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# Exact critical values of the symmetric fourth $L$ function and vector valued Siegel modular forms

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**Abstract.** Exact critical values of symmetric fourth $L$ function of the Ramanujan Delta function $\Delta$ were conjectured by Don Zagier in 1977. They are given as products of explicit rational numbers, powers of $\pi$, and the cube of the inner product of $\Delta$. In this paper, we prove that the ratio of these critical values are as conjectured by showing that the critical values are products of the same explicit rational numbers, powers of $\pi$, and the inner product of some vector valued Siegel modular form of degree two. Our method is based on the Kim-Ramakrishnan-Shahidi lifting, the pullback formulas, and differential operators which preserve automorphy under restriction of domains. We also show a congruence between a lift and a non-lift. Furthermore, we show the algebraicity of the critical values of the symmetric fourth $L$ function of any elliptic modular form and give some conjectures in general case.

## 1. Introduction.

The critical values of the symmetric $j$-th $L$ functions $L(s, f, Sym(j))$ of an elliptic modular form $f$ are interesting objects in number theory in various respects. The algebraicity of such values was conjectured by Deligne [**5**] and has been proved in the case $j \leq 3$ (cf. Shimura [**19**], Sturm [**21**], Orloff [**16**], Satoh [**18**].) On the other hand, it is also interesting to compute such values exactly. Let $\Delta$ be the Ramanujan delta function, the unique primitive cusp of weight 12 of level 1, and $L(s, \Delta, Sym(j))$ the symmetric $j$-th $L$ function of $\Delta$. In 1977, D. Zagier gave the critical values of $L(s, \Delta, Sym(j))$ for $j = 1$, 2 and proposed a conjecture on exact critical values for $j = 3$, 4 (cf. [**25**].) Mizumoto solved the conjecture for $j = 3$ in [**14**]. The topic of this paper is the case $j = 4$. We define the norm of $\Delta$ by

$$(\Delta, \Delta) = \int_{SL_2(\mathbb{Z})\backslash H} |\Delta(\tau)|^2 y^{10} dx\, dy.$$

Then Zagier's conjecture is given as follows.

Conjecture 1.1 (Zagier [**25**]). *We have*

$$((2\pi)^{-3s+33}\Gamma(11)^{-1}\Gamma(s)\Gamma(s-11)L(s, \Delta, Sym(4)) = c(s)2^{33}(\Delta, \Delta)^3$$

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*for* $s = 24, 26, 28, 30, 32$ *where* $c(s)$ *are given in the following table.*

| $s$ | $c(s)$ |
|---|---|
| 24 | $2^5 \times 3^2$ |
| 26 | $2^5 \times 3 \times 5$ |
| 28 | $2^2 \times 23 \times 691/7^2$ |
| 30 | $2^3 \times 653$ |
| 32 | $2 \times 3 \times 34981^*/7$ |

($^*$ He stated 34891 instead of the above prime 34981, but this is an obvious typo since $34891 = 23 \times 37 \times 41$.) Our main theorem in this paper is as follows.

Theorem 1.2. *For a certain explicitly defined holomorphic vector valued Siegel modular form* $F$ *of degree two of weight* $\det^{13} Sym(10)$, *we have*

$$(2\pi)^{-3r-33}\Gamma(11)^{-1}\Gamma(r+11)\Gamma(r+22)L(r,F,St) = c(r+22)(F,F)$$

*for* $r = 2, 4, 6, 8, 10$, *where* $c(*)$ *is as in Zagier's conjecture above. Here* $F$ *does not depend on* $r$ *and* $(F,F)$ *is an explicitly defined inner metric of* $F$.

The definition of $F$ and details of the above theorem will be explained in the next section and in Theorem 2.2. The proof of Theorem 2.2 is obtained inside the theory of Siegel modular forms and has nothing to do with anything related with $\Delta$. But we considered a vector valued Siegel modular form here since by virtue of [**17**] there exists a lifting from $\Delta$ to $F$ such that $L(s,\Delta,Sym(4)) = L(s-22,F,St)$ where the latter is the standard $L$ function of $F$. So the above theorem means

Corollary 1.3. *We have*

$$(2\pi)^{33-3s}\Gamma(11)^{-1}\Gamma(s)\Gamma(s-11)L(s,\Delta,Sym(4)) = c(s)(F,F)$$

*for any* $s = 24, 26, 28, 30, 32$. *In particular, five critical values of the left hand side are exactly as conjectured by Zagier up to a common non-zero constant.*

The existence of this type of lifting from modular forms of one variable to Siegel modular forms appeared first in [**13**] for generic Siegel modular forms and then was proved for holomorphic vector valued Siegel modular forms by [**17**]. The first named author had also written a preprint [**9**] on conjectures on these liftings with numerical experiments on $\Delta$ and $F$ as cited in [**17**] and the present article is partly a continuation of that preprint. Since we do not know the relation between $(\Delta,\Delta)^3$ and $(F,F)$, we cannot say that we proved Zagier's conjecture completely, but his conjecture is now interpreted into a conjecture on a relation between $(\Delta,\Delta)^3$ and $(F,F)$ (cf. Conjecture 2.3). We also give a certain congruence between eigenvalues of the above lift and non-lift (cf. Theorem 2.4). Furthermore, we will make a remark that for any elliptic modular form $f$, the algebraicity of critical values of $L(s,f,Sym(4))$ up to a constant is obtained as an easy corollary of the theorem of Ramakrishnan-Shahidi and the pullback formula of Siegel modular forms. We state this as Theorem 3.2, since we have never seen this in the literature before. The content of each section is as follows. In Section 2, we give precise definitions and main theorems. In Section 3, we review a result of Kozima on critical values using a pullback formula and differential operators, and give some technical details. In Section 4, we explain another formulation and concrete definitions of differential operators we use. In Section 5, we complete the proof of Theorem 2.2. In the appendix, we give tables of Fourier coefficients of Siegel modular forms we use.

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## 2. Main theorems.

### 2.1. Definitions and Notation.

We prepare notation and definitions. For any natural number $n$, we denote by $\mathfrak{H}_n$ the Siegel upper half space of degree $n$.

$$\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}); Z = {}^tZ = X + \sqrt{-1}Y,\ X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any ring $R$ and any natural integer $n$, we define the symplectic group of *size* $2n$ over $R$ by

$$Sp(n, R) = \{g \in M_{2n}(R);\ gJ_n{}^tg = J_n\},$$

where $J_n = \left(\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right)$. We put $\Gamma_n = Sp(n, \mathbb{Z})$ for the sake of simplicity. Now we define vector valued Siegel modular forms of $\Gamma_n$. For any irreducible representation $(\rho, V)$ of $GL(n, \mathbb{C})$, for any $V$-valued function $F$ on $\mathfrak{H}_n$, and for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$, we write

$$F|_\rho[g] = \rho(CZ + D)^{-1}F(gZ).$$

We say that $F$ is a holomorphic Siegel modular form of weight $\rho$ with respect to $\Gamma_n$ if $F$ is holomorphic on $\mathfrak{H}$ and $F|_\rho[\gamma] = F$ for any $\gamma \in \Gamma_n$ (with the extra condition of holomorphy at the cusp $\sqrt{-1}\infty$ if $n = 1$). We describe $\rho$ more precisely when $n = 2$. We denote by $\mathbb{C}[u_1, u_2]_m$ the vector space of homogeneous polynomials of degree $m$ in $u_1$, $u_2$. Then the $m$-th symmetric tensor representation $Sym_m$ of $GL(2, \mathbb{C})$ is defined by

$$Sym_m(A)(P(u_1, u_2)) = P((u_1, u_2))A)$$

for any $A \in GL(2, \mathbb{C})$ and $P \in \mathbb{C}[u_1, u_2]_m$. For any integer $k$, we denote by $\det^k$ the representation of $GL(2, \mathbb{C})$ given by $\det^k(A) = (\det(A))^k$ for any $A \in GL(2, \mathbb{C})$. We put $\rho_{k,m} = \det^k \otimes Sym_m$. Then these exhaust all the rational irreducible representations of $GL(2, \mathbb{C})$. We denote by $A_{k,m}(\Gamma_2)$ the vector space of holomorphic Siegel modular forms of weight $\rho_{k,m}$ with respect to $\Gamma_2$. The Siegel $\Phi$ operator is defined for any element $F \in A_{k,m}(\Gamma_2)$ as usual and $\Phi(A_{k,m}(\Gamma_2))$ is identified with a subspace of the space $S_{k+m}(\Gamma_1)$ of cusp forms of weight $k+m$ with respect to $\Gamma_1$ (cf. [**1**]). We say that $F$ is a cusp form if $\Phi(F)=0$ and denote by $S_{k,m}(\Gamma_2)$ the space of cusp forms in $A_{k,m}(\Gamma_2)$. In particular, we have $A_{k,m}(\Gamma_2)=0$ if $m$ is odd and $A_{k,m}(\Gamma_2)=S_{k,m}(\Gamma_2)$ if $k$ is odd. We denote by $\mathcal{H}_n(\mathbb{Z})_{\geq 0}$ the set of positive semi-definite $n \times n$ half-integral symmetric matrices and by $\mathcal{H}_n(\mathbb{Z})_{>0}$ the subset of positive definite matrices. Any element $F$ of $A_{k,m}(\Gamma_2)$ has the following Fourier expansion.

$$F(Z) = \sum_{A \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}} c_F(A;u) \exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)),$$

where $c_F(A;u)$ is a homogeneous polynomial of degree $m$ in $u=(u_1,u_2)$ with coefficients in $\mathbb{C}$ for any $A \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$, and in particular, if $F$ is a cusp form, then $c_F(A;u)=0$ unless $A \in \mathcal{H}_2(\mathbb{Z})_{>0}$. For any ring $R \subset \mathbb{C}$, we denote by $A_{k,m}(\Gamma_2)(R)$ the subspace of $F \in A_{k,m}(\Gamma_2)$ such that $c_F(A,u) \in R[u_1,u_2]_m$ for all $A \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$.

### 2.2. A lifting and vector valued Siegel modular forms.

For any primitive Hecke eigenform $f = \sum_{n=1}^{\infty} a(n)e^{2\pi\sqrt{-1}nz} \in S_k(\Gamma_1)$, we write $1-a(p)x+p^{k-1}x^2 = (1-\alpha_p x)(1-\beta_p x)$ for any prime $p$ and define the symmetric $j$-th $L$ function of $f$ by

$$L(s,f,Sym(j)) = \prod_{p:prime} \prod_{i=0}^{j} (1-\alpha_p^i \beta_p^{j-i} p^{-s})^{-1}.$$

It is proved that this $L$ function is continued meromorphically to the whole $s$ plane and that for small $j$ it satisfies the functional equation for $s \to (k-1)j+1-s$ (cf. [**23**] for example). For any $F \in S_{k,j}(\Gamma_2)$, we denote by $L(s,F,St)$ the standard $L$ function of $F$ normalized so that this satisfies the functional equation for $s \to 1-s$. We write $L(s,F,Sp)$ the spinor $L$ function which satisfies the functional equation for $s \to 2k+j-2-s$. Now we quote a part of the theorem in [**17**] we need.

THEOREM 2.1 (Ramakrishnan-Shahidi [**17**]). *For any primitive Hecke eigenform $f \in S_k(\Gamma_1)$, there exists a holomorphic Siegel modular form $F \in S_{k+1,k-2}(\Gamma_2)$ which is a Hecke eigenform such that*

$$L(s,f,Sym(3)) = L(s,F,Sp).$$

We can easily see that for the same $f$ and $F$ above, we have also

$$L(s-2k+2,f,Sym(4)) = L(s,F,St)$$

by checking the relation between Satake parameters.

Their theorem is a kind of existence theorem, and we do not know how to give $F$ explicitly in general when $f$ is given. Also this theorem is a correspondence between automorphic representations, so even if multiplicity one theorem holds, $F$ is defined only up to constants and there is no canonical way to choose normalization of $F$ at moment.

Now we would like to construct $F \in S_{13,10}(\Gamma_2)$ corresponding to $\Delta \in S_{12}(\Gamma_1)$ predicted by the above theorem. The dimension of $S_{k,m}(\Gamma_2)$ is known for $k > 4$ by Tsushima [**24**]. In particular we have $\dim A_{13,10}(\Gamma_2) = \dim S_{13,10}(\Gamma_2) = 2$. We denote by $E_8 \subset 2^{-1}\mathbb{Z}^8 \subset \mathbb{R}^8$ the positive definite even unimodular integral lattice of rank 8 which is unique up to isometry. We write $(x, y) = \sum_{i=1}^{8} x_i y_i$ for $x = (x_i)$, $y = (y_i) \in \mathbb{C}^8$. For any natural number $k$, $m$, and for any $a_1$, $a_2 \in \mathbb{C}^8$ such that $(a_1, a_1) = (a_1, a_2) = (a_2, a_2) = 0$ and $i$ with $0 \le i \le m$, define

$$\widetilde{P}_i(x, y, a_1, a_2) = (x, a_1)^{m-i}(y, a_1)^i \begin{vmatrix} (x, a_1) & (x, a_2) \\ (y, a_1) & (y, a_2) \end{vmatrix}^k .$$

We may write $\widetilde{P}_i = P_i(x, y, a_1, a_2) + \sqrt{-1}R_i(x, y, a_1, a_2)$ by polynomials $P_i$ and $R_i$ in $x$, $y$ with real coefficients. We define

$$\theta_{a_1,a_2,(k,m),i}(Z) = \sum_{x,y \in E_8} P_i(x, y, a_1, a_2) e^{\pi\sqrt{-1}((x,x)\tau + 2(x,y)z + (y,y)\omega)},$$

where we write $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathfrak{H}_2$. We put

$$\theta_{a_1,a_2,(k,m)}(Z) = \sum_{i=0}^{m} \binom{m}{i} \theta_{a_1,a_2,(k,m),i}(Z) u_1^{m-i} u_2^i.$$

Then it is well known and easy to see that $\theta_{a_1,a_2,(k,m)}(Z) \in A_{k+4,m}(\Gamma_2)$, though $\theta_{a,b}$ might be zero for many choices of $a$, $b$. (We can use $\widetilde{P}_i$ instead of $P_i$ here, but we use real valued polynomials for computational simplicity.) Now put

$$\begin{aligned} a_1 &= (2, 1, i, i, i, i, i, 0) \\ a_2 &= (1, -1, i, i, 1, -1, -i, i) \\ b_1 &= (3, 2i, i, i, i, i, i, 0) \\ b_2 &= (1, i, -1, i, 1, i, -i, 1) \end{aligned}$$

and define

$$\begin{aligned} f_{13,10a} &= \theta_{a_1,a_2,(9,10)}/(2^{33} \times 3^3 \times 5^3 \times 7^2), \\ f_{13,10b} &= \theta_{b_1,b_2,(9,10)}/(2^{33} \times 3^8 \times 5^4 \times 7). \end{aligned}$$

Then these span $A_{13,10}(\Gamma_2)$. To pick up among these a form corresponding to the lifting of $\Delta$, we calculate the action of the Hecke operators. For any natural number $n$, we put

$$T(n) = \{g \in M_4(\mathbb{Z});\ gJ_2{}^t g = nJ_2\}.$$

This is a union of double cosets with respect to $\Gamma_2$. If we regard $T(p)$, $T(p^2)$ and $p^{\pm 1}\Gamma_2$ for all primes $p$ as formal sums of such $\Gamma_2$-double cosets, these give the generators of the Hecke ring with respect to the pair $(GSp(2,\mathbb{Q}), \Gamma_2)$. For a Hecke eigenform $F \in A_{k,j}(\Gamma_2)$, we write $T(n)F = \lambda(n,F)F$. Then the Euler $p$-factor of the spinor $L$ function of $F$ is given by

$$H_p(s,F) = 1 - \lambda(p,F)T + (\lambda(p,F)^2 - \lambda(p^2,F) - p^{\mu-1})T^2 - \lambda(p,F)p^{\mu}T^3 + p^{2\mu}T^4,$$

where $\mu = 2k+m-3$ and $T = p^{-s}$ (cf. Arakawa [**1**]).

For a Hecke operator $T(n)$ and a vector valued Siegel modular form $F$ of degree two, the Fourier coefficients of $T(n)F$ is written by using the Fourier coefficients of $F$ as given in Arakawa [**1**]. So if we have enough Fourier coefficients, we can calculate the eigenvalues for each Hecke operator. In our case we can show that the Hecke eigenforms are given by

$$F_{13,10a} = (3677 f_{13,10a} + 120147 f_{13,10b})/(2^2 \times 23 \times 21800833),$$

$$F_{13,10b} = (-107841 f_{13,10a} + 21791 f_{13,10b})/(2 \times 19 \times 23 \times 21800833).$$

We omit the details of the calculation since examples of this sort of numerical calculation are found in [**8**] and we can do in the same way. Instead, in the appendix, we give explicit Fourier coefficients we need. The Euler factors at 2 and 3 of the spinor $L$ functions of $F_{13,10a}$ and $F_{13,10b}$ are given as follows.

$$\begin{aligned} H_2(s,F_{13,10a}) &= 1 - 84480T + 10611589120T^2 - 84480 \cdot 2^{33}T^3 + 2^{66}T^4 \\ &= (1 + 49152T + 2^{33}T^2)(1 - 133632T + 2^{33}T^2), \\ H_2(s,F_{13,10b}) &= 1 + 52800T - 889978880T^2 + 52800 \cdot 2^{33}T^3 + 2^{66}T^4, \\ H_3(s,F_{13,10a}) &= 1 + 73279080T + 5854043689141590T^2 + 73279080 \cdot 3^{33}T^3 + 3^{66}T^4 \\ &= (1 - 44641044T + 3^{33}T^2)(1 + 117920124T + 3^{33}T^2), \\ H_3(s,F_{13,10b}) &= 1 - 20395800T + 4889758827715990T^2 - 20395800 \cdot 3^{33}T^3 + 3^{66}T^4. \end{aligned}$$

Comparing these with $\Delta(\tau) = q - 24q + 252q^3 + \cdots$, and taking the result in [**17**] into account, we see that $L(s, F_{13,10a}, Sp) = L(s, \Delta, Sym(3))$ and $L(s-22, F_{13,10a}, St) = L(s, \Delta, Sym(4))$.

### 2.3. Main theorems.

For any non-negative integers $a$, $b$, $c$, $d$ with $a+b = c+d$, we put

$$\langle u_1^a u_2^b, u_1^c u_2^d \rangle = \frac{a!c!}{m!}\delta_{ac}$$

and define the hermitian inner product on $\mathbb{C}[u_1, u_2]_m$ by extending this linearly. Then we have $\langle \rho_{k,m}(A)x, y\rangle = \langle x, \rho_{k,m}({}^t\overline{A})y\rangle$ for any $x$, $y \in \mathbb{C}[u_1, u_2]_m$ and $A \in GL_2(\mathbb{C})$. For any vector valued Siegel modular form $F \in A_{k,m}(\Gamma_2)$, we define the inner product $(F, F)$ of $F$ by

$$(F, F) = \int_{\Gamma_2\backslash\mathfrak{H}_2} \big\langle \rho_{k,m}(\sqrt{Y})f(Z), \rho_{k,m}(\sqrt{Y})g(Z)\big\rangle \det(Y)^{-3}\, dX\, dY$$

where $Z = X + \sqrt{-1}Y$ and $dX = dx_{11}dx_{12}dx_{22}$, $dY = dy_{11}dy_{12}dy_{22}$ for $X = (x_{ij})$, $Y = (y_{ij}) \in M_2(\mathbb{R})$.

Theorem 2.2. *For $r = 2, 4, 6, 8, 10$, we have*

$$\begin{aligned}&(2\pi)^{-3r-33}\Gamma(11)^{-1}\Gamma(r+11)\Gamma(r+22)L(r+22, \Delta, Sym(4))\\ &\quad= (2\pi)^{-3r-33}\Gamma(11)^{-1}\Gamma(r+11)\Gamma(r+22)L(r, F_{13,10a}, St)\\ &\quad= 2^{18}\cdot 3^{-6}\cdot 5^{-5}\cdot 13^{-1}\times c(r+22)(F_{13,10a}, F_{13,10a})\end{aligned}$$

*where $c(r+22)$ is given as in the introduction.*

By the above theorem, Zagier's conjecture is equivalent to the following conjecture.

Conjecture 2.3. *We should have*

$$(F_{13,10a}, F_{13,10a}) = 2^{15}\times 3^6\times 5^5\times 13\times(\Delta,\Delta)^3.$$

We will discuss problems on more general modular forms in Section 3 (cf. Theorem 3.2 and Conjecture 3.3).

All the proofs of the above theorems will be given in Section 5. By the way, by the numerical data above, we see that $\lambda(n, F_{13,10a}) \equiv \lambda(n, F_{13,10b}) \bmod 13$ for $n = 2, 4, 3, 9$. We can show that this is always the case for any $n$.

Theorem 2.4. *For any natural number $n$, we have*

$$\lambda(n, F_{13,10a}) \equiv \lambda(n, F_{13,10b}) \bmod 13.$$

Proof. Since $E_8$ is an integral lattice, we see that the coefficients of $\theta_{a_1,a_2,(9,10)}$ and $\theta_{b_1,b_2,(9,10)}$ are contained in $\mathbb{Z}$. Hence the prime divisors of denominators of coefficients of $f_{13,10a}$ and $f_{13,10b}$ are at most 2, 3, 5, 7 (and it is plausible that denominators are always one). We have

$$F_{13,10a} - F_{13,10b} = 13\times(5\times 191 f_{13,10a} + 7489 f_{13,10b})/(2^2\times 19\times 21800833).$$

So all the Fourier coefficients of $F_{13,10a} - F_{13,10b}$ are divisible by 13 at least in the ring of 13-adic integers $\mathbb{Z}_{13}$. Now we see in the table of the Fourier coefficients that the Fourier coefficient at $T_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ at $u_1^6u_2^4$ is $-\binom{10}{4} = -2\cdot 3\cdot 5\cdot 7$ for each $F_{13,10a}$ or $F_{13,10b}$. For any Hecke operator $T(n)$ and any form $F$, the Fourier coefficient of $T(n)F$ is a linear combination of Fourier coefficients of $F$. So for $F = F_{13,10a}$ or $F_{13,10b}$, the eigenvalues are given by $-\binom{10}{4}^{-1}$ times the coefficients of $T(n)F$ at $T_1$ at $u_1^6u_2^4$. Hence the difference is 13-integral. □

This can also be proved by the same argument as in the proof of [**11**, Theorem 5.2], using the pullback formula, but we omit the details here. This type of congruence is often found between a lift and a non-lift. It would be interesting to ask if there always exist congruences between lifts of Kim-Ramakrishnan-Shahidi type and non-lift in general.

## 3. Siegel Eisenstein series and critical values.

### 3.1. Pullback formula.

In this section, we quote a general theoretical formula in Kozima [**15**] to give critical values of the standard $L$ function of vector valued Siegel modular forms, restricting the situation to our cases (cf. also [**6**, Section 7].) The method is roughly as follows. We prepare an Eisenstein series $E_{4,l}$ of degree 4 of weight $l$ and differential operators $\mathbb{D}$ such that the restriction of $(\mathbb{D}E_{4,l})$ to the diagonal block $\mathfrak{H}_2\times\mathfrak{H}_2\subset\mathfrak{H}_4$ is a tensor of two vector valued Siegel modular forms in $A_{k,m}(\varGamma_2)$. Then the inner product of this restriction with a Hecke eigenform $F\in A_{k,m}(\varGamma_2)$ gives the value $L(l-2,F,St)$ up to $(F,F)$, powers of $\pi$, and elementary gamma factors, depending on $l$, $k$, $m$ and $\mathbb{D}$. Here the differential operator is needed to evaluate $L(s,F,St)$ at various $s$ starting from various $l$. For this purpose, Kozima used differential operators defined in Böcherer [**2**] or Böcherer, Satoh and Yamazaki [**3**], imitating their results for scalar valued cases. Put $V_1^{(m)} = \mathbb{C}[u_1,u_2]_m$ and $V_2^{(m)} = \mathbb{C}[u_3,u_4]_m$. Denote by $\mathrm{Hol}(\mathfrak{H}_2, V_i^{(m)})$ the space of $V_i^{(m)}$-valued holomorphic functions. Then we have $A_{k,m}(\varGamma_2)\otimes A_{k,m}(\varGamma_2)\subset \mathrm{Hol}(\mathfrak{H}_2,V_1^{(m)})\otimes\mathrm{Hol}(\mathfrak{H}_2,V_2^{(m)})$. We naturally identify elements in $V_1^{(m)}\otimes V_2^{(m)}$ with polynomials in $u_1$, $u_2$, $u_3$, $u_4$. The differential operator we need is described as a product $\widetilde{L}_{k,m}\mathcal{D}_{l,(k,m)}$ of $\mathrm{Hol}(\mathfrak{H}_4, V_1^{(m)}\otimes V_2^{(m)})$- valued operator $\widetilde{L}^{k,m}$ and a scalar-valued operator $\widetilde{\mathcal{D}}_l^{k-l} = \tilde{\mathcal{D}}_{k-1}\circ\cdots\tilde{\mathcal{D}}_{l+1}\circ\tilde{\mathcal{D}}_l$, both acting on holomorphic scalar valued functions on $\mathfrak{H}_4$. The definition of $\widetilde{\mathcal{D}}_\alpha$ is fairly complicated, and we do not use this operator in the actual calculation except for a comparison of normalization with other operators we use. So we refer to section 7 of [**6**] for the definitions of $\widetilde{\mathcal{D}}_\alpha$ and $L^{(k,m)}$ and do not repeat them here. (Note that there are typos in [**2**] or [**4**] and this is corrected in the paper quoted above.) For any scalar valued holomorphic function $f$ on $\mathfrak{H}_4$, we write

$$\mathcal{D}_{l,(k,m)}(f) = \big(\tilde{L}^{k,m}\tilde{\mathcal{D}}_l^{k-l}(f)\big)\big|_{\mathfrak{H}_2\times\mathfrak{H}_2},$$

where $|_{\mathfrak{H}_2\times\mathfrak{H}_2}$ means the restriction of functions of $Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix}\in\mathfrak{H}_4$ to the set $\mathfrak{H}_2\times\mathfrak{H}_2\cong\left\{\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}; Z_i\in\mathfrak{H}_2\right\}$. Then $\mathcal{D}_{l,(k,m)}$ maps holomorphic functions on $\mathfrak{H}_4$ to $\mathrm{Hol}(\mathfrak{H}_2,V_1^{(m)})\otimes\mathrm{Hol}(\mathfrak{H}_2,V_2^{(m)})$. In particular, it preserves automorphy after restriction and maps $A_l(\varGamma_4)$ to $A_{k,m}(\varGamma_2)\otimes A_{k,m}(\varGamma_2)$. The image is contained in $S_{k,m}(\varGamma_2)\otimes S_{k,m}(\varGamma_2)$ if $k-l>0$. For an even positive integer $l$, we define the Siegel Eisenstein series $E_{4,l}(Z,s)$

of degree 4 by

$$E_{4,l}(Z,s) = \zeta(1-l-2s)\zeta(3-2l-4s)\zeta(5-2l-4s) \times \sum_{g=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{4,\infty}\backslash\Gamma_4} \det(CZ+D)^{-l}(\det(\mathrm{Im}(g(Z))))^s$$

($Z \in \mathfrak{H}_4, s \in \mathbb{C}$), where $\zeta(*)$ is Riemann's zeta function, and $\Gamma_{4\infty} = \left\{\left(\begin{smallmatrix} * & * \\ 0_4 & * \end{smallmatrix}\right) \in \Gamma_4\right\}$. This series converges for $2Re(s) + l > 5$ and is continued meromorphically to the whole plane as a function of $s$. Furthermore, for $l \geq 4$, $E_{4,l}(Z,0)$ is a holomorphic Siegel modular form of weight $l$ as a function of $Z$ (cf. [**20**]). From now on we assume that $E_{4,l}(Z,0)$ is holomorphic as a function of $Z$, and write $E_{4,l}(Z) = E_{4,l}(Z,0)$. For an integer $k \geq l$ put

$$F_{l,(k,m)}(Z_1,Z_2) = (2\pi\sqrt{-1})^{-2(k-l)-m}\mathcal{D}_{l,(k,m)}(E_{4,l})\big|_{\mathfrak{H}_2\times\mathfrak{H}_2}$$

For $F \in S_{k,m}(\Gamma_4)$, we define $\Lambda(r,F,St)$ by

$$\Lambda(r,F,St) = 2^{14-6k-r-2m}(-1)^{r/2} \times \frac{\Gamma(r+1)\Gamma(2r)\Gamma(2k+m-3)}{(k-2)(k)_m m!} \times \frac{L(r,F,St)}{\pi^{2k+m+3r-3}(F,F)},$$

where $(k)_m = k(k+1)\cdots(k+m-1)$. (This is the same as the definition in [**15**] or [**6**] for $n=2$, though apparently different looking.) Then the following result is a special case of the pullback formula for the Siegel Eisenstein series in [**2**], [**3**] and [**15**].

Proposition 3.1. *Let $k$ be an odd positive integer and let $F_1,\ldots,F_d$ be an orthogonal basis of $S_{k,m}(\Gamma_2)$ consisting of Hecke eigenforms. Assume that $F_1,\ldots,F_d$ belong to $S_{k,m}(\Gamma_2)(\mathbb{R})$. Let $l$ be an even integer such that $4 \leq l \leq k$. Then we have*

$$F_{l,(k,m)}(Z_1,Z_2) = \sum_{i=1}^{d} \Lambda(F_i, l-2, St)F_i(Z_1)\otimes F_i(Z_2).$$

Here note that each product term depending on $i$ in the above sum does not change even if we replace $F_i$ by a constant multiple. Also we can take a basis $F_1,\ldots,F_d$ of $S_{k,m}(\Gamma_2)$ so that $F_1,\ldots,F_d$ satisfy the condition in the above proposition.

For a Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$ or a Hecke eigenform $F \in S_{k,j}(\Gamma_2)$, we denote by $\mathbb{Q}(f)$ or $\mathbb{Q}(F)$, the field generated over $\mathbb{Q}$ by all the Hecke eigenvalues of $f$ or $F$, respectively.

Theorem 3.2. *For any primitive form $f \in S_k(\Gamma_1)$, there exists a constant $c(f)$ depending only on $f$ such that $L(l,f,Sym(4))/\pi^{-3k+3l+3}c(f)$ belongs to $\mathbb{Q}(f)$ for any even integer $l$ such that $2k \leq l \leq 3k-4$.*

Proof. Let $F \in S_{k+1,k-2}(\Gamma_2)$ be a Siegel modular form obtained as a lifting from $f$ by Ramakrishnan-Shahidi's theorem such that

$$L(s,f,Sym(3)) = L(s,F,Sp).$$

Then as we stated before, we have $L(s,f,Sym(4)) = L(s-2k+2,F,St)$. By Takei [**22**], we have a Hecke eigenform $G \in S_{k+1,k-2}(\Gamma_2)(\mathbb{Q}(F))$ such that eigenvalues of $G$ are the same as those of $F$. Moreover, we have $\mathbb{Q}(G) = \mathbb{Q}(F) = \mathbb{Q}(f)$. So by Theorem of Kozima [**15**], we can prove

$$L(r,F,St)/(\pi^{3k+3r-3}(G,G)) \in \mathbb{Q}(f)$$

for any even integer $r$ such that $2 \le r \le k-2$. This proves the assertion. □

According to a part of Deligne's general conjecture in [**5**], the critical values $L(l,f,Sym(4))/\pi^{3l-3k+3}(f,f)^3$ should belong to $\mathbb{Q}(f)$ for any even integer $l$ such that $2k \le l \le 3k-4$. Hence, taking Theorem 3.2 into account, we propose the following conjecture.

Conjecture 3.3. *For a primitive form $f \in S_k(\Gamma_1)$, there should exist a Hecke eigenform $G \in S_{k+1,k-2}(\Gamma_2)(\mathbb{Q}(f))$ such that $L(s,G,Sp) = L(s,f,Sym(3))$ and that $(G,G)/(f,f)^3 \in \mathbb{Q}(f)$.*

### 3.2. Calculation of critical values.

The above differential operators are convenient to see the concrete relation between actions and values of $L$ functions, but the operators themselves are sometimes very complicated for practical use since coefficients are not constant and too many terms appear after iterations which should become much simpler under the restriction to $\mathfrak{H}_2 \times \mathfrak{H}_2$. More direct characterization of linear holomorphic differential operators with constant coefficients which preserve automorphy after a restriction of the domain has been given in [**7**] and we use them here. We extract the necessary part from [**7**]. We define $2 \times 2$ matrices $R = (r_{ij})$, $S = (s_{ij})$, $T = (t_{ij})$, and assume that $R$, $S$ are symmetric. Here $r_{ij}$, $s_{ij}$ $(1 \le i \le j \le 2)$ and $t_{ij}$ $(1 \le i,j \le 2)$ are independent variables. As before, we put $V_1^{(m)} = \mathbb{C}[u_1,u_2]_m$ and $V_2^{(m)} = \mathbb{C}[u_3,u_4]_m$. For any non-negative integers $k$, $l$, $m$ with $l \le k$, we denote by $\boldsymbol{P}_{l,(k,m)}$ the linear space of $V_1^{(m)} \otimes V_2^{(m)}$ valued polynomials $P$ in $r_{ij}$, $s_{ij}$ and $t_{ij}$ satisfying the following conditions (1) and (2).
(1) $P(AR\,{}^tA, BS\,{}^tB, AT\,{}^tB) = \det(AB)^{k-l} Sym_m(A) \otimes Sym_m(B) P(R,S,T)$. (2) For any $2 \times 2l$ matrices $X = (x_{ij})$, $Y = (y_{ij})$ with variable components, put $P^*(X,Y) = P(X\,{}^tX, Y\,{}^tY, X\,{}^tY)$. Then $P^*$ is pluriharmonic with respect to each $X$ or $Y$, i.e.

$$\sum_{\mu=1}^{2l} \frac{\partial^2 P^*}{\partial x_{i\mu}\partial x_{j\mu}} = \sum_{\mu=1}^{2l} \frac{\partial^2 P^*}{\partial y_{i\mu}\partial y_{j\mu}} = 0.$$

for any $i$, $j$ with $1 \le i,j \le 2$.

When $l \geq 4$, we have $\dim_{\mathbb{C}} \boldsymbol{P}_{l,(k,m)} = 1$ for any $(l, k, m)$. We write $Z_4 = (z_{ij}) \in \mathfrak{H}_4$ and put $\partial_{ij} = ((1+\delta_{ij})/2) \times (\partial/\partial z_{ij})$. Define a differential operator associated with $P$ by

$$\mathbb{D}_P = P\Bigl( \begin{pmatrix} \partial_{11} & \partial_{12} \\ \partial_{12} & \partial_{22} \end{pmatrix}, \begin{pmatrix} \partial_{33} & \partial_{34} \\ \partial_{34} & \partial_{44} \end{pmatrix}, \begin{pmatrix} \partial_{13} & \partial_{14} \\ \partial_{23} & \partial_{24} \end{pmatrix} \Bigr).$$

Now we explain how $\mathbb{D}_P$ behaves well on Siegel modular forms. For any $V_1^{(m)} \otimes V_2^{(m)}$-valued holomorphic function $G(Z_1, Z_2)$ on $\mathfrak{H}_2 \times \mathfrak{H}_2$, and any $g_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right) \in Sp(2, \mathbb{R})$ (matrix size 4), we define

$$(G|_{k,m}[g_1, g_2])(Z_1, Z_2) = \rho_{k,m}(C_1 Z_1 + D_1)^{-1} \otimes \rho_{k,m}(C_2 Z_2 + D_2)^{-1} G(g_1 Z_1, g_2 Z_2).$$

We identify $Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R})$ with a subgroup of $Sp(4, \mathbb{R})$ (size 8) by embedding

$$\iota(g_1, g_2) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Let $\mathbb{D}$ be a $V_1^{(m)} \otimes V_2^{(m)}$ valued linear holomorphic homogeneous differential operator with constant coefficients on functions on $\mathfrak{H}_4$. We consider the following condition for $\mathbb{D}$.

Condition 3.4. *For any scalar valued holomorphic function $F(Z)$ of $\mathfrak{H}_4$, we have*

$$\bigl(\mathbb{D}(F|_{det^l}[\iota(g_1, g_2)])\bigr)\big|_{\mathfrak{H}_2 \times \mathfrak{H}_2} = \bigl((\mathbb{D}F)\big|_{\mathfrak{H}_2 \times \mathfrak{H}_2}|_{k,m}[(g_1, g_2)]\bigr).$$

Then an operator $\mathbb{D}$ which satisfies Condition 3.4 is equal to $\mathbb{D}_P$ for some $P \in \boldsymbol{P}_{l,(k,m)}$ given above. For each $(l,(k,m))$ and for a fixed non-zero $\phi = \phi_{l,(k,m)} \in \boldsymbol{P}_{l,(k,m)}$, we write $(\Phi_{l,(k,m)}F)(Z_1, Z_2) = (\mathbb{D}_\phi F)|_{\mathfrak{H}_2 \times \mathfrak{H}_2}$. (A concrete suitable choice of $\phi_{l,(k,m)}$ will be given later for special cases in next section.) Since Böcherer's operator $\mathcal{D}_{l,(k,m)}$ gives an operator of the above type, it differs from $\Phi_{l,(k,m)}$ only by a constant. So we can use $\Phi_{l,(k,m)}$ instead. We define the constant $C(l,(k,m))$ by $\Phi_{l,(k,m)} = C(l,(k,m))\mathcal{D}_{l,(k,m)}$. We also put $\widetilde{F}_{l,(k,m)}(Z_1, Z_2) = \Phi_{l,(k,m)}(E_{4,l})\big|_{\mathfrak{H}_2 \times \mathfrak{H}_2} = C(l,(k,m))F_{l,(k,m)}$ and $\widetilde{\Lambda}(r, F, St) = C(l,(k,m))\Lambda(r, F, St)$ for any $F \in A_{k,m}(\Gamma_2)$. Then the same pullback formula in 3.1 holds for $\widetilde{F}$ and $\widetilde{\Lambda}$. We explain shortly in a general setting how to obtain $\widetilde{\Lambda}(r, F, St)$ from Fourier coefficients. We write the Fourier expansion of $\widetilde{F}$ by

$$\widetilde{F}_{l,(k,m)}(Z_1, Z_2) = \sum_{R,S \in \mathcal{H}_2(\mathbb{Z})_{>0}} \epsilon_{l,(k,m)}(R, S; U) \exp\bigl(2\pi\sqrt{-1}\,\mathrm{tr}(AZ_1 + BZ_2)\bigr),$$

where $\epsilon_{l,(k,m)}(R, S; U)$ is a polynomial in $U = (u_1, u_2, u_3, u_4)$. Then for fixed $R$ and $S$, we have

$$\epsilon_{l,(k,m)}(R,S;U)=\sum_{T\in M_2(\mathbb{Z})} c_{4,l}\Big(\begin{pmatrix} R & T/2 \\ {}^tT/2 & S\end{pmatrix}\Big)\phi_{l,(k,m)}(R,S,T/2),$$

where $c_{4,l}(\mathcal{T})$ denotes the Fourier coefficient of $E_{4,l}$ at $\mathcal{T}\in\mathcal{H}_4(\mathbb{Z})_{\geq 0}$ and is regarded as zero if $\mathcal{T}$ is not positive semi-definite. For a fixed $S\in\mathcal{H}_2(\mathbb{Z})_{>0}$, we write

$$G_{l,(k,m),S}(Z_1)=\sum_{R\in\mathcal{H}_2(\mathbb{Z})_{>0}}\epsilon_{l,(k,m)}(R,S;U)\exp(2\pi\sqrt{-1}\mathrm{tr}(RZ_1)).$$

Then we have

$$\widetilde{F}_{l,(k,m)}(Z_1,Z_2)=\sum_{S\in\mathcal{H}_2(\mathbb{Z})_{>0}} G_{l,(k,m),S}(Z_1)\exp(2\pi\sqrt{-1}\mathrm{tr}(SZ_2)).$$

Assume that $A_{k,m}(\Gamma_2)$ is spanned by orthogonal Hecke eigenforms $F_i\in A_{k,m}(\Gamma_2)(\mathbb{R})$ $(1\leq i\leq d)$. For each $S\in\mathcal{H}_2(\mathbb{Z})_{>0}$, we denote by $c_i(S,v)$ the Fourier coefficient of $F_i(Z_2)$ at $S\in\mathcal{H}_2(\mathbb{Z})_{>0}$, which is a polynomial in $v=(u_3,u_4)$. Then comparing the Fourier expansions of both sides of the pullback formula, for any $S\in\mathcal{H}_2(\mathbb{Z})_{>0}$ we have

$$G_{l,(k,m),S}(Z_1)=\sum_{i=1}^{d} c_{F_i}(S;v)\widetilde{\Lambda}(l-n,F_i,St)F_i(Z_1), \tag{1}$$

where $v=(u_3,u_4)$. We assume that $c_{F_i}(S;v)\neq 0$ for any $1\leq i\leq d$. Then by comparing enough Fourier coefficients of both sides of (1) at $\exp(2\pi\sqrt{-1}\mathrm{tr}(AZ_1))$ for various $A$, we have a simultaneous equation which has the unique solution, regarding $\widetilde{\Lambda}(l-n,F_i,St)$ as unknowns. Indeed take matrices $A_1,\ldots,A_d\in\mathcal{H}_2(\mathbb{Z})_{>0}$ such that

$$\det(c_{F_i}(A_j;u)_{1\leq i,j\leq d})\neq 0,$$

where $u=(u_1,u_2)$. Then solving the simultaneous equation

$$\epsilon_{l,(k,m)}(A_i,S;U)=\sum_{i=1}^{d} c_{F_j}(A_i;U_1)\big(c_{F_j}(S;U_2)\Lambda(l-n,F_j,St)\big)$$

for $i=1,\ldots,d$ by Cramer's formula, we have

$$\widetilde{\Lambda}(l-n,F_i,St)=\frac{\det(K_i)}{c_{F_i}(S;v)\times\det(K)}.$$

where $K$ is the $d\times d$ matrix $(c_{F_i}(A_j;U_1))_{1\leq i,j\leq d}$ and $K_i$ is a matrix obtained by replacing the $i$-th column of $K$ by the column vector $(\epsilon_{l,(k,m)}(A_j,S;U))_{1\leq j\leq d}$. In order to calculate concrete values, we need explicit $\phi$, values of $\epsilon_{l,(k,m)}(A_i,S;U)$ and the Fourier coefficients $c_{F_i}(A_j,u)$. Still this is not very easy and will be explained in the following section.

## 4. Construction of differential operators.

We construct non-zero polynomials $\phi$ in $\boldsymbol{P}_{12,(13,10)}$, $\boldsymbol{P}_{10,(13,10)}$, $\boldsymbol{P}_{8,(13,10)}$, $\boldsymbol{P}_{6,(13,10)}$, $\boldsymbol{P}_{4,(13,10)}$. The first two cases has already been given in [**6**, Section 8]. The latter three is much more complicated. So here we give a polynomial in $\boldsymbol{P}_{8,(13,10)}$ in details, which is the simplest among three, and in the other cases we give only a short hint for the polynomials. We explain a slightly more general case $\boldsymbol{P}_{l,(l+5,10)}$ first. Since any $\phi \in \boldsymbol{P}_{l,(k,m)}$ is $V_1^{(m)} \otimes V_2^{(m)}$-valued, we can regard $\phi$ as a polynomial of $r_{ij}$, $s_{ij}$, $t_{ij}$ and $u_i$ $(1 \leq i \leq 4)$. For $\boldsymbol{P}_{l,(l+5,10)}$, the polynomial $\phi$ is given by a certain linear combination of the following 134 polynomials $h_i = h_i(R,S,T)$ over the polynomials in $l$. Here as for the definitions of the polynomials $P_i$ $(i = 0,1,2)$ ,$Q$, $Q_i$ $(i = 1,2)$, $m$, $s$ in $r_{ij}$, $s_{ij}$, $t_{ij}$, confer [**6**, Section 8].

| | | |
|---|---|---|
| $h_1 = QQ_1^2m^2$ | $h_2 = QQ_1^2ms^2$ | $h_3 = QQ_1^2s^4$ |
| $h_4 = QQ_2^2m^2$ | $h_5 = QQ_2^2ms^2$ | $h_6 = QQ_2^2s^4$ |
| $h_7 = \det(T)Q_1^2m^3$ | $h_8 = \det(T)Q_1^2m^2s^2$ | $h_9 = \det(T)Q_1^2ms^4$ |
| $h_{10} = \det(T)Q_1^2s^6$ | $h_{11} = \det(T)Q_2^2m^3$ | $h_{12} = \det(T)Q_2^2m^2s^2$ |
| $h_{13} = \det(T)Q_2^2ms^4$ | $h_{14} = \det(T)Q_2^2s^6$ | $h_{15} = \det(T)Q_1P_0m^4$ |
| $h_{16} = \det(T)Q_1P_0m^3s^2$ | $h_{17} = \det(T)Q_1P_0m^2s^4$ | $h_{18} = \det(T)Q_1P_0ms^6$ |
| $h_{19} = \det(T)Q_1P_0s^8$ | $h_{20} = \det(T)Q_1P_1m^4$ | $h_{21} = \det(T)Q_1P_1m^3s^2$ |
| $h_{22} = \det(T)Q_1P_1m^2s^4$ | $h_{23} = \det(T)Q_1P_1ms^6$ | $h_{24} = \det(T)Q_1P_1s^8$ |
| $h_{25} = \det(T)Q_1P_2m^4$ | $h_{26} = \det(T)Q_1P_2m^3s^2$ | $h_{27} = \det(T)Q_1P_2m^2s^4$ |
| $h_{28} = \det(T)Q_1P_2ms^6$ | $h_{29} = \det(T)Q_1P_2s^8$ | $h_{30} = \det(T)Q_2P_0m^4$ |
| $h_{31} = \det(T)Q_2P_0m^3s^2$ | $h_{32} = \det(T)Q_2P_0m^2s^4$ | $h_{33} = \det(T)Q_2P_0ms^6$ |
| $h_{34} = \det(T)Q_2P_0s^8$ | $h_{35} = \det(T)Q_2P_1m^4$ | $h_{36} = \det(T)Q_2P_1m^3s^2$ |
| $h_{37} = \det(T)Q_2P_1m^2s^4$ | $h_{38} = \det(T)Q_2P_1ms^6$ | $h_{39} = \det(T)Q_2P_1s^8$ |
| $h_{40} = \det(T)Q_2P_2m^4$ | $h_{41} = \det(T)Q_2P_2m^3s^2$ | $h_{42} = \det(T)Q_2P_2m^2s^4$ |
| $h_{43} = \det(T)Q_2P_2ms^6$ | $h_{44} = \det(T)Q_2P_2s^8$ | $h_{45} = QQ_1P_0m^3$ |
| $h_{46} = QQ_1P_0m^2s^2$ | $h_{47} = QQ_1P_0ms^4$ | $h_{48} = QQ_1P_0s^6$ |
| $h_{49} = QQ_1P_1m^3$ | $h_{50} = QQ_1P_1m^2s^2$ | $h_{51} = QQ_1P_1ms^4$ |
| $h_{52} = QQ_1P_1s^6$ | $h_{53} = QQ_1P_2m^3$ | $h_{54} = QQ_1P_2m^2s^2$ |
| $h_{55} = QQ_1P_2ms^4$ | $h_{56} = QQ_1P_2s^6$ | $h_{57} = QQ_2P_0m^3$ |
| $h_{58} = QQ_2P_0m^2s^2$ | $h_{59} = QQ_2P_0ms^4$ | $h_{60} = QQ_2P_0s^6$ |
| $h_{61} = QQ_2P_1m^3$ | $h_{62} = QQ_2P_1m^2s^2$ | $h_{63} = QQ_2P_1ms^4$ |
| $h_{64} = QQ_2P_1s^6$ | $h_{65} = QQ_2P_2m^3$ | $h_{66} = QQ_2P_2m^2s^2$ |
| $h_{67} = QQ_2P_2ms^4$ | $h_{68} = QQ_2P_2s^6$ | $h_{69} = \det(T)P_0^2m^5$ |
| $h_{70} = \det(T)P_0^2m^4s^2$ | $h_{71} = \det(T)P_0^2m^3s^4$ | $h_{72} = \det(T)P_0^2m^2s^6$ |
| $h_{73} = \det(T)P_0^2ms^8$ | $h_{74} = \det(T)P_0^2s^10$ | $h_{75} = \det(T)P_1^2m^5$ |
| $h_{76} = \det(T)P_1^2m^4s^2$ | $h_{77} = \det(T)P_1^2m^3s^4$ | $h_{78} = \det(T)P_1^2m^2s^6$ |
| $h_{79} = \det(T)P_1^2ms^8$ | $h_{80} = \det(T)P_1^2s^10$ | $h_{81} = \det(T)P_2^2m^5$ |
| $h_{82} = \det(T)P_2^2m^4s^2$ | $h_{83} = \det(T)P_2^2m^3s^4$ | $h_{84} = \det(T)P_2^2m^2s^6$ |
| $h_{85} = \det(T)P_2^2ms^8$ | $h_{86} = \det(T)P_2^2s^10$ | $h_{87} = \det(T)P_0P_1m^5$ |
| $h_{88} = \det(T)P_0P_1m^4s^2$ | $h_{89} = \det(T)P_0P_1m^3s^4$ | $h_{90} = \det(T)P_0P_1m^2s^6$ |

| | | |
|---|---|---|
| $h_{91} = \det(T)P_0P_1ms^8$ | $h_{92} = \det(T)P_0P_1s^10$ | $h_{93} = \det(T)P_0P_2m^5$ |
| $h_{94} = \det(T)P_0P_2m^4s^2$ | $h_{95} = \det(T)P_0P_2m^3s^4$ | $h_{96} = \det(T)P_0P_2m^2s^6$ |
| $h_{97} = \det(T)P_0P_2ms^8$ | $h_{98} = \det(T)P_0P_2s^10$ | $h_{99} = \det(T)P_1P_2m^5$ |
| $h_{100} = \det(T)P_1P_2m^4s^2$ | $h_{101} = \det(T)P_1P_2m^3s^4$ | $h_{102} = \det(T)P_1P_2m^2s^6$ |
| $h_{103} = \det(T)P_1P_2ms^8$ | $h_{104} = \det(T)P_1P_2s^10$ | $h_{105} = QP_0^2m^4$ |
| $h_{106} = QP_0^2m^3s^2$ | $h_{107} = QP_0^2m^2s^4$ | $h_{108} = QP_0^2ms^6$ |
| $h_{109} = QP_0^2s^8$ | $h_{110} = QP_1^2m^4$ | $h_{111} = QP_1^2m^3s^2$ |
| $h_{112} = QP_1^2m^2s^4$ | $h_{113} = QP_1^2ms^6$ | $h_{114} = QP_1^2s^8$ |
| $h_{115} = QP_2^2m^4$ | $h_{116} = QP_2^2m^3s^2$ | $h_{117} = QP_2^2m^2s^4$ |
| $h_{118} = QP_2^2ms^6$ | $h_{119} = QP_2^2s^8$ | $h_{120} = QP_0P_1m^4$ |
| $h_{121} = QP_0P_1m^3s^2$ | $h_{122} = QP_0P_1m^2s^4$ | $h_{123} = QP_0P_1ms^6$ |
| $h_{124} = QP_0P_1s^8$ | $h_{125} = QP_0P_2m^4$ | $h_{126} = QP_0P_2m^3s^2$ |
| $h_{127} = QP_0P_2m^2s^4$ | $h_{128} = QP_0P_2ms^6$ | $h_{129} = QP_0P_2s^8$ |
| $h_{130} = QP_2P_1m^4$ | $h_{131} = QP_2P_1m^3s^2$ | $h_{132} = QP_2P_1m^2s^4$ |
| $h_{133} = QP_2P_1ms^6$ | $h_{134} = QP_2P_1s^8$ | |

The coefficients of polynomials for general $l$ are polynomials in $l$ but too complicated to write them down here. Instead, we give here only the case $l = 8$, where the coefficients are numbers. We have $\phi_{8,(13,10)} = \sum_{i=1}^{134} c_ih_i \in \boldsymbol{P}_{8,(13,10)}$ if we define $c_i$ by the following table.

| | | | |
|---|---|---|---|
| $c_1 = 4232592$ | $c_2 = -112869120$ | $c_3 = 474050304$ | $c_4 = 4232592$ |
| $c_5 = -112869120$ | $c_6 = 474050304$ | $c_7 = 16633344$ | $c_8 = -1241560320$ |
| $c_9 = 15621086208$ | $c_{10} = -44244695040$ | $c_{11} = 16633344$ | $c_{12} = -1241560320$ |
| $c_{13} = 15621086208$ | $c_{14} = -44244695040$ | $c_{15} = 241922240$ | $c_{16} = -15308160000$ |
| $c_{17} = 228273454080$ | $c_{18} = -1127996620800$ | $c_{19} = 1691994931200$ | $c_{20} = 63497448$ |
| $c_{21} = -4817272320$ | $c_{22} = 70234112256$ | $c_{23} = -293668085760$ | $c_{24} = 317249049600$ |
| $c_{25} = 18659200$ | $c_{26} = -1450208256$ | $c_{27} = 26011991040$ | $c_{28} = -159308685312$ |
| $c_{29} = 301447372800$ | $c_{30} = 241922240$ | $c_{31} = -15308160000$ | $c_{32} = 228273454080$ |
| $c_{33} = -1127996620800$ | $c_{34} = 1691994931200$ | $c_{35} = 63497448$ | $c_{36} = -4817272320$ |
| $c_{37} = 70234112256$ | $c_{38} = -293668085760$ | $c_{39} = 317249049600$ | $c_{40} = 18659200$ |
| $c_{41} = -1450208256$ | $c_{42} = 26011991040$ | $c_{43} = -159308685312$ | $c_{44} = 301447372800$ |
| $c_{45} = 104415360$ | $c_{46} = -4644998400$ | $c_{47} = 43029181440$ | $c_{48} = -98699704320$ |
| $c_{49} = 18946704$ | $c_{50} = -638144640$ | $c_{51} = 4360220928$ | $c_{52} = -7049978880$ |
| $c_{53} = 6397440$ | $c_{54} = -265676544$ | $c_{55} = 2396298240$ | $c_{56} = -5514958848$ |
| $c_{57} = 104415360$ | $c_{58} = -4644998400$ | $c_{59} = 43029181440$ | $c_{60} = -98699704320$ |
| $c_{61} = 18946704$ | $c_{62} = -638144640$ | $c_{63} = 4360220928$ | $c_{64} = -7049978880$ |
| $c_{65} = 6397440$ | $c_{66} = -265676544$ | $c_{67} = 2396298240$ | $c_{68} = -5514958848$ |
| $c_{69} = 255963456$ | $c_{70} = -17620149120$ | $c_{71} = 294466486272$ | $c_{72} = -1757257592832$ |
| $c_{73} = 3996330885120$ | $c_{74} = -2797431619584$ | $c_{75} = 53189556$ | $c_{76} = -4003398000$ |
| $c_{77} = 61987994880$ | $c_{78} = -297417076992$ | $c_{79} = 444987952640$ | $c_{80} = -116559650816$ |
| $c_{81} = 2107392$ | $c_{82} = -171664640$ | $c_{83} = 4234374144$ | $c_{84} = -43949002752$ |
| $c_{85} = 197032935424$ | $c_{86} = -307800457216$ | $c_{87} = 364827904$ | $c_{88} = -23305074240$ |
| $c_{89} = 357128131584$ | $c_{90} = -1889998623744$ | $c_{91} = 3483360993280$ | $c_{92} = -1398715809792$ |
| $c_{93} = 97800192$ | $c_{94} = -6029168320$ | $c_{95} = 104368750080$ | $c_{96} = -750749819904$ |
| $c_{97} = 2372821463040$ | $c_{98} = -2677111980032$ | $c_{99} = 26151104$ | $c_{100} = -2036539344$ |
| $c_{101} = 40090197504$ | $c_{102} = -291216717312$ | $c_{103} = 762601656320$ | $c_{104} = -472818583552$ |
| $c_{105} = 124607280$ | $c_{106} = -7987203840$ | $c_{107} = 119497142016$ | $c_{108} = -588169666560$ |
| $c_{109} = 874197381120$ | $c_{110} = 17538577$ | $c_{111} = -671331360$ | $c_{112} = 5674929120$ |

| | | | |
|---|---|---|---|
| $c_{113} = -14351742720$ | $c_{114} = 9106222720$ | $c_{115} = 932960$ | $c_{116} = -57576960$ |
| $c_{117} = 922053888$ | $c_{118} = -5110118400$ | $c_{119} = 8719284224$ | $c_{120} = 163353680$ |
| $c_{121} = -8081566080$ | $c_{122} = 90033092352$ | $c_{123} = -311206210560$ | $c_{124} = 291399127040$ |
| $c_{125} = 39050564$ | $c_{126} = -1602330240$ | $c_{127} = 22595529600$ | $c_{128} = -143673707520$ |
| $c_{129} = 299624102400$ | $c_{130} = 9729440$ | $c_{131} = -461438208$ | $c_{132} = 5251018752$ |
| $c_{133} = -19116804096$ | $c_{134} = 18799943680$ | | |

By the way, gcd of coefficients $c_i$ is one.

Next, a polynomial in $\boldsymbol{P}_{6,(13,10)}$ is a linear combination of the 270 polynomials in the following set.

$$\{Q_1^3, Q_2^3\} \times Q \times \{m, s^2\}$$

$$\{Q_1^3, Q_2^3\} \times \det(T) \times \{m^2, ms^2, s^4\}$$

$$\{Q_1^2, Q_2^2\} \times Q \times \{P_0, P_1, P_2\} \times \{m^2, ms^2, s^4\}$$

$$\{Q_1^2, Q_2^2\} \times \det(T) \times \{P_0, P_1, P_2\} \times \{m^3, m^2s^2, ms^4, s^6\}$$

$$\{Q_1, Q_2\} \times Q \times \{P_0^2, P_1^2, P_2^2, P_0P_1, P_0P_2, P_1P_2\} \times \{m^3, m^2s^2, ms^4, s^6\}$$

$$\{Q_1, Q_2\} \times \det(T) \times \{P_0^2, P_1^2, P_2^2, P_0P_1, P_0P_2, P_1P_2\} \times \{m^4, m^3s^2, m^2s^4, ms^6, s^8\}$$

$$Q \times \{P_0^3, P_1^3, P_2^3, P_0^2P_1, P_0^2P_2, P_1^2P_0, P_1^2P_2, P_2^2P_0, P_2^2P_1, P_0P_1P_2\} \\ \times \{m^4, m^3s^2, m^2s^4, ms^6, s^8\}$$

$$\det(T) \times \{P_0^3, P_1^3, P_2^3, P_0^2P_1, P_0^2P_2, P_1^2P_0, P_1^2P_2, P_2^2P_0, P_2^2P_1, P_0P_1P_2\} \\ \times \{m^5, m^4s^2, m^3s^4, m^2s^6, ms^8, s^{10}\}$$

Here by the product $\times$ of the sets we mean the set obtained by taking product of each element in each set. For example, we mean

$$\{Q_1^3, Q_2^3\} \times Q \times \{m, s^2\} = \{QQ_1^3m, QQ_2^3m, QQ_1^3s^2, QQ_2^3s^2\}.$$

We denote by $\phi_{6,(13,10)}$ the unique polynomial in $\boldsymbol{P}_{6,(13,10)}$ which is a linear combination of the above 270 polynomials such that the coefficient of $\det(T)P_0^3s^{10}$ is $-2^{20} \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31$. Then all the coefficients of $\phi_{6,(13,10)}$ are integers whose gcd is one.

Next, a polynomial in $\boldsymbol{P}_{4,(13,10)}$ is a linear combination of the following 465 polynomials.

$$\{Q_1^4, Q_2^4\} \times \det(T) \times \{m, s^2\}$$

$$\{Q_1^4, Q_2^4\} \times Q$$

$$\{Q_1^3, Q_2^3\} \times \det(T) \times \{P_0, P_1, P_2\} \times \{m^2, ms^2, s^4\}$$

$$\{Q_1^3, Q_2^3\} \times Q \times \{m, s^2\} \times \{P_0, P_1, P_2\}$$

$$\{Q_1^2, Q_2^2\} \times \det(T) \times \{P_0^2, P_1^2, P_2^2, P_0P_1, P_0P_2, P_1P_2\} \times \{m^3, m^2s^2, ms^4, s^6\}$$

$$\{Q_1^2, Q_2^2\} \times Q \times \{P_0^2, P_1^2, P_2^2, P_0P_1, P_0P_2, P_1P_2\} \times \{m^2, ms^2, s^4\}$$

$$\{Q_1, Q_2\} \times \det(T) \times \{P_0^3, P_1^3, P_2^3, P_0^2P_1, P_0^2P_2, P_1^2P_0, P_1^2P_2, P_2^2P_0, P_2^2P_1, P_0P_1P_2\} \times \{m^4, m^3s^2, m^2s^4, ms^6, s^8\}$$

$$\{Q_1, Q_2\} \times Q \times \{P_0^3, P_1^3, P_2^3, P_0^2P_1, P_0^2P_2, P_1^2P_0, P_1^2P_2, P_2^2P_0, P_2^2P_1, P_0P_1P_2\} \times \{m^3, m^2s^2, ms^4, s^6\}$$

$$\det(T) \times \{P_0^4, P_1^4, P_2^4, P_0^3P_1, P_0^3P_2, P_1^3P_0, P_1^3P_2, P_2^3P_0, P_2^3P_1, P_0^2P_1^2, P_0^2P_1P_2, P_0^2P_2^2, P_1^2P_2^2, P_1^2P_0P_2, P_2^2P_0P_1\} \times \{m^5, m^4s^2, m^3s^4, m^2s^6, ms^8, s^{10}\}$$

$$Q \times \{P_0^4, P_1^4, P_2^4, P_0^3P_1, P_0^3P_2, P_1^3P_0, P_1^3P_2, P_2^3P_0, P_2^3P_1, P_0^2P_1^2, P_0^2P_1P_2, P_0^2P_2^2, P_1^2P_2^2, P_1^2P_0P_2, P_2^2P_0P_1\} \times \{m^4, m^3s^2, m^2s^4, ms^6, s^8\}$$

We denote by $\phi_{4,(13,10)}$ the unique polynomial in $\boldsymbol{P}_{4,(13,10)}$ which is a linear combination of the above 465 polynomials such that the coefficient of $\det(T)P_0^4s^{10}$ is $-2^{20}\cdot 3\cdot 5^2\cdot 7^2\cdot 13\cdot 17\cdot 19\cdot 29\cdot 31$. Then all the coefficients of $\phi_{4,(13,10)}$ are integers whose gcd is one.

## 5. Proof of Theorem 2.2.

For $(k,m)=(13,10)$ and $l=4$, 6, 8, 10, 12, we denote by $\Phi_{l,(k,m)}$ the differential operators corresponding to $\phi_{l,(k,m)}$ defined in the last section. (As for $l=10$ and 12, see the definition in [**6**].) In these cases, explicit values of $C(l,(k,m))$ can be calculated for each $l$ by comparing the actions of both operators $\Phi_{l,(k,m)}$ and $\mathcal{D}_{l,(k,m)}$ on the polynomial. $\det(Z_{12})^{k-l}({}^tuZ_{12}v)^m$ where $Z=\left(\begin{smallmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2\end{smallmatrix}\right)\in\mathfrak{H}_4$ and $u=(u_1,u_2)$ and $v=(u_3,u_4)$. The image of these actions are constant multiples of $(u,u)^m(v,v)^m$. The action of $\mathcal{D}_{l,(m,k)}$ has been given in [**6**, Section 7, Corollary 7.6]. The actions of $\Phi_{l,(k,m)}$ depends only on the coefficients of $\det(T)P_0^{(k-l-1)/2}s^m$ and the above constants are calculated by using Lemma 7.5 in [**6**]. We have the following values.

$$C(4,(13,10)) = -2^{23}\times 3^8\times 5^5\times 7^5\times 11^3\times 13\times 17^2\times 19^2$$

$$C(6,(13,10)) = -2^{23}\times 3^9\times 5^5\times 7^4\times 11^3\times 13\times 17^2\times 19^2$$

$$C(8,(13,10)) = -2^{21}\times 3^9\times 5^4\times 7^5\times 11^2\times 13^2\times 17^2\times 19^2$$

$$C(10,(13,10)) = -2^{20}\times 3^{13}\times 5^6\times 7^4\times 11^2\times 13\times 17^3\times 19^2\times 41$$

$$C(12,(13,10)) = -2^{17}\times 3^{11}\times 5^5\times 7^4\times 11^2\times 13\times 17^2\times 19^2.$$

For $\Phi_{l,(13,10)}$ ($l=4$, 6, 8, 10, 12), we have

$$(\Phi_{l,(13,10)}E_{4,l})\big|_{\mathfrak{H}_2\times\mathfrak{H}_2} = c_1(l)F_{13,10a}(Z_1)F_{13,10a}(Z_2)+c_2(l)F_{13,10b}(Z_1)F_{13,10b}(Z_2)$$

for some constant $c_1(l)$ and $c_2(l)\in\mathbb{C}$, where $Z_1$, $Z_2\in\mathfrak{H}_2$. Here we are identifying the vector valued forms with polynomials in two variables and using variables $u_1$, $u_2$ for $Z_1$ and $u_3$, $u_4$ for $Z_2$. So the product is a polynomial in $u_i$ but means the tensor product in the usual sense. Explict values of $c_i(l)$ are obtained by comparing the Fourier coefficients of both sides as we exlained in Section 3.2. For the sake of simplicity, we put

$$T_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

We define $\epsilon_{l,(13,10)}(A,B,U)$ for $\Phi_{l,(13,10)}$ as in Section 3.2 and denote this shortly by $\epsilon_l(A,B)$ for the sake of simplicity. To calculate this, we need Fourier coefficients of $E_{4,l}$. In principle, the Fourier coefficients can be calculated by using the results in [**10**], but still it is very complicated. A simpler way of calculation has been given in [**6**] Proposition 7.8 and we use this. We omit the details of the calculation but anyway we obtain the following results.

$$\begin{aligned}
\epsilon_4(T_0,T_1) &= -(3968055/32)u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(14779u_3^8-57972u_3^6u_4^2+57972u_3^2u_4^6-14779u_4^8) \\
\epsilon_4(T_0,T_2) &= -(3968055/32)u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(14779u_3^8-57972u_3^6u_4^2+57972u_3^2u_4^6-14779u_4^8) \\
\epsilon_6(T_0,T_1) &= 915705/8u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(20301u_3^8-38968u_3^6u_4^2+38968u_3^2u_4^6-20301u_4^8) \\
\epsilon_6(T_0,T_2) &= 7325640u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(283499u_3^8-184578472u_3^6u_4^2+184578472u_3^2u_4^6-283499u_4^8) \\
\epsilon_8(T_0,T_1) &= -1180242u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(52013u_3^8-76579u_3^6u_4^2+76579u_3^2u_4^6-52013u_4^8) \\
\epsilon_8(T_0,T_2) &= -37767744u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(9524109u_3^8-938928392u_3^6u_4^2+938928392u_3^2u_4^6-9524109u_4^8) \\
\epsilon_{10}(T_0,T_1) &= 135158058000u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(1693627u_3^8-2321496u_3^6u_4^2+2321496u_3^2u_4^6-1693627u_4^8) \\
\epsilon_{10}(T_0,T_2) &= 8650115712000u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(184909013u_3^8-15261048024u_3^6u_4^2 \\
&\qquad\qquad +15261048024u_3^2u_4^6-184909013u_4^8) \\
\epsilon_{12}(T_0,T_1) &= -75952235520u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(26199u_3^8-35275u_3^6u_4^2+35275u_3^2u_4^6-26199u_4^8) \\
\epsilon_{12}(T_0,T_2) &= -2430471536640u_1u_2(u_1^2+u_1u_2+u_2^2)^2(2u_1^4+5u_1^3u_2-5u_1u_2^3-2u_2^4) \\
&\quad \times u_3u_4(5941681u_3^8-471962984u_3^6u_4^2+471962984u_3^2u_4^6-5941681u_4^8).
\end{aligned}$$

We also have the following table of $\Lambda(r, F, St)$ for $F \in S_{13,10}(\Gamma_2)$.

| $r$ | $F_{13,10a}$ | $F_{13,10b}$ |
|---|---|---|
| 2 | $\dfrac{-29 \cdot 31}{2^{29} \cdot 3^7 \cdot 5^6 \cdot 7^2 \cdot 11^3 \cdot 13 \cdot 17 \cdot 19}$ | $\dfrac{-73}{2^{30} \cdot 3^6 \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13 \cdot 17}$ |
| 4 | $\dfrac{29 \cdot 31}{2^{24} \cdot 3^7 \cdot 5^6 \cdot 7^2 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19}$ | $\dfrac{6529}{2^{28} \cdot 3^7 \cdot 5^4 \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19}$ |
| 6 | $\dfrac{-23 \cdot 29 \cdot 31 \cdot 691}{2^{23} \cdot 3^9 \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19}$ | $\dfrac{-29 \cdot 9049}{2^{23} \cdot 3^9 \cdot 5^5 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17}$ |
| 8 | $\dfrac{31 \cdot 653}{2^{15} \cdot 3^9 \cdot 5^5 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19}$ | $\dfrac{312931}{2^{18} \cdot 3^7 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19}$ |
| 10 | $\dfrac{-34981}{2^{10} \cdot 3^5 \cdot 5^6 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19}$ | $\dfrac{-436307}{2^{10} \cdot 3^5 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19}$ |

Here, if we use exact values of $\epsilon_4(T_i, T_j)$ given above and the table of the Fourier coefficients in the appendix, it is a routine work to calculate $c_l(i)$ by the method explained in Section 3.2. Dividing $\widetilde{\Lambda}(l-n, F, St)$ for $F = F_{13,10a}$ or $F_{13,10b}$ by $C(l, (k, m))$, we obtain $\Lambda(r, F, St)$ for $F = F_{13,10a}$ or $F_{13,10b}$ as in the above table.

Remark. From the above table, we see that the values of $L(r, F, St)$ are all positive for $r = 2, 4, 6, 8, 10$ for $F = F_{13,10a}$ or $F_{13,10b}$. If the absolute value of the Satake parameters in $L(s, F, St)$ is one, then the Euler products of $L(s, F, St)$ converges absolutely for $Re(s) > 1$, so this is naturally expected.

Proof of Theorem 2.2. The value $L(r, F_{13,10a}, St)/(F_{13,10a}, F_{13,10a})$ is by definition a rational multiple of $\Lambda(r, F_{13,10a}, St)$ depending on $r$, and can be calculated easily for each $r$. Then multiplying $(2\pi)^{-3r-33}\Gamma(11)^{-1}\Gamma(r+11)\Gamma(r+22)$ to this, we obtain Theorem 2.2. □

## 6. Table of the Fourier coefficients.

We give below the Fourier coefficients we needed in the above calculation. These are calculated by a computer. We explain the notation. For a cusp form $F \in S_{k,m}(\mathit{\Gamma}_2)$, we write the Fourier expansion as

$$F(Z) = \sum_T c_F(T, u) e^{2\pi\sqrt{-1}\,Tr(TZ)}$$

where $T$ runs over positive definite half integral symmetric matrices and the Fourier coefficients $c_F(T, u) = \sum_{\mu=0}^{m} c_{F,\mu}(T) u_1^{m-\mu} u_2^{\mu}$ are in $V_1^{(m)} = \mathbb{C}[u_1, u_2]_m$ and $c_{F,\mu}(T) \in \mathbb{C}$. In the table, $(a, b, c, \mu)$ denotes $\binom{m}{\mu}^{-1} c_{F,\mu}(T)$, where

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

Here we multiplied $\binom{m}{\mu}^{-1}$ to make the table simpler. When $a = c$ in $T$, we have

$$c_{F,\mu}(T) = (-1)^k c_{F,m-\mu}(T)$$

by the automorphy with respect to the transformation $(\tau, z, \omega) \mapsto (\omega, z, \tau)$ for $\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathfrak{H}_2$. So we give values of $c_{F,\nu}(T)$ only for $\mu \leq m/2$ in the following table.

| | $f_{13,10a}$ | $f_{13,10b}$ | $F_{13,10a}$ | $F_{13,10b}$ |
|---|---|---|---|---|
| (1,1,0,0) | 0 | 0 | 0 | 0 |
| (1,1,0,1) | −1033482 | 131790 | 6 | 6 |
| (1,1,0,2) | 0 | 0 | 0 | 0 |
| (1,1,0,3) | −732570 | −127822 | −9 | 4 |
| (1,1,0,4) | 0 | 0 | 0 | 0 |
| (1,1,0,5) | 0 | 0 | 0 | 0 |
| (1,1,1,0) | 0 | 0 | 0 | 0 |
| (1,1,1,1) | 516741 | −65895 | −3 | −3 |
| (1,1,1,2) | 516741 | −65895 | −3 | −3 |
| (1,1,1,3) | 344494 | −43930 | −2 | −2 |
| (1,1,1,4) | 172247 | −21965 | −1 | −1 |
| (1,1,1,5) | 0 | 0 | 0 | 0 |
| (2,2,0,0) | 0 | 0 | 0 | 0 |
| (2,2,0,1) | −120565055616 | −14105233536 | −1065984 | 666240 |
| (2,2,0,2) | 0 | 0 | 0 | 0 |
| (2,2,0,3) | 41609947392 | −7401697024 | −367104 | −243968 |
| (2,2,0,4) | 0 | 0 | 0 | 0 |
| (2,2,0,5) | 0 | 0 | 0 | 0 |
| (2,2,2,0) | 0 | 0 | 0 | 0 |
| (2,2,2,1) | −28664602560 | −3353549760 | −253440 | 158400 |
| (2,2,2,2) | −28664602560 | −3353549760 | −253440 | 158400 |
| (2,2,2,3) | −19109735040 | −2235699840 | −168960 | 105600 |
| (2,2,2,4) | −9554867520 | −1117849920 | −84480 | 52800 |
| (2,2,2,5) | 0 | 0 | 0 | 0 |
| (3,3,0,0) | 0 | 0 | 0 | 0 |
| (3,3,0,1) | −22962940769520 | −6636953887920 | −439674480 | 122374800 |
| (3,3,0,2) | 0 | 0 | 0 | 0 |
| (3,3,0,3) | −12114978315120 | 11380342607280 | 659511720 | 81583200 |
| (3,3,0,4) | 0 | 0 | 0 | 0 |
| (3,3,0,5) | 0 | 0 | 0 | 0 |
| (3,3,3,0) | 0 | 0 | 0 | 0 |
| (3,3,3,1) | 33725476041021 | 481913263665 | 90697077 | −190327563 |
| (3,3,3,2) | 33725476041021 | 481913263665 | 90697077 | −190327563 |
| (3,3,3,3) | 22483650694014 | 321275509110 | 60464718 | −126885042 |
| (3,3,3,4) | 11241825347007 | 160637754555 | 30232359 | −63442521 |
| (3,3,3,5) | 0 | 0 | 0 | 0 |
| (1,3,0,0) | 0 | 0 | 0 | 0 |
| (1,3,0,1) | 2091767568 | −266742960 | −12144 | −12144 |

| | $f_{13,10a}$ | $f_{13,10b}$ | $F_{13,10a}$ | $F_{13,10b}$ |
|---|---|---|---|---|
| (1,3,0,2) | 0 | 0 | 0 | 0 |
| (1,3,0,3) | 1011105232 | 317082512 | 20848 | −5360 |
| (1,3,0,4) | 0 | 0 | 0 | 0 |
| (1,3,0,5) | 4811457440 | −1091436320 | −56560 | −28480 |
| (1,3,0,6) | 0 | 0 | 0 | 0 |
| (1,3,0,7) | −8848264032 | 6576137568 | 377712 | 57600 |
| (1,3,0,8) | 0 | 0 | 0 | 0 |
| (1,3,0,9) | −193798838736 | −5297409936 | −672624 | 1090800 |
| (1,3,0,10) | 0 | 0 | 0 | 0 |
| (1,4,0,0) | 0 | 0 | 0 | 0 |
| (1,4,0,1) | 49607136 | −6325920 | −288 | −288 |
| (1,4,0,2) | 0 | 0 | 0 | 0 |
| (1,4,0,3) | −12061925760 | −1605230976 | −118272 | 66432 |
| (1,4,0,4) | 0 | 0 | 0 | 0 |
| (1,4,0,5) | −2789152800 | 886648800 | 48000 | 16800 |
| (1,4,0,6) | 0 | 0 | 0 | 0 |
| (1,4,0,7) | −391895872320 | 4480764096 | −450048 | 2223168 |
| (1,4,0,8) | 0 | 0 | 0 | 0 |
| (1,4,0,9) | −3468463687296 | 491999011200 | 23113728 | 20193408 |
| (1,4,0,10) | 0 | 0 | 0 | 0 |
| (1,7,1,0) | 0 | 0 | 0 | 0 |
| (1,7,1,1) | −101179954764 | 12902504580 | 587412 | 587412 |
| (1,7,1,2) | −101179954764 | 12902504580 | 587412 | 587412 |
| (1,7,1,3) | −943015320576 | −125948841984 | −9273600 | 5193216 |
| (1,7,1,4) | −1784850686388 | −264800188548 | −19134612 | 9799020 |
| (1,7,1,5) | −3352044999240 | 362462653080 | 15567480 | 19386360 |
| (1,7,1,6) | −5644598259132 | 1755839682900 | 94832676 | 33955236 |
| (1,7,1,7) | −609784653744 | 863008300944 | 50579280 | 4438224 |
| (1,7,1,8) | 19805121629244 | −5368354092756 | −285274404 | −118232100 |
| (1,7,1,9) | −1035074248691868 | 81228688663284 | 2968282692 | 5951187396 |
| (1,7,1,10) | −5354649359992800 | 460040331492000 | 17741354400 | 30832250400 |
| (1,9,0,0) | 0 | 0 | 0 | 0 |
| (1,9,0,1) | −346874798034 | 44233600230 | 2013822 | 2013822 |
| (1,9,0,2) | 0 | 0 | 0 | 0 |
| (1,9,0,3) | 3023838489870 | 313300863594 | 24311403 | −16755948 |
| (1,9,0,4) | 0 | 0 | 0 | 0 |
| (1,9,0,5) | −13497275937600 | −2278657310400 | −161244000 | 73785600 |
| (1,9,0,6) | 0 | 0 | 0 | 0 |
| (1,9,0,7) | 338156932299570 | 54763902017046 | 3900489957 | −1851262452 |
| (1,9,0,8) | 0 | 0 | 0 | 0 |
| (1,9,0,9) | −11260623773839086 | 1058621009379930 | 42771014658 | 64943319618 |
| (1,9,0,10) | 0 | 0 | 0 | 0 |
| (2,5,2,0) | 0 | 0 | 0 | 0 |
| (2,5,2,1) | 25959576133872 | 3037081366512 | 229523328 | −143452080 |
| (2,5,2,2) | 25959576133872 | 3037081366512 | 229523328 | −143452080 |
| (2,5,2,3) | 14227355195730 | 1121949324882 | 93291624 | −79240554 |

| | $f_{13,10a}$ | $f_{13,10b}$ | $F_{13,10a}$ | $F_{13,10b}$ |
|---|---|---|---|---|
| (2,5,2,4) | 2495134257588 | −793182716748 | −42940080 | −15029028 |
| (2,5,2,5) | −8526618668250 | −2304796506570 | −153697140 | 45622890 |
| (2,5,2,6) | −18837903581784 | −3412892044584 | −238979556 | 102715200 |
| (2,5,2,7) | −140269573451520 | 3330960223068 | −57619629 | 797704074 |
| (2,5,2,8) | −484652481245964 | 25375189850244 | 631550340 | 2772045684 |
| (2,5,2,9) | 466463893314618 | 83030231321442 | 5828966010 | −2545124598 |
| (2,5,2,10) | 5861811443758200 | 209468524051800 | 23294331000 | −32936992200 |

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