whether $\chi_c(\overline{A_2(f_t)}) = 1 - \mu(\overline{A_2(f)})$ holds or not.

Remember that the deformation theory of varieties concerns the defining ideals, and it is important to know when these ideals define reduced spaces or not in the geometric setup. The following example shows that the reduced structure of singularities locus may not fit the context of deformation of maps.

Example 7.7. Let us consider the image of the map $g : \mathbb{C} \to \mathbb{C}^3$ defined by $s \mapsto (s^3, s^4, s^5)$, whose Milnor number μ is 4 (cf. [**4**, p. 244]). The defining ideal is

$$I_0 = \langle xz - y^2,\ yz - x^3,\ x^2y - z^2\rangle.$$

Since it defines a reduced curve, it defines a Cohen-Macaulay space. Consider the map

$$G : (\mathbb{C}^2,0) \to (\mathbb{C}^4,0) \quad \text{defined by} \quad (s,t) \mapsto (x,y,z,t) = (st + s^3, s^4, s^5, t).$$

Remark that $g_0(s) = g(s)$ where $G(s,t) = (g_t(s),t)$. The image of g_t, $t \neq 0$, is nonsingular, and its Euler characteristic is 1, which is not $1-\mu$. Let us see what happens in this example. Eliminating s from the ideal generated by

$$x - st - s^3, \quad y - s^4, \quad z - s^5,$$

we obtain the ideal

$$I = \langle z^2 - x^2y + ty^2 + txz,\ xy^2 - x^2z + tyz + t^2xy - t^3z,\ y^3 - xyz + tx^2y - t^2y^2 + t^2xz,\\ xyz + tz^2 - x^4 + 2tx^2y + 2t^2xz + t^4y,\ y^2z - xz^2 + tx^2z - 2t^2yz - t^3xy + t^4z\rangle$$

of $\mathbb{C}\{x,y,z,t\}$. Computing a free resolution of $\mathbb{C}\{x,y,z,t\}/I$ as $\mathbb{C}\{x,y,z,t\}$-module, we see that the variety X defined by the ideal I is not Cohen-Macaulay. We also remark that this defines a reduced space, but the fiber $\pi^{-1}(0)$, where $\pi : X \to \mathbb{C}$ is the projection $\pi(x,y,z,t) = t$, is not reduced, since

$$\mathbb{C}\{x,y,z,t\}/I + \langle t\rangle \simeq \mathbb{C}\{x,y,z\}/I_0 \cap \langle x,\ y^3,\ y^2z,\ z^2\rangle.$$

Sending t to zero in the free resolution of $\mathbb{C}\{x,y,z,t\}/I$, we obtain a free resolution of this module as $\mathbb{C}\{x,y,z\}$-module. This implies that $\mathbb{C}\{x,y,z,t\}/I$ has no t-torsion elements and thus that π is a flat morphism.

References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, Singularities of differentiable maps, Vol. I, The classification of critical points, caustics and wave fronts, Monogr. Math., **82**, Birkhäuser Boston, Inc., Boston, MA, 1985.

[2] A. Borel and J. C. Moore, Homology theory for locally compact spaces, Michigan Math. J., **7** (1960), 137–159.

[3] S. A. Broughton, On the topology of polynomial hypersurfaces, In: Singularities. Part 1, Arcata, Calif., 1981, (ed. P. Orlik), Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983, pp. 167–178.

[4] R.-O. Buchweitz and G.-M. Greuel, The Milnor number and deformations of complex curve singularities, Invent. Math., **58** (1980), 241–281.

[5] S. K. Donaldson and P. B. Kronheimer, The Geometry of Four-Manifolds, Oxford Math. Monogr., Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1990.

[6] A. H. Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc., **276** (1983), 517–530.

[7] J. M. Èliašberg, On singularities of folding type, Math. USSR-Izv., **4** (1970), 1119–1134.

[8] T. Fukuda, Local topological properties of differentiable mappings. I, Invent. Math., **65** (1981/82), 227–250.

[9] T. Fukuda, Local topological properties of differentiable mappings. II, Tokyo J. Math., **8** (1985), 501–520.

[10] T. Fukuda, Topology of folds, cusps and Morin singularities, In: A fête of topology, (eds. Y. Matsumoto, T. Mizutani and S. Morita), Academic Press, Boston, MA, 1988, pp. 331–353.

[11] T. Fukuda and G. Ishikawa, On the number of cusps of stable perturbations of a plane-to-plane singularity, Tokyo J. Math., **10** (1987), 375–384.

[12] T. Fukui and J. Weyman, Cohen-Macaulay properties of Thom-Boardman strata. II, The defining ideals of $\Sigma^{i,j}$, Proc. London Math. Soc., **87** (2003), 137–163.

[13] T. Gaffney and D. M. Q. Mond, Cusps and double folds of germs of analytic maps $\mathbb{C}^2 \to \mathbb{C}^2$, J. London Math. Soc. (2), **43** (1991), 185–192.

[14] K. Ikegami and O. Saeki, Cobordism of Morse maps and its application to map germs, Math. Proc. Cambridge Philos. Soc., **147** (2009), 235–254.

[15] G. M. Khimshiashvili, On the local degree of a smooth map, Soobshch. Akad. Nauk Gruz. SSR, **85** (1977), 309–311.

[16] H. I. Levine, Mappings of manifolds into the plane, Amer. J. Math., **88** (1966), 357–365.

[17] J. W. Milnor, Topology from the differentiable viewpoint, Based on notes by David W. Weaver,

The University Press of Virginia, Charlottesville, Va., 1965.

[18] I. Nakai, Characteristic classes and fiber products of smooth mappings, Preprint.

[19] I. Nakai, Elementary topology of stratified mappings, In: Singularities-Sapporo 1998, (eds. J. P. Brasselet and T. Suwa), Adv. Stud. Pure Math., **29**, Kinokuniya, Tokyo, 2000, pp. 221–243.

[20] J. R. Quine, A global theorem for singularities of maps between oriented 2-manifolds, Trans. Amer. Math. Soc., **236** (1978), 307–314.

[21] O. Saeki, Studying the topology of Morin singularities from a global viewpoint, Math. Proc. Cambridge Philos. Soc., **117** (1995), 223–235.

[22] Z. Szafraniec, On the Euler characteristic of analytic and algebraic sets, Topology, **25** (1986), 411–414.

[23] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier (Grenoble), **6** (1955–1956), 43–87.

[24] M. Tibăr and A. Zaharia, Asymptotic behaviour of families of real curves, Manuscripta Math., **99** (1999), 383–393.

[25] O. Ya. Viro, Some integral calculus based on Euler characteristic, In: Topology and Geometry – Rohlin Seminar, (ed. O. Ya. Viro), Lecture Notes in Math., **1346**, Springer-Verlag, Berlin, 1988, pp. 127–138.

[26] O. Ya. Viro, Plane real algebraic curves: constructions with controlled topology, Leningrad Math. J., **1** (1990), 1059–1134.

[27] Y. Yomdin, The structure of strata $\mu = const$ in a critical set of a complete intersection singularity, In: Singularities. Part 2, Arcata, Calif., 1981, (ed. P. Orlik), Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983, pp. 663–665.

[28] C. T. C. Wall, Topological invariance of the Milnor number mod 2, Topology, **22** (1983), 345–350.

[29] C. T. C. Wall, Transversality in families of mappings, Proc. Lond. Math. Soc., **99** (2009), 67–99.

Nicolas Dutertre
Université d'Aix-Marseille
Laboratoire d'Analyse, Topologie
et Probabilités, UMR 7353
39 rue F. Joliot-Curie
13453 Marseille Cedex 13, France
E-mail: dutertre@cmi.univ-mrs.fr

Toshizumi Fukui
Department of Mathematics
Faculty of Science
Saitama University
255 Shimo-Okubo, Sakura-ku
Saitama 338-8570, Japan
E-mail: tfukui@rimath.saitama-u.ac.jp