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The classification of real forms of simple irreducible pseudo-Hermitian symmetric spaces

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Abstract. The main purpose of this paper is to classify the real forms M of simple irreducible pseudo-Hermitian symmetric spaces G/R with G non-compact. That provides an extension of Jaffee's results (Bull. Amer. Math. Soc. '75; J. Differential Geom. '78), Leung's result (J. Differential Geom. '79) and Takeuchi's result (Tohoku Math. J. '84) concerning the classification of real forms of irreducible Hermitian symmetric spaces of the non-compact type. Moreover, that enables us to classify the pairs of simple para-Hermitian symmetric Lie algebras and their para-holomorphic involutions, which includes Kaneyuki-Kozai's result (Tokyo J. Math. '85) of the classification of simple para-Hermitian symmetric Lie algebras.

1. Introduction and the main result.

Let G/R be a simple irreducible pseudo-Hermitian symmetric space. A non-empty subset $M \subset G/R$ is said to be a *real form* of G/R, if there exists an involutive anti-holomorphic isometry $\hat{\eta}$ of G/R satisfying $M = (G/R)^{\hat{\eta}}$, where $(G/R)^{\hat{\eta}}$ denotes the fixed point set of $\hat{\eta}$ in G/R. Real forms M_1 of G/R_1 and M_2 of G/R_2 are said to be *equivalent*, if there exists a holomorphic homothety f of G/R_1 onto G/R_2 satisfying $f(M_1) = M_2$.

The main purpose of this paper is to classify the real forms M of simple irreducible pseudo-Hermitian symmetric spaces G/R with G non-compact, which provides an extension of Jaffee's results [**7**], [**8**] (see [**6**] also), Leung's result [**14**] and Takeuchi's result [**26**] concerning the classification of real forms of irreducible Hermitian symmetric spaces of the non-compact type (cf. Remarks 4.14.2 and 5.3.1):

Theorem 1.0.1. *Up to equivalence, the real forms M of simple irreducible pseudo-Hermitian symmetric spaces G/R with G non-compact are given in List* I.

A simple irreducible pseudo-Hermitian symmetric space G/R is, in fact, a pseudo-Kählerian homogeneous space, and a real form $M \subset G/R$ has several features—for example,

(1) M is a totally real submanifold of G/R and the induced metric on M from G/R is non-degenerate;
(2) M is a totally geodesic submanifold of G/R;
(3) M is a Lagrangian submanifold of G/R

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List I

G/R	M	no.
AI, cf. Proposition 4.3.5		
$SL(2n,\mathbb{R})/(SL(n,\mathbb{C})\cdot T)$	$S(GL(n,\mathbb{R})\times GL(n,\mathbb{R}))/SL(n,\mathbb{R})$	1
	$SO_0(n,n)/SO(n,\mathbb{C})$	2
$SL(4m,\mathbb{R})/(SL(2m,\mathbb{C})\cdot T)$	$(SL(2m,\mathbb{C})\cdot T)/SU^*(2m)$	3
	$Sp(2m,\mathbb{R})/Sp(m,\mathbb{C})$	4
AII, cf. Proposition 4.4.1		
$SU^*(2n)/(SL(n,\mathbb{C})\cdot T)$	$(SL(n,\mathbb{C})\cdot T)/SL(n,\mathbb{R})$	1
	$SO^*(2n)/SO(n,\mathbb{C})$	2
$SU^*(4m)/(SL(2m,\mathbb{C})\cdot T)$	$(SU^*(2m)\cdot SU^*(2m)\cdot\mathbb{R})/SU^*(2m)$	3
	$Sp(m,m)/Sp(m,\mathbb{C})$	4
AIII, cf. Proposition 4.5.1		
$SU(p,q)/$ $S(U(h,k-p)\times U(p-h,p+q-k))$	$SO_0(p,q)/$ $(SO_0(h,k-p)\cdot SO_0(p-h,p+q-k))$	1
$SU(2n,2m)/S(U(n,m)\times U(n,m))$	$S(U(n,m)\times U(n,m))/SU(n,m)$	2
$SU(2n,2m)/$ $S(U(2h,2k-2n)\times U(2n-2h,2n+2m-2k))$	$Sp(n,m)/$ $(Sp(h,k-n)\times Sp(n-h,n+m-k))$	3
$SU(p,p)/S(U(h,h)\times U(p-h,p-h))$	$Sp(p,\mathbb{R})/(Sp(h,\mathbb{R})\times Sp(p-h,\mathbb{R}))$	4
	$SO^*(2p)/(SO^*(2h)\cdot SO^*(2p-2h))$	5
$SU(p,p)/S(U(h,p-h)\times U(h,p-h))$	$(SL(p,\mathbb{C})\cdot\mathbb{R})/SU(h,p-h)$	6
BDI, cf. Proposition 4.6.1		
$SO_0(p,q)/(SO_0(p-2,q)\cdot SO(2))$	$(SO_0(h,k)\cdot SO_0(p-h,q-k))/$ $(SO_0(h-1,k)\cdot SO_0(p-h-1,q-k))$	1
$SO_0(2n,2m)/U(n,m)$	$(SO_0(n,m)\cdot SO_0(n,m))/SO_0(n,m)$	2
$SO_0(4n,4m)/U(2n,2m)$	$U(2n,2m)/Sp(n,m)$	3
$SO_0(2n,2n)/U(n,n)$	$GL(2n,\mathbb{R})/Sp(n,\mathbb{R})$	4
	$SO(2n,\mathbb{C})/SO^*(2n)$	5
DIII, cf. Proposition 4.7.1		
$SO^*(4m)/U(m,m)$	$(SO^*(2m)\cdot SO^*(2m))/SO^*(2m)$	1
	$U(m,m)/Sp(m,\mathbb{R})$	2
$SO^*(4m)/U(2k,2m-2k)$	$(SU^*(2m)\cdot\mathbb{R})/Sp(k,m-k)$	3
$SO^*(2n)/U(k,n-k)$	$SO(n,\mathbb{C})/SO_0(k,n-k)$	4
$SO^*(2n)/(SO^*(2)\cdot SO^*(2n-2))$	$SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$	5
CI, cf. Proposition 4.8.1		
$Sp(2m,\mathbb{R})/U(m,m)$	$(Sp(m,\mathbb{R})\times Sp(m,\mathbb{R}))/Sp(m,\mathbb{R})$	1
	$U(m,m)/SO^*(2m)$	2
$Sp(2m,\mathbb{R})/U(2k,2m-2k)$	$Sp(m,\mathbb{C})/Sp(k,m-k)$	3
$Sp(n,\mathbb{R})/U(k,n-k)$	$GL(n,\mathbb{R})/SO_0(k,n-k)$	4

G/R	M	no.
CII, cf. Proposition 4.9.1		
$Sp(p,q)/U(p,q)$	$U(p,q)/SO_0(p,q)$	1
$Sp(2n,2m)/U(2n,2m)$	$(Sp(n,m) \times Sp(n,m))/Sp(n,m)$	2
$Sp(p,p)/U(p,p)$	$Sp(p,\mathbb{C})/Sp(p,\mathbb{R})$	3
	$(SU^*(2p) \cdot \mathbb{R})/SO^*(2p)$	4
EII, cf. Proposition 4.10.4		
$E_{6(2)}/(SO_0(6,4) \cdot SO(2))$	$Sp(3,1)/(Sp(2) \times Sp(1,1))$	1
	$F_{4(4)}/SO_0(5,4)$	2
	$Sp(4,\mathbb{R})/(Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R}))$	3
$E_{6(2)}/(SO^*(10) \cdot SO^*(2))$	$Sp(4,\mathbb{R})/Sp(2,\mathbb{C})$	4
EIII, cf. Proposition 4.11.1		
$E_{6(-14)}/(SO^*(10) \cdot SO^*(2))$	$Sp(2,2)/Sp(2,\mathbb{C})$	1
$E_{6(-14)}/(SO_0(8,2) \cdot SO(2))$	$F_{4(-20)}/SO_0(8,1)$	2
	$Sp(2,2)/(Sp(1,1) \times Sp(1,1))$	3
$E_{6(-14)}/(SO(10) \cdot SO(2))$	$F_{4(-20)}/SO(9)$	4
	$Sp(2,2)/(Sp(2) \times Sp(2))$	5
EV, cf. Proposition 4.12.1		
$E_{7(7)}/(E_{6(2)} \cdot T)$	$(E_{6(6)} \cdot \mathbb{R})/F_{4(4)}$	1
	$SU^*(8)/Sp(3,1)$	2
	$SL(8,\mathbb{R})/Sp(4,\mathbb{R})$	3
EVI, cf. Proposition 4.13.1		
$E_{7(-5)}/(E_{6(2)} \cdot T)$	$(E_{6(2)} \cdot T)/F_{4(4)}$	1
	$SU(6,2)/Sp(3,1)$	2
	$SU(4,4)/Sp(4,\mathbb{R})$	3
$E_{7(-5)}/(E_{6(-14)} \cdot T)$	$SU(4,4)/Sp(2,2)$	4
	$(E_{6(-14)} \cdot T)/F_{4(-20)}$	5
EVII, cf. Proposition 4.14.1		
$E_{7(-25)}/(E_{6(-14)} \cdot T)$	$(E_{6(-26)} \cdot \mathbb{R})/F_{4(-20)}$	1
	$SU^*(8)/Sp(2,2)$	2
$E_{7(-25)}/(E_6 \cdot T)$	$(E_{6(-26)} \cdot \mathbb{R})/F_4$	3
	$SU^*(8)/Sp(4)$	4

(cf. Section 5). For this reason $(G/R, M)$ appears in several studies, though G/R is Hermitian; for example, the study of visible action by Kobayashi [**12**], the study of (generalized) Segal-Bargmann transform by Ólafsson [**22**], the study of Floer homology by Iriyeh-Sakai-Tasaki [**5**] and so on. We expect that the study of real forms of Hermitian symmetric spaces will advance in that of pseudo-Hermitian symmetric spaces, and then Theorem 1.0.1 will play a role in them.

As an application, we apply Theorem 1.0.1 to classifying the pairs $((\mathfrak{g}_d, \sigma), \theta)$ of para-Hermitian symmetric Lie algebras $(\mathfrak{g}_d, \sigma)$ and para-holomorphic involutions θ of $(\mathfrak{g}_d, \sigma)$, where $\mathfrak{g}_d$ are real forms of complex simple Lie algebras (cf. Theorem 6.5.1, List III). Theorem 6.5.1 includes Kaneyuki-Kozai's result [**9**] of the classification of simple para-Hermitian symmetric Lie algebras (cf. Remark 6.5.2-(iii)). From List III, one can read the graded decompositions $\mathfrak{g}_d = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ of the first kind and their further decompositions,

$$\begin{aligned} &\mathfrak{g}_d = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, && [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \\ &\mathfrak{g}_i = \mathfrak{g}_i^+ \oplus \mathfrak{g}_i^-, && [\mathfrak{g}_i^+, \mathfrak{g}_j^+] \subset \mathfrak{g}_{i+j}^+,\ [\mathfrak{g}_i^+, \mathfrak{g}_j^-] \subset \mathfrak{g}_{i+j}^-,\ [\mathfrak{g}_i^-, \mathfrak{g}_j^-] \subset \mathfrak{g}_{i+j}^+ \end{aligned}$$

(cf. Example 6.5.3). So, we guess that Theorem 6.5.1 will enable one to partially generalize the Tanaka theory [**28**], [**29**] of differential systems, in the future.

This paper is organized as follows:

Section 2 Preliminaries

We recall the definitions of pseudo-Hermitian symmetric space and pseudo-Hermitian symmetric Lie algebra, and demonstrate Theorems 2.6.1, 2.7.8, etc. Theorem 2.7.8 implies that one can assert Theorem 1.0.1, if we determine $d\mathcal{R}_\mathfrak{g}/_\sim$ for each non-compact real form $\mathfrak{g}$ of complex simple Lie algebras (see Subsection 2.7.2. for $d\mathcal{R}_\mathfrak{g}/_\sim$).

Section 3 Cartan decompositions and Root systems

We review elementary facts about a Cartan decomposition and the root theory. This section consists of three subsections. In Subsection 3.1 we construct a non-compact real form $\mathfrak{l}$ of $\mathfrak{l}_\mathbb{C}$ endowed with a Cartan decomposition $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$, from a compact real form $\mathfrak{l}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{l}_\mathbb{C}$ and $\theta \in \mathrm{Inv}(\mathfrak{l}_u)$. In Subsection 3.2 we survey how to construct an automorphism of $\mathfrak{l}_u$ from a root system $\triangle(\mathfrak{l}_\mathbb{C}, \mathfrak{c}_\mathbb{C})$. Finally in Subsection 3.3 we mention a restricted root system $\Sigma(\mathfrak{k}, \mathfrak{a})$. Here $\mathfrak{l}_\mathbb{C}$ is a complex semisimple Lie algebra and $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{l}$.

Section 4 The classification of elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_\mathfrak{g}$

This section is devoted to determining $d\mathcal{R}_\mathfrak{g}/_\sim$ for each non-compact real form $\mathfrak{g}$ of complex simple Lie algebras (cf. Propositions 4.3.5 (AI), 4.4.1 (AII), 4.5.1 (AIII), 4.6.1 (BDI), 4.7.1 (DIII), 4.8.1 (CI), 4.9.1 (CII), 4.10.4 (EII), 4.11.1 (EIII), 4.12.1 (EV), 4.13.1 (EVI) and 4.14.1 (EVII)).

Section 5 Real forms and totally real totally geodesic submanifolds

We investigate relation between real forms and totally real totally geodesic submanifolds of G/R (see Theorem 5.1.1). Theorems 1.0.1 and 5.1.1 enable us to determine every connected, totally real, complete totally geodesic submanifold M of G/R with $\dim_\mathbb{R} M = \dim_\mathbb{C} G/R$, where the induced metric on M is non-degenerate.

Section 6 Para-holomorphic involutions of simple para-Hermitian symmetric Lie algebras

As an application, we classify the pairs of simple para-Hermitian symmetric Lie algebras and their para-holomorphic involutions (cf. Theorem 6.5.1) by use of Theorem 1.0.1.

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2. Preliminaries.

2.1. Notation.

In this paper we utilize the following notation, where G is a Lie group and $\mathfrak{g}$ is a Lie algebra:

(n1) $\mathfrak{g}_{\mathbb{C}}$: the complexification of $\mathfrak{g}$,
(n2) ∇^1: the canonical connection on a symmetric space (cf. Definition 2.2.1-(iii)),
(n3) $[\phi, \psi] = 0$ means that $\phi \circ \psi = \psi \circ \phi$, where ϕ and ψ are automorphisms,
(n4) G_0: the identity component of G,
(n5) M^f: the fixed point set of an involutive map f in a set M,
(n6) $Z(G)$, $z(\mathfrak{g})$: the center of G, $\mathfrak{g}$,
(n7) $B_{\mathfrak{g}}$: the Killing form of $\mathfrak{g}$,
(n8) $\mathrm{Lie}\, G$: the Lie algebra of G,
(n9) Ad_G, $\mathrm{ad}_{\mathfrak{g}}$ (or Ad, ad): the adjoint representation of G, $\mathfrak{g}$,
(n10) $C_G(T) := \{g \in G \mid \mathrm{Ad}(g)T = T\}$, for $T \in \mathrm{Lie}\, G$,
(n11) $\mathfrak{c}_{\mathfrak{g}}(T) := \{X \in \mathfrak{g} \mid [T, X] = 0\}$, for $T \in \mathfrak{g}$,
(n12) A_g: the inner automorphism of G by $g \in G$,
(n13) τ_g: a diffeomorphism of a homogeneous space G/H defined by $\tau_g : aH \mapsto gaH$ for $aH \in G/H$, where $g \in G$,
(n14) $f|_A$: the restriction of a map f to a set A,
(n15) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,
(n16) o: the origin of a homogeneous space,
(n17) T_pM: the tangent space of a manifold M at $p \in M$,
(n18) $\mathfrak{X}(M)$: the set of vector fields on a manifold M,
(n19) $\mathrm{Aut}(G)$, $\mathrm{Aut}(\mathfrak{g})$: the group of automorphisms of G, $\mathfrak{g}$,
(n20) $\mathrm{Int}(G)$, $\mathrm{Int}(\mathfrak{g})$: the group of inner automorphisms of G, $\mathfrak{g}$,
(n21) $\mathrm{Inv}(G)$, $\mathrm{Inv}(\mathfrak{g})$: the set of involutions of G, $\mathfrak{g}$,
(n22) $\mathrm{Aut}(\mathfrak{g}, \psi_1, \ldots, \psi_n) := \{\phi \in \mathrm{Aut}(\mathfrak{g}) \mid [\psi_k, \phi] = 0 \text{ for all } 1 \leq k \leq n\}$, where $\psi_1, \ldots, \psi_n \in \mathrm{Aut}(\mathfrak{g})$,
(n23) $\mathrm{Int}(\mathfrak{g}, \psi_1, \ldots, \psi_n)$: similar to (n22),
(n24) $i := \sqrt{-1}$,
(n25) $\mathfrak{m} \perp \mathfrak{n}$ means that $\mathfrak{m}$ is perpendicular to $\mathfrak{n}$ with respect to a non-degenerate symmetric bilinear form, where $\mathfrak{m}$ and $\mathfrak{n}$ are vector spaces.

In addition, we sometimes denote the Lie algebra of a Lie group by the corresponding German small letter.

2.2. Definitions.

We recall the definitions of pseudo-Hermitian symmetric space and pseudo-Hermitian symmetric Lie algebra. In order to do so, we first recall

Definition 2.2.1 (cf. Nomizu [**21**, p. 52, p. 54, p. 56]). (i) Let U be a connected Lie group, and H a closed subgroup of U. The homogeneous space U/H is said to be an *affine symmetric space* or a *symmetric space*, if there exists a $\sigma \in \mathrm{Inv}(U)$ satisfying

$$(U^\sigma)_0 \subset H \subset U^\sigma.$$

(ii) A symmetric space U/H is called *effective* (resp. *almost effective*), if U is effective (resp. almost effective) on U/H as a transformation group.

(iii) A symmetric space $(U/H, \sigma)$ is furnished with a unique U-invariant affine connection ∇^1 such that

$$ds_o(\nabla^1_X Y) = \nabla^1_{(ds_o X)}(ds_o Y) \text{ for any } X, Y \in \mathfrak{X}(U/H),$$

where $s_o(uH) := \sigma(u)H$ for $uH \in U/H$. This connection ∇^1 is called the *canonical* (*affine*) *connection* on $(U/H, \sigma)$.

(iv) Let L be a connected semisimple Lie group. An almost effective, semisimple symmetric space $(L/H, \sigma)$ is called *irreducible*, if the action of $\mathrm{ad}\,\mathfrak{h}$ on $\mathfrak{m}$ is irreducible. Here, $\mathfrak{h} := \mathfrak{l}^{d\sigma}$ and $\mathfrak{m} := \mathfrak{l}^{-d\sigma}$.

Definition 2.2.2 (cf. Berger [**1**, p. 94]). (i) A symmetric space U/R is said to be *pseudo-Hermitian*, if it admits a U-invariant complex structure J and a U-invariant pseudo-Hermitian metric $\boldsymbol{g}$ (with respect to J).

(ii) A symmetric Lie algebra $(\mathfrak{u}, \mathfrak{r}, \sigma)$ is called *pseudo-Hermitian*, if there exist an $\mathrm{ad}\,\mathfrak{r}$-invariant complex structure $\boldsymbol{j}$ on $\mathfrak{q}$ and an $\mathrm{ad}\,\mathfrak{r}$-invariant pseudo-Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle$ (with respect to $\boldsymbol{j}$) on $\mathfrak{q}$. Here $\mathfrak{q} := \mathfrak{u}^{-\sigma}$.

Remark 2.2.3. (i) A simple pseudo-Hermitian symmetric space G/R is irreducible if and only if $\mathfrak{g}$ is a real form of a complex simple Lie algebra (cf. Shapiro [**25**, p. 532]).

(ii) Throughout this paper, we assume a Hermitian symmetric space to be one of the pseudo-Hermitian symmetric spaces.

2.3. A fundamental lemma and proposition.

We will need the following Lemma 2.3.1 and Proposition 2.3.3 later. Here, Shapiro [**25**] has proved almost all of statements on the proposition.

Lemma 2.3.1. *Let $(L_a/H_a, \sigma_a)$ be an effective, semisimple irreducible symmetric space $(a = 1, 2)$, and $\hat{\phi}$ an affine diffeomorphism of $(L_1/H_1, \nabla^1_1)$ onto $(L_2/H_2, \nabla^1_2)$ such that $\hat{\phi}(o_1) = o_2$. Then, there exists a unique isomorphism ϕ of L_1 onto L_2 satisfying*

$$\text{(i) } \phi \circ \sigma_1 = \sigma_2 \circ \phi, \quad \text{(ii) } \phi(H_1) = H_2, \quad \text{(iii) } \hat{\phi} \circ \mathrm{Pr}_1 = \mathrm{Pr}_2 \circ \phi.$$

In particular, ϕ is involutive if $L_1 = L_2$, $H_1 = H_2$, $\sigma_1 = \sigma_2$ and $\hat{\phi}$ is involutive. Here, we denote by o_a the origin of L_a/H_a, by ∇^1_a the canonical connection on $(L_a/H_a, \sigma_a)$, and by Pr_a the projection from L_a onto L_a/H_a.

Proof. Let us prove the uniqueness of ϕ. Let ϕ' be an isomorphism of L_1 onto L_2 satisfying (i′) $\phi' \circ \sigma_1 = \sigma_2 \circ \phi'$ and (iii′) $\hat{\phi} \circ \mathrm{Pr}_1 = \mathrm{Pr}_2 \circ \phi'$. On the one hand, (i), (i′) and $\mathrm{Pr}_2 \circ \phi = \mathrm{Pr}_2 \circ \phi'$ assure that $d\phi(X) = d\phi'(X)$ for every $X \in \mathfrak{m}_1$, where $\mathfrak{m}_1 := \mathfrak{l}_1^{-d\sigma_1}$. On the other hand, $\mathfrak{l}_1 = [\mathfrak{m}_1, \mathfrak{m}_1] \oplus \mathfrak{m}_1$ follows from Nomizu [**21**, p. 56, (16.2)]. Consequently we have $d\phi = d\phi'$ on $\mathfrak{l}_1$, and thus $\phi = \phi'$ because L_1 is connected. See Lemma 6.3.2 [**2**, p. 115] and its proof for the rest of proof. □

Remark 2.3.2 (cf. Nomizu [**21**, p. 55]). Let U/H be a symmetric space. An invariant pseudo-Riemannian metric on U/H, if there exists any, induces the canonical connection.

Proposition 2.3.3 (cf. Shapiro [**25**, p. 534, p. 532]; [**2**, p. 17, p. 114]). *Let $(L/R, \sigma, J, \boldsymbol{g})$ be an almost effective, semisimple pseudo-Hermitian symmetric space, $(\mathfrak{l}, \mathfrak{r}, d\sigma, \boldsymbol{j}, \langle\,\cdot\,,\,\cdot\,\rangle)$ its symmetric Lie algebra and $\mathfrak{q} := \mathfrak{l}^{-d\sigma}$. Then, there exists a unique $T \in \mathfrak{r}$ satisfying*

$$\text{(i) } R = C_L(T) = (L^\sigma)_0, \quad \text{(ii) } d\sigma = \exp \pi \operatorname{ad} T, \quad \text{(iii) } \boldsymbol{j} = \operatorname{ad} T|_{\mathfrak{q}}, \quad \text{(iv) } \mathfrak{q} = [T, \mathfrak{l}].$$

In addition; *if L/R is simple irreducible* (*i.e., $\mathfrak{l}_{\mathbb{C}}$ is a complex simple Lie algebra*), *then* (v) $z(\mathfrak{r}) = \mathrm{span}_{\mathbb{R}}\{T\}$ *and* (vi) *there exists a non-zero $\lambda \in \mathbb{R}$ such that*

$$\langle X, Y\rangle = \lambda \cdot B_{\mathfrak{l}}(X, Y) \textit{ for any } X, Y \in \mathfrak{q}.$$

Remark 2.3.4. (i) The element T in Proposition 2.3.3 is called the *canonical central element* of $\mathfrak{r} = \mathfrak{l}^\sigma$ (cf. Shapiro [**25**, p. 533]).

(ii) The element T becomes a non-zero semisimple element of $\mathfrak{l}$ such that the eigenvalue of $\operatorname{ad} T$ in $\mathfrak{l}$ is $\pm i$ or zero. Such an element is said to be an *Spr-element* of $\mathfrak{l}$ (cf. [**2**, p. 14 or p. 16]).

(iii) For each simple irreducible pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma)$, the canonical central element of $\mathfrak{g}^\sigma$ is unique up to sign $\pm$ because the complex structure $\boldsymbol{j}$ of $(\mathfrak{g}, \sigma)$ is unique up to sign (cf. Shapiro [**25**, p. 534]).

We end this subsection with stating two lemmas.

Lemma 2.3.5. *Let $(\mathfrak{g}, \sigma_a)$ be a simple irreducible pseudo-Hermitian symmetric Lie algebra, let T_a be its canonical central element* ($a = 1, 2$), *and let $\psi \in \mathrm{Aut}(\mathfrak{g})$. If $\psi \circ \sigma_1 = \sigma_2 \circ \psi$, then $\psi(T_1) = T_2$ or $\psi(T_1) = -T_2$.*

Proof. Both $\operatorname{ad} \psi(T_1)$ and $\operatorname{ad} T_2$ induce complex structures of $(\mathfrak{g}, \sigma_2)$. Hence $\operatorname{ad} \psi(T_1) = \pm \operatorname{ad} T_2$ on $\mathfrak{q}_2$, where $\mathfrak{q}_2 := \mathfrak{g}^{-\sigma_2}$, because the complex structure of $(\mathfrak{g}, \sigma_2)$ is unique up to sign. Therefore, $\operatorname{ad} \psi(T_1) = \pm \operatorname{ad} T_2$ on $\mathfrak{g} = [\mathfrak{q}_2, \mathfrak{q}_2] \oplus \mathfrak{q}_2$. This yields $\psi(T_1) = \pm T_2$, since $\mathfrak{g}$ is (semi)simple. □

Lemma 2.3.6. *Let U/R be a pseudo-Hermitian symmetric space, and M a real form of U/R. Then, there exist a real form M_0 of U/R and a holomorphic isometry f of U/R such that* (i) $o \in M_0$ *and* (ii) $f(M) = M_0$.

Proof. Since M is a real form, there exists an involutive antiholomorphic isometry $\hat{\eta}$ of U/R satisfying $M = (U/R)^{\hat{\eta}}$. Take any $aR \in M$ and set $\hat{\eta}_0 := \tau_{a^{-1}} \circ \hat{\eta} \circ \tau_a$. In this case $\hat{\eta}_0$ is an involutive antiholomorphic isometry of U/R because τ_a is a holomorphic isometry. We get the conclusion by $M_0 := (U/R)^{\hat{\eta}_0}$ and $f := \tau_{a^{-1}}$. □

2.4. Connected components.

Our aim in this subsection is to establish Theorem 2.4.1 and Corollaries 2.4.5 and 2.4.6 which will play roles in Subsections 2.6, 2.7 and 5.3.

Theorem 2.4.1. *Let $(U/H, \sigma)$ be a symmetric space, let $\eta \in \mathrm{Inv}(U)$ such that $[\sigma, \eta] = 0$ and $\eta(H) = H$, and let C_o denote the connected component of $(U/H)^{\hat{\eta}}$ containing o. Then, $C_o = (U^\eta)_0/(H \cap (U^\eta)_0)$. Here $\hat{\eta}(uH) := \eta(u)H$ for $uH \in U/H$.*

Let us prepare Proposition 2.4.2 and Lemma 2.4.4 for proof of Theorem 2.4.1.

Proposition 2.4.2 (cf. Kobayashi-Nomizu [**11**, p. 235]). *Let N' and N'' be connected, complete totally geodesic submanifolds of an affine manifold (N, ∇). If at one point of $N' \cap N''$ the tangent space of N' coincides with that of N'', then $N' = N''$.*

Remark 2.4.3. In this paper we refer to Kobayashi-Nomizu [**10**, p. 180] for the definition of totally geodesic submanifold. For this reason Lemma 14.3 [**4**, p. 79] does not hold in this paper—that is, a totally geodesic submanifold M of N is not always complete even if N is complete. For example, $\mathbb{R}^2 \setminus \{0\}$ is a totally geodesic submanifold of $\mathbb{C}^2$ and is not complete. This example also implies that in general, a totally geodesic submanifold M of a symmetric space $(U/H, \nabla^1)$ cannot be any symmetric space with respect to the induced connection $\nabla^1|_M$.

Lemma 2.4.4. *With the same setting as in Theorem 2.4.1; C_o is a closed connected complete totally geodesic submanifold of $(U/H, \nabla^1)$.*

Proof. It follows from $[\sigma, \eta] = 0$ and $\eta(H) = H$ that $\hat{\eta}$ is an involutive affine transformation of $(U/H, \nabla^1)$. Hence, C_o is a closed connected totally geodesic submanifold of $(U/H, \nabla^1)$ (cf. Kobayashi-Nomizu [**11**, p. 61]). Remark that the topology of C_o accords with the induced topology from U/H. The rest of proof is to demonstrate that the totally geodesic submanifold C_o is complete with respect to the induced connection $\nabla^1|_{C_o}$. It suffices to verify that

$$s_x(C_o) = C_o \text{ for any } x \in C_o \tag{2.4.1}$$

because C_o has the induced topology, and hence (2.4.1) assures that $s_x|_{C_o}$ is an affine transformation of C_o with respect to $\nabla^1|_{C_o}$. Here, s_x denotes the symmetry of $(U/H, \nabla^1)$ at x. We devote ourselves to verifying (2.4.1) henceforth. First, let us confirm that

$$\hat{\eta}(s_x(y)) = s_x(y) \text{ for every } x, y \in C_o. \tag{2.4.2}$$

Let Pr denote the projection from U onto U/H. Take any $z = \mathrm{Pr}(g) \in C_o$ $(\subset (U/H)^{\hat{\eta}})$, where $g \in U$. One has $\mathrm{Pr}(g) = z = \hat{\eta}(z) = \mathrm{Pr}(\eta(g))$. Hence

$$g^{-1}\cdot\eta(g)\in H \text{ for every } \mathrm{Pr}(g)\in C_o. \tag{2.4.3}$$

This, together with $H\subset U^\sigma$ and $[\sigma,\eta]=0$, yields $g^{-1}\cdot\eta(g)=\sigma(g^{-1}\cdot\eta(g))=\sigma(g)^{-1}\cdot\eta(\sigma(g))$. Therefore we have

$$\begin{cases} g\cdot\sigma(g)^{-1}=\eta(g\cdot\sigma(g)^{-1}), \\ \sigma(g)\cdot g^{-1}=\eta(\sigma(g)\cdot g^{-1}) \end{cases} \quad \text{for any } \mathrm{Pr}(g)\in C_o. \tag{2.4.4}$$

The symmetry s_x of $(U/H,\nabla^1)$ at $x=\mathrm{Pr}(a)$ is expressed as $s_x=\tau_a\circ\hat{\sigma}\circ\tau_{a^{-1}}$, where $\hat{\sigma}(uH):=\sigma(u)H$ for $uH\in U/H$. So, it is natural that for $x=\mathrm{Pr}(a)$, $y=\mathrm{Pr}(b)$,

$$s_x(y)=(\tau_a\circ\hat{\sigma}\circ\tau_{a^{-1}})(\mathrm{Pr}(b))=\mathrm{Pr}\left(a\cdot\sigma(a)^{-1}\cdot\sigma(b)\right).$$

A direct computation enables us to have

$$\begin{aligned}\hat{\eta}(s_x(y)) &= \hat{\eta}\big(\mathrm{Pr}(a\cdot\sigma(a)^{-1}\cdot\sigma(b))\big)=\mathrm{Pr}\big(\eta\big(a\cdot\sigma(a)^{-1}\cdot\sigma(b)\big)\big) \\ &\underset{(2.4.4)}{=} \mathrm{Pr}\big(a\cdot\sigma(a)^{-1}\cdot\eta(\sigma(b))\big)=\mathrm{Pr}\big(a\cdot\sigma(a)^{-1}\cdot\eta\big(\sigma(b)\cdot b^{-1}\cdot b\big)\big) \\ &\underset{(2.4.4)}{=} \mathrm{Pr}\big(a\cdot\sigma(a)^{-1}\cdot\sigma(b)\cdot b^{-1}\cdot\eta(b)\big)\underset{(2.4.3)}{=}\mathrm{Pr}\big(a\cdot\sigma(a)^{-1}\cdot\sigma(b)\big) \\ &= s_x(y)\end{aligned}$$

for every $x=\mathrm{Pr}(a), y=\mathrm{Pr}(b)\in C_o$. Therefore (2.4.2) follows. On the one hand, (2.4.2) means that $s_x(C_o)\subset(U/H)^{\hat{\eta}}$ for any $x\in C_o$. On the other hand, both C_o and $s_x(C_o)$ are the connected components of $(U/H)^{\hat{\eta}}$ containing x because of $s_x(x)=x$. Accordingly, (2.4.1) holds. □

Now, we are going to prove Theorem 2.4.1 by utilizing the same notation as in the proof of Lemma 2.4.4.

Proof of Theorem 2.4.1. From $[\sigma,\eta]=0$ and $(U^\eta)_0$ being the identity component of U^η, it follows that $(U^\eta)_0$ is invariant under σ, so that $(U^\eta)_0/(H\cap(U^\eta)_0)$ is a symmetric closed subspace of $(U/H,\sigma)$. Consequently, Theorem 4.1 [**11**, p. 234] implies that $(U^\eta)_0/(H\cap(U^\eta)_0)$ is a connected complete totally geodesic submanifold of $(U/H,\nabla^1)$. This, combined with Lemma 2.4.4, enables us to assert that both C_o and $(U^\eta)_0/(H\cap(U^\eta)_0)$ are connected complete totally geodesic submanifolds of $(U/H,\nabla^1)$ containing o. In addition, both T_oC_o and $T_o((U^\eta)_0/(H\cap(U^\eta)_0))$ coincide with $\mathfrak{m}\cap\mathfrak{u}^{d\eta}$, where we identify $T_o(U/H)$ with $\mathfrak{m}:=\mathfrak{u}^{-d\sigma}$. Accordingly $C_o=(U^\eta)_0/(H\cap(U^\eta)_0)$ comes from Proposition 2.4.2. □

Theorem 2.4.1 leads to

Corollary 2.4.5. *With the same setting as in Theorem* 2.4.1; *let C_p denote the connected component of $(U/H)^{\hat{\eta}}$ containing $p=\mathrm{Pr}(a)$, where $a\in U$. Then, $C_p=\mathrm{Pr}((U^\eta)_0\cdot a)$.*

PROOF. Let $h := a^{-1} \cdot \eta(a)$. From $\hat{\eta}(p) = p$ one obtains $\Pr(\eta(a)) = \hat{\eta}(p) = p = \Pr(a)$; and therefore $h \in H$ and $\eta(h) = h^{-1}$. This implies that $\eta' := \mathrm{A}_h \circ \eta$ is an involution of U and satisfies $[\sigma, \eta'] = 0$ and $\eta'(H) = H$ by virtue of $[\sigma, \eta] = 0$ and $\eta(H) = H$. Consequently, one can get an involutive diffeomorphism $\hat{\eta}'$ of U/H by setting $\hat{\eta}'(uH) := \eta'(u)H$ for $uH \in U/H$. Needless to say, $\hat{\eta}'(o) = o$. Define a homeomorphism f of $(U/H)^{\hat{\eta}'}$ onto $(U/H)^{\hat{\eta}}$ by

$$f(x) := \tau_a(x) \text{ for } x \in (U/H)^{\hat{\eta}'}.$$

Let C'_o denote the connected component of $(U/H)^{\hat{\eta}'}$ containing o. Then, $p = \Pr(a)$ yields $f(C'_o) = C_p$. Theorem 2.4.1 implies that $C'_o = \Pr((U^{\eta'})_0)$, and so

$$\begin{aligned} C_p &= f(C'_o) = f\big(\Pr((U^{\eta'})_0)\big) = \Pr\big(a \cdot (U^{\eta'})_0\big) \\ &= \Pr\big(a \cdot (U^{\mathrm{A}_h \circ \eta})_0\big) = \Pr\big(\mathrm{A}_a((U^{\mathrm{A}_h \circ \eta})_0) \cdot a\big) = \Pr\big((U^\eta)_0 \cdot a\big), \end{aligned}$$

where we remark that $\mathrm{A}_a|_{U^{\mathrm{A}_h \circ \eta}}$ is a Lie group isomorphism of $U^{\mathrm{A}_h \circ \eta}$ onto U^η since $h = a^{-1} \cdot \eta(a)$. $\square$

COROLLARY 2.4.6. *Let $(L/H, \sigma)$ be an effective, semisimple irreducible symmetric space, $\hat{\eta}$ an involutive affine transformation of $(L/H, \nabla^1)$ such that $\hat{\eta}(o) = o$, and C_o the connected component of $(L/H)^{\hat{\eta}}$ containing o. Then, there exists a unique $\eta \in \mathrm{Inv}(L)$ satisfying*

$$\text{(i) } [\sigma, \eta] = 0, \quad \text{(ii) } \eta(H) = H, \quad \text{(iii) } \hat{\eta} \circ \Pr = \Pr \circ \eta, \quad \text{(iv) } C_o = (L^\eta)_0/(H \cap (L^\eta)_0),$$

where Pr *denotes the projection from L onto L/H.*

PROOF. Cf. Lemma 2.3.1 and Theorem 2.4.1. $\square$

2.5. (I) Connectedness of real forms.

In the next subsection we will demonstrate that any real form of G/R is connected (cf. Theorem 2.6.1). To do so we recall the following (cf. Lemma 4.1 [**27**]):

PROPOSITION 2.5.1 (cf. Tanaka-Tasaki [**27**, p. 164]). *Any real form of a compact Kähler manifold of positive holomorphic sectional curvature is connected.*

Proposition 2.5.1 leads to

COROLLARY 2.5.2. *Let $(K/R, \sigma, J, \boldsymbol{g})$ be a Hermitian symmetric space of a connected compact Lie group K. Suppose that there exists a $T \in \mathfrak{r}$ satisfying $R = C_K(T)$. Then, any real form of K/R is connected.*

PROOF. Let M be the real form of K/R by an involutive antiholomorphic isometry $\hat{\eta}$. Express K as $K = K_{\mathrm{ss}} \cdot Z(K)$, where K_{ss} denotes the semisimple part of K. Define a K_{ss}-equivariant diffeomorphism ι of $K_{\mathrm{ss}}/C_{K_{\mathrm{ss}}}(T)$ onto $K/C_K(T) = K/R$ by $\iota : aC_{K_{\mathrm{ss}}}(T) \mapsto aC_K(T)$. Then $(K_{\mathrm{ss}}/C_{K_{\mathrm{ss}}}(T), \sigma', J', \boldsymbol{g}')$ is a Hermitian symmetric space of the compact type, $\hat{\eta}'$ is an involutive antiholomorphic isometry of $K_{\mathrm{ss}}/C_{K_{\mathrm{ss}}}(T)$ and

$$\iota\bigl(\bigl(K_{\mathrm{ss}}/C_{K_{\mathrm{ss}}}(T)\bigr)^{\hat{\eta}'}\bigr) = M,$$

where $\sigma' := \sigma|_{K_{\mathrm{ss}}}$, $J' := d\iota^{-1} \circ J \circ d\iota$, $\boldsymbol{g}' := \iota^* \boldsymbol{g}$ and $\hat{\eta}' := \iota^{-1} \circ \hat{\eta} \circ \iota$. By Proposition 2.5.1 $(K_{\mathrm{ss}}/C_{K_{\mathrm{ss}}}(T))^{\hat{\eta}'}$ is connected, and so is M. $\square$

2.6. (II) Connectedness of real forms.

This subsection is devoted to demonstrating

THEOREM 2.6.1. *Let $(G/R, \sigma, J, \boldsymbol{g})$ be a simple irreducible pseudo-Hermitian symmetric space, where $Z(G)$ is trivial. Then, any real form of G/R is connected.*

PROOF. Let M be the real form of G/R by an involutive antiholomorphic isometry $\hat{\eta}$. Lemma 2.3.6 allows us to assume $\hat{\eta}(o) = o$. By Lemma 2.3.1 there exists a unique $\eta \in \mathrm{Inv}(G)$ such that

$$\text{(i) } [\sigma, \eta] = 0, \quad \text{(ii) } \eta(R) = R, \quad \text{(iii) } \hat{\eta} \circ \mathrm{Pr} = \mathrm{Pr} \circ \eta.$$

Now, let us prove that $M = (G/R)^{\hat{\eta}}$ is connected. It suffices to confirm (S2), (S3) and (S4):

(S1) There exists a Cartan involution θ of G satisfying $[\theta, \sigma] = [\theta, \eta] = 0$.
(S2) $\hat{\eta}(K/(K \cap R)) \subset K/(K \cap R)$, where $K := G^{\theta}$.
(S3) $(K/(K \cap R))^{\hat{\eta}}$ is connected.
(S4) Any $q \in M$ can be joined to an element of $(K/(K \cap R))^{\hat{\eta}}$ by an arc in M.

(S1) Since $Z(G) = \{e\}$ we have $G = \mathrm{Int}(\mathfrak{g})$. Hence, for any $\bar{\phi} \in \mathrm{Aut}(\mathfrak{g})$ there exists a unique $\phi \in \mathrm{Aut}(G)$ satisfying $d\phi = \bar{\phi}$, indeed one can get ϕ by setting $\phi := \bar{\phi} \circ g \circ \bar{\phi}^{-1}$ for $g \in G$. Consequently, the above (i) and Lemma 2.7 [**20**, p. 71] assure that there exists a Cartan involution θ of G satisfying

$$[\theta, \sigma] = [\theta, \eta] = 0. \tag{2.6.1}$$

(S2) It is immediate from $[\theta, \eta] = 0$, $K = G^{\theta}$ and (ii) that $\eta(K) \subset K$ and $\eta(K \cap R) \subset K \cap R$. Hence $\hat{\eta}(K/(K \cap R)) \subset K/(K \cap R)$ follows from (iii).

(S3) By Proposition 2.3.3 there exist the canonical central element $T \in \mathfrak{r}$ and a non-zero $\lambda \in \mathbb{R}$ which satisfy

$$\text{(iv) } R = C_G(T), \quad \text{(v) } \boldsymbol{j} = \mathrm{ad}\, T|_{\mathfrak{q}},$$
$$\text{(vi) } \langle X, Y \rangle = \lambda \cdot B_{\mathfrak{g}}(X, Y) \text{ for any } X, Y \in \mathfrak{q}.$$

Then Lemma 2.3.5, combined with (2.6.1) and (i), enables us to have $d\theta(T) = \pm T$ and $d\eta(T) = \pm T$; and therefore

$$d\theta(T) = T, \quad d\eta(T) = -T \tag{2.6.2}$$

because T is non-zero elliptic, $\hat{\eta}$ is antiholomorphic and (v). Now, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition by $d\theta$, where $\mathfrak{k} := \mathfrak{g}^{d\theta}$ and $\mathfrak{p} := \mathfrak{g}^{-d\theta}$. It follows from $Z(G) = \{e\}$ and Theorem 1.1 [**4**, p. 252–253] that

(vii) $K = G^\theta$ is a connected compact subgroup of G with Lie algebra $\mathfrak{k}$;
(viii) a map $\mu : (k, Y) \mapsto k \cdot \exp Y$ is a diffeomorphism of $K \times \mathfrak{p}$ onto G.

Therefore (iv), (v), (2.6.1) and (2.6.2) imply that

$(K/(K \cap R), \sigma_k, J_k, \boldsymbol{g}_k)$ is a compact Hermitian symmetric space satisfying $T \in \mathfrak{k} \cap \mathfrak{r}$ and $K \cap R = C_K(T)$, where $\sigma_k := \sigma|_K$, $J_k := J|_{K/(K\cap R)}$ and $\boldsymbol{g}_k := \boldsymbol{g}|_{K/(K\cap R)}$;
$\hat{\eta}|_{K/(K\cap R)}$ is an involutive antiholomorphic isometry of $(K/(K \cap R), J_k, \boldsymbol{g}_k)$.

Consequently, $\big(K/(K \cap R)\big)^{\hat{\eta}}$ is connected due to Corollary 2.5.2. Remark here that the metric $\boldsymbol{g}_k = \boldsymbol{g}|_{K/(K\cap R)}$ is positive or negative definite in view of (vi), and that $K/(K \cap R) = \{o\}$ when G/R is a Hermitian symmetric space.

(S4) The rest of proof is to confirm (S4). We will confirm it after proving

Lemma 2.6.2. *With the setting above; for any $q \in M = (G/R)^{\hat{\eta}}$ there exists a $b \in G$ such that $q = \mathrm{Pr}(b)$ and $b^{-1} \cdot \eta(b) \in K \cap R$.*

Proof. Since $q \in G/R$ there exists an $a \in G$ satisfying $q = \mathrm{Pr}(a)$. Let $r := a^{-1} \cdot \eta(a)$. Then $\hat{\eta}(q) = q$ and $\hat{\eta}(q) = \mathrm{Pr}(\eta(a))$ (cf. (iii)) yield

$$r = a^{-1} \cdot \eta(a) \in R \ (\subset G^\sigma), \quad \eta(r) = r^{-1}. \tag{2.6.3}$$

By (viii), there exist a unique $k \in K$ and a unique $Y \in \mathfrak{p}$ satisfying

$$r = k \cdot \exp Y.$$

From (2.6.3) we obtain $\sigma(r) = r$, and $\sigma(k) \cdot \exp d\sigma(Y) = \sigma(r) = r = k \cdot \exp Y$. Hence,

$$\sigma(k) = k, \quad d\sigma(Y) = Y \tag{2.6.4}$$

because it follows from $[\theta, \sigma] = 0$ and $K = G^\theta$ that $\sigma(k) \in K$ and $d\sigma(Y) \in \mathfrak{p}$. Since (2.6.4) one has $\exp(-Y) \in R$; and hence $k \in K$ and (2.6.3) yield

$$k = r \cdot \exp(-Y) \in K \cap R. \tag{2.6.5}$$

In a similar way, one can conclude that

$$\eta(k) = k^{-1}, \quad d\eta(Y) = -\operatorname{Ad}(k)Y \tag{2.6.6}$$

by virtue of $\eta(r) = r^{-1}$, $[\theta, \eta] = 0$ and $\operatorname{Ad}(k)Y \in \mathfrak{p}$. Now, set

$$r' := k \cdot \exp(Y/2).$$

By (2.6.4) we deduce $\exp(Y/2) \in R$, and thus (2.6.5) implies that $r' = k \cdot \exp(Y/2) \in R$. Therefore

$$q = \Pr(a) = \Pr(a \cdot r'). \tag{2.6.7}$$

A direct computation, together with $\eta(a) = a \cdot r$, provides

$$\begin{aligned} \eta(a \cdot r') &= a \cdot r \cdot \eta(r') \underset{(2.6.6)}{=} a \cdot r \cdot \big(k^{-1} \cdot \exp(-\operatorname{Ad}(k)Y/2)\big) \\ &= a \cdot (k \cdot \exp Y) \cdot \big(k^{-1} \cdot \exp(-\operatorname{Ad}(k)Y/2)\big) = a \cdot \exp\big(\operatorname{Ad}(k)Y/2\big) \\ &= (a \cdot r') \cdot k^{-1}. \end{aligned}$$

This, (2.6.5) and (2.6.7) enable us to conclude Lemma 2.6.2 by $b := a \cdot r'$. □

Now, we are in a position to confirm (S4): *any $q \in M = (G/R)^{\hat{\eta}}$ can be joined to an element of $(K/(K \cap R))^{\hat{\eta}}$ by an arc in M.* By Lemma 2.6.2 there exists a $b \in G$ satisfying $q = \Pr(b)$ and $b^{-1} \cdot \eta(b) \in K \cap R$. Let $k := b^{-1} \cdot \eta(b)$. Needless to say, $k \in K \cap R$. By (viii) there exist a unique $k' \in K$ and a unique $Y \in \mathfrak{p}$ such that

$$b = k' \cdot \exp Y.$$

Then $\eta(k') \cdot \exp d\eta(Y) = \eta(b) = b \cdot k = k' \cdot \exp Y \cdot k = (k' \cdot k) \cdot \exp(\operatorname{Ad}(k^{-1})Y)$. So,

$$\eta(k') = k' \cdot k, \quad d\eta(Y) = \operatorname{Ad}(k^{-1})Y \tag{2.6.8}$$

follows from (viii), $\eta(k'), k' \cdot k \in K$ and $d\eta(Y), \operatorname{Ad}(k^{-1})Y \in \mathfrak{p}$. This gives

$$d\eta\big(\operatorname{Ad}(k')Y\big) = \operatorname{Ad}\big(\eta(k')\big)\big(d\eta(Y)\big) = \operatorname{Ad}(k' \cdot k)\big(\operatorname{Ad}(k^{-1})Y\big) = \operatorname{Ad}(k')Y,$$

and $\operatorname{Ad}(k')Y \in \mathfrak{g}^{d\eta}$. Hence, the whole 1-parameter subgroup $\{\exp(-t\operatorname{Ad}(k')Y) \mid t \in \mathbb{R}\}$ lies in $(G^{\eta})_0$. Set $c(t) := \exp(-t\operatorname{Ad}(k')Y)$. Now, denote by C_q the connected component of $M = (G/R)^{\hat{\eta}}$ containing $q = \Pr(b)$. It can be expressed as $C_q = \Pr((G^{\eta})_0 \cdot b)$ (cf. (i), (ii), (iii), Corollary 2.4.5). Hence $\Pr\big(c(t) \cdot b\big)$ lies in C_q for all $t \in \mathbb{R}$, and satisfies $\Pr\big(c(0) \cdot b\big) = \Pr(b) = q$ and

$$\begin{aligned} \Pr\big(c(1) \cdot b\big) &= \Pr\big(\exp(-\operatorname{Ad}(k')Y) \cdot b\big) \\ &= \Pr\big(\exp(-\operatorname{Ad}(k')Y) \cdot k' \cdot \exp Y\big) = \Pr(k') \in \Pr(K). \end{aligned}$$

Since $k \in K \cap R$, (2.6.8) and (iii), we have $\hat{\eta}(\Pr(k')) = \Pr(k' \cdot k) = \Pr(k')$. This means that $\Pr(k')$ belongs to $(K/(K \cap R))^{\hat{\eta}}$ because $k \in K \cap R$. For this reason, $q = \Pr(b)$ can be joined to $\Pr(k') \in (K/(K \cap R))^{\hat{\eta}}$ by the arc $\Pr(c(t) \cdot b)$ in C_q ($\subset M$). This completes the proof of Theorem 2.6.1. □

2.7. Reduction.

In this subsection, we see that one can achieve the classification of real forms M of G/R by arguments in the Lie algebra level (cf. Theorem 2.7.8). Here G/R are simple irreducible pseudo-Hermitian symmetric spaces, where $Z(G)$ are trivial.

2.7.1. An equivalence relation on $\mathcal{R}_G$.

Throughout Subsection 2.7, we always regard G as a connected simple Lie group such that

(1) $Z(G)$ is trivial,
(2) $\mathfrak{g} = \operatorname{Lie} G$ is a real form of a complex simple Lie algebra.

Remark 2.7.1. (i) In view of (1), we may identify $\operatorname{Aut}(G)$ with $\operatorname{Aut}(\mathfrak{g})$. We impose this identification on Subsection 2.7.
(ii) Any pseudo-Hermitian symmetric space G/R of G is effective and irreducible by virtue of (1) and (2) (cf. Remark 2.2.3-(i)).

Denote by $\mathcal{R}_G$ the set of pairs $(G/R_i, M_i)$ of pseudo-Hermitian symmetric spaces G/R_i and real forms M_i of G/R_i. Let us recall the equivalence relation on $\mathcal{R}_G$ (cf. Section 1).

Definition 2.7.2. $(G/R_1, M_1), (G/R_2, M_2) \in \mathcal{R}_G$ are said to be *equivalent*, if there exists a holomorphic homothety f of G/R_1 onto G/R_2 satisfying $f(M_1) = M_2$. Henceforth, we denote by $\mathcal{R}_G/_{\simeq}$ the set of equivalence classes on $\mathcal{R}_G$.

2.7.2. An equivalence relation on $d\mathcal{R}_{\mathfrak{g}}$.

Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra. Consider a triplet $(\mathfrak{g}, T, \eta)$ consisting of an *Spr*-element T of $\mathfrak{g}$ (see Remark 2.3.4-(ii)) and $\eta \in \operatorname{Inv}(\mathfrak{g})$ such that $\eta(T) = -T$; and denote by $d\mathcal{R}_{\mathfrak{g}}$ the set of such triplets.

Remark 2.7.3. The following items hold for any $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ (cf. Lemma 3.1.1-(1), (2) in [**2**, p. 22] and its proof):

(1) $\sigma_T := \exp \pi \operatorname{ad} T$ is involutive;
(2) $\mathfrak{c}_{\mathfrak{g}}(T)$ and $[T, \mathfrak{g}]$ coincide with $\mathfrak{g}^{\sigma_T}$ and $\mathfrak{g}^{-\sigma_T}$, respectively;
(3) $(\mathfrak{g}, \sigma_T, \boldsymbol{j}, \langle \cdot\,, \cdot \rangle)$ is a simple irreducible pseudo-Hermitian symmetric Lie algebra, where $\boldsymbol{j} := \operatorname{ad} T|_{[T,\mathfrak{g}]}$ and $\langle X, Y \rangle := B_{\mathfrak{g}}(X, Y)$ for $X, Y \in [T, \mathfrak{g}]$;
(4) T is the canonical central element of $\mathfrak{g}^{\sigma_T}$;
(5) $[\sigma_T, \eta] = 0$;
(6) η is an involutive antiholomorphic isometry of $(\mathfrak{g}, \sigma_T, \boldsymbol{j}, \langle \cdot\,, \cdot \rangle)$.

Here (5) and (6) follow from $\eta(T) = -T$.

Let us define an equivalence relation on $d\mathcal{R}_{\mathfrak{g}}$ as follows:

Definition 2.7.4. For $(\mathfrak{g}, T_1, \eta_1), (\mathfrak{g}, T_2, \eta_2) \in d\mathcal{R}_{\mathfrak{g}}$ we say that they are *equivalent*, if there exists a $\phi \in \operatorname{Aut}(\mathfrak{g})$ satisfying

$$\text{(i) } \phi(T_1) = T_2, \quad \text{(ii) } \phi \circ \eta_1 \circ \phi^{-1} = \eta_2.$$

Henceforth, we denote by $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ the set of equivalence classes on $d\mathcal{R}_{\mathfrak{g}}$.

2.7.3. A correspondence between $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ and $\mathcal{R}_G/_{\simeq}$.

We aim to obtain a bijective map $F : d\mathcal{R}_{\mathfrak{g}}/_{\sim} \to \mathcal{R}_G/_{\simeq}$. First, let us introduce a way to construct an element of $\mathcal{R}_G$ from $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$:

LEMMA 2.7.5. *For any $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$, the following four steps provide an element $(G/R, M) \in \mathcal{R}_G$, where G denotes the connected Lie group such that $Z(G)$ is trivial and* $\operatorname{Lie} G = \mathfrak{g}$:

(s1) *Define $R := C_G(T)$;*
(s2) *Equip G/R with a pseudo-Hermitian structure $\{J, \boldsymbol{g}\}$ induced by* $\operatorname{ad} T$ *and $B_{\mathfrak{g}}$;*
(s3) *Define $\hat{\eta}(gR) := \eta(g)R$ for $gR \in G/R$;*
(s4) *Then, $(G/R, M)$ belongs to $\mathcal{R}_G$, where $M := (G/R)^{\hat{\eta}}$.*

Here, we identify $\operatorname{Aut}(G)$ *with* $\operatorname{Aut}(\mathfrak{g})$ *in view of $Z(G) = \{e\}$.*

PROOF. $(G/R, \sigma_T, J, \boldsymbol{g})$ is a pseudo-Hermitian symmetric space by Lemma 3.1.1-(3) in [**2**, p. 22]. Therefore, the rest of proof is to verify that $\hat{\eta}$ is an involutive antiholomorphic isometry of G/R. However, that is immediate from $\eta(T) = -T$ and (s2). □

Lemma 2.7.6 (below) allows us to define a map $F : d\mathcal{R}_{\mathfrak{g}}/_{\sim} \to \mathcal{R}_G/_{\simeq}$ by

$$F : [(\mathfrak{g}, T, \eta)] \mapsto [(G/R, M)], \tag{2.7.1}$$

where $(G/R, M)$ denotes the element of $\mathcal{R}_G$ constructed from $(\mathfrak{g}, T, \eta)$ in the way of Lemma 2.7.5.

LEMMA 2.7.6. *Let $(\mathfrak{g}, T_a, \eta_a) \in d\mathcal{R}_{\mathfrak{g}}$ and let $(G/R_a, M_a)$ denote the element of $\mathcal{R}_G$ constructed from $(\mathfrak{g}, T_a, \eta_a)$ in the way of Lemma 2.7.5, for $a = 1, 2$. If $(\mathfrak{g}, T_1, \eta_1)$ is equivalent to $(\mathfrak{g}, T_2, \eta_2)$, then $(G/R_1, M_1)$ is equivalent to $(G/R_2, M_2)$.*

PROOF. By Definition 2.7.4 there exists a $\phi \in \operatorname{Aut}(\mathfrak{g})$ such that (i) $\phi(T_1) = T_2$ and (ii) $\phi \circ \eta_1 \circ \phi^{-1} = \eta_2$. Define a map $f : G/R_1 \to G/R_2$ by $f(gR_1) := \phi(g)R_2$, where f is well-defined by (i). Then f is a holomorphic isometry of G/R_1 onto G/R_2 satisfying $f(M_1) = M_2$ by virtue of (i), (ii). □

The following lemma assures that $F : [(\mathfrak{g}, T, \eta)] \mapsto [(G/R, M)]$ is injective:

LEMMA 2.7.7. *Let $(\mathfrak{g}, T_a, \eta_a) \in d\mathcal{R}_{\mathfrak{g}}$ and let $(G/R_a, M_a)$ denote the element of $\mathcal{R}_G$ constructed from $(\mathfrak{g}, T_a, \eta_a)$ in the way of Lemma 2.7.5, for $a = 1, 2$. If $(G/R_1, M_1)$ is equivalent to $(G/R_2, M_2)$, then $(\mathfrak{g}, T_1, \eta_1)$ is equivalent to $(\mathfrak{g}, T_2, \eta_2)$.*

PROOF. By Definition 2.7.2 there exists a holomorphic homothety f_1 of G/R_1 onto G/R_2 satisfying $f_1(M_1) = M_2$. We want to first show that

there exists a holomorphic homothety f_2 of G/R_1 onto G/R_2 satisfying $f_2(M_1) = M_2$ and $f_2(o_1) = o_2$, (2.7.2) where o_a is the origin of G/R_a. Note that $[\sigma_a, \eta_a] = 0$, $\eta_a(R_a) = R_a$ and $o_a \in M_a$ for all $a = 1, 2$, where $\sigma_a := \sigma_{T_a}$. Theorems 2.4.1 and 2.6.1 imply that $M_2 = \mathrm{Pr}_2((G^{\eta_2})_0)$. Since $f_1(o_1), o_2 \in M_2$ there exists a $g \in (G^{\eta_2})_0$ satisfying $\tau_g(f_1(o_1)) = o_2$. Accordingly one deduces (2.7.2) by setting $f_2 := \tau_g \circ f_1$. Since (2.7.2) and Lemma 2.3.1 there exists a unique $\phi \in \mathrm{Aut}(G)$ satisfying

$$\text{(i) } \phi \circ \sigma_1 = \sigma_2 \circ \phi, \quad \text{(ii) } \phi(R_1) = R_2, \quad \text{(iii) } f_2 \circ \mathrm{Pr}_1 = \mathrm{Pr}_2 \circ \phi.$$

From Lemma 2.3.5 and (i) we obtain $\phi(T_1) = T_2$ or $\phi(T_1) = -T_2$. Therefore $\phi(T_1) = T_2$ follows from (iii), f_2 being holomorphic and Lemma 2.7.5-(s2). Consequently, the rest of proof is to conclude that

$$\phi \circ \eta_1 = \eta_2 \circ \phi. \tag{2.7.3}$$

Denote by $\mathfrak{q}_a$ the -1-eigenspace of σ_a in $\mathfrak{g}$, and decompose it as $\mathfrak{q}_a = \mathfrak{q}_a^+ \oplus \mathfrak{q}_a^-$, where $\mathfrak{q}_a^\pm := \{Z \in \mathfrak{q}_a \mid \eta_a(Z) = \pm Z\}$. Then

$$T_{o_a}(G/R_a) = \mathfrak{q}_a, \quad T_{o_a} M_a = \mathfrak{q}_a^+, \quad B_{\mathfrak{g}}(\mathfrak{q}_a^+, \mathfrak{q}_a^-) = \{0\} \text{ for each } a = 1, 2.$$

Therefore the above (iii) and (2.7.2) imply that $\phi(\mathfrak{q}_1^+) = \mathfrak{q}_2^+$. Moreover, $\phi(\mathfrak{q}_1^-) = \mathfrak{q}_2^-$ follows from $\phi(\mathfrak{q}_1) = \mathfrak{q}_2$, $\mathfrak{q}_a = \mathfrak{q}_a^+ \oplus \mathfrak{q}_a^-$ and $B_{\mathfrak{g}}$ being non-degenerate on $\mathfrak{q}_a$. These $\phi(\mathfrak{q}_1^+) = \mathfrak{q}_2^+$, $\phi(\mathfrak{q}_1^-) = \mathfrak{q}_2^-$ assure that $\phi \circ \eta_1 = \eta_2 \circ \phi$ on $\mathfrak{q}_1 = \mathfrak{q}_1^+ \oplus \mathfrak{q}_1^-$. Hence (2.7.3) holds because $\mathfrak{g} = [\mathfrak{q}_1, \mathfrak{q}_1] \oplus \mathfrak{q}_1$ (cf. Nomizu [**21**, p. 56, (16.2)]). $\square$

Now, let us verify that F is bijective:

Theorem 2.7.8. *The map* $F : [(\mathfrak{g}, T, \eta)] \mapsto [(G/R, M)]$ *is a bijection from* $d\mathcal{R}_{\mathfrak{g}}/_\sim$ *onto* $\mathcal{R}_G/_\simeq$ *(see* (2.7.1) *for* F*).*

Proof. Lemma 2.7.7 implies that F is injective. So, we only prove that F is surjective. Take any $[(G/R, M)] \in \mathcal{R}_G/_\simeq$. First, let us construct an element $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ from $(G/R, M)$. Let σ be the involution of $\mathfrak{g} = \mathrm{Lie}\, G$ satisfying $\mathfrak{r} = \mathfrak{g}^\sigma$, and $\{J, \boldsymbol{g}\}$ the pseudo-Hermitian structure on G/R. On the one hand, Proposition 2.3.3 allows us to assume that

$$\text{(1) } R = C_G(T), \quad \text{(2) } J \text{ is induced by } \mathrm{ad}\, T, \quad \text{(3) } \boldsymbol{g} \text{ is induced by } B_{\mathfrak{g}},$$

up to holomorphic homothety, where T denotes the canonical central element of $\mathfrak{g}^\sigma$. On the other hand, Lemma 2.3.6 allows us to assume $o \in M$ up to holomorphic homothety. Since M is a real form, there exists an involutive antiholomorphic isometry $\hat{\eta}$ of G/R satisfying $M = (G/R)^{\hat{\eta}}$. Then $\hat{\eta}(o) = o$ and Lemma 2.3.1 enable us to have a unique $\eta \in \mathrm{Inv}(G)$ satisfying (4) $[\sigma, \eta] = 0$, (5) $\eta(R) = R$ and (6) $\hat{\eta}(gR) = \eta(g)R$ for all $gR \in G/R$. By (4) and Lemma 2.3.5 one has $\eta(T) = T$ or $\eta(T) = -T$, and hence $\eta(T) = -T$ follows from (2), (6) and $\hat{\eta}$ being antiholomorphic. Consequently $(\mathfrak{g}, T, \eta)$ belongs to $d\mathcal{R}_{\mathfrak{g}}$. Furthermore, up to holomorphic homothety, $(G/R, M)$ coincides with the element of $\mathcal{R}_G$ constructed from $(\mathfrak{g}, T, \eta)$ in the way of Lemma 2.7.5 because (1)–(3) and (6). Therefore $F([(\mathfrak{g}, T, \eta)]) = [(G/R, M)]$, and hence F is surjective. $\square$

The main purpose of this paper is to classify the real forms M of simple irreducible pseudo-Hermitian symmetric spaces G/R with G non-compact (cf. Theorem 1.0.1). Theorem 2.7.8 implies that we can accomplish the purpose by classifying the elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ for each non-compact real form $\mathfrak{g}$ of complex simple Lie algebras.

3. Cartan decompositions and Root systems.

In this section we review elementary facts about a Cartan decomposition and the root theory.

3.1. Cartan decompositions.

Let $\mathfrak{l}_{\mathbb{C}}$ be a complex semisimple Lie algebra, let $\mathfrak{l}_u$ be a compact real form of $\mathfrak{l}_{\mathbb{C}}$, and let $\theta \in \mathrm{Inv}(\mathfrak{l}_u)$. Decompose $\mathfrak{l}_u$ as

$$\mathfrak{l}_u = \mathfrak{k} \oplus i\mathfrak{p},$$

where $\mathfrak{k} := \mathfrak{l}_u^{\theta}$ and $i\mathfrak{p} := \mathfrak{l}_u^{-\theta}$. Then one can get a non-compact real form $\mathfrak{l}$ of $\mathfrak{l}_{\mathbb{C}}$ by

$$\mathfrak{l} := \mathfrak{k} \oplus \mathfrak{p}.$$

It is known that $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition and $\mathfrak{k}$ $(= \mathfrak{l} \cap \mathfrak{l}_u)$ is a maximal compact subalgebra of $\mathfrak{l}$ (cf. Helgason [**4**, p. 235]).

Remark 3.1.1. Henceforth, we assume that each non-compact real form $\mathfrak{l} \subset \mathfrak{l}_{\mathbb{C}}$ is related with a compact real form $\mathfrak{l}_u \subset \mathfrak{l}_{\mathbb{C}}$ and $\theta \in \mathrm{Inv}(\mathfrak{l}_u)$ as in the formulae

$$\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{l}_u = \mathfrak{k} \oplus i\mathfrak{p}, \quad \mathfrak{k} = \mathfrak{l}_u^{\theta}, \quad i\mathfrak{p} = \mathfrak{l}_u^{-\theta}.$$

Moreover, we assume that $\mathrm{Aut}(\mathfrak{l}) = \{\phi \in \mathrm{Aut}(\mathfrak{l}_{\mathbb{C}}) \mid \phi(\mathfrak{l}) \subset \mathfrak{l}\}$ and $\mathrm{Aut}(\mathfrak{l}_u) = \{\psi \in \mathrm{Aut}(\mathfrak{l}_{\mathbb{C}}) \mid \psi(\mathfrak{l}_u) \subset \mathfrak{l}_u\}$. Under this assumption, θ is a Cartan involution of $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$.

We will need the following proposition later:

Proposition 3.1.2 (cf. Murakami [**17**, p. 106]). *With the setting above*; *for any* $\psi \in \mathrm{Aut}(\mathfrak{l}_u)$, *the following* (i), (ii) *and* (iii) *are equivalent*:

$$\text{(i) } \psi \circ \theta = \theta \circ \psi, \quad \text{(ii) } \psi \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u), \quad \text{(iii) } \psi(\mathfrak{k}) \subset \mathfrak{k}.$$

3.2. Root systems and compact real forms.

Let $\mathfrak{l}_{\mathbb{C}}$ be a complex semisimple Lie algebra, $\mathfrak{c}_{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{l}_{\mathbb{C}}$, and $\triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ the set of non-zero roots of $\mathfrak{l}_{\mathbb{C}}$ with respect to $\mathfrak{c}_{\mathbb{C}}$. For each $\alpha \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$, there exists a unique $H_\alpha \in \mathfrak{c}_{\mathbb{C}}$ such that $B_{\mathfrak{l}_{\mathbb{C}}}(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{c}_{\mathbb{C}}$. Then there exists a basis (so-called, *Weyl basis*) $\{X_\alpha \mid \alpha \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})\}$ of $\mathfrak{l}_{\mathbb{C}}$ mod $\mathfrak{c}_{\mathbb{C}}$ such that

$$\begin{aligned}[X_\alpha, X_{-\alpha}] &= H_\alpha, && [H, X_\alpha] = \alpha(H)\cdot X_\alpha \text{ for any } H \in \mathfrak{c}_{\mathbb{C}};\\ [X_\alpha, X_\beta] &= 0 && \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}});\\ [X_\alpha, X_\beta] &= N_{\alpha,\beta}\cdot X_{\alpha+\beta} && \text{if } \alpha+\beta \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}),\end{aligned}$$

where the non-zero real constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ (cf. Helgason [**4**, p. 176]). The Weyl basis gives rise to a compact real form $\mathfrak{l}_u$ of $\mathfrak{l}_{\mathbb{C}}$ as follows:

$$\mathfrak{l}_u := i\mathfrak{c}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\triangle(\mathfrak{l}_{\mathbb{C}},\mathfrak{c}_{\mathbb{C}})} \mathrm{span}_{\mathbb{R}}\{X_\alpha - X_{-\alpha}\} \oplus \mathrm{span}_{\mathbb{R}}\{i(X_\alpha + X_{-\alpha})\} \tag{3.2.1}$$

(cf. Helgason [**4**, p. 182, (2)]), where $\mathfrak{c}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H_\alpha \mid \alpha \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})\}$.

From now on, we are going to state Proposition 3.2.1 and Lemma 3.2.2 needed later. First, let us recall

Proposition 3.2.1 (e.g. Murakami [**19**, p. 295]). *Let $\{\alpha_i\}_{i=1}^l$ and $\{\beta_i\}_{i=1}^l$ be fundamental systems of $\triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$, and $\bar\phi$ a linear isomorphism of $\mathfrak{c}_{\mathbb{C}}$ defined by ${}^t\bar\phi(\beta_i) := \alpha_i$ for $1 \leq i \leq l$. Suppose that for every $1 \leq i, j \leq l$,*

$$B_{\mathfrak{l}_{\mathbb{C}}}(H_{\alpha_i}, H_{\alpha_j})/B_{\mathfrak{l}_{\mathbb{C}}}(H_{\alpha_j}, H_{\alpha_j}) = B_{\mathfrak{l}_{\mathbb{C}}}(H_{\beta_i}, H_{\beta_j})/B_{\mathfrak{l}_{\mathbb{C}}}(H_{\beta_j}, H_{\beta_j}).$$

Then, there exists a unique $\phi \in \mathrm{Aut}(\mathfrak{l}_{\mathbb{C}})$ satisfying

$$\text{(i) } \phi|_{\mathfrak{c}_{\mathbb{C}}} = \bar\phi, \quad \text{(ii) } \phi(X_{\pm\alpha_i}) = X_{\pm\beta_i} \textit{ for any } 1 \leq i \leq l.$$

Next, we prove

Lemma 3.2.2. *Let ϕ be the automorphism of $\mathfrak{l}_{\mathbb{C}}$ in Proposition* 3.2.1*, and $\mathfrak{l}_u$ the compact real form of $\mathfrak{l}_{\mathbb{C}}$ given by* (3.2.1)*. Then, $\phi(\mathfrak{l}_u) \subset \mathfrak{l}_u$.*

Proof. Let τ denote the complex conjugation of $\mathfrak{l}_{\mathbb{C}}$ with respect to $\mathfrak{l}_u$. Then $\phi(\mathfrak{l}_u) \subset \mathfrak{l}_u$ follows from $\phi\circ\tau = \tau\circ\phi$. Therefore it suffices to show that

$$\phi\big(\tau(X_{\pm\alpha_i})\big) = \tau\big(\phi(X_{\pm\alpha_i})\big) \text{ for all } 1 \leq i \leq l.$$

It is known by Helgason [**4**, p. 421] that $\tau(X_\alpha) = -X_{-\alpha}$ for all $\alpha \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$. This gives us $\tau(\phi(\tau(X_{\pm\alpha_i}))) = -\tau(\phi(X_{\mp\alpha_i})) = -\tau(X_{\mp\beta_i}) = X_{\pm\beta_i} = \phi(X_{\pm\alpha_i})$. □

3.3. Restricted root systems of maximal compact subalgebras.

In Section 4 we will classify the elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ by considering restricted root systems $\Sigma(\mathfrak{k}, \mathfrak{a})$, where $\mathfrak{k}$ are maximal compact subalgebras of $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. So, we are going to review elementary facts about restricted roots.

Let $\mathfrak{k}$ be a compact Lie algebra, where $\mathfrak{k}$ is not necessary semisimple, and let $\eta \in \mathrm{Inv}(\mathfrak{k})$. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{k}^{-\eta}$ and denote by $\Sigma(\mathfrak{k}, \mathfrak{a})$ the set of non-zero restricted roots of $\mathfrak{k}$ with respect to $\mathfrak{a}$ (cf. Loos [**16**, p. 58]). Here, a non-zero linear form $\gamma : \mathfrak{a} \to \mathbb{R}$ belongs to $\Sigma(\mathfrak{k}, \mathfrak{a})$ if and only if there exists a non-zero $Z \in \mathfrak{k}_{\mathbb{C}}$ such that $[A, Z] = i\gamma(A) \cdot Z$ for all $A \in \mathfrak{a}$. Now, let $\mathfrak{t}$ denote a maximal abelian subalgebra of $\mathfrak{k}$ containing $\mathfrak{a}$, let $\triangle(\mathfrak{k}, \mathfrak{t})$ denote the set of non-zero roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$, and let $W_{\triangle}$ denote the Weyl group of $\triangle(\mathfrak{k}, \mathfrak{t})$. Then, the Weyl group W_{Σ} of $\Sigma(\mathfrak{k}, \mathfrak{a})$ is related with $W_{\triangle}$ as in the formula

$$W_{\Sigma} = \{\overline{w} \mid w \in W_{\eta}\} \tag{3.3.1}$$

(cf. Satake [**24**, p. 107]), where $W_{\eta} := \{w \in W_{\triangle} \mid [\eta, w] = 0\}$ and $\overline{w} := w|_{\mathfrak{a}}$.

Lemma 3.3.1. *With the setting above; let $\Pi(\mathfrak{k}, \mathfrak{a}) = \{\gamma_j\}_{j=1}^{p}$ be a restricted fundamental system of $\Sigma(\mathfrak{k}, \mathfrak{a})$ and let*

$$C_{\Pi(\mathfrak{k},\mathfrak{a})} := \{A \in \mathfrak{a} \mid \gamma_j(A) \geq 0 \textit{ for all } 1 \leq j \leq p\}.$$

Then, for any $T \in \mathfrak{k}^{-\eta}$ there exists a $\varphi \in \mathrm{Int}(\mathfrak{k}, \eta)$ such that $\varphi(T) \in C_{\Pi(\mathfrak{k},\mathfrak{a})}$.

Proof. Although $\mathfrak{k}$ is not necessary semisimple, one can assert that

$$\mathfrak{k}^{-\eta} = \bigcup \{\psi(\mathfrak{a}) \mid \psi \in \mathrm{Int}_{\mathfrak{k}}(\mathfrak{k}^{\eta})\} \tag{3.3.2}$$

(cf. Loos [**16**, p. 56]), where $\mathrm{Int}_{\mathfrak{k}}(\mathfrak{k}^{\eta})$ denotes the connected subgroup of $\mathrm{Int}(\mathfrak{k})$ which corresponds to the subalgebra $\mathrm{ad}_{\mathfrak{k}}(\mathfrak{k}^{\eta})$ of $\mathrm{ad}_{\mathfrak{k}}(\mathfrak{k})$. Since $\mathrm{Int}_{\mathfrak{k}}(\mathfrak{k}^{\eta}) \subset \mathrm{Int}(\mathfrak{k}, \eta)$ and (3.3.2) we may assume $T \in \mathfrak{a}$ from the beginning. Then, there exists a $\varphi \in W_{\Sigma}$ satisfying $\varphi(T) \in C_{\Pi(\mathfrak{k},\mathfrak{a})}$ because W_{Σ} acts transitively on the set of Weyl chambers. Remark that $\varphi \in \mathrm{Int}(\mathfrak{k}, \eta)$ follows from (3.3.1). □

4. The classification of elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$.

Our goal in this section is to determine $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ for each non-compact real form $\mathfrak{g}$ of complex simple Lie algebras. That and Theorem 2.7.8 provide us with the main Theorem 1.0.1.

4.1. Reduction.

According to Berger [**1**, Tableau II, p. 157–161], there does not exist any pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma)$ when $\mathfrak{g} = \mathfrak{sl}(2n{+}1, \mathbb{R})$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{g}_{2(2)}$[1]. For this reason we will only consider the case where $\mathfrak{g}$ is one of the following:

List II									
AI	$\mathfrak{sl}(2n, \mathbb{R})$	AII	$\mathfrak{su}^*(2n)$	AIII	$\mathfrak{su}(p, q)$	BDI	$\mathfrak{so}(p, q)$	DIII	$\mathfrak{so}^*(2n)$
CI	$\mathfrak{sp}(n, \mathbb{R})$	CII	$\mathfrak{sp}(p, q)$						
EII	$\mathfrak{e}_{6(2)}$	EIII	$\mathfrak{e}_{6(-14)}$	EV	$\mathfrak{e}_{7(7)}$	EVI	$\mathfrak{e}_{7(-5)}$	EVII	$\mathfrak{e}_{7(-25)}$

[1]There is a minor misprint in [**1**, p. 161, ↑ 15]: Read "E_7^3 $E_6^4 + \mathbb{R}$ $\boldsymbol{g}_1(E_6^4) \times \mathbb{R}$ réd." instead of "E_7^3 $E_6^4 + \mathbb{R}$ $\boldsymbol{g}_1(E_6^4) \times \mathbb{R}$ réd. 1/2kähl."

4.2. A way to determine $d\mathcal{R}_{\mathfrak{g}}/_\sim$.

We will start on determining $d\mathcal{R}_{\mathfrak{g}}/_\sim$ from the next subsection. Our aim in this subsection is to introduce a way to determine $d\mathcal{R}_{\mathfrak{g}}/_\sim$ (cf. Theorem 4.2.2). First, let us prove

LEMMA 4.2.1. *For any $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$, there exists a Cartan involution θ of $\mathfrak{g}$ satisfying $\theta(T) = T$ and $[\theta, \eta] = 0$.*

PROOF. Remark 2.7.3 implies that $\sigma_T, \eta \in \mathrm{Inv}(\mathfrak{g})$ and $[\sigma_T, \eta] = 0$. So, Lemma 2.7 [**20**, p. 71] enables us to have a Cartan involution θ of $\mathfrak{g}$ such that $[\theta, \sigma_T] = [\theta, \eta] = 0$. Lemma 2.3.5 and $[\theta, \sigma_T] = 0$ yield $\theta(T) = T$ or $\theta(T) = -T$. Hence $\theta(T) = T$, since T is non-zero elliptic. $\square$

THEOREM 4.2.2. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. The following seven steps enable us to determine $d\mathcal{R}_{\mathfrak{g}}/_\sim$:*

(S1) *Take a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_{\mathbb{C}}$ and $\theta \in \mathrm{Inv}(\mathfrak{g}_u)$ which satisfy $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ (see Subsection 3.1 for detail).*
(S2) *Settle an $\eta \in \mathrm{Inv}(\mathfrak{g})$ by following Berger's classification* [**1**, Tableau II, p. 157–161].
(S3) *Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{k}^{-\eta}$.*
(S4) *Choose every element A such that the eigenvalue of $\mathrm{ad}\, A$ in $\mathfrak{k}$ is $\pm i$ or zero, from among all elements in $C_{\Pi(\mathfrak{k},\mathfrak{a})}$ (see Lemma 3.3.1 for $C_{\Pi(\mathfrak{k},\mathfrak{a})}$).*
(S5) *Select every element T such that the eigenvalue of $\mathrm{ad}\, T$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\pm i$ or zero, from among the elements chosen in* (S4). *Then, $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$.*
(S6) *Find out all triplets which are equivalent to each other, among the triplets $(\mathfrak{g}, T, \eta)$ selected in* (S5).
(S7) *Repeat* (S2)–(S6) *until all involutions of $\mathfrak{g}$ are exhausted from* Tableau II.

Here we remark that one can perform (S3) *because η satisfies $\eta(\mathfrak{k}) \subset \mathfrak{k}$.*

PROOF. It suffices to confirm that any $(\mathfrak{g}, T_1, \eta_1) \in d\mathcal{R}_{\mathfrak{g}}$ is equivalent to $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ which satisfies conditions

(a) η is an involution in Tableau II;
(b) $T \in C_{\Pi(\mathfrak{k},\mathfrak{a})}$;
(c) the eigenvalue of $\mathrm{ad}\, T$ in $\mathfrak{k}$ is $\pm i$ or zero;
(d) the eigenvalue of $\mathrm{ad}\, T$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\pm i$ or zero.

By Lemma 4.2.1 there exists a Cartan involution θ' of $\mathfrak{g}$ such that $\theta'(T_1) = T_1$ and $[\theta', \eta_1] = 0$. Theorem 7.2 [**4**, p. 183] enables us to have a $\psi_1 \in \mathrm{Int}(\mathfrak{g})$ satisfying $\psi_1 \circ \theta' = \theta \circ \psi_1$. Then $(\mathfrak{g}, T_1, \eta_1)$ is equivalent to $(\mathfrak{g}, T_2, \eta_2)$ which satisfies

$$(1)\ \theta(T_2) = T_2, \quad (2)\ [\theta, \eta_2] = 0,$$

where $T_2 := \psi_1(T_1)$ and $\eta_2 := \psi_1 \circ \eta_1 \circ \psi_1^{-1}$. Due to Berger's classification [**1**] there exists a $\phi_2 \in \mathrm{Aut}(\mathfrak{g})$ such that $\phi_2 \circ \eta_2 \circ \phi_2^{-1}$ lies in Tableau II. By $[\theta, \eta_2] = 0$ and considering Berger's process of getting Tableau II, one may assume that ϕ_2 also satisfies $[\theta, \phi_2] = 0$. Hence, $(\mathfrak{g}, T_2, \eta_2)$ is equivalent to $(\mathfrak{g}, T_3, \eta)$ which satisfies

$$(1)\ \theta(T_3) = T_3, \quad (2)\ [\theta, \eta] = 0, \quad (a)\ \eta \text{ is an involution in Tableau II},$$

where $T_3 := \phi_2(T_2)$ and $\eta := \phi_2 \circ \eta_2 \circ \phi_2^{-1}$. Remark that $\mathrm{Int}(\mathfrak{k}, \eta) \subset \mathrm{Int}(\mathfrak{g}, \theta, \eta)$. Since $T_3 \in \mathfrak{k}^{-\eta}$ and Lemma 3.3.1, there exists a $\psi_3 \in \mathrm{Int}(\mathfrak{g}, \theta, \eta)$ satisfying $\psi_3(T_3) \in C_{\Pi(\mathfrak{k},\mathfrak{a})}$. Then $(\mathfrak{g}, T_3, \eta)$ is equivalent to $(\mathfrak{g}, T, \eta)$ which satisfies

$$(b)\ T \in C_{\Pi(\mathfrak{k},\mathfrak{a})}, \quad (2)\ [\theta, \eta] = 0, \quad (a)\ \eta \text{ is an involution in Tableau II},$$

where $T := \psi_3(T_3)$, in terms of $\eta = \psi_3 \circ \eta \circ \psi_3^{-1}$. The rest of proof is to conclude (c) and (d). (d) is immediate from T being an *Spr*-element of $\mathfrak{g}$. (c) comes from $T \in \mathfrak{k}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $[T, \mathfrak{k}] \subset \mathfrak{k}$ and (d). □

The following lemma enables us to perform (S5) in Theorem 4.2.2 more easily:

LEMMA 4.2.3. *Let $\mathfrak{l}$ be a real semisimple Lie algebra, T a semisimple element of $\mathfrak{l}$, and $\mathfrak{c}$ a Cartan subalgebra of $\mathfrak{l}$ containing T. Then, the following* (i), (ii) *and* (iii) *are equivalent*:

(i) *The eigenvalue of* $\mathrm{ad}\, T$ *in* $\mathfrak{l}$ *is* $\pm i$ *or zero*;
(ii) $[T, [T, [T, X]]] = -[T, X]$ *for all* $X \in \mathfrak{l}$;
(iii) $\alpha(T) = \pm i$ *or zero for each* $\alpha \in \triangle(\mathfrak{l}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$.

PROOF. Since T is semisimple, one can decompose $\mathfrak{l}$ as

$$\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(T) \oplus [T, \mathfrak{l}].$$

Hence, (i) if and only if $\mathrm{ad}\, T|_{[T,\mathfrak{l}]}$ is a complex structure of the vector space $[T, \mathfrak{l}]$ if and only if (ii). It is clear that (i) is equivalent to (iii). □

Now, we are in a position to classify the elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ for each Lie algebra $\mathfrak{g}$ in List II (see p. 55).

REMARK 4.2.4. In Subsections 4.3 through 4.14, we refer to Helgason [**4**, p. 444, p. 186] for notation $I_{p,q}$, J_n, $K_{p,q}$ and E_{ij}. In addition, we utilize the following notation:

$$A' := \begin{pmatrix} O & A \\ A & O \end{pmatrix}, \quad A \times B := \begin{pmatrix} A & O \\ O & B \end{pmatrix} \text{ for matrices } A, B;$$

$$S_m := \left.\begin{pmatrix} J_1 & & O \\ & \ddots & \\ O & & J_1 \end{pmatrix}\right\} 2m; \qquad \mathrm{adiag}(a_1, \ldots, a_n) := \begin{pmatrix} O & & a_1 \\ & \cdot^{\textstyle\cdot^{\textstyle\cdot}} & \\ a_n & & O \end{pmatrix}.$$

4.3. Type AI, $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$.

Our goal in this subsection is to determine $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ for $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$ (cf. Proposition 4.3.5). The arguments below will be mainly based on Theorem 4.2.2.

Let $\mathfrak{g}_u := \mathfrak{su}(2n)$ and

$$\theta(X) := -{}^tX \, (= \overline{X}) \text{ for } X \in \mathfrak{g}_u. \tag{4.3.1}$$

Construct a non-compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from them. Then $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(2n)$. Now, let us settle $\eta \in \mathrm{Inv}(\mathfrak{g})$ by following Berger's classification. According to Berger [**1**, p. 157], η is conjugate to one of the following:

$$\eta_1 := \mathrm{Ad}\, I_{p,q}, \quad \eta_2 := \theta \circ \eta_1, \quad \eta_3 := \mathrm{Ad}\, J_n, \quad \eta_4 := \theta \circ \eta_3, \tag{4.3.2}$$

where $p + q = 2n$ and $p \leq q$. Here

$$\begin{aligned} \mathfrak{g}^{\eta_1} &= \mathfrak{sl}(p, \mathbb{R}) \oplus \mathfrak{sl}(q, \mathbb{R}) \oplus \mathbb{R}, & \mathfrak{g}^{\eta_2} &= \mathfrak{so}(p, q), \\ \mathfrak{g}^{\eta_3} &= \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}, & \mathfrak{g}^{\eta_4} &= \mathfrak{sp}(n, \mathbb{R}). \end{aligned} \tag{4.3.3}$$

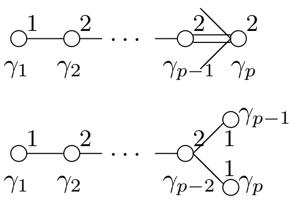
Let us consider a restricted root system $\Sigma(\mathfrak{k}, \mathfrak{a})$ for each η_a.

4.3.1. Case $\eta = \eta_1$.

Let $\eta := \eta_1$ (cf. (4.3.2)). It follows from (4.3.3) that $(\mathfrak{k}, \mathfrak{k}^\eta) = (\mathfrak{so}(2n), \mathfrak{so}(p) \oplus \mathfrak{so}(q))$, where $p + q = 2n$ and $p \leq q$; and hence the rank of $(\mathfrak{k}, \mathfrak{k}^\eta)$ equals p. So, a maximal abelian subspace $\mathfrak{a}_p \subset \mathfrak{k}^{-\eta}$ is given by the matrices

$$H_* = \left(\begin{array}{c|c} O_p & \mathrm{diag}(x_1, \ldots, x_p) \;\; 0 \\ \hline -\mathrm{diag}(x_1, \ldots, x_p) & \\ 0 & O_q \end{array} \right),$$

where $x_j \in \mathbb{R}$ and $\mathrm{diag}(x_1, \ldots, x_p)$ denotes the diagonal matrix with components $x_1, \ldots, x_p$. The Dynkin diagram of $\Sigma(\mathfrak{k}, \mathfrak{a}_p)$ is



B_p when $p < n$;

D_p when $p = n$. (4.3.4)

Here $\gamma_b(H_*) := x_b - x_{b+1}$ for $1 \leq b \leq p-1$ and $\gamma_p(H_*) := x_p$ if $p < n$ (resp. $\gamma_p(H_*) := x_{p-1} + x_p$ if $p = n$). Let us investigate two cases $p < n$ and $p = n$ individually.

Case $\eta = \eta_1$ and $p < n$. Let $\{T_j\}_{j=1}^p$ be the dual basis of $\Pi(\mathfrak{k}, \mathfrak{a}_p) = \{\gamma_j\}_{j=1}^p$, and A an element of $C_{\Pi(\mathfrak{k}, \mathfrak{a}_p)}$ such that the eigenvalue of $\mathrm{ad}\, A$ in $\mathfrak{k}$ is $\pm i$ or zero. Then $A = T_1$ only follows from (4.3.4). Here

$$T_1 = \left(\begin{array}{c|c} O_p & \mathrm{diag}(1, 0, \ldots, 0) \;\; 0 \\ \hline -\mathrm{diag}(1, 0, \ldots, 0) & \\ 0 & O_q \end{array} \right) = E_{1,p+1} - E_{p+1,1}. \tag{4.3.5}$$

Lemma 4.2.3 implies that the eigenvalue of $\operatorname{ad} T_1$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ cannot be $\pm i$ or zero because $X := E_{1,p+1} + E_{p+1,1}$ belongs to $\mathfrak{g}$ and $[T_1, [T_1, [T_1, X]]] \neq -[T_1, X]$. Thus, $d\mathcal{R}_{\mathfrak{g}} = \emptyset$ in case $\eta = \eta_1$ and $p < n$.

Case $\eta = \eta_1$ and $p = n$. Let $\{T_j\}_{j=1}^n$ be the dual basis of $\Pi(\mathfrak{k}, \mathfrak{a}_n) = \{\gamma_j\}_{j=1}^n$, and A an element of $C_{\Pi(\mathfrak{k},\mathfrak{a}_n)}$ such that the eigenvalue of $\operatorname{ad} A$ in $\mathfrak{k}$ is $\pm i$ or zero. Then $A = T_1$, T_{n-1} or T_n in view of (4.3.4). Here T_1 is the same as (4.3.5), and

$$T_{n-1} = \frac{1}{2}\left(\begin{array}{c|c} O_n & -I_{n-1,1} \\ \hline I_{n-1,1} & O_n \end{array}\right), \quad T_n = \frac{1}{2}\left(\begin{array}{c|c} O_n & I_n \\ \hline -I_n & O_n \end{array}\right) = \frac{1}{2} J_n. \tag{4.3.6}$$

We have already known that the eigenvalue of $\operatorname{ad} T_1$ in $\mathfrak{g}$ cannot be $\pm i$ or zero. By direct computation and Lemma 4.2.3-(ii) we see that the eigenvalue of $\operatorname{ad} T_{n-1}$ (resp. $\operatorname{ad} T_n$) in $\mathfrak{g}$ is $\pm i$ or zero. For this reason $(\mathfrak{g}, T_{n-1}, \eta), (\mathfrak{g}, T_n, \eta) \in d\mathcal{R}_{\mathfrak{g}}$. Now, define a $\varphi \in \operatorname{Aut}(\mathfrak{g})$ by $\varphi := \eta \circ \operatorname{Ad} I_{n-1,n+1}$. It is clear that $[\theta, \varphi] = [\eta, \varphi] = 0$ and

$$\varphi(T_{n-1}) = T_n.$$

Hence $(\mathfrak{g}, T_{n-1}, \eta)$ is equivalent to $(\mathfrak{g}, T_n, \eta)$. Consequently, one can assert the following lemma by the arguments above and Theorem 4.2.2:

Lemma 4.3.1 (AI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_1$ is fixed (cf. (4.3.2)) :*

$$(1)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, T_n, \operatorname{Ad} I_{n,n}),$$

where T_n is given in (4.3.6). Moreover, $\sigma := \sigma_T$ satisfies

$$(1)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}, \mathfrak{sl}(n, \mathbb{R})). \end{cases}$$

4.3.2. Case $\eta = \eta_2$.

Let $\eta := \eta_2$ (cf. (4.3.2)). By $\eta = \theta \circ \eta_1$ we deduce that η coincides with η_1 on $\mathfrak{k}$. Accordingly one can have the following result similar to Lemma 4.3.1:

Lemma 4.3.2 (AI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_2$ is fixed (cf. (4.3.2)) :*

$$(2)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, T_n, \theta \circ \operatorname{Ad} I_{n,n}),$$

where T_n is given in (4.3.6). Moreover, $\sigma := \sigma_T$ satisfies

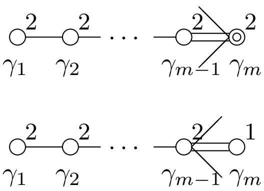
$$(2)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbb{C})). \end{cases}$$

4.3.3. Case $\boldsymbol{\eta = \eta_3}$.

Let $\eta := \eta_3$ (cf. (4.3.2)). From (4.3.3) we obtain $(\mathfrak{k}, \mathfrak{k}^\eta) = (\mathfrak{so}(2n), \mathfrak{u}(n))$. The rank of $(\mathfrak{k}, \mathfrak{k}^\eta)$ equals m, where $n = 2m+1$ or $n = 2m$. Hence, a maximal abelian subspace $\mathfrak{a}_m \subset \mathfrak{k}^{-\eta}$ is given by the matrices

$$H_* = \left(\begin{array}{c|c} A & O_n \\ \hline O_n & -A \end{array}\right), \quad A = \sum_{a=1}^{m} x_a (E_{2a-1,2a} - E_{2a,2a-1}),$$

where $x_a \in \mathbb{R}$. The Dynkin diagram of $\Sigma(\mathfrak{k}, \mathfrak{a}_m)$ is



BC_m when $n = 2m+1$; $\qquad$ (4.3.7)

C_m when $n = 2m$.

Here $\gamma_b(H_*) := x_b - x_{b+1}$ for $1 \leq b \leq m-1$ and $\gamma_m(H_*) := x_m$ if $n = 2m+1$ (resp. $\gamma_m(H_*) := 2x_m$ if $n = 2m$). Let us investigate two cases $n = 2m+1$ and $n = 2m$ individually.

Case $\eta = \eta_3$ and $n = 2m+1$. (4.3.7) implies that there does not exist any $A \in C_{\Pi(\mathfrak{k},\mathfrak{a}_m)}$ such that the eigenvalue of $\operatorname{ad} A$ in $\mathfrak{k}$ is $\pm i$ or zero. Therefore $d\mathcal{R}_{\mathfrak{g}} = \emptyset$ in case $\eta = \eta_3$ and $n = 2m+1$.

Case $\eta = \eta_3$ and $n = 2m$. Let $\{T_j\}_{j=1}^m$ be the dual basis of $\Pi(\mathfrak{k}, \mathfrak{a}_m) = \{\gamma_j\}_{j=1}^m$, and A an element of $C_{\Pi(\mathfrak{k},\mathfrak{a}_m)}$ such that the eigenvalue of $\operatorname{ad} A$ in $\mathfrak{k}$ is $\pm i$ or zero. Then $A = T_m$ only follows from (4.3.7). Here

$$T_m = \frac{1}{2}\left(\begin{array}{c|c} S_m & O_{2m} \\ \hline O_{2m} & -S_m \end{array}\right). \tag{4.3.8}$$

A direct computation, together with Lemma 4.2.3-(ii), enables us to confirm that the eigenvalue of $\operatorname{ad} T_m$ in $\mathfrak{g}$ is $\pm i$ or zero. Hence $(\mathfrak{g}, T_m, \eta) \in d\mathcal{R}_{\mathfrak{g}}$, and

Lemma 4.3.3 (AI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_3$ is fixed* (*cf.* (4.3.2)) :

$$(3) \ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, T_m, \eta_3),$$

where $n = 2m$ and T_m is given in (4.3.8). *Moreover, $\sigma := \sigma_T$ satisfies*

$$(3) \ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(4m, \mathbb{R}), \mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}, \mathfrak{su}^*(2m)). \end{cases}$$

4.3.4. Case $\eta = \eta_4$.

By Lemma 4.3.3 and arguments similar to those in Subsection 4.3.2. we conclude

LEMMA 4.3.4 (AI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_4$ is fixed (cf. (4.3.2)) :*

$$(4)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, T_m, \eta_4),$$

where $n = 2m$ and T_m is given in (4.3.8). Moreover, $\sigma := \sigma_T$ satisfies

$$(4)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(4m, \mathbb{R}), \mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(2m, \mathbb{R}), \mathfrak{sp}(m, \mathbb{C})). \end{cases}$$

Four Lemmas 4.3.1–4.3.4 and (4.3.2) lead to

PROPOSITION 4.3.5 (AI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$ and its Cartan involution is defined as θ in (4.3.1) :*

$\mathfrak{g}$	T	η	no.
AI	T_n *in* (4.3.6)	$\mathrm{Ad}\, I_{n,n}$	(1)
	the same as in (1)	$\theta \circ \mathrm{Ad}\, I_{n,n}$	(2)
	T_m *in* (4.3.8)	$\mathrm{Ad}\, J_n$, *where* $n = 2m$	(3)
	the same as in (3)	$\theta \circ \mathrm{Ad}\, J_n$, *where* $n = 2m$	(4)

Moreover, $\sigma := \sigma_T$ satisfies

$$(1) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}, \mathfrak{sl}(n, \mathbb{R})); \end{cases}$$

$$(2) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbb{C})); \end{cases}$$

$$(3) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{sl}(4m, \mathbb{R}), \mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}, \mathfrak{su}^*(2m)); \end{cases}$$

$$(4) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (3), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(2m, \mathbb{R}), \mathfrak{sp}(m, \mathbb{C})). \end{cases}$$

4.4. Type AII, $\mathfrak{g} = \mathfrak{su}^*(2n)$.

Let $\mathfrak{g}_u := \mathfrak{su}(2n)$ and

$$\theta := \tau \circ \mathrm{Ad}\, J_n, \tag{4.4.1}$$

where $\tau(X) := -{}^tX\ (= \overline{X})$ for $X \in \mathfrak{g}_u$. From $\mathfrak{g}_u$ and θ, we construct a non-compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{su}^*(2n)$ and $\mathfrak{k} = \mathfrak{sp}(n)$. By arguments similar to those in

Subsection 4.3 one can demonstrate

Proposition 4.4.1 (AII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{su}^*(2n)$ and its Cartan involution is defined as θ in* (4.4.1) :

$\mathfrak{g}$	T	η	no.
AII	$(i/2) \cdot I'_n$	$\mathrm{Ad}\, J_n$	(1)
	the same as in (1)	$\theta \circ \mathrm{Ad}\, J_n$	(2)
	$(1/2) \cdot (J_m \times J_m)$	$\mathrm{Ad}\, K_{m,m}$, *where* $n = 2m$	(3)
	the same as in (3)	$\theta \circ \mathrm{Ad}\, K_{m,m}$, *where* $n = 2m$	(4)

Moreover, $\sigma := \sigma_T$ satisfies

$$(1) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{su}^*(2n), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}, \mathfrak{sl}(n, \mathbb{R})); \end{cases}$$

$$(2) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \text{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{so}^*(2n), \mathfrak{so}(n, \mathbb{C})); \end{cases}$$

$$(3) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{su}^*(4m), \mathfrak{sl}(2m, \mathbb{C}) \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{su}^*(2m) \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}, \mathfrak{su}^*(2m)); \end{cases}$$

$$(4) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \text{ is the same as in } (3), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(m, m), \mathfrak{sp}(m, \mathbb{C})). \end{cases}$$

4.5. Type AIII, $\mathfrak{g} = \mathfrak{su}(p, q)$.

Let $\mathfrak{g}_u := \mathfrak{su}(p + q)$ and

$$\theta := \mathrm{Ad}\, I_{p,q}. \tag{4.5.1}$$

Construct a non-compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from them. Then $\mathfrak{g} = \mathfrak{su}(p, q)$ and $\mathfrak{k} = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{t}$. Set

$$T_a := \frac{i}{p+q} \mathrm{diag}\big(\underbrace{p + q - a, \ldots, p + q - a}_{a}, \underbrace{-a, \ldots, -a}_{p+q-a}\big). \tag{4.5.2}$$

With this notation we can state

Proposition 4.5.1 (AIII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{su}(p, q)$ and its Cartan involution is defined as θ in* (4.5.1) :

$\mathfrak{g}$	T	η	no.
AIII	$T_h - T_p + T_k$ *with* $h < p$ & $p \le k < p+q$	$\eta_1(X) := -{}^tX$ *for* $X \in \mathfrak{g}$	(1)
	$(i/2) \cdot (I'_n \times I'_m)$	$\mathrm{Ad}(I_{n,n} \times I_{m,m})$ *where* $(p,q) = (2n, 2m)$	(2)
	$T_h - T_n + T_{n+h} - T_{2n} + T_{n+k} - T_{2n+m} + T_{n+m+k}$ *with* $h < n$ & $n \le k < n+m$	$\eta_1 \circ \mathrm{Ad}(J_n \times J_m)$ *where* $(p,q) = (2n, 2m)$	(3)
	$T_h - T_p + T_{p+h}$ *with* $1 \le h < p$	$\eta_1 \circ \mathrm{Ad}\, J_p$, *where* $p = q$	(4)
	the same as in (4)	$\theta \circ \eta_1 \circ \mathrm{Ad}\, J_p$, *where* $p = q$	(5)
	$T_h - T_{p+h}$ *with* $1 \le h < p$	$\mathrm{Ad}\, J_p$, *where* $p = q$	(6)

where T_a *is given by* (4.5.2). *Moreover,* $\sigma := \sigma_T$ *satisfies*

$$(1) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{su}(p,q), \mathfrak{s}\big(\mathfrak{u}(h, k-p) \oplus \mathfrak{u}(p-h, p+q-k)\big)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(p,q), \mathfrak{so}(h, k-p) \oplus \mathfrak{so}(p-h, p+q-k)\big); \end{cases}$$

$$(2) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{su}(2n,2m), \mathfrak{s}\big(\mathfrak{u}(n,m) \oplus \mathfrak{u}(n,m)\big)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{s}\big(\mathfrak{u}(n,m) \oplus \mathfrak{u}(n,m)\big), \mathfrak{su}(n,m)\big); \end{cases}$$

$$(3) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{su}(2n,2m), \mathfrak{s}\big(\mathfrak{u}(2h, 2k-2n) \oplus \mathfrak{u}(2n-2h, 2n+2m-2k)\big)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sp}(n,m), \mathfrak{sp}(h, k-n) \oplus \mathfrak{sp}(n-h, n+m-k)\big); \end{cases}$$

$$(4) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{su}(p,p), \mathfrak{s}\big(\mathfrak{u}(h,h) \oplus \mathfrak{u}(p-h, p-h)\big)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sp}(p,\mathbb{R}), \mathfrak{sp}(h,\mathbb{R}) \oplus \mathfrak{sp}(p-h,\mathbb{R})\big); \end{cases}$$

$$(5) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (4), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}^*(2p), \mathfrak{so}^*(2h) \oplus \mathfrak{so}^*(2p-2h)\big); \end{cases}$$

$$(6) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{su}(p,p), \mathfrak{s}\big(\mathfrak{u}(h, p-h) \oplus \mathfrak{u}(h, p-h)\big)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{R}, \mathfrak{su}(h, p-h)\big). \end{cases}$$

4.6. Type BDI, $\mathfrak{g} = \mathfrak{so}(p,q)$.

Let $\mathfrak{g}_u := \mathfrak{so}(p+q)$ and

$$\theta := \mathrm{Ad}\, I_{p,q}. \tag{4.6.1}$$

Then we get a real form $\mathfrak{g}$ of $\mathfrak{g}_\mathbb{C}$ by setting $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} := \mathfrak{g}_u^\theta$ and $i\mathfrak{p} := \mathfrak{g}_u^{-\theta}$. We regard $\mathfrak{g}$ as $\mathfrak{so}(p,q)$, since $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p,q)$. With this setting, we can demonstrate the following by arguments similar to those in Subsection 4.3:

Proposition 4.6.1 (BDI). *The elements* $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_\mathfrak{g}$ *are classified as follows, where* $\mathfrak{g} = \mathfrak{so}(p,q)$ *and its Cartan involution is defined as* θ *in* (4.6.1) :

$\mathfrak{g}$	T	η	no.
BDI	$E_{1,p} - E_{p,1}$	$\mathrm{Ad}(I_{h,p-h} \times I_{k,q-k})$ *with* $1 \leq h < p$ & $k \leq q$	(1)
	$E_{p+1,p+q} - E_{p+q,p+1}$	$\mathrm{Ad}(I_{h,p-h} \times I_{k,q-k})$ *with* $h \leq p$ & $1 \leq k < q$	(1′)
	$(1/2) \cdot (A_n \times A_m)$ $A_i := \begin{pmatrix} O_i & \mathrm{adiag}(1,\dots,1) \\ -\mathrm{adiag}(1,\dots,1) & O_i \end{pmatrix}$	$\mathrm{Ad}(I_{n,n} \times I_{m,m})$ *where* $(p,q) = (2n, 2m)$	(2)
	$(1/2) \cdot (S'_n \times S'_m)$	$\mathrm{Ad}(J_{2n} \times J_{2m})$ *where* $(p,q) = (4n, 4m)$	(3)
	$(1/2) \cdot (S_n \times (-S_n))$	$\mathrm{Ad}\, J_{2n}$*, where* $p = q = 2n$	(4)
	the same as in (4)	$\theta \circ \mathrm{Ad}\, J_{2n}$*, where* $p = q = 2n$	(5)

Moreover, $\sigma := \sigma_T$ *satisfies*

$$(1)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}(p,q), \mathfrak{so}(p-2,q) \oplus \mathfrak{so}(2)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(h,k) \oplus \mathfrak{so}(p-h,q-k), \mathfrak{so}(h-1,k) \oplus \mathfrak{so}(p-h-1,q-k)\big); \end{cases}$$

$$(1')\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}(p,q), \mathfrak{so}(p,q-2) \oplus \mathfrak{so}(2)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(h,k) \oplus \mathfrak{so}(p-h,q-k), \mathfrak{so}(h,k-1) \oplus \mathfrak{so}(p-h,q-k-1)\big); \end{cases}$$

$$(2)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}(2n,2m), \mathfrak{u}(n,m)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(n,m) \oplus \mathfrak{so}(n,m), \mathfrak{so}(n,m)\big); \end{cases}$$

$$(3)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}(4n,4m), \mathfrak{u}(2n,2m)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{u}(2n,2m), \mathfrak{sp}(n,m)\big); \end{cases}$$

$$(4)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}(2n,2n), \mathfrak{u}(n,n)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sl}(2n,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{sp}(n,\mathbb{R})\big); \end{cases}$$

$$(5)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \text{ is the same as in } (4), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{so}^*(2n)\big). \end{cases}$$

Let us comment on the above proposition.

REMARK 4.6.2 (Proposition 4.6.1). The item (1) is the same as (1′) if one identifies $\mathfrak{so}(p,q)$ with $\mathfrak{so}(q,p)$.

4.7. Type DIII, $\mathfrak{g} = \mathfrak{so}^*(2n)$.

Let $\mathfrak{g}_u := \mathfrak{so}(2n)$ and

$$\theta := \mathrm{Ad}\, J_n. \tag{4.7.1}$$

Construct a non-compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from them. Then $\mathfrak{g} = \mathfrak{so}^*(2n)$, $\mathfrak{k} = \mathfrak{u}(n)$ and

Proposition 4.7.1 (DIII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified asfollows, where $\mathfrak{g} = \mathfrak{so}^*(2n)$ and its Cartan involution is defined as θ in* (4.7.1) :

$\mathfrak{g}$	T	η	no.
DIII	$\frac{1}{2}\begin{pmatrix} O_{2m} & I'_m \\ -I'_m & O_{2m} \end{pmatrix}$	$\mathrm{Ad}(I_{m,m} \times I_{m,m})$ *where* $n = 2m$	(1)
	the same as in (1)	$\theta \circ \mathrm{Ad}(I_{m,m} \times I_{m,m})$ *where* $n = 2m$	(2)
	$\frac{1}{2}\begin{pmatrix} O_{2m} & -(I_{k,m-k} \times I_{k,m-k}) \\ I_{k,m-k} \times I_{k,m-k} & O_{2m} \end{pmatrix}$ *with* $2k \leq m$	$\mathrm{Ad}(J_m \times (-J_m))$ *where* $n = 2m$	(3)
	$\frac{1}{2}\begin{pmatrix} O_n & -I_{k,n-k} \\ I_{k,n-k} & O_n \end{pmatrix}$ *with* $2k \leq n$	$\mathrm{Ad}\, I_{n,n}$	(4)
	$E_{1,n+1} - E_{n+1,1}$	*the same as in* (4)	(5)

Moreover, $\sigma := \sigma_T$ *satisfies*

$$(1)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}^*(4m), \mathfrak{u}(m,m)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}^*(2m) \oplus \mathfrak{so}^*(2m), \mathfrak{so}^*(2m)\big); \end{cases}$$

$$(2)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \text{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{u}(m,m), \mathfrak{sp}(m,\mathbb{R})\big); \end{cases}$$

$$(3)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}^*(4m), \mathfrak{u}(2k, 2m-2k)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{su}^*(2m) \oplus \mathbb{R}, \mathfrak{sp}(k, m-k)\big); \end{cases}$$

$$(4)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}^*(2n), \mathfrak{u}(k, n-k)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(k, n-k)\big); \end{cases}$$

$$(5)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{so}^*(2n), \mathfrak{so}^*(2) \oplus \mathfrak{so}^*(2n-2)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C})\big). \end{cases}$$

4.8. Type CI, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$.

Let $\mathbb{H}$ denote the division algebra of quaternions with basis $\{1, i_1, i_2, i_3\}$. We realize $\mathfrak{sp}(n)$ by use of $\mathbb{H}$ and denote it by $\mathfrak{g}_u$, i.e., $\mathfrak{g}_u := \{A \in \mathfrak{gl}(n, \mathbb{H}) \mid A^* = -A\}$. Now, let

$$\theta := \mathrm{Ad}(i_1 I_n). \tag{4.8.1}$$

Construct a non-compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g}_u$ and θ. Then, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{u}(n)$. In this setting we have

Proposition 4.8.1 (CI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ and its Cartan involution is defined as θ in* (4.8.1) :

$\mathfrak{g}$	T	η	no.
CI	$(i_1/2)\cdot I'_m$	$\operatorname{Ad} I_{m,m}$, *where* $n=2m$	(1)
	the same as in (1)	$\theta\circ\operatorname{Ad} I_{m,m}$ *where* $n=2m$	(2)
	$\dfrac{i_1}{m}\left(\begin{array}{c\|c} (m-k)I_k\times(-k)I_{m-k} & O_m \\ \hline O_m & (m-k)I_k\times(-k)I_{m-k}\end{array}\right)$ *with* $2k\le m$	$\operatorname{Ad}(i_2J_m)$ *where* $n=2m$	(3)
	$(i_1/n)\cdot\big((n-k)I_k\times(-k)I_{n-k}\big)$ *with* $2k\le n$	$\operatorname{Ad}(i_2I_n)$	(4)

where $\{1,i_1,i_2,i_3\}$ *is the basis of* $\mathbb{H}$. *Moreover,* $\sigma:=\sigma_T$ *satisfies*

$$(1)\ \begin{cases}(\mathfrak{g},\mathfrak{g}^\sigma)=\big(\mathfrak{sp}(2m,\mathbb{R}),\mathfrak{u}(m,m)\big),\\ (\mathfrak{g}^\eta,\mathfrak{g}^\sigma\cap\mathfrak{g}^\eta)=\big(\mathfrak{sp}(m,\mathbb{R})\oplus\mathfrak{sp}(m,\mathbb{R}),\mathfrak{sp}(m,\mathbb{R})\big);\end{cases}$$

$$(2)\ \begin{cases}(\mathfrak{g},\mathfrak{g}^\sigma)\ \textit{is the same as in}\ (1),\\ (\mathfrak{g}^\eta,\mathfrak{g}^\sigma\cap\mathfrak{g}^\eta)=\big(\mathfrak{u}(m,m),\mathfrak{so}^*(2m)\big);\end{cases}$$

$$(3)\ \begin{cases}(\mathfrak{g},\mathfrak{g}^\sigma)=\big(\mathfrak{sp}(2m,\mathbb{R}),\mathfrak{u}(2k,2m-2k)\big),\\ (\mathfrak{g}^\eta,\mathfrak{g}^\sigma\cap\mathfrak{g}^\eta)=\big(\mathfrak{sp}(m,\mathbb{C}),\mathfrak{sp}(k,m-k)\big);\end{cases}$$

$$(4)\ \begin{cases}(\mathfrak{g},\mathfrak{g}^\sigma)=\big(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{u}(k,n-k)\big),\\ (\mathfrak{g}^\eta,\mathfrak{g}^\sigma\cap\mathfrak{g}^\eta)=\big(\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R},\mathfrak{so}(k,n-k)\big).\end{cases}$$

4.9. Type CII, $\mathfrak{g}=\mathfrak{sp}(p,q)$.

Realize $\mathfrak{sp}(p+q)$ by the same way as in Subsection 4.8. Let $\mathfrak{g}_u:=\mathfrak{sp}(p+q)$ and

$$\theta:=\operatorname{Ad} I_{p,q}. \tag{4.9.1}$$

Construct a real form $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from them. Then it follows that $\mathfrak{g}=\mathfrak{sp}(p,q)$, $\mathfrak{k}=\mathfrak{sp}(p)\oplus\mathfrak{sp}(q)$ and

Proposition 4.9.1 (CII). *The elements* $(\mathfrak{g},T,\eta)\in d\mathcal{R}_{\mathfrak{g}}$ *are classified as follows, where* $\mathfrak{g}=\mathfrak{sp}(p,q)$ *and its Cartan involution is defined as* θ *in* (4.9.1) :

$\mathfrak{g}$	T	η	no.
CII	$(i_2/2)\cdot I_{p+q}$	$\operatorname{Ad}(i_1I_{p+q})$	(1)
	the same as in (2) *at Proposition* 4.6.1(BDI)	$\operatorname{Ad}(I_{n,n}\times I_{m,m})$ *where* $(p,q)=(2n,2m)$	(2)
	$(i_1/2)\cdot I_{p,p}$	$\theta\circ\operatorname{Ad} J_p$, *where* $p=q$	(3)
	the same as in (3)	$\operatorname{Ad} J_p$, *where* $p=q$	(4)

where $\{1,i_1,i_2,i_3\}$ *is the basis of* $\mathbb{H}$. *Moreover,* $\sigma:=\sigma_T$ *satisfies*

$$(1)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{sp}(p,q), \mathfrak{u}(p,q)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{u}(p,q), \mathfrak{so}(p,q)\big); \end{cases}$$

$$(2)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{sp}(2n,2m), \mathfrak{u}(2n,2m)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sp}(n,m) \oplus \mathfrak{sp}(n,m), \mathfrak{sp}(n,m)\big); \end{cases}$$

$$(3)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = \big(\mathfrak{sp}(p,p), \mathfrak{u}(p,p)\big), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{sp}(p,\mathbb{C}), \mathfrak{sp}(p,\mathbb{R})\big); \end{cases}$$

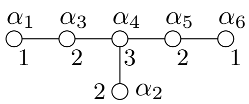
$$(4)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (3), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = \big(\mathfrak{su}^*(2p) \oplus \mathbb{R}, \mathfrak{so}^*(2p)\big). \end{cases}$$

4.10. Type EII, $\mathfrak{g} = \mathfrak{e}_{6(2)}$.

Our aim in this subsection is to determine $d\mathcal{R}_{\mathfrak{g}}/_\sim$ for $\mathfrak{g} = \mathfrak{e}_{6(2)}$ (cf. Proposition 4.10.4). One can accomplish the aim by arguments similar to those in Subsection 4.3. However, we are going to construct arguments in detail, since we need to treat an exceptional Lie algebra.

4.10.1. Setting.

Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{e}_6^{\mathbb{C}}$. We assume that the Dynkin diagram of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ is as follows (cf. Bourbaki [**3**, p. 276]):

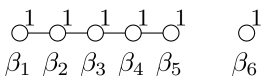


(4.10.1)

Denote by $\mathfrak{g}_u$ the compact real form of $\mathfrak{g}_{\mathbb{C}}$ given by (3.2.1). Let $\{Z_j\}_{j=1}^6$ be the dual basis of $\{\alpha_j\}_{j=1}^6 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ and

$$\theta := \exp \pi \operatorname{ad} i(Z_2 + Z_3 + Z_5). \tag{4.10.2}$$

Then $\theta \in \mathrm{Inv}(\mathfrak{g}_u)$, $\theta(\mathfrak{c}_{\mathbb{C}}) \subset \mathfrak{c}_{\mathbb{C}}$ and the Dynkin diagram of $\mathfrak{k} := \mathfrak{g}_u^\theta$ (with respect to $i\mathfrak{c}_{\mathbb{R}}$) is as follows:



(4.10.3)

where $\beta_1 := \alpha_2 + \alpha_3 + \alpha_4$, $\beta_2 := \alpha_1$, $\beta_3 := \alpha_3 + \alpha_4 + \alpha_5$, $\beta_4 := \alpha_6$, $\beta_5 := \alpha_2 + \alpha_4 + \alpha_5$ and $\beta_6 := \alpha_4$. This provides us with $\mathfrak{k} = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$ and $\mathfrak{g} = \mathfrak{e}_{6(2)}$, where $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$.

4.10.2. Reduction.

Berger [**1**, p. 160] has shown that any involution of $\mathfrak{g} = \mathfrak{e}_{6(2)}$ is conjugate to $\eta \in \mathrm{Inv}(\mathfrak{g})$ such that $\mathfrak{g}^\eta$ is one of the following:

$$\begin{array}{llll} (1)\ \mathfrak{sp}(3,1), & (2)\ \mathfrak{f}_{4(4)}, & (3)\ \mathfrak{sp}(4,\mathbb{R}), & \\ (4)\ \mathfrak{so}^*(10)\oplus\mathfrak{t}, & (5)\ \mathfrak{so}(6,4)\oplus\mathfrak{t}, & (6)\ \mathfrak{su}(4,2)\oplus\mathfrak{su}(2), & (4.10.4) \\ (7)\ \mathfrak{su}(3,3)\oplus\mathfrak{sl}(2,\mathbb{R}), & (8)\ \mathfrak{su}(6)\oplus\mathfrak{su}(2). & & \end{array}$$

We will investigate the involutions only for (1), (2) and (3), because the real forms of compact Hermitian symmetric Lie algebra $(\mathfrak{e}_6, \mathfrak{so}(10)\oplus\mathfrak{t})$ are exhausted by

$$(\mathfrak{sp}(4), \mathfrak{sp}(2)\oplus\mathfrak{sp}(2)), \quad (\mathfrak{f}_4, \mathfrak{so}(9))$$

(cf. Leung [**14**, p. 182] or Takeuchi [**26**]). The involutions are given by

$$\begin{array}{ll} (1)\ \eta_1 := \tau\circ\exp\pi\,\mathrm{ad}(iZ_2), & (2)\ \eta_2 := \theta\circ\eta_1, \\ (3)\ \eta_3,\ \text{where } {}^t\eta_3(\alpha_j) := -\alpha_j \text{ for } 1\le j\le 6. & \end{array} \tag{4.10.5}$$

Here τ is an involution of $\mathfrak{g}_u$ defined as follows:

$$\begin{cases} {}^t\tau(\alpha_1) := \alpha_6, & {}^t\tau(\alpha_2) := \alpha_2, \quad {}^t\tau(\alpha_3) := \alpha_5, \\ {}^t\tau(\alpha_4) := \alpha_4, & {}^t\tau(\alpha_5) := \alpha_3, \quad {}^t\tau(\alpha_6) := \alpha_1 \end{cases} \tag{4.10.6}$$

(cf. Proposition 3.2.1, Lemma 3.2.2). Now, let us consider a restricted root system $\Sigma(\mathfrak{k},\mathfrak{a})$ for each η_a.

4.10.3. Case $\eta = \eta_1$.

Let $\eta := \eta_1$ (cf. (4.10.5)). By (4.10.4) we see that $(\mathfrak{k},\mathfrak{k}^\eta) = (\mathfrak{su}(6)\oplus\mathfrak{su}(2), \mathfrak{sp}(3)\oplus\mathfrak{sp}(1))$, and its rank equals $2+0=2$. Hence the following $\mathfrak{a}_2$ is a maximal abelian subspace in $\mathfrak{k}^{-\eta}$:

$$\mathfrak{a}_2 := \mathrm{span}_{\mathbb{R}}\{T_1, T_2\},$$

where $T_1 := i(Z_3 - Z_5)$ and $T_2 := i(Z_1 - Z_6)$. The Dynkin diagram of $\Sigma(\mathfrak{k},\mathfrak{a}_2)$ is

$$\overset{1}{\underset{\overline{\beta}_1}{\circ}}\!\!-\!\!\!-\!\!\overset{1}{\underset{\overline{\beta}_2}{\circ}} \quad A_2 \tag{4.10.7}$$

where $\overline{\beta}_1 := \beta_1|_{\mathfrak{a}_2}$ and $\overline{\beta}_2 := \beta_2|_{\mathfrak{a}_2}$. Remark that $\{T_1, T_2\}$ becomes the dual basis of $\{\overline{\beta}_1, \overline{\beta}_2\} = \Pi(\mathfrak{k},\mathfrak{a}_2)$. Now, let A be an element of $C_{\Pi(\mathfrak{k},\mathfrak{a}_2)}$ such that the eigenvalue of $\mathrm{ad}\,A$ in $\mathfrak{k}$ is $\pm i$ or zero. Then $A = T_1, T_2$ follows from (4.10.7). One can deduce that the eigenvalue of $\mathrm{ad}\,T_1$ (resp. $\mathrm{ad}\,T_2$) in $\mathfrak{g}$ is also $\pm i$ or zero, by direct computation, (4.10.1), Lemma 4.2.3-(iii) and $\alpha_i(Z_j) = \delta_{i,j}$. For this reason $(\mathfrak{g}, T_1, \eta), (\mathfrak{g}, T_2, \eta) \in d\mathcal{R}_{\mathfrak{g}}$. From now on, let us show that

$$(\mathfrak{g}, T_1, \eta) \text{ is equivalent to } (\mathfrak{g}, T_2, \eta). \tag{4.10.8}$$

Denote by $S_{\overline{\beta}_{12}}$ the reflection along $\overline{\beta}_{12} := \overline{\beta}_1 + \overline{\beta}_2 \in \Sigma(\mathfrak{k}, \mathfrak{a}_2)$. In this case $S_{\overline{\beta}_{12}} \in \mathrm{Int}(\mathfrak{k}, \eta) \subset \mathrm{Int}(\mathfrak{g}, \theta, \eta)$ (cf. (3.3.1)), and satisfies $S_{\overline{\beta}_{12}}(T_1) = -T_2$ because of $\overline{\beta}_i(T_j) = \delta_{i,j}$. Consequently $\varphi := \eta \circ S_{\overline{\beta}_{12}}$ satisfies $[\theta, \varphi] = [\eta, \varphi] = 0$ and $\varphi(T_1) = T_2$; and so (4.10.8) holds. Hence, one can assert the following lemma by the arguments above and Theorem 4.2.2:

Lemma 4.10.1 (EII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_1$ is fixed (cf. (4.10.5)) :*

$$(1)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, i(Z_1 - Z_6), \eta_1),$$

where $\{Z_j\}_{j=1}^6$ is the dual basis of $\{\alpha_j\}_{j=1}^6 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.10.1)). Moreover, $\sigma := \sigma_T$ satisfies

$$(1)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(3,1), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1,1)). \end{cases}$$

4.10.4. Case $\boldsymbol{\eta = \eta_2}$.

By Lemma 4.10.1 and arguments similar to those in Subsection 4.3.2., we have

Lemma 4.10.2 (EII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_2$ is fixed (cf. (4.10.5)) :*

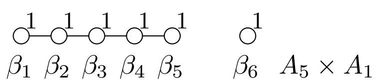
$$(2)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, i(Z_1 - Z_6), \eta_2),$$

where $\{Z_j\}_{j=1}^6$ is the dual basis of $\{\alpha_j\}_{j=1}^6 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.10.1)). Moreover, $\sigma := \sigma_T$ satisfies

$$(2)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4)). \end{cases}$$

4.10.5. Case $\boldsymbol{\eta = \eta_3}$.

Let $\eta := \eta_3$ (cf. (4.10.5)). Notice that η gives rise to a normal real form of $\mathfrak{g}_{\mathbb{C}}$. So, $\mathfrak{a}_6 := \mathrm{span}_{\mathbb{R}}\{iZ_j\}_{j=1}^6$ $(= i\mathfrak{c}_{\mathbb{R}})$ is a maximal abelian subspace in $\mathfrak{k}^{-\eta}$ and the Dynkin diagram of $\Sigma(\mathfrak{k}, \mathfrak{a}_6)$ is the same as (4.10.3).



Denote by $\{T_j\}_{j=1}^6$ the dual basis of $\{\beta_j\}_{j=1}^6$. Then, we obtain

$$T_1 = i(Z_2 + Z_3 - Z_5)/2, \quad T_2 = iZ_1, \quad T_3 = i(-Z_2 + Z_3 + Z_5)/2,$$
$$T_4 = iZ_6, \quad T_5 = i(Z_2 - Z_3 + Z_5)/2, \quad T_6 = i(-Z_2 - Z_3 + 2Z_4 - Z_5)/2$$

from $\alpha_i(Z_j) = \delta_{i,j}$. Let A be an element of $C_{\Pi(\mathfrak{k},\mathfrak{a}_6)}$ such that the eigenvalue of $\operatorname{ad} A$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\pm i$ or zero. Then one has $A = T_2,\ T_4,\ (T_1 + T_6),\ (T_5 + T_6)$ by direct computation, (4.10.1), Lemma 4.2.3-(iii) and $\alpha_i(Z_j) = \delta_{i,j}$. Moreover, $(\mathfrak{g}, T_2, \eta)$ and $(\mathfrak{g}, T_1 + T_6, \eta)$ are equivalent to $(\mathfrak{g}, T_4, \eta)$ and $(\mathfrak{g}, T_5 + T_6, \eta)$, respectively. Thereby

LEMMA 4.10.3 (EII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, in the case where $\eta = \eta_3$ is fixed* (*cf.* (4.10.5)) :

$$(3)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, iZ_1, \eta_3), \quad (4)\ (\mathfrak{g}, T, \eta) = (\mathfrak{g}, i(Z_4 - Z_5), \eta_3),$$

where $\{Z_j\}_{j=1}^6$ is the dual basis of $\{\alpha_j\}_{j=1}^6 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.10.1)). Moreover, $\sigma := \sigma_T$ satisfies

$$(3)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(4,\mathbb{R}), \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R})); \end{cases}$$

$$(4)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}^*(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(4,\mathbb{R}), \mathfrak{sp}(2,\mathbb{C})). \end{cases}$$

By (4.10.5) and three Lemmas 4.10.1–4.10.3, we conclude

PROPOSITION 4.10.4 (EII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{e}_{6(2)}$ and its Cartan involution is defined as θ in* (4.10.2) :

$\mathfrak{g}$	T	η	no.
EII	$i(Z_1 - Z_6)$	η_1 *in* (4.10.5)	(1)
	the same as in (1)	$\theta \circ \eta_1$	(2)
	iZ_1	η_3 *in* (4.10.5)	(3)
	$i(Z_4 - Z_5)$	*the same as in* (3)	(4)

where $\{Z_j\}_{j=1}^6$ is the dual basis of $\{\alpha_j\}_{j=1}^6 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.10.1)). Moreover, $\sigma := \sigma_T$ satisfies

$$(1)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(3,1), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1,1)); \end{cases}$$

$$(2)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \text{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4)); \end{cases}$$

$$(3)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(4,\mathbb{R}), \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R})); \end{cases}$$

$$(4)\ \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}^*(2)), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sp}(4,\mathbb{R}), \mathfrak{sp}(2,\mathbb{C})). \end{cases}$$

4.11. Type EIII, $\mathfrak{g} = \mathfrak{e}_{6(-14)}$.

Let $\mathfrak{g}_u$ be the same Lie algebra as in Subsection 4.10. Define $\theta \in \mathrm{Inv}(\mathfrak{g}_u)$ by

$$\theta := \exp \pi \,\mathrm{ad}\, i(Z_1 - Z_6), \tag{4.11.1}$$

where $\{Z_j\}_{j=1}^{6}$ denotes the dual basis of $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}) = \{\alpha_j\}_{j=1}^{6}$ (cf. (4.10.1)). From them we construct a real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$. Then, $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ and $\mathfrak{k} = \mathfrak{so}(10) \oplus \mathfrak{t}$. In this setting, one can prove the following by arguments similar to those in Subsection 4.10:

PROPOSITION 4.11.1 (EIII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ and its Cartan involution is defined as θ in* (4.11.1) :

$\mathfrak{g}$	T	η	no.
EIII	$i(Z_1 - Z_3)$	η_3 *in* (4.10.5)	(1)
	$i(Z_3 - Z_5)$	τ *in* (4.10.6)	(2)
	iZ_1	*the same as in* (1)	(3)
	$i(Z_1 - Z_6)$	*the same as in* (2)	(4)
	the same as in (4)	*the same as in* (1)	(5)

where $\{Z_j\}_{j=1}^{6}$ is the dual basis of $\{\alpha_j\}_{j=1}^{6} = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.10.1)). *Moreover, $\sigma := \sigma_T$ satisfies*

$$(1) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}^*(2)), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{sp}(2,2), \mathfrak{sp}(2,\mathbb{C})); \end{cases}$$

$$(2) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(8,1)); \end{cases}$$

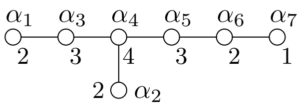
$$(3) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{sp}(2,2), \mathfrak{sp}(1,1) \oplus \mathfrak{sp}(1,1)); \end{cases}$$

$$(4) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \oplus \mathfrak{so}(2)), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(9)); \end{cases}$$

$$(5) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) \textit{ is the same as in } (4), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{sp}(2,2), \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)). \end{cases}$$

4.12. Type EV, $\mathfrak{g} = \mathfrak{e}_{7(7)}$.

Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{e}_7^{\mathbb{C}}$ and let $\mathfrak{g}_u$ be the compact real form of $\mathfrak{g}_{\mathbb{C}}$ given by (3.2.1), where the Dynkin diagram of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ is as follows (cf. Bourbaki [**3**, p. 280]):



(4.12.1)

Denote by $\{Z_j\}_{j=1}^7$ the dual basis of $\{\alpha_j\}_{j=1}^7$ and define $\theta \in \mathrm{Inv}(\mathfrak{g}_u)$ by

$$\theta := \exp \pi \,\mathrm{ad}(iZ_2). \tag{4.12.2}$$

We construct a real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g}_u$ and θ. In this case $\mathfrak{g} = \mathfrak{e}_{7(7)}$ and $\mathfrak{k} = \mathfrak{su}(8)$. Now, let us prepare $\eta_1, \eta_3 \in \mathrm{Inv}(\mathfrak{g})$ for stating Proposition 4.12.1 (below). Proposition 3.2.1 and Lemma 3.2.2 enable us to get $\eta_1, \eta_3 \in \mathrm{Inv}(\mathfrak{g}_u)$ by

$$\begin{cases} {}^t\eta_1(\alpha_1) := \alpha_6, \quad {}^t\eta_1(\alpha_2) := \alpha_2, \quad {}^t\eta_1(\alpha_3) := \alpha_5, \quad {}^t\eta_1(\alpha_4) := \alpha_4, \\ {}^t\eta_1(\alpha_5) := \alpha_3, \quad {}^t\eta_1(\alpha_6) := \alpha_1, \quad {}^t\eta_1(\alpha_7) := -\widetilde{\alpha}; \end{cases} \tag{4.12.3}$$

$${}^t\eta_3(\alpha_j) := -\alpha_j \text{ for } 1 \leq j \leq 7, \tag{4.12.4}$$

where $\widetilde{\alpha} := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. It follows from (4.12.2) that $[\theta, \eta_1] = [\theta, \eta_3] = 0$, so that $\eta_1, \eta_3 \in \mathrm{Inv}(\mathfrak{g}) \cap \mathrm{Inv}(\mathfrak{g}_u)$ by Proposition 3.1.2. With this setting we can state

PROPOSITION 4.12.1 (EV). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{e}_{7(7)}$ and its Cartan involution is defined as θ in* (4.12.2) :

$\mathfrak{g}$	T	η	no.
EV	iZ_7	η_1 *in* (4.12.3)	(1)
	the same as in (1)	$\theta \circ \eta_1$	(2)
	the same as in (1)	η_3 *in* (4.12.4)	(3)

where $\{Z_j\}_{j=1}^7$ is the dual basis of $\{\alpha_j\}_{j=1}^7 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.12.1)). *Moreover, $\sigma := \sigma_T$ satisfies*

$$(1) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{e}_{6(6)} \oplus \mathbb{R}, \mathfrak{f}_{4(4)}); \end{cases}$$

$$(2) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{su}^*(8), \mathfrak{sp}(3,1)); \end{cases}$$

$$(3) \quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (1), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{sl}(8,\mathbb{R}), \mathfrak{sp}(4,\mathbb{R})). \end{cases}$$

4.13. Type EVI, $\mathfrak{g} = \mathfrak{e}_{7(-5)}$.

Let $\mathfrak{g}_u$ be the same Lie algebra as in Subsection 4.12 and

$$\theta := \exp \pi \,\mathrm{ad}\, i(Z_2 + Z_7), \tag{4.13.1}$$

where $\{Z_j\}_{j=1}^7$ is the dual basis of $\{\alpha_j\}_{j=1}^7 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.12.1)). Construct a real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g}_u$ and θ. Then $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ and $\mathfrak{k} = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$. Define

$\eta_5 \in \mathrm{Inv}(\mathfrak{g}_u)$ by

$$\eta_5 := \phi \circ \eta_1 \circ \phi^{-1}, \tag{4.13.2}$$

where η_1 in (4.12.3) and ϕ is an automorphism of $\mathfrak{g}_u$ defined by

$$\begin{cases} {}^t\phi(\alpha_1) := -\sum_{a=1}^{4} \alpha_a, & {}^t\phi(\alpha_2) := \alpha_4, & {}^t\phi(\alpha_3) := \alpha_1, \\ {}^t\phi(\alpha_4) := \alpha_3, & {}^t\phi(\alpha_5) := \alpha_2 + \sum_{b=4}^{7} \alpha_b, & {}^t\phi(\alpha_6) := -\alpha_6 - \alpha_7, \\ {}^t\phi(\alpha_7) := \alpha_6. \end{cases}$$

Proposition 3.1.2 and (4.13.1) imply that $\eta_5 \in \mathrm{Inv}(\mathfrak{g})$, and moreover $\eta_1, \eta_3 \in \mathrm{Inv}(\mathfrak{g})$ (see (4.12.4) for η_3). In this setting, we assert

PROPOSITION 4.13.1 (EVI). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ and its Cartan involution is defined as θ in* (4.13.1) :

$\mathfrak{g}$	T	η	no.
EVI	iZ_7	η_1 *in* (4.12.3)	(1)
	the same as in (1)	$\theta \circ \eta_1$	(2)
	the same as in (1)	η_3 *in* (4.12.4)	(3)
	$i(Z_6 - Z_7)$	*the same as in* (3)	(4)
	the same as in (4)	η_5 *in* (4.13.2)	(5)

where $\{Z_j\}_{j=1}^{7}$ is the dual basis of $\{\alpha_j\}_{j=1}^{7} = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.12.1)*). Moreover, $\sigma := \sigma_T$ satisfies*

$$(1)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{e}_{6(2)} \oplus \mathfrak{t}, \mathfrak{f}_{4(4)}); \end{cases}$$

$$(2)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) \textit{ is the same as in } (1), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{su}(6,2), \mathfrak{sp}(3,1)); \end{cases}$$

$$(3)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) \textit{ is the same as in } (1), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{su}(4,4), \mathfrak{sp}(4,\mathbb{R})); \end{cases}$$

$$(4)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{su}(4,4), \mathfrak{sp}(2,2)); \end{cases}$$

$$(5)\quad \begin{cases} (\mathfrak{g}, \mathfrak{g}^{\sigma}) \textit{ is the same as in } (4), \\ (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}, \mathfrak{f}_{4(-20)}). \end{cases}$$

4.14. Type EVII, $\mathfrak{g} = \mathfrak{e}_{7(-25)}$.
Let $\mathfrak{g}_u$ be the same Lie algebra as in Subsection 4.12 and

$$\theta := \exp \pi \operatorname{ad}(iZ_7), \tag{4.14.1}$$

where $\{Z_j\}_{j=1}^7$ is the dual basis of $\{\alpha_j\}_{j=1}^7 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$. Construct a real form $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g}_u$ and θ. Then $\mathfrak{g} = \mathfrak{e}_{7(-25)}$, $\mathfrak{k} = \mathfrak{e}_6 \oplus \mathfrak{t}$ and

PROPOSITION 4.14.1 (EVII). *The elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ are classified as follows, where $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ and its Cartan involution is defined as θ in* (4.14.1) :

$\mathfrak{g}$	T	η	no.
EVII	$i(Z_1 - Z_6 + Z_7)$	η_1 *in* (4.12.3)	(1)
	$i(Z_6 - Z_7)$	η_3 *in* (4.12.4)	(2)
	iZ_7	*the same as in* (1)	(3)
	the same as in (3)	*the same as in* (2)	(4)

where $\{Z_j\}_{j=1}^7$ is the dual basis of $\{\alpha_j\}_{j=1}^7 = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ (cf. (4.12.1)*). Moreover, $\sigma := \sigma_T$ satisfies*

$$(1) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{e}_{6(-26)} \oplus \mathbb{R}, \mathfrak{f}_{4(-20)}); \end{cases}$$

$$(2) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{su}^*(8), \mathfrak{sp}(2,2)); \end{cases}$$

$$(3) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{t}), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{e}_{6(-26)} \oplus \mathbb{R}, \mathfrak{f}_4); \end{cases}$$

$$(4) \begin{cases} (\mathfrak{g}, \mathfrak{g}^\sigma) \textit{ is the same as in } (3), \\ (\mathfrak{g}^\eta, \mathfrak{g}^\sigma \cap \mathfrak{g}^\eta) = (\mathfrak{su}^*(8), \mathfrak{sp}(4)). \end{cases}$$

We end this section with commenting on the main theorem.

REMARK 4.14.2. Let us enumerate our comments on Theorem 1.0.1.

(1) One can realize all the real forms $M \subset G/R$ in List I by taking twelve Propositions 4.3.5 (AI), 4.4.1 (AII), 4.5.1 (AIII), 4.6.1 (BDI), 4.7.1 (DIII), 4.8.1 (CI), 4.9.1 (CII), 4.10.4 (EII), 4.11.1 (EIII), 4.12.1 (EV), 4.13.1 (EVI) and 4.14.1 (EVII). Here we utilize the notation in Remark 4.2.4.
(2) By G we mean the adjoint group of $\operatorname{Lie} G$, for G/R in Theorem 1.0.1.
(3) The conjecture of Boumuki [**2**, p. 118] is not true, because the table in [**2**, p. 118–121] is missing the real forms of nos. 4, 5 and 6 of type AIII, nos. 4 and 5 of type BDI, no. 3 of type DIII, and nos. 3 and 4 of type CII.

5. Real forms and totally real totally geodesic submanifolds.

5.1. Theorem 5.1.1.

Our goal in this section is to establish the following theorem (see Subsection 5.3 for its proof):

THEOREM 5.1.1. *Let $(G/R, \sigma, J, \boldsymbol{g})$ be a simple irreducible pseudo-Hermitian symmetric space, where $Z(G)$ is trivial, and let N be a subset of G/R containing o. Then, the following* (i) *and* (ii) *are equivalent*:

(i) *N is a real form of G/R*;
(ii) *N is a connected, totally real, complete totally geodesic submanifold of G/R with $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} G/R$, where the induced metric $\boldsymbol{g}|_N$ is non-degenerate.*

Here, we recall that a submanifold N of G/R is said to be *totally real* when $\boldsymbol{g}_p\bigl(J_p(T_pN), T_pN\bigr) = \{0\}$ for each $p \in N$.

5.2. Totally real submanifolds.

To prove Theorem 5.1.1 we need Lemma 5.2.1 and Proposition 5.2.3.

LEMMA 5.2.1. *Let $(G/R, \sigma, J, \boldsymbol{g})$ be a simple irreducible pseudo-Hermitian symmetric space, and N a connected totally real, complete totally geodesic submanifold of G/R with $o \in N$ and $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} G/R$, where $\boldsymbol{g}|_N$ is non-degenerate. Then, $\mathfrak{n} := T_oN$ satisfies*

$$(1)\ \mathfrak{q} = \mathfrak{n} \oplus \boldsymbol{j}(\mathfrak{n}), \qquad (2)\ \bigl[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}\bigr] \subset \mathfrak{n}, \qquad (3)\ \bigl[[\boldsymbol{j}(\mathfrak{n}), \boldsymbol{j}(\mathfrak{n})], \boldsymbol{j}(\mathfrak{n})\bigr] \subset \boldsymbol{j}(\mathfrak{n}),$$

$$(4)\ \bigl[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], \mathfrak{n}\bigr] \subset \boldsymbol{j}(\mathfrak{n}), \quad (5)\ \bigl[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], \boldsymbol{j}(\mathfrak{n})\bigr] \subset \mathfrak{n}.$$

Here $\boldsymbol{j} := J_o$ and $\mathfrak{q} := T_o(G/R)$.

REMARK 5.2.2. Lemma 5.2.1 means that the above $\mathfrak{n}$ is a *reflective subspace* of $\mathfrak{q}$ in the sense of Leung [**13**, p. 156] (see Leung [**14**, p. 180] also), though the terminology "reflective subspace" is used only for a Riemannian symmetric space in the original sense.

PROOF OF LEMMA 5.2.1. (1) Let $\mathfrak{n}^{\perp}$ denote the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{q}$ with respect to $\boldsymbol{g}_o$. Then one has $\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{n}^{\perp}$ since $\boldsymbol{g}|_N$ is non-degenerate. Therefore we conclude (1), if $\boldsymbol{j}(\mathfrak{n}) = \mathfrak{n}^{\perp}$. Since N is totally real, we obtain $\boldsymbol{j}(\mathfrak{n}) \subset \mathfrak{n}^{\perp}$. This gives

$$\boldsymbol{j}(\mathfrak{n}) = \mathfrak{n}^{\perp}$$

because $\dim_{\mathbb{R}} \mathfrak{n}^{\perp} = \dim_{\mathbb{R}} \mathfrak{q} - \dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}} \boldsymbol{j}(\mathfrak{n})$ follows from $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} G/R$ and $\boldsymbol{j} : \mathfrak{q} \to \mathfrak{q}$ being linear isomorphic.

(2) Theorem 4.3 [**11**, p. 237] provides (2).

(3) By Proposition 2.3.3 there exist the canonical central element $T \in \mathfrak{r}$ and a non-zero $\lambda \in \mathbb{R}$ satisfying

$$R = C_G(T), \quad \boldsymbol{j} = \operatorname{ad} T|_{\mathfrak{q}}, \quad \boldsymbol{g}_o = \lambda \cdot B_{\mathfrak{g}}|_{\mathfrak{q} \times \mathfrak{q}}.$$

Needless to say, $\boldsymbol{j}(\mathfrak{n}) = [T, \mathfrak{n}]$ holds. First, let us show

$$[\boldsymbol{j}(\mathfrak{n}), \boldsymbol{j}(\mathfrak{n})] \subset [\mathfrak{n}, \mathfrak{n}]. \tag{5.2.1}$$

By the Jacobi identity, $\boldsymbol{j} = \operatorname{ad} T|_{\mathfrak{q}}$ and $\boldsymbol{j}^2 = -\operatorname{id}_{\mathfrak{q}}$ one deduces

$$[\boldsymbol{j}(\mathfrak{n}), \boldsymbol{j}(\mathfrak{n})] \subset \big[[T, \mathfrak{n}], \boldsymbol{j}(\mathfrak{n})\big] \subset \big[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], T\big] + \big[[\boldsymbol{j}(\mathfrak{n}), T], \mathfrak{n}\big] \subset \big[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], T\big] + [\mathfrak{n}, \mathfrak{n}].$$

It follows from $[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})] \subset \mathfrak{r}$ and $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(T)$ that $[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], T] = \{0\}$, so that (5.2.1) holds. Since (5.2.1), (2) and $[[\mathfrak{n}, \mathfrak{n}], T] \subset [\mathfrak{r}, T] = \{0\}$, we have

$$\begin{aligned}\big[[\boldsymbol{j}(\mathfrak{n}), \boldsymbol{j}(\mathfrak{n})], \boldsymbol{j}(\mathfrak{n})\big] &\subset \big[[\mathfrak{n}, \mathfrak{n}], \boldsymbol{j}(\mathfrak{n})\big] \subset \big[[\mathfrak{n}, \mathfrak{n}], [T, \mathfrak{n}]\big] \subset \big[T, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]\big] + \big[\mathfrak{n}, [[\mathfrak{n}, \mathfrak{n}], T]\big] \\ &\subset [T, \mathfrak{n}] + \big[\mathfrak{n}, [[\mathfrak{n}, \mathfrak{n}], T]\big] \subset [T, \mathfrak{n}] = \boldsymbol{j}(\mathfrak{n}).\end{aligned}$$

(4) It suffices to confirm that

$$B_{\mathfrak{g}}\big([[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], \mathfrak{n}], \mathfrak{n}\big) = \{0\}$$

by virtue of $[[\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})], \mathfrak{n}] \subset \mathfrak{q}$, $\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{n}^{\perp}$, $\mathfrak{n}^{\perp} = \boldsymbol{j}(\mathfrak{n})$ (cf. (1)) and $\boldsymbol{g}_o = \lambda \cdot B_{\mathfrak{g}}|_{\mathfrak{q} \times \mathfrak{q}}$. That is immediate from (2) and $B_{\mathfrak{g}}([X, Y], Z) = -B_{\mathfrak{g}}(Y, [X, Z])$ for all $X, Y, Z \in \mathfrak{g}$.

(5) We conclude (5) by arguments similar to those stated above and (3). □

Let us prove

Proposition 5.2.3. *With the same setting as in Lemma* 5.2.1; *suppose that $Z(G)$ is trivial. Then, N is also a real form of G/R.*

Proof. It suffices to construct an involutive antiholomorphic isometry $\hat{\varrho}$ of G/R such that $N = (G/R)^{\hat{\varrho}}$. Our first aim is to verify

$$\mathfrak{r} = [\mathfrak{n}, \mathfrak{n}] \oplus [\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})]. \tag{5.2.2}$$

Nomizu [**21**, p. 56, (16.2)] shows $\mathfrak{r} = [\mathfrak{q}, \mathfrak{q}]$. So, Lemma 5.2.1-(1) and (5.2.1) yield

$$\mathfrak{r} = [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})].$$

Furthermore, it follows from Lemma 5.2.1-(2) and $\boldsymbol{j}(\mathfrak{n}) = \mathfrak{n}^{\perp}$ that

$$B_{\mathfrak{g}}\big([\mathfrak{n}, \mathfrak{n}], [\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})]\big) \subset B_{\mathfrak{g}}\big([\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]], \boldsymbol{j}(\mathfrak{n})\big) \subset B_{\mathfrak{g}}\big(\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})\big) = \lambda^{-1} \cdot \boldsymbol{g}_o\big(\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})\big) = \{0\}$$

(see Proof of Lemma 5.2.1-(1) for $\boldsymbol{j}(\mathfrak{n}) = \mathfrak{n}^{\perp}$). Consequently, (5.2.2) holds because $B_{\mathfrak{g}}$ is non-degenerate on $\mathfrak{r}$. Lemma 5.2.1-(1) and (5.2.2) allow us to define a linear map ϱ of $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{q}$ by

$$\varrho(X) := \begin{cases} X & \text{if } X \in [\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{n}, \\ -X & \text{if } X \in [\mathfrak{n}, \boldsymbol{j}(\mathfrak{n})] \oplus \boldsymbol{j}(\mathfrak{n}). \end{cases}$$

Then $\varrho \in \mathrm{Inv}(\mathfrak{g})$ by virtue of Lemma 5.2.1 and the Jacobi identity (cf. Leung [**13**, p. 157]). Our next aim is to show

$$\varrho(T) = -T. \tag{5.2.3}$$

On the one hand, (5.2.2) yields $\varrho(\mathfrak{r}) \subset \mathfrak{r}$, and hence $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(T)$ implies that

$$[\varrho(T), X] = \varrho\big([T, \varrho(X)]\big) = 0 = -[T, X] \text{ for any } X \in \mathfrak{r}.$$

On the other hand, by $\boldsymbol{j} = \mathrm{ad}\, T|_{\mathfrak{q}}$ and $\boldsymbol{j}^2 = -\mathrm{id}$ we deduce $[\varrho(T), N_1 + \boldsymbol{j}(N_2)] = \varrho([T, N_1 - \boldsymbol{j}(N_2)]) = \varrho(\boldsymbol{j}(N_1) + N_2) = -\boldsymbol{j}(N_1) + N_2 = -[T, N_1 + \boldsymbol{j}(N_2)]$ for any $N_1, N_2 \in \mathfrak{n}$. This and Lemma 5.2.1-(1) imply that

$$[\varrho(T), Y] = -[T, Y] \text{ for any } Y \in \mathfrak{q}.$$

Consequently $[\varrho(T), Z] = -[T, Z]$ for all $Z \in \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{q}$, and hence (5.2.3) follows from $\mathfrak{g}$ being semisimple. Now, by $Z(G) = \{e\}$ there exists a unique involution of G whose differential coincides with ϱ. Denote the involution by the same ϱ. It follows from (5.2.3) and $R = C_G(T)$ that $\varrho(R) = R$. Hence, one can get an involutive antiholomorphic isometry $\hat{\varrho}$ of G/R by setting

$$\hat{\varrho}(gR) := \varrho(g)R \text{ for } gR \in G/R$$

because $\boldsymbol{j} = \mathrm{ad}\, T|_{\mathfrak{q}}$, (5.2.3) and $\boldsymbol{g}_o = \lambda \cdot B_{\mathfrak{g}}|_{\mathfrak{q} \times \mathfrak{q}}$. The rest of proof is to confirm $N = (G/R)^{\hat{\varrho}}$. Lemma 2.4.4 and Theorem 2.6.1 imply that $(G/R)^{\hat{\varrho}}$ is a connected complete totally geodesic submanifold of $(G/R, \nabla^1)$. For this reason Proposition 2.4.2 and Remark 2.3.2 enable us to conclude $N = (G/R)^{\hat{\varrho}}$ because $o \in N$, $o \in (G/R)^{\hat{\varrho}}$ and $T_oN = \mathfrak{n}$ coincides with $T_o(G/R)^{\hat{\varrho}}$. □

5.3. Proof of Theorem 5.1.1.

We are now in a position to demonstrate Theorem 5.1.1.

Proof of Theorem 5.1.1. (i) $\Longrightarrow$ (ii): Suppose that N is a real form of G/R. Then, there exists an involutive antiholomorphic isometry $\hat{\eta}$ of G/R satisfying $N = (G/R)^{\hat{\eta}}$. Take any $p \in N = (G/R)^{\hat{\eta}}$ and decompose $T_p(G/R)$ as

$$T_p(G/R) = T_p^+(G/R) \oplus T_p^-(G/R), \quad \boldsymbol{g}_p\big(T_p^+(G/R), T_p^-(G/R)\big) = \{0\}. \tag{5.3.1}$$

Here $T_p^{\pm}(G/R)$ denotes the ± 1-eigenspace of $(d\hat{\eta})_p$ in $T_p(G/R)$. Since $d\hat{\eta} \circ J = -J \circ d\hat{\eta}$ at p, we have $J_p(T_p^{\pm}(G/R)) = T_p^{\mp}(G/R)$. Therefore $T_p^+(G/R) = T_pN$ yields $\boldsymbol{g}_p(T_pN, J_p(T_pN)) = \{0\}$ and $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} G/R$ in terms of (5.3.1). By (5.3.1) and

$T_p^+(G/R) = T_pN$ we confirm that $\boldsymbol{g}|_N$ is non-degenerate. Consequently,

(a) N is a totally real submanifold of G/R, (b) $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} G/R$,
(c) $\boldsymbol{g}|_N$ is non-degenerate.

Theorem 2.6.1 assures that (d) N is connected. It is immediate from $o \in N = (G/R)^{\hat{\eta}}$ that $\hat{\eta}(o) = o$. Hence, Corollary 2.4.6 and Lemma 2.4.4 imply that (e) N is a complete totally geodesic submanifold of G/R. We have completed the proof of (i) $\Longrightarrow$ (ii) by virtue of (a)–(e).

(ii) $\Longrightarrow$ (i): cf. Proposition 5.2.3. □

REMARK 5.3.1. Theorem 5.1.1 means that Leung's classification [**14**] of real forms is equivalent to Takeuchi's classification [**26**] of totally real totally geodesic submanifolds.

6. Para-holomorphic involutions of simple para-Hermitian symmetric Lie algebras.

The main purpose of this section is to demonstrate Theorem 6.5.1 which is a classification of the pairs $((\mathfrak{g}_d, \sigma), \theta)$ of simple para-Hermitian symmetric Lie algebras $(\mathfrak{g}_d, \sigma)$ and their para-holomorphic involutions θ, where $\mathfrak{g}_d$ are real forms of complex simple Lie algebras.

6.1. An equivalence relation on $\mathrm{Inv}(\mathfrak{g})^{p,p}$.

We first recall

DEFINITION 6.1.1 (cf. Kaneyuki-Kozai [**9**, p. 88]). (i) A symmetric Lie algebra $(\mathfrak{u}, \sigma)$ is called *para-Hermitian* when there exist an $\mathrm{ad}\,\mathfrak{u}^\sigma$-invariant para-complex structure $\boldsymbol{i}$ on $\mathfrak{u}^{-\sigma}$ and an $\mathrm{ad}\,\mathfrak{u}^\sigma$-invariant para-Hermitian form $\langle \cdot\,, \cdot \rangle$ (with respect to $\boldsymbol{i}$) on $\mathfrak{u}^{-\sigma}$.

(ii) Let $(\mathfrak{u}, \sigma)$ be a para-Hermitian symmetric Lie algebra with para-complex structure $\boldsymbol{i}$. Then $\vartheta \in \mathrm{Inv}(\mathfrak{u})$ is said to be a *para-holomorphic involution* of $(\mathfrak{u}, \sigma)$, if $[\sigma, \vartheta] = 0$ and $\vartheta \circ \boldsymbol{i} = \boldsymbol{i} \circ \vartheta$ on $\mathfrak{u}^{-\sigma}$. Remark that we assume id and σ to be one of the para-holomorphic involutions of $(\mathfrak{u}, \sigma)$.

REMARK 6.1.2. About a para-Hermitian symmetric Lie algebra $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}_{\mathbb{C}}$ simple, Kaneyuki-Kozai [**9**] enables us to assert statements analogous with those on Proposition 2.3.3, Remark 2.3.4 and Lemma 2.3.5. In this paper, we do not describe these statements in detail.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$. Denote by $\mathrm{Inv}(\mathfrak{g})^{p,p}$ the set of pairs $((\mathfrak{g}, \sigma), \vartheta)$ of para-Hermitian symmetric Lie algebras $(\mathfrak{g}, \sigma)$ and para-holomorphic involutions ϑ of $(\mathfrak{g}, \sigma)$. Define an equivalence relation on $\mathrm{Inv}(\mathfrak{g})^{p,p}$ by

DEFINITION 6.1.3. For $((\mathfrak{g}, \sigma_1), \vartheta_1), ((\mathfrak{g}, \sigma_2), \vartheta_2) \in \mathrm{Inv}(\mathfrak{g})^{p,p}$ we say that they are *equivalent*, if there exists a $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\text{(i) } \phi \circ \sigma_1 \circ \phi^{-1} = \sigma_2, \quad \text{(ii) } \phi \circ \vartheta_1 \circ \phi^{-1} = \vartheta_2.$$

Henceforth, we denote by $\mathrm{Inv}(\mathfrak{g})^{p,p}/_{\mathrm{Aut}(\mathfrak{g})}$ the set of equivalence classes on $\mathrm{Inv}(\mathfrak{g})^{p,p}$.

We will determine $\mathrm{Inv}(\mathfrak{g}_d)^{p,p}/_{\mathrm{Aut}(\mathfrak{g}_d)}$ for all real forms $\mathfrak{g}_d$ of complex simple Lie algebras $\mathfrak{g}_{\mathbb{C}}$, by considering

$$\begin{array}{ccccccccc} d\mathcal{R}_{\mathfrak{g}}/_{\sim} & \xrightarrow{F_1} & \widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim} & \xleftrightarrow{F} & \widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}/_{\approx} & \xleftarrow{F_2} & d\mathcal{P}_{\mathfrak{g}_d}/_{\approx} & \xleftrightarrow{f} & \mathrm{Inv}(\mathfrak{g}_d)^{p,p}/_{\mathrm{Aut}(\mathfrak{g}_d)} \\ [(\mathfrak{g},T,\eta)] & & [(\mathfrak{g},T,\eta,\theta)] & & [(\mathfrak{g}_d,iT,\theta,\eta)] & & [(\mathfrak{g}_d,iT,\theta)] & & [(\mathfrak{g}_d,\exp\pi\,\mathrm{ad}\,T,\theta)] \end{array} \tag{6.1.1}$$

(cf. four Lemmas 6.4.1–6.4.4). Note that we have already known the left-hand side of (6.1.1), where $\mathfrak{g}$ are non-compact real forms of $\mathfrak{g}_{\mathbb{C}}$.

6.2. An equivalence relation on $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}$.

Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra. Consider a triplet $(\mathfrak{g},Z,\vartheta)$ consisting of (1) a non-zero semisimple element $Z\in\mathfrak{g}$ such that the eigenvalue of $\mathrm{ad}\,Z$ in $\mathfrak{g}$ is ±1 or zero and (2) $\vartheta\in\mathrm{Inv}(\mathfrak{g})$ such that $\vartheta(Z)=Z$; and denote by $d\mathcal{P}_{\mathfrak{g}}$ the set of such triplets. Let us define an equivalence relation '$\approx$' on $d\mathcal{P}_{\mathfrak{g}}$ similarly to Definition 2.7.4.

Remark 6.2.1. About any $(\mathfrak{g},Z,\vartheta)\in d\mathcal{P}_{\mathfrak{g}}$, one can assert statements analogous with those on Remark 2.7.3 and Lemma 4.2.1.

Now, let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. Denote by $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}$ the set of quartets $(\mathfrak{g},Z,\vartheta,\tau)$, where $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$, $(\mathfrak{g},Z,\vartheta)\in d\mathcal{P}_{\mathfrak{g}}$ and τ is a Cartan involution of $\mathfrak{g}$ such that $\tau(Z)=-Z$ and $[\tau,\vartheta]=0$. We extend the equivalence relation '$\approx$' on $d\mathcal{P}_{\mathfrak{g}}$ to $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}$ as follows:

Definition 6.2.2. For $(\mathfrak{g}_1,Z_1,\vartheta_1,\tau_1),(\mathfrak{g}_2,Z_2,\vartheta_2,\tau_2)\in\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}$ we say that they are *equivalent*, if there exists an isomorphism ϕ of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ satisfying

$$\text{(i) } \phi(Z_1)=Z_2,\quad \text{(ii) } \phi\circ\vartheta_1=\vartheta_2\circ\phi,\quad \text{(iii) } \phi\circ\tau_1=\tau_2\circ\phi.$$

Henceforth, we denote by $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}/_{\approx}$ the set of equivalence classes on $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}$.

6.3. An equivalence relation on $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}$.

Denote by $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}$ the set of quartets $(\mathfrak{g},T,\eta,\theta)$ which consist of (1) real forms $\mathfrak{g}\subset\mathfrak{g}_{\mathbb{C}}$, (2) $(\mathfrak{g},T,\eta)\in d\mathcal{R}_{\mathfrak{g}}$ and (3) Cartan involutions θ of $\mathfrak{g}$ satisfying $\theta(T)=T$ and $[\theta,\eta]=0$. We extend the equivalence relation '$\sim$' on $d\mathcal{R}_{\mathfrak{g}}$ to $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}$ as follows:

Definition 6.3.1. For $(\mathfrak{g}_1,T_1,\eta_1,\theta_1),(\mathfrak{g}_2,T_2,\eta_2,\theta_2)\in\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}$ we say that they are *equivalent*, if there exists an isomorphism ϕ of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ satisfying

$$\text{(i) } \phi(T_1)=T_2,\quad \text{(ii) } \phi\circ\eta_1=\eta_2\circ\phi,\quad \text{(iii) } \phi\circ\theta_1=\theta_2\circ\phi.$$

Henceforth, we denote by $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$ the set of equivalence classes on $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}$.

6.4. Relation among $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$, $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$, etc.

We are going to clarify relation among $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$, $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$, $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}/_{\approx}$, $d\mathcal{P}_{\mathfrak{g}}/_{\approx}$ and $\mathrm{Inv}(\mathfrak{g})^{p,p}/_{\mathrm{Aut}(\mathfrak{g})}$. The following comes from Proposition 3.1 [**20**, p. 76] and Proposition 3.1.2:

Lemma 6.4.1. *For a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, the following F is a bijection of $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$ onto $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}/_{\approx}$:*

$$F : [(\mathfrak{g}, T, \eta, \theta)] \mapsto [(\mathfrak{g}_d, iT, \theta, \eta)],$$

where $(\mathfrak{g}_d, \theta)$ is the Berger dual of symmetric Lie algebra $(\mathfrak{g}, \eta)$.

Let us study relation between $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ and $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$:

Lemma 6.4.2. *For a real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, the following F_1 is an injection of $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ into $\widehat{d\mathcal{R}_{\mathfrak{g}_{\mathbb{C}}}}/_{\sim}$:*

$$F_1 : [(\mathfrak{g}, T, \eta)] \mapsto [(\mathfrak{g}, T, \eta, \theta)],$$

where θ is any Cartan involution of $\mathfrak{g}$ such that $\theta(T) = T$ and $[\theta, \eta] = 0$ (see Lemma 4.2.1 for the existence of θ).

Proof. We only prove that F_1 is well-defined. Take $(\mathfrak{g}, T_1, \eta_1), (\mathfrak{g}, T_2, \eta_2) \in d\mathcal{R}_{\mathfrak{g}}$ and suppose that they are equivalent via $\phi \in \mathrm{Aut}(\mathfrak{g})$,

$$\text{(i) } \phi(T_1) = T_2, \quad \text{(ii) } \phi \circ \eta_1 \circ \phi^{-1} = \eta_2.$$

Let θ_a be any Cartan involution of $\mathfrak{g}$ such that $\theta_a(T_a) = T_a$ and $[\theta_a, \eta_a] = 0$ $(a = 1, 2)$. Both $\theta_1' := \phi \circ \theta_1 \circ \phi^{-1}$ and θ_2 are Cartan involutions, and satisfy $\theta_1'(T_2) = \theta_2(T_2) = T_2$ and $[\theta_1', \eta_2] = [\theta_2, \eta_2] = 0$. This implies that $\theta_2 \circ \theta_1' \circ \theta_2 \circ \theta_1'$ commutes with both σ_{T_2} and η_2. Therefore, the proof of Theorem 2.1 [**15**, p. 153] enables us to have a $\psi = \exp \mathrm{ad}_{\mathfrak{g}} X$ such that $\sigma_{T_2}(X) = X$, $\eta_2(X) = X$ and $\psi \circ \theta_1' \circ \psi^{-1} = \theta_2$. Note that ψ satisfies

$$\psi \circ \theta_1' \circ \psi^{-1} = \theta_2, \quad \psi(T_2) = T_2, \quad [\psi, \eta_2] = 0$$

because $\mathfrak{c}_{\mathfrak{g}}(T_2) = \mathfrak{g}^{\sigma_{T_2}}$. Then $(\mathfrak{g}, T_1, \eta_1, \theta_1)$ is equivalent to $(\mathfrak{g}, T_2, \eta_2, \theta_2)$ via $\psi \circ \phi$, in the sense of Definition 6.3.1. □

Similarly one can conclude

Lemma 6.4.3. *For a real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, the following F_2 is an injection of $d\mathcal{P}_{\mathfrak{g}}/_{\approx}$ into $\widehat{d\mathcal{P}_{\mathfrak{g}_{\mathbb{C}}}}/_{\approx}$:*

$$F_2 : [(\mathfrak{g}, Z, \vartheta)] \mapsto [(\mathfrak{g}, Z, \vartheta, \tau)],$$

where τ is any Cartan involution of $\mathfrak{g}$ such that $\tau(Z) = -Z$ and $[\tau, \vartheta] = 0$.

There exists a one-to-one correspondence between $d\mathcal{P}_{\mathfrak{g}}/_{\approx}$ and $\mathrm{Inv}(\mathfrak{g})^{p,p}/_{\mathrm{Aut}(\mathfrak{g})}$:

LEMMA 6.4.4. *For a real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$, the following f is a bijection of $d\mathcal{P}_{\mathfrak{g}}/_{\approx}$ onto* $\mathrm{Inv}(\mathfrak{g})^{p,p}/_{\mathrm{Aut}(\mathfrak{g})}$:

$$f : [(\mathfrak{g}, Z, \vartheta)] \mapsto [(\mathfrak{g}, \exp \pi \,\mathrm{ad}(iZ), \vartheta)].$$

PROOF. We only show that f is injective. Take $(\mathfrak{g}, Z_1, \vartheta_1), (\mathfrak{g}, Z_2, \vartheta_2) \in d\mathcal{P}_{\mathfrak{g}}$ and suppose that $(\mathfrak{g}, \sigma_1, \vartheta_1)$ is equivalent to $(\mathfrak{g}, \sigma_2, \vartheta_2)$ via $\phi \in \mathrm{Aut}(\mathfrak{g})$,

$$\text{(i) } \phi \circ \sigma_1 \circ \phi^{-1} = \sigma_2, \quad \text{(ii) } \phi \circ \vartheta_1 \circ \phi^{-1} = \vartheta_2,$$

where $\sigma_a := \exp \pi \,\mathrm{ad}(iZ_a)$ for $a = 1, 2$. On the one hand,

$$\phi(Z_1) = Z_2 \text{ or } \phi(Z_1) = -Z_2$$

follows from (i) because the para-complex structure $\boldsymbol{i}_2$ of $(\mathfrak{g}, \sigma_2)$ is unique up to sign $\pm$ and both $\mathrm{ad}\,\phi(Z_1)$ and $\mathrm{ad}\,Z_2$ induce para-complex structures of $(\mathfrak{g}, \sigma_2)$. On the other hand, there exists a Cartan involution τ of $\mathfrak{g}$ such that $\tau(Z_2) = -Z_2$ and $[\tau, \vartheta_2] = 0$. Hence, $(\mathfrak{g}, Z_1, \vartheta_1)$ is equivalent to $(\mathfrak{g}, Z_2, \vartheta_2)$ via ϕ in case of $\phi(Z_1) = Z_2$, and via $\tau \circ \phi$ in case of $\phi(Z_1) = -Z_2$. This means that f is injective. □

6.5. The classification of elements $((\mathfrak{g}, \sigma), \vartheta) \in \mathrm{Inv}(\mathfrak{g})^{p,p}$.

Four Lemmas 6.4.1–6.4.4 and (6.1.1) imply that the classification of elements $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ for each real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ is equivalent to that of $((\mathfrak{g}_d, \sigma), \theta) \in \mathrm{Inv}(\mathfrak{g}_d)^{p,p}$ for each real form $\mathfrak{g}_d \subset \mathfrak{g}_{\mathbb{C}}$. We have already determined $d\mathcal{R}_{\mathfrak{g}}/_{\sim}$ for each non-compact real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ (cf. Section 4). Thus, one can assert the following (cf. Remark 6.5.2-(ii)):

THEOREM 6.5.1. *Up to equivalence, the pairs $((\mathfrak{g}_d, \sigma), \theta)$ of para-Hermitian symmetric Lie algebras $(\mathfrak{g}_d, \sigma)$ and para-holomorphic involutions θ of $(\mathfrak{g}_d, \sigma)$ are given in List* III (*see p.* 84)*, where $\mathfrak{g}_d$ are real forms of complex simple Lie algebras.*

PROOF. We only explain how to obtain the pairs $((\mathfrak{g}_d, \mathfrak{g}_d^{\sigma}), (\mathfrak{g}_d^{\theta}, \mathfrak{g}_d^{\sigma} \cap \mathfrak{g}_d^{\theta}))$ in List III from the triplets $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_{\mathfrak{g}}$ which correspond to $(G/R, M)$ in List I. It suffices to give an example.

Let $(\mathfrak{g}, T, \eta)$ denote the triplet in Proposition 4.10.4-(1). Then $T = i(Z_1 - Z_6)$, $\eta = \eta_1$, and θ in (4.10.2) is a Cartan involution of $\mathfrak{g}$ such that $\theta(T) = T$ and $[\theta, \eta] = 0$. Furthermore,

$$(\mathfrak{g}, \mathfrak{g}^{\sigma}) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{so}(2)), \quad (\mathfrak{g}^{\eta}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta}) = (\mathfrak{sp}(3,1), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1,1)),$$

where $\sigma := \sigma_T$. Denote by $(\mathfrak{g}_d, \theta)$ the Berger dual of symmetric Lie algebra $(\mathfrak{g}, \eta)$. It follows from $\mathfrak{g}_u = \mathfrak{e}_6$ and $\mathfrak{g}_u^{\eta} = \mathfrak{sp}(4)$ (resp. $\mathfrak{g}_u^{\sigma} = \mathfrak{so}(10) \oplus \mathfrak{so}(2)$ and $\mathfrak{g}_u^{\sigma} \cap \mathfrak{g}_u^{\eta} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$) that

$$\mathfrak{g}_d = \mathfrak{e}_{6(6)} \text{ (resp. } \mathfrak{g}_d^\sigma = \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)).$$

Similarly, it follows from $\mathfrak{g}_u^\theta = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$ and $\mathfrak{g}_u^\theta \cap \mathfrak{g}_u^\eta = \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$ (resp. $\mathfrak{g}_u^\theta \cap \mathfrak{g}_u^\sigma = \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(2)$ and $\mathfrak{g}_u^\theta \cap \mathfrak{g}_u^\sigma \cap \mathfrak{g}_u^\eta = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$) that

$$\mathfrak{g}_d^\theta = \mathfrak{su}^*(6) \oplus \mathfrak{su}^*(2) \text{ (resp. } \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta = \mathfrak{su}^*(4) \oplus \mathfrak{su}^*(2) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2)).$$

Hence, we have obtained the pair $((\mathfrak{g}_d, \mathfrak{g}_d^\sigma), (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta))$ of no. EII.1 in List III from $(\mathfrak{g}, T, \eta)$ in Proposition 4.10.4-(1). □

Remark 6.5.2. (i) In List III, the notation

X.i	$\begin{cases}(\mathfrak{g}_d, \mathfrak{g}_d^\sigma), \\ (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta)\end{cases}$

means that the pair $((\mathfrak{g}_d, \sigma), \theta)$ corresponds to that of the pseudo-Hermitian symmetric space G/R of type X and the real form $M \subset G/R$ of no. i in List I.

(ii) Let $((\mathfrak{g}_d, \sigma), \theta)$ denote an element of $\mathrm{Inv}(\mathfrak{g}_d)^{p,p}$ which corresponds to $(\mathfrak{g}, T, \eta) \in d\mathcal{R}_\mathfrak{g}$. If $\mathfrak{g}$ is compact, then $\theta = \mathrm{id}$, $(\mathfrak{g}_d, \mathfrak{g}_d^\sigma) = (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta)$, and $(\mathfrak{g}_d, \mathfrak{g}_d^\sigma)$ is the same as $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ in Theorem 4.6 [**20**, p. 80].

(iii) One can achieve the classification of simple para-Hermitian symmetric Lie algebras by collecting all $(\mathfrak{g}_d, \mathfrak{g}_d^\sigma)$ in List III and the complexifications of $(\mathfrak{g}_d, \mathfrak{g}_d^\sigma)$ without overlap.

We end this paper with giving an example which enables one to read the graded decompositions $\mathfrak{g}_d = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ of the first kind and their further decompositions $\mathfrak{g}_j = \mathfrak{g}_j^+ \oplus \mathfrak{g}_j^-$ $(j = 0, \pm 1)$ from List III.

Example 6.5.3. Let $((\mathfrak{g}_d, \sigma), \theta)$ denote the pair of no. AIII.1 in List III. Then $\sigma = \exp \pi \,\mathrm{ad}\, T$, $\theta = \mathrm{Ad}\, I_{p,q}$ and

$$\mathfrak{g}_d = \mathfrak{sl}(p+q, \mathbb{R}) = \{A \in M_{p+q,p+q}(\mathbb{R}) \mid \mathrm{Tr}(A) = 0\},$$

where $T := T_h - T_p + T_k$ and we denote by $M_{a,b}(\mathbb{R})$ the set of $a \times b$ real matrices (cf. Proposition 4.5.1-(1)). Remark that a para-complex structure $\boldsymbol{i}$ of $(\mathfrak{g}_d, \sigma)$ is induced by $\mathrm{ad}(iT)$, and that (4.5.2) yields

$$T = T_h - T_p + T_k = \frac{i}{x-y} \mathrm{diag}(\underbrace{x, \dots, x}_{h}, \underbrace{y, \dots, y}_{p-h}, \underbrace{x, \dots, x}_{k-p}, \underbrace{y, \dots, y}_{p+q-k}),$$

where $x := 2p + q - h - k$ and $y := p - h - k$. Now, let $\mathfrak{g}_j$ denote the j-eigenspace of $\mathrm{ad}(iT)$ in $\mathfrak{g}_d$, where $j = 0, \pm 1$. A direct computation provides us with

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{cc|cc} O & O & O & O \\ M_{p-h,h} & O & M_{p-h,k-p} & O \\ \hline O & O & O & O \\ M_{p+q-k,h} & O & M_{p+q-k,k-p} & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{0} = \left\{ \left(\begin{array}{cc|cc} M_{h,h} & O & M_{h,k-p} & O \\ O & M_{p-h,p-h} & O & M_{p-h,p+q-k} \\ \hline M_{k-p,h} & O & M_{k-p,k-p} & O \\ O & M_{p+q-k,p-h} & O & M_{p+q-k,p+q-k} \end{array} \right) \in \mathfrak{g}_d \, \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{+1} = \left\{ \left(\begin{array}{cc|cc} O & M_{h,p-h} & O & M_{h,p+q-k} \\ O & O & O & O \\ \hline O & M_{k-p,p-h} & O & M_{k-p,p+q-k} \\ O & O & O & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\}.$$

Since $\theta(T) = T$, all $\mathfrak{g}_j$ are invariant under θ. So, each $\mathfrak{g}_j$ can be further decomposed as $\mathfrak{g}_j = \mathfrak{g}_j^+ \oplus \mathfrak{g}_j^-$, where $\mathfrak{g}_j^\pm$ denote the ± 1-eigenspaces of θ in $\mathfrak{g}_j$, respectively:

$$\mathfrak{g}_{-1}^{+} = \left\{ \left(\begin{array}{cc|cc} O & O & O & O \\ M_{p-h,h} & O & O & O \\ \hline O & O & O & O \\ O & O & M_{p+q-k,k-p} & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{-1}^{-} = \left\{ \left(\begin{array}{cc|cc} O & O & O & O \\ O & O & M_{p-h,k-p} & O \\ \hline O & O & O & O \\ M_{p+q-k,h} & O & O & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{0}^{+} = \left\{ \left(\begin{array}{cc|cc} M_{h,h} & O & O & O \\ O & M_{p-h,p-h} & O & O \\ \hline O & O & M_{k-p,k-p} & O \\ O & O & O & M_{p+q-k,p+q-k} \end{array} \right) \in \mathfrak{g}_d \, \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{0}^{-} = \left\{ \left(\begin{array}{cc|cc} O & O & M_{h,k-p} & O \\ O & O & O & M_{p-h,p+q-k} \\ \hline M_{k-p,h} & O & O & O \\ O & M_{p+q-k,p-h} & O & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{+1}^{+} = \left\{ \left(\begin{array}{cc|cc} O & M_{h,p-h} & O & O \\ O & O & O & O \\ \hline O & O & O & M_{k-p,p+q-k} \\ O & O & O & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\},$$

$$\mathfrak{g}_{+1}^{-} = \left\{ \left(\begin{array}{cc|cc} O & O & O & M_{h,p+q-k} \\ O & O & O & O \\ \hline O & M_{k-p,p-h} & O & O \\ O & O & O & O \end{array} \right) \middle| \, M_{a,b} \in M_{a,b}(\mathbb{R}) \right\}.$$

The above subspaces satisfy

$$\begin{aligned} &\mathfrak{g}_d = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \\ &\mathfrak{g}_i = \mathfrak{g}_i^{+} \oplus \mathfrak{g}_i^{-}, \qquad [\mathfrak{g}_i^{+}, \mathfrak{g}_j^{+}] \subset \mathfrak{g}_{i+j}^{+}, \ [\mathfrak{g}_i^{+}, \mathfrak{g}_j^{-}] \subset \mathfrak{g}_{i+j}^{-}, \ [\mathfrak{g}_i^{-}, \mathfrak{g}_j^{-}] \subset \mathfrak{g}_{i+j}^{+} \end{aligned}$$

for any $i, j = 0, \pm 1$.

List III

no.	$\begin{cases} (\mathfrak{g}_d, \mathfrak{g}_d^{\sigma}), \\ (\mathfrak{g}_d^{\theta}, \mathfrak{g}_d^{\sigma} \cap \mathfrak{g}_d^{\theta}) \end{cases}$
AI	
AIII.1	$\begin{cases} \big(\mathfrak{sl}(p+q,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(h+k-p,\mathbb{R}) \oplus \mathfrak{gl}(2p+q-h-k,\mathbb{R}))\big), \\ \big(\mathfrak{sl}(p,\mathbb{R}) \oplus \mathfrak{sl}(q,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{s}(\mathfrak{gl}(h,\mathbb{R}) \oplus \mathfrak{gl}(p-h,\mathbb{R})) \\ \quad \oplus \mathfrak{s}(\mathfrak{gl}(k-p,\mathbb{R}) \oplus \mathfrak{gl}(p+q-k,\mathbb{R})) \oplus \mathbb{R}\big) \end{cases}$
AI.2	$\begin{cases} \big(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(n,\mathbb{R}) \oplus \mathfrak{gl}(n,\mathbb{R}))\big), \\ \big(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$
AII.2	$\begin{cases} \big(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(n,\mathbb{R}) \oplus \mathfrak{gl}(n,\mathbb{R}))\big), \\ \big(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$
AIII.5	$\begin{cases} \big(\mathfrak{sl}(2p,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(2h,\mathbb{R}) \oplus \mathfrak{gl}(2p-2h,\mathbb{R}))\big), \\ \big(\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{t}, \mathfrak{s}(\mathfrak{gl}(h,\mathbb{C}) \oplus \mathfrak{gl}(p-h,\mathbb{C})) \oplus \mathfrak{t}\big) \end{cases}$
AII	
AIII.3	$\begin{cases} \big(\mathfrak{su}^*(2n+2m), \mathfrak{su}^*(2h+2k-2n) \oplus \mathfrak{su}^*(4n+2m-2h-2k) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}^*(2n) \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}, \mathfrak{su}^*(2h) \oplus \mathfrak{su}^*(2n-2h) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2k-2n) \\ \quad \oplus \mathfrak{su}^*(2n+2m-2k) \oplus \mathbb{R} \oplus \mathbb{R}\big) \end{cases}$
AIII.4	$\begin{cases} \big(\mathfrak{su}^*(2p), \mathfrak{su}^*(2h) \oplus \mathfrak{su}^*(2p-2h) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{t}, \mathfrak{s}(\mathfrak{gl}(h,\mathbb{C}) \oplus \mathfrak{gl}(p-h,\mathbb{C})) \oplus \mathfrak{t}\big) \end{cases}$
AI.4	$\begin{cases} \big(\mathfrak{su}^*(4m), \mathfrak{su}^*(2m) \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}^*(4m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
AII.4	$\begin{cases} \big(\mathfrak{su}^*(4m), \mathfrak{su}^*(2m) \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(m,m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
AIII	
AI.1	$\begin{cases} \big(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$
AII.1	$\begin{cases} \big(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$

no.	$\begin{cases} (\mathfrak{g}_d, \mathfrak{g}_d^\sigma), \\ (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta) \end{cases}$
AIII.2	$\begin{cases} \big(\mathfrak{su}(n+m, n+m), \mathfrak{sl}(n+m, \mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}(n,n) \oplus \mathfrak{su}(m,m) \oplus \mathfrak{t}, \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{t}\big) \end{cases}$
AIII.6	$\begin{cases} \big(\mathfrak{su}(p,p), \mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{R}, \mathfrak{s}(\mathfrak{gl}(h,\mathbb{C}) \oplus \mathfrak{gl}(p-h,\mathbb{C})) \oplus \mathbb{R}\big) \end{cases}$
AI.3	$\begin{cases} \big(\mathfrak{su}(2m,2m), \mathfrak{sl}(2m,\mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}^*(4m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
AII.3	$\begin{cases} \big(\mathfrak{su}(2m,2m), \mathfrak{sl}(2m,\mathbb{C}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(m,m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
BDI	
BDI.1	$\begin{cases} \big(\mathfrak{so}(h+k, p+q-h-k), \mathfrak{so}(h+k-1, p+q-h-k-1) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(h, p-h) \oplus \mathfrak{so}(k, q-k), \mathfrak{so}(h-1, p-h-1) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(k, q-k)\big) \end{cases}$
DIII.5	$\begin{cases} \big(\mathfrak{so}(n,n), \mathfrak{so}(n-1,n-1) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{s}(\mathfrak{gl}(n-1,\mathbb{R}) \oplus \mathfrak{gl}(1,\mathbb{R})) \oplus \mathbb{R}\big) \end{cases}$
BDI.2	$\begin{cases} \big(\mathfrak{so}(n+m, n+m), \mathfrak{sl}(n+m, \mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(n,n) \oplus \mathfrak{so}(m,m), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$
DIII.4	$\begin{cases} \big(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{s}(\mathfrak{gl}(k,\mathbb{R}) \oplus \mathfrak{gl}(n-k,\mathbb{R})) \oplus \mathbb{R}\big) \end{cases}$
BDI.5	$\begin{cases} \big(\mathfrak{so}(2n,2n), \mathfrak{sl}(2n,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}\big) \end{cases}$
DIII.1	$\begin{cases} \big(\mathfrak{so}(2m,2m), \mathfrak{sl}(2m,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}(m,m) \oplus \mathfrak{t}, \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{t}\big) \end{cases}$
DIII	
BDI.3	$\begin{cases} \big(\mathfrak{so}^*(4n+4m), \mathfrak{su}^*(2n+2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}^*(4n) \oplus \mathfrak{so}^*(4m), \mathfrak{su}^*(2n) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
DIII.3	$\begin{cases} \big(\mathfrak{so}^*(4m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}^*(2m) \oplus \mathbb{R}, \mathfrak{su}^*(2k) \oplus \mathfrak{su}^*(2m-2k) \oplus \mathbb{R} \oplus \mathbb{R}\big) \end{cases}$
BDI.4	$\begin{cases} \big(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}\big) \end{cases}$
DIII.2	$\begin{cases} \big(\mathfrak{so}^*(4m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}(m,m) \oplus \mathfrak{t}, \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{t}\big) \end{cases}$
CI	
CI.4	$\begin{cases} \big(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}, \mathfrak{s}(\mathfrak{gl}(k,\mathbb{R}) \oplus \mathfrak{gl}(n-k,\mathbb{R})) \oplus \mathbb{R}\big) \end{cases}$
CII.1	$\begin{cases} \big(\mathfrak{sp}(p+q,\mathbb{R}), \mathfrak{sl}(p+q,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sp}(q,\mathbb{R}), \mathfrak{sl}(p,\mathbb{R}) \oplus \mathbb{R} \oplus \mathfrak{sl}(q,\mathbb{R}) \oplus \mathbb{R}\big) \end{cases}$
CI.2	$\begin{cases} \big(\mathfrak{sp}(2m,\mathbb{R}), \mathfrak{sl}(2m,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}(m,m) \oplus \mathfrak{t}, \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{t}\big) \end{cases}$

no.	$\begin{cases} (\mathfrak{g}_d, \mathfrak{g}_d^\sigma), \\ (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta) \end{cases}$
CII.4	$\begin{cases} \big(\mathfrak{sp}(2p,\mathbb{R}), \mathfrak{sl}(2p,\mathbb{R}) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(p,\mathbb{C}), \mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{C}\big) \end{cases}$
CII	
CI.3	$\begin{cases} \big(\mathfrak{sp}(m,m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}^*(2m) \oplus \mathbb{R}, \mathfrak{su}^*(2k) \oplus \mathfrak{su}^*(2m-2k) \oplus \mathbb{R} \oplus \mathbb{R}\big) \end{cases}$
CII.2	$\begin{cases} \big(\mathfrak{sp}(n+m,n+m), \mathfrak{su}^*(2n+2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(n,n) \oplus \mathfrak{sp}(m,m), \mathfrak{su}^*(2n) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2m) \oplus \mathbb{R}\big) \end{cases}$
CI.1	$\begin{cases} \big(\mathfrak{sp}(m,m), \mathfrak{su}^*(2m) \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}(m,m) \oplus \mathfrak{t}, \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R} \oplus \mathfrak{t}\big) \end{cases}$
CII.3	$\begin{cases} \big(\mathfrak{sp}(p,p), \mathfrak{su}^*(2p) \oplus \mathbb{R}\big), \\ \big(\mathfrak{sp}(p,\mathbb{C}), \mathfrak{sl}(p,\mathbb{C}) \oplus \mathbb{C}\big) \end{cases}$
EI	
EII.1	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{su}^*(6) \oplus \mathfrak{su}^*(2), \mathfrak{su}^*(4) \oplus \mathfrak{su}^*(2) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2)\big) \end{cases}$
EII.3	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(4,\mathbb{R}) \oplus \mathfrak{gl}(2,\mathbb{R})) \oplus \mathfrak{sl}(2,\mathbb{R})\big) \end{cases}$
EII.4	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(5,\mathbb{R}) \oplus \mathfrak{gl}(1,\mathbb{R})) \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EIII.1	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1), \mathfrak{sl}(5,\mathbb{R}) \oplus \mathbb{R} \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EIII.3	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1), \mathfrak{so}(4,4) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EIII.5	$\begin{cases} \big(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1), \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EIV	
EII.2	$\begin{cases} \big(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{su}^*(6) \oplus \mathfrak{su}^*(2), \mathfrak{su}^*(4) \oplus \mathfrak{su}^*(2) \oplus \mathbb{R} \oplus \mathfrak{su}^*(2)\big) \end{cases}$
EIII.2	$\begin{cases} \big(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1), \mathfrak{so}(8) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EIII.4	$\begin{cases} \big(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1)\big), \\ \big(\mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1), \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1)\big) \end{cases}$
EV	
EV.2	$\begin{cases} \big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}^*(8), \mathfrak{su}^*(6) \oplus \mathfrak{su}^*(2) \oplus \mathbb{R}\big) \end{cases}$
EVI.2	$\begin{cases} \big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}^*(12) \oplus \mathfrak{su}(2), \mathfrak{su}^*(6) \oplus \mathbb{R} \oplus \mathfrak{su}(2)\big) \end{cases}$

no.	$\begin{cases}(\mathfrak{g}_d, \mathfrak{g}_d^\sigma), \\ (\mathfrak{g}_d^\theta, \mathfrak{g}_d^\sigma \cap \mathfrak{g}_d^\theta)\end{cases}$
EV.3	$\begin{cases}\big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{sl}(8,\mathbb{R}), \mathfrak{s}(\mathfrak{gl}(6,\mathbb{R}) \oplus \mathfrak{gl}(2,\mathbb{R}))\big)\end{cases}$
EVI.3	$\begin{cases}\big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{sl}(6,\mathbb{R}) \oplus \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})\big)\end{cases}$
EVI.4	$\begin{cases}\big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)\big)\end{cases}$
EVII.2	$\begin{cases}\big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{e}_{6(6)} \oplus \mathbb{R}, \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1) \oplus \mathbb{R}\big)\end{cases}$
EVII.4	$\begin{cases}\big(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{e}_{6(6)} \oplus \mathbb{R}, \mathfrak{e}_{6(6)} \oplus \mathbb{R}\big)\end{cases}$
EVII	
EV.1	$\begin{cases}\big(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{su}^*(8), \mathfrak{su}^*(6) \oplus \mathfrak{su}^*(2) \oplus \mathbb{R}\big)\end{cases}$
EVI.1	$\begin{cases}\big(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}^*(12) \oplus \mathfrak{su}(2), \mathfrak{su}^*(6) \oplus \mathbb{R} \oplus \mathfrak{su}(2)\big)\end{cases}$
EVI.5	$\begin{cases}\big(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{so}(10,2) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,1)\big)\end{cases}$
EVII.1	$\begin{cases}\big(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{e}_{6(-26)} \oplus \mathbb{R}, \mathfrak{so}(9,1) \oplus \mathfrak{so}(1,1) \oplus \mathbb{R}\big)\end{cases}$
EVII.3	$\begin{cases}\big(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big), \\ \big(\mathfrak{e}_{6(-26)} \oplus \mathbb{R}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R}\big)\end{cases}$

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