J. Math. Soc. Japan
Vol. 66, No. 1 (2014) pp. 25–36
doi: 10.2969/jmsj/06610025

On the enhancement to the Milnor number of a class of mixed polynomials

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(Received Sep. 14, 2011)
(Revised Mar. 22, 2012)

Abstract. The enhancement to the Milnor number is an invariant of the homotopy classes of fibered links in the sphere S^{2n-1} and belongs to $\mathbb{Z}/r\mathbb{Z}$, where $r = 0$ if $n = 2$ and $r = 2$ if $n = 2$. Mixed polynomials are polynomials in complex variables $z_1, \ldots, z_n$ and their conjugates $\bar{z}_1, \ldots, \bar{z}_n$. M. Oka showed that mixed polynomials have Milnor fibrations under the strongly non-degeneracy condition. In this present paper, we study fibered links which are defined by a certain class of mixed polynomials which admit Milnor fibrations and show that any element of $\mathbb{Z}/r\mathbb{Z}$ is realized by the enhancement to the Milnor number of such a fibered link.

1. Introduction.

Let (S^{2n-1}, K) be a *link*, i.e., K is an oriented codimension-two closed smooth submanifold in the $(2n-1)$-sphere S^{2n-1}. A link K is *fibered* if there are a 2-dimensional disk bundle neighborhood $N(K)$ of K in S^{2n-1} with a trivialization $\phi_0 : N(K) \to D^2$ and a fibration of the link exterior $E(K) = S^{2n-1} \setminus \mathrm{Int}(N(K))$, $\phi_1 : E(K) \to S^1$ such that $\phi_0|\partial N(K) = \phi_1|\partial N(K)$. This fibration is also called an *open book decomposition* of S^{2n-1}. A fibered link K is *simple* if K is $(n-3)$-connected and its fiber surface, which by definition is a fiber of ϕ_1, is $(n-2)$-connected. The *Milnor number* $\mu(K)$ of a simple fibered link (S^{2n-1}, K) is defined as the rank of the $(n-1)$-th homology group of the fiber of ϕ_1.

A link of an isolated singularity of a complex hypersurface is a typical example of a simple fibered link. Let $f(\boldsymbol{z})$ be a complex polynomial of n-complex variables $\boldsymbol{z} = (z_1, \ldots, z_n)$ such that $f(\boldsymbol{o}) = 0$, where $\boldsymbol{o} = (0, \ldots, 0)$ is the origin of $\mathbb{C}^n$. J. Milnor proved in [**10**] that there exists $\varepsilon > 0$ such that

$$f/|f| \colon S_{\varepsilon}^{2n-1} \setminus K \to S^1$$

is a locally trivial fibration, where S_{ε}^{2n-1} is the $(2n-1)$-dimensional sphere centered at the origin of $\mathbb{C}^n$ of radius ε and $K = S_{\varepsilon}^{2n-1} \cap f^{-1}(0)$. If the origin is a regular point or an isolated singularity, K is a simple fibered link and the rank of the $(n-1)$-th homology group of the fiber is called the Milnor number. This is the reason why $\mu(K)$ is called the Milnor number in general case.

In [**11**], [**12**], [**19**], W. Neumann and L. Rudolph studied an oriented $(2n-2)$-plane

2010 *Mathematics Subject Classification.* Primary 14J17; Secondary 37C27, 58K45.
Key Words and Phrases. mixed polynomial, enhanced Milnor number, fibered link.

field in $TS^{2n-1} \oplus \mathbb{R}$ which is defined as follows: on $S^{2n-1} \setminus N(K)$ it is the tangent field to the fibration for K, on K it is the tangent field to $K \oplus \mathbb{R}$ and on $N(K)$ it interpolates between $\partial N(K)$ and K to make a smooth plane field. This plane field defines a map $\Lambda : S^{2n-1} \to G(2n-2, 2n)$, where $G(2n-2, 2n)$ is the Grassman manifold of oriented $(2n-2)$-planes in $\mathbb{R}^{2n}$. Neumann and Rudolph observed that $\pi_{2n-1}(G(2n-2, 2n))$ is isomorphic to the direct sum of $\pi_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$ and $\pi_{2n-1}(S^{2n-2}) \cong \mathbb{Z}/r\mathbb{Z}$, where $r = 0$ if $n = 2$ and $r = 2$ if $n > 2$. The homotopy class of Λ has the form $((-1)^n\mu(K), \lambda(K))$ in $\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$. This pair $((-1)^n\mu(K), \lambda(K))$ is called the *enhanced Milnor number* and $\lambda(K)$ is called the *enhancement* to the Milnor number. Note that if K is a simple fibered link coming from an isolated singularity of a complex hypersurface, $\lambda(K)$ always vanishes.

In this paper, we study the enhancement $\lambda(K)$ of a simple fibered link K coming from a real polynomial map $(g, h) : \mathbb{R}^{2n} \to \mathbb{R}^2$, where $g(\boldsymbol{x}, \boldsymbol{y})$ and $h(\boldsymbol{x}, \boldsymbol{y})$ are real polynomials with variables $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$. Two real polynomials $g(\boldsymbol{x}, \boldsymbol{y})$ and $h(\boldsymbol{x}, \boldsymbol{y})$ can define a polynomial with complex and complex-conjugate variables $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$

$$f(\boldsymbol{z}, \bar{\boldsymbol{z}}) := g\left(\frac{\boldsymbol{z} + \bar{\boldsymbol{z}}}{2}, \frac{\boldsymbol{z} - \bar{\boldsymbol{z}}}{2i}\right) + ih\left(\frac{\boldsymbol{z} + \bar{\boldsymbol{z}}}{2}, \frac{\boldsymbol{z} - \bar{\boldsymbol{z}}}{2i}\right),$$

where $z_j = x_j + iy_j$ $(j = 1, \dots, n)$. The polynomial $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ of variables $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ is called a *mixed polynomial*. Let K_f be the intersection of S^{2n-1} and $f^{-1}(0)$. In [**14**], [**15**], Oka introduced several classes of mixed polynomials which are called *strongly non-degenerate* and *polar weighted homogeneous* mixed polynomials. He proved that those polynomials guarantee the existence of the Milnor fibration. The main theorem in this paper is the following.

THEOREM 1. *For any $k \in \mathbb{Z}/r\mathbb{Z}$, there exists a mixed polynomial $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ whose Milnor fibration $f/|f| : S^{2n-1}_\varepsilon \setminus K_f \to S^1$ satisfies $\lambda(K_f) = k$, where $r = 0$ if $n = 2$ and $r = 2$ if $n > 2$.*

This paper is organized as follows. In Section 2 we give the definitions of the enhancements to Milnor numbers and mixed polynomials, and introduce several classes of mixed polynomials which define the fibration $S^{2n-1}_\varepsilon \setminus K_f \to S^1$. In Section 3 we calculate the enhancement $\lambda(K_f)$ and prove Theorem 1. In Section 4 we make a few comments on convenient strongly non-degenerate and polar weighted homogeneous mixed polynomials and the contact structures compatible with their Milnor fibrations $S^{2n-1}_\varepsilon \setminus K_f \to S^1$.

The author would like to thank Professor Masaharu Ishikawa for precious comments.

2. Preliminaries.

2.1. Enhanced Milnor numbers.

We first introduce the definition of the *enhanced Milnor number* for $n = 2$ which was defined by Neumann and Rudolph in [**11**], [**12**], [**13**], [**19**]. Let (S^3, K) be a fibered link. To define the enhanced Milnor number, we first construct a nowhere zero vector field $\xi(K)$ on S^3. The construction of $\xi(K)$ is the following. On $E(K)$, $\xi(K)$ is a transverse field to the fiber surfaces of the fibration, the same direction of the monodromy of the fibration; on K, $\xi(K)$ is the tangent field of K; on the rest of $N(K)$, $\xi(K)$ interpolates reasonably between its values on K and on $\partial N(K)$ to make a smooth vector field on S^3. The reasonable interpolation means that $\xi(K)$ is transverse to any torus parallel to $\partial N(K)$ in $\operatorname{Int} N(K) \setminus K$. As mentioned in [**13**, Lemma 3.5], the homotopy class of $\xi(K)$ does not depend on a reasonable choice for this interpolation. So $\xi(K)$ can be taken as $r(\partial/\partial\theta) + (1-r^2)(\partial/\partial\phi)$ on $N(K)$, where (r,θ) are the coordinates of the meridian disk of $N(K) \cong D^2 \times S^1$ and ϕ is the coordinate of the longitude of $N(K)$.

Next we consider the map $p : S^3 \to \mathbb{C}P^1$ defined by $p(z_1, z_2) = [z_1 : z_2]$. Since the 2-sphere S^2 is diffeomorphic to $\mathbb{C}P^1$, we obtain a smooth map $S^3 \to S^2$. This map is called *the Hopf fibration.* Let ψ be a vector field which is homotopic to the field of tangent vectors to the fibers of the Hopf fibration and define two subsets $\Delta^+(K)$ and $\Delta^-(K)$ in S^3 by

$$\Delta^\pm(K) := \{x \in S^3 \mid \psi(x) = \pm t\, \xi(K)(x) \ \text{ for some } t > 0\}.$$

Now we assume $\xi(K)$ and ψ are vector fields on S^3 in general position so that $\Delta^\pm(K)$ are compact oriented 1-manifolds in S^3.

The orientation of each connected component of $\Delta^\pm(K)$ is determined as follows. It is well-known that S^3 is parallelizable. If we fix a trivialization of the tangent bundle of S^3, then each vector field v on S^3 can be represented by the map $S^3 \to \mathbb{R}^3$. In particular, the unit vector fields corresponding to ψ and $\xi(K)$ are represented as $p(\psi) : S^3 \to S^2$ and $p(\xi(K)) : S^3 \to S^2$ respectively. We fix a trivialization of the tangent bundle of S^3 such that ψ becomes a constant vector field. In this case, $\Delta^+(K)$ and $\Delta^-(K)$ become the preimages of regular values of $p(\xi(K))$. Let Δ_0 be a connected component of $\Delta^\pm(K)$ and $b \in S^2$ a regular value of $p(\xi(K))$ such that $\Delta_0 = p(\xi(K))^{-1}(b)$. Take a point $a \in p(\xi(K))^{-1}(b)$. Let T_aS^3 denote the tangent space to S^3 at a and T_bS^2 the tangent space to S^2 at b. We fix a Riemannian metric on S^3. Let W be the orthogonal complement of the tangent space $T_a\Delta_0$ in T_aS^3. Since b is a regular value of $p(\xi(K))$, the induced map $dp(\xi(K))_a : W \to T_bS^2$ is an isomorphism. Fix the orientation of W such that $dp(\xi(K))_a$ is orientation preserving and then fix the orientation of $T_a\Delta_0$ such that the orientation of $T_a\Delta_0 \times W$ coincides with that of T_aS^3, which determines the orientation of Δ_0.

Since $\Delta^+(K)$ and $\Delta^-(K)$ are disjoint, we can consider their linking number $\operatorname{link}(\Delta^+(K), \Delta^-(K))$. We call it the *enhancement* to the Milnor number and denote it by

$$\lambda(K) := \operatorname{link}(\Delta^+(K), \Delta^-(K)) \in \mathbb{Z}.$$

The pair $(\mu(K), \lambda(K))$ is called the *enhanced Milnor number* of K. Note that, as mentioned in [**11**], $\lambda(K)$ is regarded as the Hopf invariant of $p(\xi(K))$ (cf. [**19**]).

Finally we introduce the definition of the *enhanced Milnor number* of K as follows. Let K be a simple fibered link and $\phi_0 \cup \phi_1 : S^{2n-1} \to D^2$ an open book decomposition of S^{2n-1} which is determined by K. This map can be extended to a map $\Lambda : D^{2n} \to D^2$ which is a smooth submersion except for an isolated singularity at the origin $0 \in D^{2n}$ and a corner along $\partial N(K)$ [9]. Such a Λ is called a *trivial unfolding.* Then the map $x \mapsto$

$\operatorname{Ker} D\Lambda(x)$ is continuous on $D^{2n}\backslash\{0\}$, where $D\Lambda$ is differential of Λ and $\operatorname{Ker} D\Lambda(x) = \{v \in T_xD^{2n} \mid D\Lambda(v) = 0\}$. Since $\operatorname{Ker} D\Lambda(x)$ is an oriented $(2n-2)$-plane in $\mathbb{R}^{2n}$, we can get a map $k(K) : S^{2n-1} \to \mathrm{Gr}(2n-2, 2n)$ which is defined by the restriction of this map to the $(2n-1)$-sphere S^{2n-1} around 0. This map does not depend on the choice of sphere or the trivial unfolding. The homotopy group $\pi_{2n-1}(\mathrm{Gr}(2n-2, 2n))$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/r\mathbb{Z}$, where $r = 0$ if $n = 2$ and $r = 2$ if $n > 2$ (cf. [**6**], [**11**], [**12**]). Neumann and Rudolph showed that the first component of $[k(K)] \in \pi_{2n-1}(\mathrm{Gr}(2n-2, 2n))$ is $(-1)^n\mu(K)$, where $\mu(K)$ is the Milnor number of K [**11**], [**12**]. Thus the homotopy class $[k(K)]$ has the form $((-1)^n\mu(K), \lambda(K))$. We call $((-1)^n\mu(K), \lambda(K))$ the *enhanced Milnor number* and $\lambda(K)$ the *enhancement* to the Milnor number.

2.2. Mixed polynomials.

Let $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ be a polynomial expanded in a convergent power series of variables $\boldsymbol{z} = (z_1, \dots, z_n)$ and $\bar{\boldsymbol{z}} = (\bar{z}_1, \dots, \bar{z}_n)$

$$f(\boldsymbol{z}, \bar{\boldsymbol{z}}) := \sum_{\nu,\mu} c_{\nu,\mu}\boldsymbol{z}^{\nu}\bar{\boldsymbol{z}}^{\mu},$$

where $\boldsymbol{z}^{\nu} = z_1^{\nu_1}\cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\boldsymbol{z}}^{\mu} = \bar{z}_1^{\mu_1}\cdots\bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$). $\bar{z}_j$ represents the complex conjugate of z_j. A polynomial $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ of this form is called a *mixed polynomial.*

Assume that $f(\boldsymbol{o}) = 0$, where $\boldsymbol{o}$ is the origin of $\mathbb{C}^n$. We are interested in the topology of the singularities of maps $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ at the origin. The set $K_f := S_{\varepsilon}^{2n-1} \cap f^{-1}(0)$ is called the *link* of $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ at $\boldsymbol{o}$. When $\boldsymbol{o}$ is an isolated singularity of $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$, then K_f is a $(2n-1)$-manifold [**10**, Corollary 2.9].

If a mixed polynomial does not have complex-conjugate variables, i.e., it is a complex polynomial, we have the Milnor fibration

$$f/|f| \colon S_{\varepsilon}^{2n-1} \setminus K_f \to S^1$$

where $S_{\varepsilon}^{2n-1} := \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 = \varepsilon\}$. It is known that such a fibration also exists for some specific mixed polynomials. Certain restricted classes of polynomials in the variables $\boldsymbol{z}, \bar{\boldsymbol{z}}$ which admit a Milnor fibration had been considered by J. Seade, see for instance [**20**], [**21**]. Oka introduced the notation of the *Newton boundary* for mixed polynomials and proposed a wide class of mixed polynomials which guarantee the existence of Milnor fibration.

In this subsection, we introduce several classes of mixed polynomials which admit Milnor fibrations as given by Oka in [**14**], [**15**]. The *radial Newton polygon* $\Gamma_+(f; \boldsymbol{z}.\bar{\boldsymbol{z}})$ is defined by the convex hull of

$$\bigcup_{(\nu,\mu)} \{(\nu + \mu) + \mathbb{R}_+^n \mid c_{\nu,\mu} \neq 0\},$$

where $\nu + \mu$ is the sum of the multi-indices of $\boldsymbol{z}^{\nu}\bar{\boldsymbol{z}}^{\mu}$, i.e., $\nu + \mu = (\nu_1 + \mu_1, \dots, \nu_n + \mu_n)$. *The Newton boundary* $\Gamma(f; \boldsymbol{z}, \bar{\boldsymbol{z}})$ is the union of compact faces of $\Gamma_+(f; \boldsymbol{z}, \bar{\boldsymbol{z}})$. If

$f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is a complex polynomial, this definition of the Newton boundary of a mixed polynomial agrees with the definition of the ordinary Newton boundary. The *strongly non-degeneracy* is defined from the Newton boundary as follows: let $\Delta_1, \ldots, \Delta_m$ be the faces of $\Gamma(f; \boldsymbol{z}, \bar{\boldsymbol{z}})$. For each face Δ_k, the *face function* $f_{P_k}(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is defined by $f_{P_k}(\boldsymbol{z}, \bar{\boldsymbol{z}}) := \sum_{(\nu+\mu)\in\Delta_k} c_{\nu,\mu}\boldsymbol{z}^\nu \bar{\boldsymbol{z}}^\mu$, where P_k is the positive vector which is perpendicular to Δ_k. If $f_{P_k}(\boldsymbol{z}, \bar{\boldsymbol{z}}) : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical point, we say that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is *strongly non-degenerate* for P_k. If $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is strongly non-degenerate for any P_k for $k = 1, \ldots, m$, we say that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is *strongly non-degenerate*. Oka showed that the singular point $\boldsymbol{o}$ of a strongly non-degenerate mixed polynomial $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ has the following fibration.

THEOREM 2 ([**15**]). *Let $f(\boldsymbol{z}, \bar{\boldsymbol{z}}) : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ be a strongly non-degenerate mixed polynomial. Then there exists $\varepsilon > 0$ such that*

$$f(\boldsymbol{z}, \bar{\boldsymbol{z}})/|f(\boldsymbol{z}, \bar{\boldsymbol{z}})| \colon S_\varepsilon^{2n-1} \setminus K_f \to S^1$$

is a locally trivial fibration for any ε with $0 < \varepsilon \leq \varepsilon_0$.

Remark that K_f becomes the fibered link if and only if the singularity $\boldsymbol{o}$ is an isolated singularity. In this paper, we always assume that $f_{P_k}^{-1} \cap \mathbb{C}^{*n}$ is non-empty for $k = 1, \ldots, m$. If $f(0, \ldots, 0, z_j, 0, \ldots, 0)$ is non-zero for each $j = 1, \ldots, n$, then we say that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is *convenient*. If $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is convenient, K_f is a fibered link [**15**].

We introduce another class of mixed polynomials which admit an S^1-action. Let $p_1, \ldots, p_n$ be integers such that $\gcd(p_1, \ldots, p_n) = 1$. We define an S^1-action on $\mathbb{C}^n$ as follows:

$$s \circ \boldsymbol{z} = (s^{p_1} z_1, \ldots, s^{p_n} z_n), \quad s \in S^1.$$

If there exists a positive integer d_p such that the mixed polynomial $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ satisfies

$$f(s^{p_1} z_1, \ldots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z_1}, \ldots, \bar{s}^{p_1} \bar{z_n}) = s^{d_p} f(\boldsymbol{z}, \bar{\boldsymbol{z}}), \quad s \in S^1,$$

we say that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is *polar weighted homogeneous*. This notation was first introduced by Ruas-Seade-Verjovsky [**18**] and J. L. Cisneros-Molina [**1**]. In this case, K_f is fibered and its monodromy is given by

$$(z_1, \ldots, z_n) \mapsto \left(\exp\left(\frac{2p_1\pi i}{d_p}\right) z_1, \ldots, \exp\left(\frac{2p_n\pi i}{d_p}\right) z_n \right),$$

see [**14**], [**15**]. We will use this S^1-action to prove Theorem 1.

3. Proof of main theorem.

We divide the proof of Theorem 1 into the 3-dimensional case and high-dimensional cases.

3.1. The case of $n = 2$.

We focus on the following type of mixed polynomials

$$f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}}) := \prod_{j=1}^{m_+}(z_1^p + \alpha_j z_2^q) \prod_{j=m_++1}^{m_++m_-} \overline{(z_1^p + \alpha_j z_2^q)} \quad (m_+ > m_-),$$

where $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$, $\overline{z_1^p + \alpha_j z_2^q}$ represents the complex-conjugate of $z_1^p + \alpha_j z_2^q$ and p and q are coprime positive integers. Since $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$, the mixed polynomial $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is strongly non-degenerate. Remark that $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ has m_+ holomorphic factors and m_- complex-conjugate factors. Such a type of mixed polynomials is a special case of polynomials of forms $(f\bar{g}, \boldsymbol{o})$ studied by A. Pichon and J. Seade in [**16**], [**17**], where $(f, \boldsymbol{o})$ and $(g, \boldsymbol{o})$ are complex polynomials with isolated singularities at $\boldsymbol{o}$ and with no common branches. The origin $\boldsymbol{o}$ is an isolated singularity of $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ and $K_{f_{p,q,m_+,m_-}} := S^3_\varepsilon \cap f^{-1}_{p,q,m_+,m_-}(0)$ is an oriented fibered link in the 3-sphere S^3_ε. The S^1-action on S^3_ε is

$$s \circ (z_1, z_2) = (s^q z_1, s^p z_2), \quad s \in S^1$$

and $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ satisfies

$$f_{p,q,m_+,m_-}(s \circ \boldsymbol{z}, \overline{s \circ \boldsymbol{z}}) = s^{pq(m_+-m_-)} f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}}).$$

Since $m_+ > m_-$, $pq(m_+ - m_-)$ is a positive integer. So $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is polar weighted homogeneous.

The enhancement $\lambda(K_{f_{p,q,m_+,m_-}})$ of this singularity is determined by the following formula.

Lemma 1. $\lambda(K_{f_{p,q,m_+,m_-}}) = (-pqm_- + p + q)m_-$.

In the proofs in this section, $f_{p,q,m_+,m_-}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is abbreviated to $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ for short. We denote by $-K$ the link obtained by reversing orientation of K.

Proof. To calculate $\lambda(K_f)$, we have to determine $\Delta^+(K_f)$ and $\Delta^-(K_f)$. We choose ψ to be the vector field on S^3_ε which determines the S^1-action. That is, ψ is a tangent field of the orbit of the S^1-action. Let B be the orbit space of S^3_ε under the S^1-action. Then B is homeomorphic to the 2-sphere [**8**].

We consider the orbit map $S^3_\varepsilon \to B$. Since $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ is polar weighted homogeneous, K_f is an invariant set for the S^1-action. So the image of the components of K_f by the orbit map are the points in B. Let $p_1, \ldots, p_{m_++m_-}$ be the image of the components of K_f and D^2_j the sufficiently small 2-disk centered at p_j. To construct the vector field $\xi(K_f)$, we prepare a vector field $\eta(K_f)$ on B such that $p_1, \ldots, p_{m_++m_-}$ are zero points as follows. Let N and S represent the North and South poles in B respectively. Since the Euler characteristic of B is 2, essentially by the theorem of Poincaré-Hopf, we choose a vector field $\eta(K_f)$ on B which satisfies the following properties:

- N and S are repellor points.
- There exist $m_+ + m_-$ mutually disjoint disks on B and, on each disk, $\eta(K_f)$ has exactly one attractor point and one saddle point.
- $\eta(K_f)$ has no zero points outside these disks except for N and S.

See Figure 1.

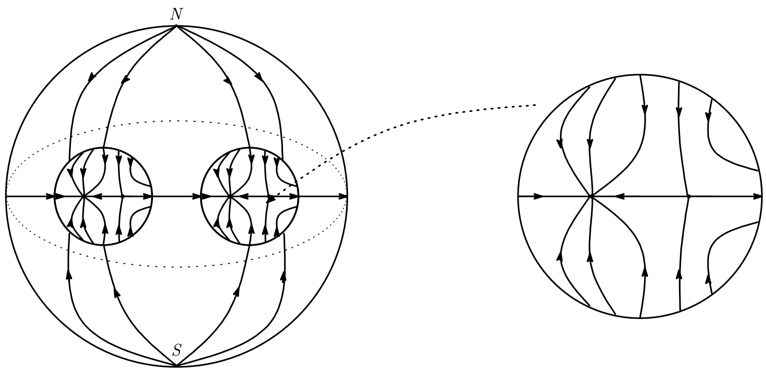

Figure 1. The vector field $\eta(K_f)$ and a 2-disk which has one attractor point and one saddle point.

Since $\xi(K)$ is transverse to any torus parallel to $\partial N(K_f)$ in $\operatorname{Int} N(K_f) \setminus K_f$, the image of $\xi(K_f)$ by the orbit map is the vector field which has only attractor points as zero points on D_j^2. Thus we choose that $p_1, \ldots, p_{m_+ + m_-}$ are the attractor points of $\eta(K_f)$. Now we construct $\xi(K_f)$ as follows. $\xi(K_f)$ is constructed by lifting $\eta(K_f)$ to the vector field on $N(K_f)$. Then $\xi(K_f)$ satisfies the following properties on $N(K_f)$:

- The image of $N(K_f)$ by the orbit map is a union of disjoint 2-disks $D_1^2, \ldots, D_{m_+ + m_-}^2$ each of which has only one attractor point.
- The image of $\xi(K_f)$ by the orbit map is $\eta(K_f)$ on $D_1^2, \ldots, D_{m_+ + m_-}^2$.
- $\xi(K_f)$ is transverse to the fiber surfaces on $\partial N(K_f)$.

Since $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is polar weighted homogeneous, the monodromy of the fibration of K_f is given by the S^1-action. On $E(K_f)$, $\xi(K_f)$ is constructed by a perturbation of ψ such that it is transverse to the fiber surfaces and its image by the orbit map is $\eta(K_f)$ on $B \setminus \bigcup_{j=1}^{m_+ + m_-} \operatorname{Int} D_j^2$. By the construction of $\xi(K_f)$, $\Delta^+(K_f)$ and $\Delta^-(K_f)$ are the preimages of zero points of $\eta(K_f)$.

Now we describe the link type of K_f with the orientation. Since K_f is an invariant set for the S^1-action $s \circ (z_1, z_2) = (s^q z_1, s^p z_2)$, each component of K_f is a (p, q)-torus knot as an unoriented link. In [**16**], Pichon showed that the link of the conjugation of a holomorphic function is opposite to the orientation of the link of the same holomorphic function. In our case, K_f has m_+ link components defined by the holomorphic factors, denoted by K^+, and m_- link components defined by the complex-conjugate factors, denoted by K^- respectively. The orientation of K^+ is the same as that of the S^1-action and K^- has the opposite orientation given by the S^1-action. By the construction, K_f is contained in $\Delta^{\pm}(K_f)$ as an unoriented link. In the previous section, we explained how to define the orientation of a component Δ_0 of $\Delta^{\pm}(K_f)$. Since K_f corresponds to the points which are attractors, the orientation of the link components of $\Delta^{\pm}(K_f)$ corresponding to K_f coincides with that of the S^1-action. Thus $\Delta^+(K_f)$ contains K^+ and $\Delta^-(K_f)$ contains $-K^-$ as an oriented link.

By the construction, $\xi(K_f)$ and ψ are in the same direction along $\Delta^{\pm}(K_f)$ on $E(K_f)$. Thus any component of $\Delta^{\pm}(K_f)$ contained in $E(K_f)$ belongs to $\Delta^+(K_f)$. On $E(K_f)$, $\Delta^+(K_f)$ corresponds to N, S and the saddle points on $\eta(K_f)$. Let Δ^+_N, Δ^+_S and Δ^+_{saddle} denote the components of $\Delta^+(K_f)$ corresponding to N, S and the saddle points respectively. Since N and S are repellor points, the orientations of Δ^+_N and Δ^+_S coincide with that of the S^1-action. The orientations of Δ^+_{saddle} are opposite to that of the orbit of the S^1-action.

Thus $\lambda(K_f)$ can be calculated as follows:

$$\begin{aligned}\lambda(K_f) &= \operatorname{link}(\Delta^+(K_f), \Delta^-(K_f)) \\ &= \operatorname{link}(\Delta^+_N \cup \Delta^+_S \cup \Delta^+_{\text{saddle}} \cup K^+, -K^-) \\ &= pm_- + qm_- - pq(m_+ + m_-)m_- + pqm_+m_- \\ &= (-pqm_- + p + q)m_-. \qquad \square\end{aligned}$$

Lemma 2. *For any integer k less than 2, there exists a mixed polynomial $f_{p,q,m_+,m_-}(\boldsymbol{z}, \bar{\boldsymbol{z}})$ such that $\lambda(K_{f_{p,q,m_+,m_-}}) = k$.*

Proof. In the case $(m_+, m_-, q) = (2, 1, 2)$, the enhancement $\lambda(K_f)$ is equal to $-p + 2$. Since p is a positive integer such that $\gcd(p, 2) = 1$, $\lambda(K_f)$ realizes all odd integers less than 2. In the case $(m_+, m_-, q) = (3, 2, 1)$, the enhancement $\lambda(K_f)$ is equal to $-2p + 2$, which realizes all even integers less than 1. The assertion follows. $\square$

The remaining integers greater than 1 can be realized by the *mirror image* of K_f, denoted $\operatorname{mir} K_f$. This is obtained by conjugating one of the complex variables $\boldsymbol{z}$, i.e., $\operatorname{mir} K_f = K_{f\circ\iota}$, where $\iota : \mathbb{C}^n \to \mathbb{C}^n$ is the smooth map defined by $\iota(z_1, \ldots, z_{n-1}, z_n) = (z_1, \ldots, z_{n-1}, \bar{z}_n)$ and $K_{f\circ\iota} = S^{2n-1}_{\varepsilon} \cap (f\circ\iota)^{-1}(0)$. In particular, $f\circ\iota$ is also a mixed polynomial by definition. The enhancement to the Milnor number of $\operatorname{mir} K_f$ is represented as follows.

Lemma 3. $\lambda(\operatorname{mir}(K_{f_{p,q,m_+,m_-}})) = (pqm_+ - p - q)m_+ + 1$. *In particular, for any $k \geq 0$, there exist p, q, m_+, m_- such that $\lambda(\operatorname{mir} K_{f_{p,q,m_+,m_-}}) = k$.*

Proof. The mixed polynomial $f\circ\iota(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is represented as

$$f\circ\iota(\boldsymbol{z}, \bar{\boldsymbol{z}}) = \prod_{j=1}^{m_+}(z_1^p + \alpha_j \bar{z}_2^q) \prod_{j=m_++1}^{m_++m_-} \overline{(z_1^p + \alpha_j \bar{z}_2^q)} \quad (m_+ > m_-).$$

The S^1-action on $\mathbb{C}^2$ is defined as

$$s \circ (z_1, z_2) = (s^q z_1, s^{-p} z_2)$$

with parameter $s \in S^1$ and we have the equation

$$f \circ \iota(s \circ \boldsymbol{z}, \overline{s \circ \boldsymbol{z}}) = s^{pq(m_+ - m_-)} f \circ \iota(\boldsymbol{z}, \bar{\boldsymbol{z}}).$$

So $f \circ \iota(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is also a strongly non-degenerate polar weighted homogeneous mixed polynomial and $K_{f\circ\iota} = \operatorname{mir} K_f$ is a fibered link in S^3_ε.

The enhancement of $K_{f\circ\iota}$ can be calculated as in the proof of Lemma 1. Since K_f is the invariant set for the S^1-action, the Euler characteristic of the fiber surface of the fibration $S^3_\varepsilon \setminus K_f \to S^1$ is equal to $-\{pq(m_+ + m_-) - p - q\}(m_+ - m_-)$ which can be calculated from the splice diagram of Eisenbud and Neumann [**3**]. Thus the Milnor number $\mu(K_f)$ is equal to $\{pq(m_+ + m_-) - p - q\}(m_+ - m_-) + 1$. In [**13**], [**19**], Neumann and Rudolph showed that $\lambda(\operatorname{mir} K_f)$ is equal to $\mu(K_f) - \lambda(K_f)$. The enhancement $\lambda(\operatorname{mir} K_f)$ is calculated as follows:

$$\begin{aligned}\lambda(\operatorname{mir} K_f) &= \mu(K_f) - \lambda(K_f) \\ &= \{pq(m_+ + m_-) - p - q\}(m_+ - m_-) + 1 - (-pqm_- + p + q)m_- \\ &= (pqm_+ - p - q)m_+ + 1.\end{aligned}$$

In the case $(m_+, m_-, q) = (2, 1, 1)$, the enhancement $\lambda(\operatorname{mir} K_f)$ is equal to $2p - 1$, which realizes all odd integers greater than 0. In the case $(m_+, m_-, q) = (1, 0, 2)$, the enhancement $\lambda(\operatorname{mir} K_f)$ is equal to $p - 1$. Since p is a positive integer such that $\gcd(p, 2) = 1$, $\lambda(K_f)$ realizes all non-negative even integers. Thus the assertion follows. □

PROOF OF THEOREM 1 FOR $n = 2$. The theorem follows from Lemma 1, 2 and 3. □

3.2. The case of $n > 2$.

We prove Theorem 1 for $n > 2$. This fact had been showed by Neumann and Rudolph [**12**].

PROOF OF THEOREM 1 FOR $n > 2$. Let (S^{2n-1}, K) be a simple fibered link. Neumann and Rudolph showed that the enhancement $\lambda(K)$ is determined modulo 2 by the Seifert form L_K:

$$(-1)^{\lambda(K)} = \det((-1)^{n(n-1)/2} L_K),$$

see for instance [**11**], [**12**], [**13**], where the Seifert form L_K of K is non-singular bilinear form

$$L_K : H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$$

on the $(n-1)$-th homology group $H_{n-1}(F)$ of the fiber of the fibration, with respect to a basis of $H_{n-1}(F)$. Note that L_K becomes an invertible integer matrix.

We take two strongly and true non-degenerate mixed polynomials $f_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = z_1^2 + \cdots + z_n^2$ and $f_2(\boldsymbol{z}, \bar{\boldsymbol{z}}) = \bar{z}_1^2 + z_2^2 + \cdots + z_n^2$. Then the links K_{f_1} and K_{f_2} are the simple fibered link and the Seifert forms of K_1 and K_2 are 1×1 matrices $(-1)^{n(n-1)/2}$ and $-(-1)^{n(n-1)/2}$ respectively (cf. [**2**]).

We calculate the determinants of $(-1)^{n(n-1)/2} L_{K_{f_1}}$ and $(-1)^{n(n-1)/2} L_{K_{f_2}}$:

$$(-1)^{\lambda(K_{f_1})} = (-1)^{n(n-1)/2} \cdot (-1)^{n(n-1)/2} = 1$$
$$(-1)^{\lambda(K_{f_2})} = (-1)^{n(n-1)/2} \cdot (-(-1)^{n(n-1)/2}) = -1.$$

Thus $\lambda(K_{f_1}) = 0$ and $\lambda(K_{f_2}) = 1$. Hence the enhancements $\lambda(K_{f_1})$ and $\lambda(K_{f_2})$ realize any element of $\mathbb{Z}/2\mathbb{Z}$. $\square$

4. Remarks.

In this section, let $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ be a convenient strongly non-degenerate and polar weighted homogeneous mixed polynomial. We consider a polynomial map $f(\boldsymbol{z}, \bar{\boldsymbol{z}}) : \mathbb{C}^2 \to \mathbb{C}$. Then $K_f = S^3_\varepsilon \cap f^{-1}(0)$ is a fibered link whose monodromy is given by the S^1-action

$$(z_1, z_2) \mapsto (s^q z_1, s^p z_2),$$

where $(z_1, z_2) \in S^3_\varepsilon$, $s \in S^1$ and $\gcd(p, q) = 1$. We denote the number of link components of K_f whose orientations coincide with (resp. are opposite to) the orientations of the orbits of the S^1-action by m_+ (resp. m_-). We observe that the enhancement $\lambda(K_f)$ is as in Lemma 1.

Theorem 3. *Let $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ be a convenient strongly non-degenerate and polar weighted homogeneous mixed polynomial with $m_+ > m_-$. Then $\lambda(K_f) = (-pqm_- + p + q)m_-$.*

Proof. The link K_f is isotopic to $K_{f_{p,q,m_+,m_-}}$ and the monodromy is given by the same S^1-action as that of $f_{p,q,m_+,m_-}(\boldsymbol{z}, \bar{\boldsymbol{z}})$ [**14**], [**15**]. Thus the proof is analogous to that of Lemma 1. $\square$

Corollary 1. *Suppose that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is a convenient strongly non-degenerate and polar weighted homogeneous mixed polynomial. Then $\lambda(K_f) \leq 1$. Moreover, $\lambda(K_f) = 0$ if and only if $m_- = 0$ or $(p, q, m_-) = (1, 1, 2)$.*

Proof. By Theorem 3, the enhancement $\lambda(K_f)$ is equal to $(-pqm_- + p + q)m_-$. If $m_- = 0$ then $\lambda(K_f) = 0 \leq 1$. In the case $m_- \geq 1$, we have the inequality

$$-pqm_- + p + q \leq -pq + p + q = -(p-1)(q-1) + 1.$$

Since p and q are coprime positive integers, we see that $\lambda(K_f) \leq 1$.

The enhancement $\lambda(K_f) = 0$ if and only if $-pqm_- + p + q = 0$ or $m_- = 0$. In the case $-pqm_- + p + q = 0$, the integer m_- is equal to $1/p + 1/q$. If $1/p + 1/q$ is an integer, the pair (p, q) is $(1, 1)$ or $(2, 2)$. Since p and q are coprime positive integers, the pair (p, q) is $(1, 1)$, and thus m_- is equal to 2. □

We close this paper with a corollary concerning contact structures compatible with the fibrations of mixed singularities. We introduce the definition of compatible contact structures. Let M be a closed, oriented, smooth 3-manifold. A *contact structure* on M is the 2-plane field locally given by the kernel of a 1-form α satisfying $\alpha \wedge d\alpha \neq 0$ everywhere on M. A disk D in M is called *overtwisted* if D is tangent to $\ker \alpha$ at each point on the boundary of D. If M has an overtwisted disk then we say $\ker \alpha$ is *overtwisted* and otherwise we say $\ker \alpha$ is *tight*. Two contact structures $\ker \alpha_1$ and $\ker \alpha_2$ are said to be *contactomorphic* if there exists an automorphism of M which maps $\ker \alpha_1$ to $\ker \alpha_2$. Such a map is called a *contactomorphism*.

Suppose that M has an open book decomposition. An open book decomposition of M is said to be *compatible* with a contact structure $\ker \alpha$ on M if

- the fibered link of the open book decomposition is transverse to $\ker \alpha$;
- $d\alpha$ is a volume form on each fiber surface;
- the orientation of the fibered link coincides with that of $\ker \alpha$ determined by α.

It is well known that any open book decomposition of M admits a compatible contact structure [**22**]. E. Giroux showed that if two contact structures are compatible with the same open book decomposition then they are contactomorphic, i.e., there exists an automorphism which maps one contact structure to the other [**5**]. Thus the compatible contact structure is an invariant of the open book decomposition. It is known in [**7**] that the contact structure compatible with $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is overtwisted if and only if $m_- > 0$. If the homotopy class of the 2-plane field corresponds to the non-zero element of $\mathbb{Z} \cong \pi_3(S^2)$, the contact structure which is determined by the 2-plane field is overtwisted [**4**]. Thus we can recover this result except for the case $(p, q, m_-) = (1, 1, 2)$.

Corollary 2. *Suppose that $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is a convenient strongly non-degenerate and polar weighted homogeneous mixed polynomial with $m_- > 0$ and $(p, q, m_-) \neq (1, 1, 2)$. Then the contact structure compatible with a locally trivial fibration $S^3_\varepsilon \setminus K_f \to S^1$ is overtwisted.*

Proof. In [**4**], Y. Eliashberg classified contact structures on S^3. He showed that a tight contact structure on S^3 is unique up to contactomorphism and the homotopy class of the 2-plane field of a tight contact structure on S^3 corresponds to $0 \in \mathbb{Z} \cong \pi_3(S^2)$. Since $m_- > 0$ and $(p, q, m_-) \neq (1, 1, 2)$, we have $\lambda(K_f) \neq 0$. Thus the contact structure compatible with $S^3_\varepsilon \setminus K_f \to S^1$ is overtwisted. □

References

[1] J. L. Cisneros-Molina, Join theorem for polar weighted homogeneous singularities, In: Singularities II, (eds. J.-P. Brasselet, J. L. Cisneros-Molina, D. Massey, J. Seade and B. Teissier), Contemp, Math., **475**, Amer. Math. Soc., Providence, RI, 2008, pp. 43–59.

[2] A. H. Durfee, Fibered knots and algebraic singularities, Topology, **13** (1974), 47–59.

[3] D. Eisenbud and W. Neumann, Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Ann. of Math. Stud., **110**, Princeton University Press, Princeton, NJ, 1985.

[4] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds, Invent. Math., **98** (1989), 623–637.

[5] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, In: Proceedings of the International Congress of Mathematicians. Vol. II, Beijing, 2002, (eds. L. Tatsien), Higher Ed. Press, Beijing, 2002, pp. 405–414.

[6] S.-T. Hu, Homotopy Theory, Pure Appl. Math. (Amst.), **8**, Academic Press, New York, London, 1959.

[7] M. Ishikawa, On the contact structure of a class of real analytic germs of the form $f\bar{g}$, In: Singularities–Niigata-Toyama 2007, (eds. J.-P. Brasselet, S. Ishii, T. Suwa and M. Vaquie), Adv. Stud. Pure Math., **56**, 2009, pp. 201–223.

[8] R. Jacoby, One-parameter transformation groups of the three-sphere, Proc. Amer. Math. Soc., **7** (1956), 131–142.

[9] L. H. Kauffman and W. D. Neumann, Products of knots, branched fibrations and sums of singularities, Topology, **16** (1977), 369–393.

[10] J. Milnor, Singular Points of Complex Hypersurfaces, Ann. Math. Stud., **61**, Princeton University Press, Princeton, NJ, 1968.

[11] W. Neumann and L. Rudolph, Unfoldings in knot theory, Math. Ann., **278** (1987), 409–439.

[12] W. Neumann and L. Rudolph, The enhanced Milnor number in higher dimensions, In: Differential Topology, Siegen, Germany, 1987, (ed. U. Koschorke), Lecture Notes in Math., **1350**, Springer-Verlag, 1988, pp. 109–121.

[13] W. Neumann and L. Rudolph, Difference index of vectorfields and the enhanced Milnor number, Topology, **29** (1990), 83–100.

[14] M. Oka, Topology of polar weighted homogeneous hypersurfaces, Kodai Math. J., **31** (2008), 163–182.

[15] M. Oka, Non-degenerate mixed functions, Kodai Math. J., **33** (2010), 1–62.

[16] A. Pichon, Real analytic germs $f\bar{g}$ and open-book decompositions of the 3-sphere, Internat. J. Math., **16** (2005), 1–12.

[17] A. Pichon and J. Seade, Fibred multilinks and singularities $f\bar{g}$, Math. Ann., **342** (2008), 487–514.

[18] M. A. S. Ruas, J. Seade and A. Verjovsky, On real singularities with a Milnor fibration, In: Trends in Singularities, (eds. A. Libgober and M. Tibăr), Trends Math., Birkhäuser, Basel, 2002, pp. 191–213.

[19] L. Rudolph, Isolated critical points of mappings from $\mathbf{R}^4$ to $\mathbf{R}^2$ and a natural splitting of the Milnor number of a classical fibered link. Part 1: Basic theory; examples, Comment. Math. Helv., **62** (1987), 630–645.

[20] J. Seade, Fibred links and a construction of real singularities via complex geometry, Bol. Soc. Brasil. Mat. (N.S.), **27** (1996), 199–215.

[21] J. Seade, On the Topology of Isolated Singularities in Analytic Spaces, Progr. Math., **241**, Birkhäuser Verlag, Basel, 2006.

[22] W. P. Thurston and H. E. Winkelnkemper, On the existence of contact forms, Proc. Amer. Math. Soc., **52** (1975), 345–347.

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