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Nondegenerate SDE's with jumps and their hypoelliptic properties

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Abstract. We study 'nondegenerate' SDE's with jumps. These include SDE satisfying 'point-wise positive' condition and that satisfying (nonstationary) Hörmander's condition. We show that solutions of these SDE's have hypoelliptic properties. Our result is based on the Malliavin calculus on the Wiener-Poisson space. In case of continuous SDE, it extends and refines works based on the Malliavin calculus on the Wiener space.

1. Introduction and main results.

We will study a nonstationary (time-dependent) jump-diffusion on Euclidean space associated with a stochastic differential equation (SDE) with jumps on $\mathbb{R}^d$ written by

$$d\xi_t = b(\xi_{t-}, t)dt + \sigma(\xi_t, t)dW(t) + \int_{\mathbb{R}_0^m} g(\xi_{t-}, t, z)\tilde{N}(dtdz), \tag{1.1}$$

where $W(t)$ is an m-dimensional standard Brownian motion and $\tilde{N}(dtdz)$ is a compensated Poisson random measure on $\mathbb{R}_0^m = \mathbb{R}^m - \{0\}$ with intensity measure $dt\nu(dz)$, which are mutually independent. Here, ν is a Lévy measure having finite moments of any order. Coefficients $b(x,t) = (b^i(x,t))$, $\sigma(x,t) = (\sigma^{ij}(x,t))$ are smooth in x and coefficients $g(x,t,z) = (g^i(x,t,z))$ is smooth in x, z, as will be stated in Section 2.1.

The solution is a nonstationary jump-diffusion. Its generator is given by

$$\begin{aligned} A(t)\varphi &= \frac{1}{2}\sum_{i,j} a^{ij}(x,t)\varphi_{ij} + \sum_i b^i(x,t)\varphi_i \\ &\quad + \int_{\mathbb{R}_0^m} \Big\{\varphi(x + g(x,t,z)) - \varphi(x) - \sum_i g^i(x,t,z)\varphi_i(x)\Big\}\nu(dz), \end{aligned} \tag{1.2}$$

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where $a^{ij}(x,t) = \sum_k \sigma^{ik}(x,t)\sigma^{jk}(x,t)$.

In the previous papers [**8**] and [**9**], we studied a 'uniformly positive' jump-diffusion. Let us recall some facts stated in these papers. We assume that ν is a Lévy measure satisfying the order condition with exponent $\alpha \in (0,2)$, namely, the function $\varphi(\rho) := \int_{|z|\le\rho} |z|^2\nu(dz)$, $\rho > 0$ satisfies $\liminf_{\rho\to 0}\varphi(\rho)/\rho^\alpha > 0$. For $\rho > 0$, we set

$$B_\rho = \left(\frac{\int_{|z|\le\rho} z_i z_j \nu(dz)}{\varphi(\rho)}\right). \tag{1.3}$$

Let B be a symmetric nonnegative $m\times m$-matrix such that the matrix inequality $B \le B_\rho$ (positive definite order) holds for all $0<\rho<\rho_0$, where ρ_0 is a certain positive number. Such B is not unique. It will be fixed in this paper. Associated with the jump coefficients, we set $\tilde\sigma^{ij}(x,t) = \partial_{z_j} g^i(x,t,z)|_{z=0}$ and $\tilde\sigma(x,t) = (\tilde\sigma^{ij}(x,t))$ ($d\times m$-matrices). We define a matrix by

$$C(x,t) := \sigma(x,t)\sigma(x,t)^T + \tilde\sigma(x,t)B\tilde\sigma(x,t)^T. \tag{1.4}$$

A *uniformly positive condition* is stated as follows:

Condition (UP). *There exists* $n_0 \in \mathbb{N}$ *and* $c_0 > 0$ *such that*

$$v^T C(x,t) v \ge \frac{c_0}{(1+|x|)^{n_0}}|v|^2, \quad \forall x,t,v. \tag{1.5}$$

Note that the above condition does not mean that coefficient matrix $(a^{ij}(x,t))$ of the integro-differential operator $A(t)$ is positive definite. Hence the operator $A(t)$ might not be elliptic. However, the following *hypoelliptic properties* I and II hold ([**8**] and [**9**]).

I. *Analytic property.* Let $c(x,t)$ be a bounded smooth function and let $T>0$ be a terminal time. Consider the Cauchy problem of the backward heat equation:

$$\begin{cases} \left(\dfrac{\partial}{\partial s} + A(s) + c(x,s)\right)u(x,s) = 0, & 0<s<T,\ x\in\mathbb{R}^d, \\ \lim\limits_{s\uparrow T} u(x,s) = f(x), & \text{(terminal condition)}. \end{cases} \tag{1.6}$$

1. For any slowly increasing continuous function f, the equation has a unique slowly increasing $C^{\infty,1}$-solution $u(x,s)$.
2. It has a fundamental solution: There exists a function $p(s,x;t,y)$ satisfying the

following properties. i) For any t, y, it is a $C^{\infty,1}$-function of (x, s) and satisfies

$$\left(\frac{\partial}{\partial s} + A(s)_x + c(x,s)\right)p(s,x;t,y) = 0, \quad 0 < s < t,\ x \in \mathbb{R}^d. \tag{1.7}$$

ii) For any $x, s < t$, it is a rapidly decreasing C^∞-function of y and satisfies

$$u(x,s) = \int p(s,x;T,y)f(y)dy. \tag{1.8}$$

II. *Probabilistic property.* Let $\xi_{s,t}(x)$, $t \geq s$ be the solution of the SDE starting from x at time s. Define the Feynman-Kac operator $P^c_{s,t}f$ by

$$P^c_{s,t}f(x) := E\left[\exp\left\{\int_s^t c(\xi_{s,u}(x),u)du\right\}f(\xi_{s,t}(x))\right]. \tag{1.9}$$

1. The solution of the Cauchy problem (1.6) is given by $u(x,s) = P^c_{s,T}f(x)$.
2. The Feynman-Kac operator $P^c_{s,t}f(x)$ can be extended from a smooth function f of polynomial growth to any tempered distribution Φ. The extended function $u(x,s) = P^c_{s,t}\Phi(x)$ is of $\mathcal{C}^{\infty,1}$-class with respect to $(x,s) \in \mathbb{R}^d \times (0,t)$ and satisfies the backward heat equation

$$\left(\frac{\partial}{\partial s} + A(s) + c(x,s)\right)u(x,s) = 0. \tag{1.10}$$

3. The function $p(s,x;t,y) := P^c_{s,t}\delta_y(x)$ is the fundamental solution of the Cauchy problem (1.6).

In this paper, we will show that hypoelliptic properties I and II are valid for a wider class of SDE's. An SDE is called *nondegenerate* if the family of solutions $\{\xi_{s,t}(x); |x| \leq N\}$ is uniformly nondegenerate for any $s < t$ and $N > 1$, i.e, the Malliavin covariances $\Pi(x)$ of $\xi_{s,t}(x)$ are invertible and satisfies

$$\sup_{|x|\leq N}\ \sup_{v\in S_{d-1},\boldsymbol{u}\in A(1)^k} E\big[(v^T\Pi(x)v)^{-p}\circ\epsilon^+_{\boldsymbol{u}}\big] < \infty, \quad \forall p > 1,$$

for any $s < t$ and $k \in \mathbb{N}$ (For the precise meaning, see Section 2). We will show in Section 2 that *any nondegenerate SDE has hypoelliptic properties* I *and* II (Theorem 2.1).

In Sections 3–5, we will study nondegenerate SDE more explicitly. We will relax the above Condition (UP) as follows.

Condition (P). *The matrix $C(x,t)$ is positive definite for any x,t.*

Instead, we assume that the growth of jumps are not too big:

Condition (G). *g and its derivative are of uniformly linear growth: There exists a positive constant K such that*

$$\sup_{x,t} \frac{|g(x,t,z)| + |\nabla g(x,t,z)|}{1+|x|} \le K, \quad a.e.\ z\ (\nu). \tag{1.11}$$

We will show in Section 3 that *any SDE satisfying Conditions* (P) *and* (G) *is nondegenerate* (Theorem 3.1).

In Sections 4 and 5, we will study SDE's which do not satisfy Condition (P). Associated with matrix functions $\sigma(x,t)$ and $\tilde{\sigma}(x,t)$, we define time-dependent vector fields:

$$V_j(x,t) = \sigma^{\cdot j}(x,t), \qquad j = 1,\dots,m, \tag{1.12}$$

$$\tilde{V}_j(x,t) = \sum_k \tilde{\sigma}^{\cdot k}(x,t)\tau_{kj}, \quad j = 1,\dots,m, \tag{1.13}$$

where (τ_{kj}) is a symmetric nonnegative definite square root of the matrix B. Then SDE satisfies Condition (P) if and only if the family of time-dependent vector fields

$$\Sigma_0 = \big\{V_j(t), \tilde{V}_j(t), j = 1,\dots,m\big\}, \tag{1.14}$$

spans $\mathbb{R}^d$ for any x,t. Now, assuming that

$$b_0(x,t) := \lim_{\delta\to 0} \int_{|z|>\delta} g(x,t,z)\nu(dz) \tag{1.15}$$

exists, we define another time-dependent vector field by

$$V_0(x,t) = b(x,t) - \frac{1}{2}\sum_{l,j} \frac{\partial \sigma^{\cdot j}(x,t)}{\partial x_l}\sigma^{lj}(x,t) - b_0(x,t). \tag{1.16}$$

Then, we define families of time-dependent vector fields for $k = 1,2,\dots$ by

$$\Sigma_k = \big\{V_t(t) + [V_0(t), V(t)], [V_j(t), V(t)], [\tilde{V}_j(t), V(t)];\ j = 1,\dots,m, V(t) \in \Sigma_{k-1}\big\},$$

where $V_t(t)$ is the derivative of $V(t)$ with respect to t and $[\ ,\ ]$ denotes the Lie bracket.

The (time-dependent) *strong Hörmander condition* is stated as follows.

Condition (SH). *There exists* $n_0 \in \mathbb{N}$ *such that the family of time-dependent vector fields* $\bigcup_{k=0}^{n_0} \Sigma_k$ *spans* $\mathbb{R}^d$ *for all* x, t.

One of our goals is to prove that *an SDE is nondegenerate if it satisfies Condition* (SH) *and Condition* (G). It will be completed at Section 5. For this purpose, we will introduce in Section 4 other types of Hörmander conditions called *modified strong Hörmander condition* etc. The nondegenerate property will be proved for SDE's with modified strong Hörmander condition etc, making use of estimates of Norris type for a certain semimartingale with jumps. A main result is Theorem 4.1. Then, in Section 5, we apply the theorem for proving the nondegenerate property for SDE's stated above. See Theorem 5.1.

In Section 6, we will consider an SDE where the Lévy measure is lacking the order condition, together with a continuous SDE. For such an SDE, we neglect vector fields $\tilde{V}_j(x,t)$ of (1.13). Define $\hat{\Sigma}_0 = \{V_j(t), j = 1, \dots, m\}$ and

$$\hat{\Sigma}_k = \big\{V_t(t) + [V_0(t), V(t)], [V_j(t), V(t)];\ j = 1, \dots, m, V(t) \in \hat{\Sigma}_{k-1}\big\}.$$

Then Conditions ($\hat{\mathrm{SH}}$) is defined similarly, using $\hat{\Sigma}_k, k = 0, 1 \dots$ instead of $\Sigma_k, k = 0, 1, \dots$. It will be shown that hypoelliptic properties I and II are valid under Condition ($\hat{\mathrm{SH}}$) and Condition (G).

If the SDE is continuous, the condition for hypoelliptic properties I and II is relaxed. It can be shown under the *Hörmander condition*:

Condition ($\hat{\mathrm{H}}$). *The family of time-dependent vector fields* $\bigcup_{k=0}^{\infty} \hat{\Sigma}_k$ *spans* $\mathbb{R}^d$ *for all* x, t.

See Theorem 6.3. Our result extends and refines some well known results for stationary continuous SDE. If coefficient vector fields $V_j, j = 1, \dots, m$ of the continuous SDE do not depend on time t (stationary), the above $\hat{\Sigma}_k$ is rewritten as

$$\hat{\Sigma}_k = \{[V_0, V], [V_j, V];\ j = 1, \dots, m, V \in \hat{\Sigma}_{k-1}\}.$$

Then Condition ($\hat{\mathrm{H}}$) coincides with the usual Hörmander condition. Under this condition, the existence of the C^∞-density for a stationary continuous SDE was shown by Kusuoka-Stroock [**10**], [**11**] and others: The existence of the fundamental solution was pointed out by Watanabe [**21**], making use of the composition of a smooth functional and a tempered distribution. These results are now extended to time-dependent case. It seems to be new that the fundamental solution $p(s, x; t, y)$ is smooth with respect to s, x and satisfies equation (1.7).

Taniguchi [**20**] may be the first work on the Malliavin calculus for nonstationary continuous SDE. He proved the existence of the smooth density for the law of the solution under the restricted Hörmander condition. See Section 5.

In the final section (Appendix), we will give an estimate of Norris' type, making use of Komatsu-Takeuchi's estimate of semimartingale with jumps [**5**]. The estimate stated in Theorem 7.1 is much simpler than the corresponding estimate of Norris' type given in [**6**].

The study of the smooth densities for jump processes satisfying the uniform Hörmander's condition was initiated by Léandre [**13**]. His approach is based on Bismut's work on Malliavin calculus for jump process [**1**], where the smooth density of the Lévy measure is assumed. The author [**6**] studied the similar problem for canonical SDE with jumps. The latter approach is based on the Malliavin calculus on the Wiener-Poisson space due to Picard [**18**], [**19**] and Ishikawa-Kunita [**4**]. Our Theorem 5.1 will cover these works.

2. Nondegenerate functionals on Wiener-Poisson space and nondegenerate SDE with jumps.

2.1. Smooth functionals on Wiener-Poisson space.

Let T be a positive number and let $\mathbb{T} = [0, T]$. Let $\boldsymbol{W}$ be the set of all continuous maps $w : \mathbb{T} \to \mathbb{R}^m$ such that $w(0) = 0$ and let $\mathcal{B}(\boldsymbol{W})$ be the smallest σ-field of $\boldsymbol{W}$ with respect to which $\{w(t), t \in \mathbb{T}\}$ are measurable. Let P_1 be a probability measure on $(\boldsymbol{W}, \mathcal{B}(\boldsymbol{W}))$ such that
(1) $W(t) := w(t)$ is a standard Brownian motion.

Let $\mathbb{R}_0^m = \mathbb{R}^m - \{0\}$ and let $\mathcal{B}(\mathbb{R}_0^m)$ be its Borel field. By a point function on $\mathbb{R}_0^m$ we mean a map $q : \mathbb{D}_q \to \mathbb{R}_0^m$, where $\mathbb{D}_q$ is a countable subset of $\mathbb{T}$. A counting measure of the point function q is defined by

$$N(E, q) = \sharp\{t \in \mathbb{D}_q : \ (t, q(t)) \in E\},$$

where E is a Borel subset of $\mathbb{U} = \mathbb{T} \times \mathbb{R}_0^m$. Let Ξ be the set of all point functions on $\mathbb{R}_0^m$. We denote by $\mathcal{B}(\Xi)$ the smallest σ-field with respect to which $N(E), E \in \mathcal{B}(\mathbb{U})$ are measurable.

Let n be a measure on $\mathbb{U}$ given by $n(E) = \int_E dt\nu(dz)$, where ν is a Lévy measure on $\mathbb{R}_0^m$ satisfying $\int |z|^2/(1 + |z|^2)\nu(dz) < \infty$. A probability measure P_2 on $(\Xi, \mathcal{B}(\Xi))$ is called a *Poisson measure with characteristic* n, if the following conditions are satisfied.
(2) If $E_1, \ldots, E_n$ are disjoint, $N(E_1), \ldots, N(E_n)$ are independent.
(3) If $0 < n(E) < \infty$, $N(E)$ is Poisson distributed with intensity $n(E)$.

Let $\Omega = \boldsymbol{W} \times \Xi$ and $\mathcal{B} = \mathcal{B}(W) \otimes \mathcal{B}(\Xi)$. Elements of Ω are denoted by

$\omega = (w, q)$. A probability measure on $(\Omega, \mathcal{B})$ is called a *Wiener-Poisson measure with characteristic* n, if w and q are independent and satisfies (1)–(3). We denote by $\mathcal{F}$ the completion of $\mathcal{B}$. The triple $(\Omega, \mathcal{F}, P)$ is again called a *Wiener-Poisson space with characteristic* n. P is given by $P_1 \times P_2$.

In the following, through the paper, we will fix a Wiener-Poisson space $(\Omega, \mathcal{F}, P)$ with characteristic $dn = dsd\nu$, and we will discuss functionals defined over it.

We will introduce some notations following [**4**], [**8**] and [**9**]. Let D_t, $t \in \mathbb{T}$ be the Malliavin-Shigekawa's derivative operator acting on the first variable w. For $\boldsymbol{t} = (t_1, \ldots, t_j) \in \mathbb{T}^j$, we set $D_{\boldsymbol{t}} = D^j_{\boldsymbol{t}} = D^j_{t_1,\ldots,t_j} = D_{t_1} \cdots D_{t_j}$. We shall introduce difference operators $\tilde{D}_u$, $u \in \mathbb{U}$, acting on the Poisson space. For each $u = (t, z) \in \mathbb{U}$, we define a transformation $\varepsilon^+_u : \Xi \to \Xi$ by setting $\mathbb{D}_{\varepsilon^+_u q} = \mathbb{D}_q \cup \{t\}$ and

$$\begin{aligned}
(\varepsilon^+_u q)(s) &= q(s), \quad \text{if } s \in \mathbb{D}_q,\ s \neq t, \\
&= z, \qquad\ \ \text{if } s = t.
\end{aligned}$$

It is extended to $\omega = (w, q)$ by setting $\varepsilon^+_u \omega = (w, \varepsilon^+_u q)$. The difference operators $\tilde{D}_u$ for a smooth Poisson functional Y is defined after Picard [**18**] by

$$\tilde{D}_u Y = Y \circ \varepsilon^+_u - Y.$$

Let $\boldsymbol{u} = (u_1, \ldots, u_k) = ((t_1, z^1), \ldots, (t_k, z^k)) = (\boldsymbol{t}, \boldsymbol{z})$. We define $\varepsilon^+_{\boldsymbol{u}} = \varepsilon^+_{u_1} \circ \cdots \circ \varepsilon^+_{u_k}$ and $\tilde{D}_{\boldsymbol{u}} = \tilde{D}^k_{\boldsymbol{u}} = \tilde{D}_{u_1} \cdots \tilde{D}_{u_k}$.

For $k, l \in \mathbb{N}$ and $p > 1$, Sobolev's norms $|\ \ |_{k,l,p}$ over Wiener-Poisson functionals are defined making use of derivative operators $D^{l'}_{\boldsymbol{t}}$, $0 \leq l' \leq l$ and the difference operators $\tilde{D}^{k'}_{\boldsymbol{u}}$, $0 \leq k' \leq k$. We set

$$|X|_{k,l,p} = \left\{ \sum_{k'=0}^{k} \sum_{l'=0}^{l} E\left[\int_{\mathbb{U}^{k'}} \left(\int_{\mathbb{T}^{l'}} \left| \frac{D^{l'}_{\boldsymbol{t}} \tilde{D}^{k'}_{\boldsymbol{u}} X}{\gamma(\boldsymbol{u})} \right|^2 d\boldsymbol{t} \right)^{p/2} \hat{m}_{k'}(d\boldsymbol{u}) \right] \right\}^{1/p},$$

where

$$\hat{m}(du) := \frac{\gamma(u)^2 1_{(0,1]}(|u|) n(du)}{\int_{\mathbb{U}} \gamma(u)^2 1_{(0,1]}(|u|) n(du)}, \quad \hat{m}_k(d\boldsymbol{u}) = \hat{m}(du_1) \cdots \hat{m}(du_k),$$

and $\gamma(\boldsymbol{u}) = \gamma(\boldsymbol{z}) = |z^1| \cdots |z^k|$. The set of functionals G such that $|G|_{k,l,p} < \infty$ is denoted by $\boldsymbol{D}_{k,l,p}$. We set $\boldsymbol{D}_\infty = \cap_{k,l,p} \boldsymbol{D}_{k,l,p}$ and denote the d-fold product of $\boldsymbol{D}_\infty$ by $\boldsymbol{D}^d_\infty$. Elements of $\boldsymbol{D}_\infty$ or $\boldsymbol{D}^d_\infty$ are called *smooth functionals*. See [**4**].

We denote by $\hat{\boldsymbol{D}}_\infty^d$ the set of all $F \in \boldsymbol{D}_\infty^d$ satisfying the following properties: i) $\tilde{D}_{t,z}F$ is twice differentiable with respect to z and derivativers are uniformly continuous in $z \in \{z \in \mathbb{R}_0^m; |z| \leq 1\}$. ii) the functional

$$\Phi(F) := \sum_{i=1}^m \sup_{|z|\leq 1, t\in\mathbb{T}} \left|\partial_{z_i}\tilde{D}_{t,z}F\right| + \sum_{i,j=1}^m \sup_{|z|\leq 1, t\in\mathbb{T}} \left|\partial_{z_i}\partial_{z_j}\tilde{D}_{t,z}F\right|$$

satisfies $\sup_{\boldsymbol{u}\in A(1)^k} E[|\Phi(F)\circ \epsilon_{\boldsymbol{u}}^+|^p] < \infty$ for any $k \in \mathbb{N}$ and $p > 1$, where $A(1) = \{u = (t,z) \in \mathbb{U}; |z| \leq 1\}$.

2.2. Nondegenerate smooth functional and its law.

In the following, throughout Sections 2–5, we will assume that the Lévy measure associated with the Poisson random measure satisfies the order condition. Let F be a smooth functional in $\hat{\boldsymbol{D}}_\infty^d$. In [**4**], we defined the Malliavin covariance of F. In this paper, we present a slightly simpler Malliavin's covariance. It is defined by

$$\Pi^F = \int_0^T (D_tF)(D_tF)^T dt + \int_0^T (\partial\tilde{D}_{t,0}F)B(\partial\tilde{D}_{t,0}F)^T dt, \tag{2.1}$$

where B is a lower bound of $B_\rho, 0 < \rho < \rho_0$ and

$$\partial\tilde{D}_{t,0}F = \lim_{z\to 0}(\partial_{z_1}\tilde{D}_{t,z}F, \ldots, \partial_{z_m}\tilde{D}_{t,z}F).$$

$F \in \hat{\boldsymbol{D}}_\infty^d$ is called *nondegenerate* if its Malliavin covariance is invertible a.s. and the inverse $(v\Pi^F v)^{-1}$ $(v \neq 0)$ satisfies

$$\sup_{v\in S_{d-1}, \boldsymbol{u}\in A(1)^k} E\left[(v\Pi^F v)^{-p}\circ\epsilon_{\boldsymbol{u}}^+\right] < \infty \tag{2.2}$$

for any $p > 1$ and $k \in \mathbb{N}$. Our definition of a nondegenerate functional is slightly stronger than that adopted in [**4**] and [**8**].

Let us recall some facts about nondegenerate functionals following [**8**]. Suppose that $F \in \hat{\boldsymbol{D}}_\infty^d$ is a nondegenerate functional and let $G \in \boldsymbol{D}_\infty$. We will consider the (inverse) Fourier transform of the signed measure $\mu_G(dy) = E[G1_{F\in dy}]$. It is written as

$$\varphi_G(v) := \int e^{i(v,y)}\mu_G(dy) = E[e^{i(v,F)}G]. \tag{2.3}$$

We call it the *weighted characteristic function* of the random variable F with respect to G. We are interested in the property of the polynomial decay of the weighted characteristic function as $|v| \to \infty$. The property of the polynomial decay will imply that the signed measure μ_G has a rapidly decreasing C^∞-density function.

The polynomial decay of the weighted characteristic function was studied in [**8**]. We will quote an estimate given in [**8**], Theorem 2.5. For any given $n \in \mathbb{N}$, there exist $k = k_n$, $l = l_n \in \mathbb{N}$, $p = p_n > 1$ and $C_n > 0$ such that the inequality

$$\left|E[e^{i(v,F)}G]\right| \le C_n(1+|v|^2)^{-q_0 n/2}|G|_{k,l,p}\Theta_n(F), \quad \forall v \in \mathbb{R}^d \tag{2.4}$$

holds for any $G \in \boldsymbol{D}_\infty$ and any nondegenerate $F \in \hat{\boldsymbol{D}}^d_\infty$. Here $q_0 > 0$ is an absolute constant determined from the exponent of the order condition of the Lévy measure. The last term $\Theta_n(F)$ is finite for any n, if F is nondegenerate. Thus we get the polynomial decay of $\varphi_G(v)$.

In later discussions, we want to show the polynomial decay property for $F(x) = \xi_{s,t}(x)$, uniformly with respect to parameter x, where $\xi_{s,t}(x)$ is the flow of solutions of SDE. For this purpose, we will write down $\Theta_n(F)$ explicitly.

$$\begin{aligned}\Theta_n(F) &= |F|^{3n}_{k+1,l+1,2(l+1)(k+1)p}\\
&\quad\times\prod_{i=1}^{2^k}\left(1+E\left[\int \Phi(F)^{2(l+1)(k+1)q}\circ\epsilon^+_{\boldsymbol{u}_i}\hat{m}_i(d\boldsymbol{u}_i)\right]^{1/8(k+1)q}\right)^n\\
&\quad\times\prod_{i=1}^{2^k}\left(1+\sup_{v\in S_{d-1}}E\left[\int (v^T\Pi^F v)^{-8(l+1)(k+1)p}\circ\epsilon^+_{\boldsymbol{u}_i}\hat{m}_i(d\boldsymbol{u}_i)\right]^{1/4(k+1)p}\right)^n,\end{aligned} \tag{2.5}$$

where $q > 1$ is a certain constant.

2.3. SDE with jumps.

Let us return to SDE (1.1). We will use the following notations. For a function $f(x)$ on $\mathbb{R}^d$, we set $\nabla f(x) = (\partial_{x_1}f(x),\dots,\partial_{x_d}f(x))$ and for a vector function $f(x) = (f_1(x),\dots,f_e(x))$, we set $\nabla f(x) = (\partial_{x_i}f_j(x))$. For a multi-index $\alpha = (\alpha_1,\dots,\alpha_d)$ of nonnegative integers, we set $\nabla^\alpha = (\partial/\partial_{x_1})^{\alpha_1}\cdots(\partial/\partial_{x_d})^{\alpha_d}$ if $\alpha \ne (0,\dots,0)$. A function $f(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{T}$ is said to be a $\mathcal{C}^\infty_b$-function, if it is infinitely continuously differentiable with respect to x and $\nabla^\alpha f(x,t)$ are bounded continuous in (x,t) for all α.

A function $g(x,t,z)$, $x \in \mathbb{R}^d$, $t \in \mathbb{T}$, $z \in \mathbb{R}^m$ is said to belong to the class $\mathcal{C}^\infty_b(0)$ or to be a $\mathcal{C}^\infty_b(0)$-function, if it satisfies

i) For any z, it is a C_b^∞-function.
ii) For any α, $\nabla^\alpha g(x,t,z)$ is twice continuously differentiable with respect to z, of linear growth with respect to z and satisfies $\nabla^\alpha g(x,t,0)=0$.

For the coefficients of the equation, we assume that b,σ are $\mathcal{C}_b^\infty$-functions and g is a $\mathcal{C}_b^\infty(0)$-function. For the Lévy measure ν, we assume $\int |z|^p\nu(dz)<\infty$ for any $p\geq 2$. We set

$$X(x,t)=\int_0^t b(x,s)ds+\int_0^t \sigma(x,s)dW(s)+\int_0^t\int_{\mathbb{R}_0^m} g(x,s,z)\tilde{N}(dsdz). \tag{2.6}$$

Let $\{\mathcal{F}_t\}$ be the filtration of sub σ-fields of $\mathcal{F}$ generated by the Brownian motion $W(t)$ and the Poisson random measure N. Given an $\mathcal{F}_{t_0}$-measurable random variable ξ_0, an $\{\mathcal{F}_t\}$-adapted cadlag process ξ_t, $t\in[t_0,T]$ with values in $\mathbb{R}^d$ is called a solution of equation (1.1) starting from ξ_0 at time t_0, if it satisfies

$$\xi_t=\xi_0+\int_{t_0}^t X(\xi_{r-},dr),\quad t\geq t_0. \tag{2.7}$$

Equation (2.7) has a unique solution. The solution such that $\xi_0=x$ and $t_0=s$ is denoted by $\xi_{s,t}(x)$. Then for any $s<t$, maps $\xi_{s,t}:\mathbb{R}^d\to\mathbb{R}^d$ are C^∞-maps a.s. Further it satisfies $\xi_{s,u}=\xi_{t,u}\circ\xi_{s,t}$ for any $s<t<u$ a.s. $\{\xi_{s,t}\}$ is called a *nonstationary Lévy flow* associated with $X(x,t)$ of (2.6).

For fixed s,x, the stochastic process $\xi_t=\xi_{s,t}(x)$ is a jump-diffusion with generator $A(t)$ given by (1.2). Let $c(x,t)$ be a bounded $\mathcal{C}_b^\infty$-function. The transition operator weighted by c or Feynman-Kac operator $P_{s,t}^c\varphi(x)$ is defined by (1.9) for any C^∞-function φ of polynomial growth. For any $0<t\leq T$, it is a $C^{\infty,1}$-function of $(x,s)\in\mathbb{R}^d\times(0,t)$. Further, $u(x,s)=P_{s,t}^c\varphi(x)$ satisfies Kolmogorov's backward equation (1.10). See [**9**].

2.4. Nondegenerate SDE with jumps and its hypoelliptic properties.

It is known ([**8**], [**9**]) that for any $s<t$ and x, $F(x)=\xi_{s,t}(x)$ belongs to $\hat{\boldsymbol{D}}_\infty^d$ and the Malliavin covariance $\Pi(x)$ of $F(x)$ is written by

$$\Pi(x)=\int_s^t \nabla\xi_{u,t}(x)C(\xi_{s,u}(x),u)\nabla\xi_{u,t}(x)^T du, \tag{2.8}$$

where $C(x,t)$ is a matrix function defined by (1.4). The SDE is called *nondegenerate* if the family of solutions $\{\xi_{s,t}(x);|x|\leq N\}$ is 'uniformly' nondegenerate for any $s<t$ and $N>1$, i.e., if the following inequality holds.

$$\sup_{|x|<N} \sup_{v \in S_{d-1}, \boldsymbol{u} \in A(1)^k} E\big[(v^T \Pi(x) v)^{-p} \circ \epsilon_{\boldsymbol{u}}^+\big] < \infty, \tag{2.9}$$

for any $N > 1$, $p > 1$ and $k \in \mathbb{N}$.

We should remark that nondegenerate SDE in this paper is more general than nondegenerate SDE in [**8**] and nondegenerate Lévy flow in [**9**]. Indeed, these coincide with SDE satisfying Condition (UP) in this paper.

LEMMA 2.1. *Let $\xi_{s,t}(x)$ be solutions of a nondegenerate SDE. For any $n \in \mathbb{N}$, there exist $k, l \in \mathbb{N}$ and $p > 1$ such that the family of solutions $\{\xi_{s,t}(x); |x| \leq N\}$ $(s, t$ being fixed$)$ satisfies*

$$\sup_{|x| \leq N} \big| E[e^{i(v, \xi_{s,t}(x))} G] \big| \leq C_N (1 + |v|^2)^{-q_0 n/2} |G|_{k,l,p}, \quad \forall v, \ \forall G \in \boldsymbol{D}_{k,l,p}, \tag{2.10}$$

for any $N > 1$, where C_N is a positive constant.

PROOF. We will apply inequality (2.4) to $F = F(x) = \xi_{s,t}(x)$. We can check by a direct computation that the first and the second terms of $\Theta_n(F(x))$ given by (2.5) involving $F(x)$ and $\Phi(F(x))$, respectively, are bounded for $|x| \leq N$. Further we have from (2.9)

$$\sup_{|x| \leq N, v \in S_{d-1}} E\bigg[\int (v^T \Pi(x) v)^{-8(l+1)(k+1)p} \circ \epsilon_{\boldsymbol{u}_i}^+ \hat{m}_i(d\boldsymbol{u}_i)\bigg] < \infty. \tag{2.11}$$

Therefore $\Theta_n(F(x))$ is bounded for $|x| \leq N$. Then inequality (2.10) follows from (2.4). $\square$

Now, let $\mathcal{S}$ be the space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ equipped with seminorms

$$\|\varphi\|_{2j} = \bigg(\int_{\mathbb{R}^d} \sum_{\alpha + \beta \leq j} (1 + |y|^2)^\alpha |(1 - \Delta)^\beta \varphi(y)|^2 dy \bigg)^{1/2}, \quad j = 1, 2, \ldots$$

Denote the completion of $\mathcal{S}$ by the norm $\| \ \|_{2j}$ by $\mathcal{S}_{2j}$. Let $\mathcal{S}_{-2j}$ be the dual space of $\mathcal{S}_{2j}$ equipped with the dual norm $\| \ \|_{-2j}$. Then $\mathcal{S}' = \bigcup_j \mathcal{S}_{-2j}$ coincides with the space of tempered distributions.

The formula (2.10) of polynomial decay enables us to get the estimate of Feynman-Kac operator (1.9): For any $s < t$, $j \in \mathbb{N}$, α and $N > 1$, there exists a positive constant C such that

$$\sup_{|x|\le N} \left|\nabla^\alpha P^c_{s,t}\varphi(x)\right| \le C\|\varphi\|_{-2j}, \quad \forall \varphi \in \mathcal{S}.$$

See Lemma 7.1 in [**9**]. Therefore the domain of the operator $P^c_{s,t}$ can be extended to $\mathcal{S}_{-2j}$ and the extended function $P^c_{s,t}\Phi(x)$ is a C^∞-function of x for any $\Phi \in \mathcal{S}_{-2j}$. The extended function $P^c_{s,t}\Phi(x)$ satisfies for $s < s+h < t$

$$P^c_{s+h,t}\Phi(x) - P^c_{s,t}\Phi(x) = -\int_s^{s+h} (A(u) + c(x,u))P^c_{u,t}\Phi(x)du,$$

since the equality is valid for a smooth function $\phi = \Phi$. Therefore the extended function $u(x,s) = P^c_{s,t}\Phi(x)$ is a $C^{\infty,1}$-function of $(x,s) \in \mathbb{R}^d \times (0,t)$ and it satisfies the backward heat equation (1.10).

Theorem 2.1 ([**9**]). *Any nondegenerate SDE has hypoelliptic properties* I *and* II *stated in Section* 1.

Proof. Consider first $P^c_{s,t}\Phi(x)$. Let us take a continuous function f of polynomial growth in place of Φ. Then $u(x,s) := P^c_{s,T}f(x)$ satisfies the backward heat equation (1.10) together with the terminal condition $\lim_{s\uparrow T} u(x,s) = f(x)$. Hence $u(x,s)$ ia a solution of the Cauchy problem (1.6).

We will show the uniqueness of the solution of equation (1.6). Suppose that $v(x,s)$ is a $C^{\infty,1}$-function of polynomial growth satisfying (1.6) with the terminal condition $v(x,T) = 0$. We want to prove that $v(x,s) \equiv 0$. Our discussion is close to [**7**]. Let ξ_t be a solution of the SDE. Then in view of Itô's formula, we have for any $t > s$,

$$\begin{aligned} &e^{\int_s^t c(\xi_u,u)du} v(\xi_t,t) \\ &\quad = v(\xi_s,s) + \int_s^t e^{\int_s^u c(\xi_u,u)du}\left(A(u) + c(u) + \frac{\partial}{\partial u}\right)v(\xi_u,u)du + M_t - M_s, \end{aligned}$$

where M_t is a local martingale. Let $t = T$. Then we have $v(\xi_T,T) = 0$. Since $v(x,t)$ satisfies equation (1.6), we get the equality

$$0 = v(\xi_s,s) + M_T - M_s.$$

Since M_t is a local martingale, we get $v(\xi_s,s) + M_t - M_s = 0$ for any $s < t < T$. This implies $v(\xi_s,s) = 0$ for any s. Now take $\xi_{s-h,s}(x)$ in place of ξ_s and take the expectation. Then we have $E[v(\xi_{s-h,s}(x),s)] = 0$ for any s,h,x. Let h tend to 0. Then we get $v(x,s) = 0$ for any x,s, proving the uniqueness of the solution of the

Cauchy problem.

The fact that $p(s, x; t, y) := P^c_{s,t}\delta_y(x)$ is the fundamental solution will be obvious. We have thus shown hypoelliptic property I.

Hypoelliptic property II has already been proved. □

3. SDE with positive condition.

3.1. Stopping time of order h.

In this section, we want to show that any SDE satisfying Conditions (P) and (G) is nondegenerate. We begin with a discussion of certain stopping times which will help us to prove this. Let $\xi_t(x) = \xi_{0,t}(x)$ be the flow associated with the SDE (1.1). For a given $N > 1$, let $\tau(x)$ be the stopping time such that

$$\tau(x) = \inf\{t > 0;\ |\xi_t(x)| \geq N + 1\} \wedge T.$$

If $\xi_t(x)$ is a diffusion process, for any $h \in \mathbb{N}$ there exists a positive constant $c = c_h > 0$ such that the inequality holds

$$\sup_{|x|\leq N} P(\tau(x) < \epsilon) \leq c\epsilon^h, \quad 0 < \forall\epsilon < 1. \tag{3.1}$$

See Nualart [**17**]. However, such an inequality does not hold for jump-diffusion. We will modify it as follows.

Lemma 3.1. *Assume Condition* (G). *Let $N > 1, k \in \mathbb{N}$ be given numbers. Define for any $h \in \mathbb{N}$ the stopping time $T_h(x) = T_{h,N,k}(x)$ by*

$$T_h(x) = \inf\{t \in [0, T]; |\xi_t(x) - x| \geq M_h,\ \text{or}\ \ |\nabla\xi_t(x) - I| \geq M_h\} \wedge T, \tag{3.2}$$

where

$$M_h = M_{h,N,k} := (N + 1)(K + 1)^{k+h}, \tag{3.3}$$

and K is a constant given by (1.11). *Then there exists a positive constant $c = c_h$ such that*

$$\sup_{|x|\leq N, \boldsymbol{u}\in A(1)^k} P\big(T_h(x) \circ \epsilon^+_{\boldsymbol{u}} < \epsilon\big) \leq c\epsilon^h, \quad 0 < \forall\epsilon < 1. \tag{3.4}$$

We call $T_h(x)$ a stopping time of *order h* (*with respect to* N, k).

Proof. Define a sequence of stopping times for $\xi_t = \xi_t(x)$ by $\tau_0 = 0$ and

$$\tau_1 = \inf\big\{t \in [0,T]; |\xi_t - x| > 1,\ \text{or } |\nabla\xi_t - I| > 1\big\} \wedge T,$$
$$\cdots$$
$$\tau_h = \inf\big\{t \in [\tau_{h-1},T]; |\xi_t - \xi_{\tau_{h-1}}| > 1,\ \text{or } |\nabla\xi_t - \nabla\xi_{\tau_{h-1}}| > 1\big\} \wedge T.$$

Then if $t < \tau_h \circ \epsilon_{\boldsymbol{u}}^+$, we have

$$\sup_{\boldsymbol{u}\in A(1)^k} |\xi_t \circ \epsilon_{\boldsymbol{u}}^+| \vee \sup_{\boldsymbol{u}\in A(1)^k} |\nabla\xi_t \circ \epsilon_{\boldsymbol{u}}^+| \le M_h. \tag{3.5}$$

Therefore we have $T_h(x) \circ \epsilon_{\boldsymbol{u}}^+ \ge \tau_h \circ \epsilon_{\boldsymbol{u}}^+$ a.s.

For the proof of (3.4), it is sufficient to prove that there exists a positive constant c such that for any $|x| \le N$, $\boldsymbol{u} \in A(1)^k$, the inequality

$$P(\tau_h \circ \epsilon_{\boldsymbol{u}}^+ < \epsilon) \le c\epsilon^h, \quad 0 < \forall\epsilon < 1 \tag{3.6}$$

holds. In the following, we drop $\epsilon_{\boldsymbol{u}}^+$ for the simplicity. Since

$$\begin{aligned}\{\tau_h < \epsilon\} \subset \bigcap_{j=1}^h \big[&\big\{|\xi_{\tau_j} - \xi_{\tau_{j-1}}|1_{\tau_j-\tau_{j-1}<\epsilon} > 1\big\}\\ &\cup \big\{|\nabla\xi_{\tau_j} - \nabla\xi_{\tau_{j-1}}|1_{\tau_j-\tau_{j-1}<\epsilon} > 1\big\}\big],\end{aligned}$$

we have, by using the strong Markov property of ξ_t,

$$\begin{aligned}P(\tau_h < \epsilon) &\le \prod_{j=1}^h \big[P\big(|\xi_{\tau_j} - \xi_{\tau_{j-1}}|1_{\tau_j-\tau_{j-1}<\epsilon} > 1\big)\\ &\qquad + P\big(|\nabla\xi_{\tau_j} - \nabla\xi_{\tau_{j-1}}|1_{\tau_j-\tau_{j-1}<\epsilon} > 1\big)\big]\\ &\le \prod_{j=1}^h \big[E[|\xi_{\tau_j} - \xi_{\tau_{j-1}}|^2 1_{\tau_j-\tau_{j-1}<\epsilon}]\\ &\qquad + E[|\nabla\xi_{\tau_j} - \nabla\xi_{\tau_{j-1}}|^2 1_{\tau_j-\tau_{j-1}<\epsilon}]\big].\end{aligned}$$

Note

$$|\xi_{\tau_j} - \xi_{\tau_{j-1}}|^2 \le 2\bigg\{\bigg(\int_{\tau_{j-1}}^{\tau_j} b\,dr\bigg)^2 + \bigg(\int_{\tau_{j-1}}^{\tau_j} \sigma dW(r)\bigg)^2 + \bigg(\int_{\tau_{j-1}}^{\tau_j}\int g\tilde{N}(drdz)\bigg)^2\bigg\},$$

and note that the functionals $|b|, |\sigma|, \int |g|^2\nu(dz)$ are bounded by a positive constant c_1 on each time intervals (τ_{j-1}, τ_j). Then we have the inequality

$$E\big[|\xi_{\tau_j} - \xi_{\tau_{j-1}}|^2 1_{\tau_j - \tau_{j-1} < \epsilon}\big] \le 6c_1^2 E\big[(\tau_j - \tau_{j-1}) 1_{\tau_j - \tau_{j-1} < \epsilon}\big] \le 6c_1^2 \epsilon.$$

A similar estimate is valid for $E[|\nabla\xi_{\tau_j} - \nabla\xi_{\tau_{j-1}}|^2 1_{\tau_j - \tau_{j-1} < \epsilon}]$. Then we get the inequality (3.6). □

3.2. Estimate of the Malliavin covariance.

A goal of this section is to prove the following.

THEOREM 3.1. *Any SDE* (1.1) *satisfying Conditions* (P) *and* (G) *is nondegenerate.*

The proof will be divided into two cases. In the first case, we assume:

Condition (D). *There exists* $0 < c < 1$ *such that*

$$|\nabla g(x,t,z)| \le c, \quad \forall x, t, z.$$

The above condition is stronger than Condition (G). In the next lemma, we will prove the assertion of the theorem under Condition (D). Then the condition will be relaxed to Condition (G) in Section 3.3.

Under Condition (D), the maps $\phi_{t,z}(x) := x + g(x,t,z); \mathbb{R}^d \to \mathbb{R}^d$ are diffeomorphsims for all t, z. Then, the solution $\xi_{s,t}$ defines a flow of diffeomorphisms. Further, Jacobian matrices $\nabla\xi_{s,t}(x)$ are invertible and $\Psi_t^{x,s} = \nabla\xi_{s,t}(x)^{-1} \circ \epsilon_{\boldsymbol{u}}^+$, $\boldsymbol{u} = ((t_1, z^1), \ldots, (t_k, z^k))$ satisfies a linear SDE;

$$\Psi^{x,s}(t) = I + \int_s^t \Psi^{x,s}(r)\nabla Z(\xi_{s,r-}(x), dr),$$

where

$$\begin{aligned}\nabla Z(x,t) = &\int_0^t (\nabla\sigma\nabla\sigma^T - \nabla b)(x,s)ds - \int_0^t \nabla\sigma(x,s)dW(s)\\ &- \int_0^t \int \nabla h(x,s,z)\tilde{N}(dsdz) - \sum_{i=1}^k \nabla h(x,t_i,z^i)1_{[t_i,T]}(t)\end{aligned}$$

and $(I + \nabla g(x,s,z))^{-1} = I - \nabla h(s,x,z)$. Then we get the estimate

$$\sup_{|x|\le N, \boldsymbol{u}\in A(1)^k} E\big[|\nabla\xi_{s,t}(x)^{-1}\circ\epsilon_{\boldsymbol{u}}^+|^p\big] < \infty, \tag{3.7}$$

for any $p > 1$. See Section 3 in [**8**].

Lemma 3.2. *Assume Conditions* (P) *and* (D). *Then the SDE is nondegenerate.*

Proof. We will give the proof of (2.9) in the case where $s = 0$ and $t = T$. We first take arbitrary $N > 1$ and $k \in \mathbb{N}$. For any $h \in \mathbb{N}$, there exists $c_0 > 0$ such that $vC(x,t)v^T/|v|^2 \ge c_0$ holds for any $|x| \le M_h = M_{h,N,k}$.

To avoid complicated notations, we will drop the transformations $\circ\epsilon_{\boldsymbol{u}}^+$. Given $h \in \mathbb{N}$, let $T_h(x)$ be the stopping time of order h with respect to N, k, defined in Lemma 3.1. Then we have

$$v^T\Pi(x)v \ge \int_0^{T_h(x)} \frac{|v^T\nabla\xi_{r,T}C(\xi_r,r)(\nabla\xi_{r,T})^Tv|}{|v^T\nabla\xi_{r,T}|^2}\big|v^T\nabla\xi_{r,T}\big|^2 dr.$$

It holds for $r < T_h(x)$,

$$\frac{|v^T\nabla\xi_{r,T}C(\xi_r,r)(\nabla\xi_{r,T})^Tv|}{|v^T\nabla\xi_{r,T}|^2} > c_0,$$

$$\big|v^T\nabla\xi_{r,T}\big| = \big|v^T\nabla\xi_{0,T}\nabla\xi_{0,r}^{-1}\big| \ge \frac{|v^T\nabla\xi_{0,T}|}{|\nabla\xi_{0,r}|} \ge \frac{|v^T\nabla\xi_{0,T}|}{M_h+1}.$$

Therefore we have

$$(v^T\Pi(x)v)^{-1} \le \frac{(M_h+1)^2}{c_0|v|^2}|\nabla\xi_{0,T}(x)^{-1}|^2 T_h(x)^{-1}. \tag{3.8}$$

Consequently, by Hölder's inequality, we have for any $v \in S_{d-1}$,

$$E\big[((v^T\Pi(x)v)\circ\epsilon_{\boldsymbol{u}}^+)^{-h/2}\big] \le CE\big[|\nabla\xi_{0,T}(x)^{-1}\circ\epsilon_{\boldsymbol{u}}^+|^{3h/2}\big]^{1/3} E\big[(T_h(x)\circ\epsilon_{\boldsymbol{u}}^+)^{-3h/4}\big]^{2/3}.$$

Since $E[(T_h(x)\circ\epsilon_{\boldsymbol{u}}^+)^{-3h/4}]^{2/3}$ is bounded for $|x| \le N$, $\boldsymbol{u} \in A(1)^k$ by Lemma 3.1, we get the inequality (2.9) for $p \le h/2$. □

3.3. A method of perturbation.

In this subsection, we want to remove Condition (D) in Lemma 3.2. We recall that the Lévy flow $\{\xi_{s,t}(x)\}$ is the solution of SDE (2.7), where $X(x,t)$ is a Lévy process with spatial parameter defined by (2.6). For $0 < c < 1$, there exists a positive number δ_0 such that $\sup_{x.t} |\nabla g(x,t,z)| < c$ holds for any $|z| \le \delta_0$. Let $0 < \delta < \delta_0$ and let $X^\delta(x,t)$ be a Lévy process with spatial parameter x given by

$$X^\delta(x,t) = \int_0^t b^\delta(x,t)dt + \int_0^t \sigma(x,t)dW(t) + \int_0^t \int_{|z|\le\delta} g(x,t,z)\tilde{N}(dtdz), \quad (3.9)$$

where

$$b^\delta(x,t) = b(x,t) - \int_{|z|>\delta} g(x,t,z)\nu(dz).$$

Let $\{\xi^\delta_{s,t}(x)\}$ be the Lévy flow associated with $X^\delta(x,t)$. It has the same drift and the same diffusion coefficients as that of $\{\xi_{s,t}(x)\}$. Since the function $g1_{|z|\le\delta}$ satisfies Condition (D), $\{\xi^\delta_{s,t}(x)\}$ is a flow of diffeomorphisms.

Given a point process q, let q'' be the restriction of q to the subdomain $\mathbb{D}_{q''} = \{t \in \mathbb{D}_q; |q(t)| > \delta\}$. It is a discrete subset and we may write it as $\{0 < \tau_1 < \tau_2 < \cdots\} = \mathbb{D}_{q''}$. It holds

$$X(x,t) = X^\delta(x,t) \circ \epsilon^+_{q''} = X^\delta(x,t) + \sum_{i;\tau_i\le t} g(x,\tau_i,q''(\tau_i)).$$

Therefore, the solution $\xi_{s,t}(x)$ associated with $X(x,t)$ is decomposed as

$$\begin{aligned}\xi_{s,t}(x) &= \xi^\delta_{s,t}(x) \circ \epsilon^+_{q''} \\ &= \xi^\delta_{\tau_{n-1},t} \circ \phi_{\tau_{n-1},q''(\tau_{n-1})} \circ \cdots \circ \phi_{\tau_i,q''(\tau_i)} \circ \xi^\delta_{s,\tau_i}(x),\end{aligned}$$

where $\tau_{i-1} < s < \tau_i$ and $\tau_{n-1} < t < \tau_n$. Consequently we may regard that $\xi_{s,t}(x)$ is obtained from $\xi^\delta_{s,t}(x)$ by the perturbation of adding jumps $g(x,\tau_j,q''(\tau_j))$, $i < j < n$ to the solution $\xi^\delta_{s,t}(x)$. We call $\xi^\delta_{s,t}(x)$ as a *truncated process* of $\xi_{s,t}(x)$. It is independent of the point process q''.

Proof of Theorem 3.1. Let $\Pi^\delta(x)$ be the Malliavin covariance of the truncated random variable $\xi^\delta_{0,T}(x)$. Take arbitrary $N > 1$ and $k \in \mathbb{N}$. Let $T^\delta_h(x)$ be the stopping time of order h with respect to N,k associated with the process $\xi^\delta_t(x)$. Then it holds $\Pi(x) = \Pi^\delta(x) \circ \epsilon^+_{q''}$ and $T_h(x) = T^\delta_h(x) \circ \epsilon^+_{q''}$ a.s. Consequently we have by (3.8)

$$\begin{aligned}&E\big[(v^T\Pi(x)v)^{-h/2} \circ \epsilon^+_{\boldsymbol{u}}\big] \le E\big[(v^T\Pi^\delta(x)v)^{-h/2} \circ \epsilon^+_{\boldsymbol{u}} \circ \epsilon^+_{q''}\big] \\ &\quad \le CE\big[|\nabla\xi^\delta_{0,T}(x)^{-1}|^{3h/2} \circ \epsilon^+_{\boldsymbol{u}} \circ \epsilon^+_{q''}\big]^{1/3} E\big[(T^\delta_h(x) \circ \epsilon^+_{\boldsymbol{u}} \circ \epsilon^+_{q''})^{-3h/4}\big]^{2/3}.\end{aligned}$$

We can show

$$\sup_{|x|\le N, v\in S_{d-1}, \boldsymbol{u}\in A(1)^k} E\big[|\nabla \xi^{\delta}_{0,T}(x)^{-1}|^{3h/2}\circ \epsilon^+_{\boldsymbol{u}}\circ \epsilon^+_{q''}\big] < \infty.$$

See the proof of Lemma 3.4 in [**8**]. Further, since $T^{\delta}_h(x)\circ\epsilon^+_{\boldsymbol{u}}\circ\epsilon^+_{q''} = T_h(x)\circ\epsilon^+_{\boldsymbol{u}}$ a.s.,

$$\sup_{|x|\le N, v\in S_{d-1}, \boldsymbol{u}\in A(1)^k} E\big[(T^{\delta}_h(x)\circ\epsilon^+_{\boldsymbol{u}}\circ\epsilon^+_{q''})^{-3h/4}\big] < \infty$$

holds by Lemma 3.1. Then we get the inequality (2.9) for $p \le h/2$. □

Remark. If jumps do not satisfy Condition (G), the above perturbation might (or might not) break hypoelliptic properties. However, if Condition (UP) is satisfied, the above perturbation does not break it even if jumps do not satisfy Condition (G). See [**9**]. In Section 5, we will prove the similar fact for SDE satisfying Condition (UH).

4. SDE with conditions of Hörmander's type.

4.1. SDE with modified strong Hörmander's condition.

In this section, we assume that coefficients $b(x,t), \sigma(x,t)$ and $g(x,t,z)$ of SDE (1.1) are infinitely differentiable with respect to t and derivatives $\partial_t^n b(x,t)$, $\partial_t^n \sigma(x,t)$ are $\mathcal{C}_b^\infty$-functions and $\partial_t^n g(x,t,z)$ belongs to the class $\mathcal{C}_b^\infty(0)$ for any $n = 0,1,\dots$.

We are interested in SDE's which do not satisfy Condition (P) but may have hypoelliptic properties. We shall first rewrite equation (1.1). As was shown in Section 3.3, there exists $\delta_0 > 0$ such that maps $\phi_{t,z}(x) := x + g(x,t,z); \mathbb{R}^d \to \mathbb{R}^d$ are diffeomorphisms for any $t \in \mathbb{T}$ and $0 < |z| < \delta_0$. We will fix such δ_0. We will take $0 < \delta < \delta_0$ and define an another time-dependent vector field V_0^δ by

$$V_0^\delta(x,t) = b(x,t) - \frac{1}{2}\sum_{l,j}\frac{\partial\sigma^{\cdot j}(x,t)}{\partial x_l}\sigma^{lj}(x,t) - \int_{|z|>\delta} g(x,t,z)\nu(dz). \tag{4.1}$$

Then, using these vector fields, equation (1.1) is rewritten as

$$\begin{aligned} d\xi_t &= V_0^\delta(\xi_{t-},t)dt + \sum_j V_j(\xi_t,t)\circ dW^j(t) \\ &\quad + \int_{0<|z|\le\delta} g(\xi_{t-},t,z)\tilde{N}(dtdz) + \int_{|z|>\delta} g(\xi_{t-},t,z)N(dtdz), \end{aligned} \tag{4.2}$$

where $V_j \circ dW^j(t)$ are Stratonovitch integrals.

Associated with a diffeomorphism ψ of $\mathbb{R}^d$, the pull-back $\psi^* V$ of a smooth vector field V is defined by $\psi^* V f(x) = V(f \circ \psi)(\psi^{-1}(x))$. We define a linear map $\mathcal{L}^\delta$ of a space of time dependent vector fields into itself by

$$\mathcal{L}^\delta V(t) = V_t(t) + [V_0^\delta(t), V(t)] + \frac{1}{2}\sum_{j=1}^m [V_j(t), [V_j(t), V(t)]]$$
$$+ \int_{0<|z|\le\delta} \{(\phi_{t,z}^{-1})^* V(t) - V(t) - [\tilde{V}_z(t), V(t)]\}\nu(dz), \tag{4.3}$$

where $V_t(t) = (d/dt)V(t)$ and $[V_1, V_2]$ is the Lie bracket of two vector fields V_1, V_2, and $\tilde{V}_z(t)$ are vector fields such that their coefficients coincide with $g(x,t,z)$.

We set $\Upsilon_0^\delta = \Sigma_0$. For $k = 1, 2, \ldots$ we define families of time-dependent vector fields by

$$\Upsilon_k^\delta = \{\mathcal{L}^\delta V(t), [V_j(t), V(t)], [\tilde{V}_j(t), V(t)];\ j = 1, \ldots, m, V \in \Upsilon_{k-1}^\delta\}. \tag{4.4}$$

A *modified uniform Hörmander's condition* and *modified strong Hörmander's condition* are stated as follows, respectively.

Condition $(MUH)_\delta$. *There exists* $n_1 \in \mathbb{N}$ *and* $c_1 > 0$ *such that*

$$\sum_{k=0}^{n_1} \sum_{V \in \Upsilon_k^\delta} |v^T V(x,t)|^2 \ge c_1 |v|^2, \quad \forall x, t, v. \tag{4.5}$$

Condition $(MSH)_\delta$. *There exists* $n_1 \in \mathbb{N}$ *such that* $\bigcup_{k=0}^{n_1} \Upsilon_k^\delta$ *spans* $\mathbb{R}^d$ *for all* x, t.

A main result of this section is the following.

THEOREM 4.1. *Consider SDE* (4.2). 1) *If it satisfies Conditions* (G) *and* $(MSH)_\delta$ *for some* $0 < \delta < \delta_0$, *then it is nondegenerate.*

2) *If it satisfies* $(MUH)_\delta$ *for some* $0 < \delta < \delta_0$, *then it is nondegenerate.*

Another type of modified Hörmander condition was introduced in Komatsu-Takeuchi [**5**], where they discussed the existence of the smooth density for the law of the solution of an SDE.

For the proof of the theorem, we need estimates of Malliavin's covariance. In Sections 4.2 and 4.3, we assume Condition (D). Then solutions $\xi_{s,t}(x)$ of the SDE define a flow of diffeomorphisms of $\mathbb{R}^d$.

A *modified Malliavin covariance* of $\xi_{s,t}(x)$ is defined by

$$\Xi(x) = \sum_{V\in\Sigma_0} \int_s^t (\xi_{s,r}^{-1})^* V(x,r)(\xi_{s,r}^{-1})^* V(x,r)^T dr, \tag{4.6}$$

where $(\xi_{s,t}^{-1})^* V(x)$ is the pull back of V by the diffeomorphism $\xi_{s,t}^{-1}$. Then the Malliavin covariance $\Pi(x)$ of $\xi_{s,t}(x)$ is computed by the formula

$$\Pi(x) = \nabla\xi_{s,t}(x)\Xi(x)\nabla\xi_{s,t}(x)^T. \tag{4.7}$$

If the modified Malliavin's covariance satisfies the inequality

$$\sup_{|x|<N} \sup_{v\in S_{d-1},\boldsymbol{u}\in A(1)^k} E\big[(v^T\Xi(x)v)^{-2p}\circ\epsilon_{\boldsymbol{u}}^+\big] < \infty \tag{4.8}$$

for some $p>1$, then the Malliavin covariance $\Pi(x)$ satisfies

$$\sup_{|x|<N} \sup_{v\in S_{d-1},\boldsymbol{u}\in A(1)^k} E\big[(v^T\Pi(x)v)^{-p}\circ\epsilon_{\boldsymbol{u}}^+\big] < \infty. \tag{4.9}$$

Indeed, (4.8) implies $\sup E[|\det\Xi(x)|^{-2p/d}\circ\epsilon_{\boldsymbol{u}}^+] < \infty$. Since

$$|\det\Pi(x)| = |\det\nabla\xi_{s,t}|^2|\det\Xi(x)|,$$

and $\sup E[|\det\nabla\xi_{s,t}(x)|^{-4p/d}\circ\epsilon_{\boldsymbol{u}}^+] < \infty$ holds, we get the inequality

$$\sup_{|x|\le N} \sup_{\boldsymbol{u}\in A(1)^k} E\big[|\det\Pi(x)|^{-p/d}\circ\epsilon_{\boldsymbol{u}}^+\big] < \infty.$$

This is equivalent to (4.9).

In Section 4.2, we will get a chain rule for $(\xi_{s,t}^{-1})^*V(t)$ with respect to t. In Section 4.3, it will be applied for getting estimates of modified Malliavin covariance. The proof of Theorem 4.1 will be given at Section 4.4.

4.2. An Itô's formula for time dependent vector fields.

Lemma 4.1 (c.f. Lemma 3.1 in [**6**]). *Assume Condition* (D). *Let $V(t)$ be a time-dependent C^∞-vector field, differentiable with respect to t. Then the pull back $(\xi_t^{-1})^*V(x,t) = (\xi_{s,t}^{-1})^*V(x,t)$ satisfies*

$$
\begin{aligned}
(\xi_t^{-1})^* V(x,t) = V(x,s) &+ \int_s^t (\xi_r^{-1})^* \mathcal{L}^\delta V(x,r) dr \\
&+ \sum_j \int_s^t (\xi_r^{-1})^* [V_j(r), V(r)](x) dW^j(r) \\
&+ \int_s^t \int_{|z|\le\delta} (\xi_{r-}^{-1})^* \big\{ (\phi_{r,z}^{-1})^* V(x,r) - V(x,r) \big\} \tilde{N}(drdz) \\
&+ \int_s^t \int_{|z|>\delta} (\xi_{r-}^{-1})^* \big\{ (\phi_{r,z}^{-1})^* V(x,r) - V(x,r) \big\} N(drdz), \qquad (4.10)
\end{aligned}
$$

for any $0 < \delta < \delta_0$.

PROOF. In view of Itô's formula for semimartingale with jumps, we have

$$
\begin{aligned}
V(\xi_t, t) = V(x,s) &+ \int_s^t V_t(\xi_r, r) dr + \int_s^t \nabla V(\xi_r, r) V_0^\delta(\xi_r, r) dr \\
&+ \int_s^t \int_{0<|z|\le\delta} \{ V(\phi_{r,z}\circ\xi_r, r) - V(\xi_r, r) - \tilde{V}_z(\xi_r, r)\nabla V(\xi_r, r)\} dr d\nu(z) \\
&+ \sum_j \int_0^t \nabla V(\xi_r, r) V_j(\xi_r, r) \circ dW^j(r) \\
&+ \int_s^t \int_{0<|z|\le\delta} (V(\phi_{r,z}\circ\xi_{r-}, r) - V(\xi_{r-}, r)) \tilde{N}(drdz) \\
&+ \int_s^t \int_{|z|>\delta} (V(\phi_{r,z}\circ\xi_{r-}, r) - V(\xi_{r-}, r)) N(drdz).
\end{aligned}
$$

Further, the inverse matrix $(\nabla\xi_t)^{-1}$ satisfies a.s.

$$
\begin{aligned}
(\nabla\xi_t)^{-1} = I &- \int_s^t (\nabla\xi_r)^{-1} \nabla V_0^\delta(\xi_r, r) dr \\
&+ \int_s^t \int_{0<|z|\le\delta} (\nabla\xi_r)^{-1} \{ \nabla\phi_{r,z}(\xi_r)^{-1} - I + \nabla\tilde{V}_z(\xi_r, r)\} dr d\nu(z) \\
&- \sum_j \int_s^t (\nabla\xi_r)^{-1} \nabla V_j(\xi_r, r) \circ dW^j(r)
\end{aligned}
$$

$$+\int_s^t \int_{0<|z|\le\delta} (\nabla\xi_{r-})^{-1}\{\nabla\phi_{r,z}(\xi_{r-})^{-1} - I\}\tilde{N}(drdz)$$

$$+\int_s^t \int_{|z|>\delta} (\nabla\xi_{r-})^{-1}\{\nabla\phi_{r,z}(\xi_{r-})^{-1} - I\}N(drdz).$$

Apply Itô's formula to the product of two semimartingales $X_t = (\nabla\xi_t)^{-1}$ and $Y_t = V(\xi_t, t)$. Then we get

$$\begin{aligned}(\nabla\xi_t)^{-1}V(\xi_t,t) = V(x,s) &+ \int_s^t (\nabla\xi_r)^{-1}\mathcal{L}^\delta V(\xi_r, r)dr \\ &+ \sum_j \int_s^t (\nabla\xi_r)^{-1}[V_j(r), V(r)](\xi_r, r)dW^j(r) \\ &+ \int_s^t \int_{|z|\le\delta} (\nabla\xi_{r-})^{-1}\big\{(\phi_{r,z}^{-1})^* V(\xi_{r-}, r) - V(\xi_{r-}, r)\big\}\tilde{N}(drdz) \\ &+ \int_s^t \int_{|z|>\delta} (\nabla\xi_{r-})^{-1}\big\{(\phi_{r,z}^{-1})^* V(\xi_{r-}, r) - V(\xi_{r-}, r)\big\}N(drdz).\end{aligned}$$

It is rewritten as (4.10). □

4.3. Estimate of modified Malliavin's covariance.

In the following, we will fix a constant β such that $\beta > \max\{4/(2-\alpha), 8\}$, where $0 < \alpha < 2$ is the exponent of the order condition for the Lévy measure. We first take arbitrary $N > 1$, $k \in \mathbb{N}$ and $h \in \mathbb{N}$. Let $M_h = M_{h,N,k}$ be the positive number given by (3.3). In view of Condition $(MSH)_\delta$, there exists $n_1 \in \mathbb{N}$ and $c_1 > 0$ such that

$$H(x,t) := \sum_{k=0}^{n_1} \sum_{V\in\Upsilon_k^\delta} V(x,t)V(x,t)^T \tag{4.11}$$

satisfies

$$v^T H(x,t)v \ge c_1|v|^2, \quad \forall |x| \le M_h, \quad \forall t.$$

Let $T_h(x)$ be the stopping time of order h with respect to N, k associated with $\xi_t(x)$. Then it satisfies

$$\sup_{|x|\le N, \boldsymbol{u}\in A(1)^k} P(T_h(x)\circ\epsilon_{\boldsymbol{u}}^+ < \gamma) \le c_p\epsilon^h, \quad 0 < \forall\epsilon < 1 \tag{4.12}$$

by Lemma 3.1.

LEMMA 4.2. *Assume Conditions* (D) *and* $(MSH)_\delta$ *for some* $0 < \delta < \delta_0$. *Then the modified Malliavin's covariance* $\Xi(x)$ *is invertible a.s. Further, for any* N, k *and sufficiently large* h, *it satisfies the inequality* (4.8) *for all* $1 < p < (1/2)\beta^{-2n_1}h$.

PROOF. We will prove (4.8) in the case $s = 0$ and $t = T$.

Step 1: We will first prove the inequality (4.8) in the case where $k = 0$, namely the inequality:

$$\sup_{|x|<N} \sup_{v \in S_{d-1}} E[(v^T \Xi(x) v)^{-2p}] < \infty. \tag{4.13}$$

For a given time dependent vector field $V(x,t)$, we set

$$Y_V^{x,v}(t) := v^T (\xi_t^{-1})^* V(x,t).$$

Let $\epsilon_1 = \min_{0 \le i \le n_1} (1/k_i)^{\beta^{(2i+1)}}$, where k_i is the number of elements of the set Υ_i^δ. For $|x| \le N$ and $0 < \epsilon < \epsilon_1$, we define events by

$$E_i = E_i^{x,v} = \left\{ \sum_{V \in \Upsilon_i^\delta} \int_0^T |Y_V^{x,v}(t)|^2 dt < \epsilon^{\beta^{-2i}} \right\}, \quad i = 0, 1, 2, \ldots, n_1.$$

Then we have the decomposition

$$E_0 = \big(E_0 \cap E_1^c\big) \cup \big(E_1 \cap E_2^c\big) \cup \cdots \cup \big(E_{n_1-1} \cap E_{n_1}^c\big) \cup F,$$

where $F = E_0 \cap E_1 \cap \cdots \cap E_{n_1}$. Therefore,

$$P(E_0) \le \sum_{i=0}^{n_1-1} P(E_i \cap E_{i+1}^c) + P(F).$$

We claim that for any $h \in \mathbb{N}$ there exist $C > 0$, $C' > 0$ such that

$$\sup_{|x| \le N, v \in S_{d-1}} P(E_i \cap E_{i+1}^c) < C\epsilon^h, \quad i = 0, \ldots, n_1 - 1, \tag{4.14}$$

$$\sup_{|x| \le N, v \in S_{d-1}} P(F) < C' \epsilon^{\beta^{-2n_1}h}, \tag{4.15}$$

hold for any $0 < \epsilon < \epsilon_1$. If the above $n_1 + 1$ inequalities are verified, then we get

$$\sup_{|x|\le N, v\in S_{d-1}} P(E_0) < C''\epsilon^{\beta^{-2n_1}h}, \quad \forall 0 < \epsilon < \epsilon_1.$$

Since $E_0 = \{v\Xi(x)v < \epsilon\}$, the above will imply (4.13) for any $p < (1/2)\beta^{-2n_1}h$.

We will first prove (4.14). Our discussion is close to Lemmas 4.2 and 4.3 in Kunita [**6**]. We need an inequality of Norris type, which will be discussed in Section 7, Appendix. Note that the semimartingale $Y_V^{x,v}(t)$ is represented by (4.10). It may be rewritten as (7.1), where coefficients of the drift term, diffusion term and jump term of the semimartingale $Y_V^{x,v}(t)$ are given by

$$\begin{aligned} a_V^{x,v}(s) &= a_V(s) = v^T(\xi_s^{-1})^*\mathcal{L}^\delta V(x,s), \\ f_V^{x,v}(s) &= f_V(s) = \big(v^T(\xi_s^{-1})^*[V_1(s), V(s)](x), \ldots, v^T(\xi_s^{-1})^*[V_m(s), V(s)](x)\big), \\ g_V^{x,v}(s,z) &= g_V(s,z) = v^T(\xi_s^{-1})^*\big\{(\phi_{s,z}^{-1})^*V(x,s) - V(x,s)\big\}, \end{aligned}$$

respectively. Let $\tilde{g}_V^{x,v}(s) = \tilde{g}_V(s)$ be the process given by (7.4). Then we have

$$|\tilde{g}_V(s)|^2 = \sum_j \big|v^T(\xi_s^{-1})^*[\tilde{V}_j(s), V(s)](x)\big|^2.$$

Now for a given $V \in \Upsilon_i^\delta$, we set for $0 < \epsilon < \epsilon_1$

$$\begin{aligned} A_V(\epsilon) &= \left\{ \int_0^T |f_V(t)|^2 dt < \epsilon^{\beta^{-2i}} \right\}, \\ B_V(\epsilon) &= \left\{ \int_0^T |a_V(t)|^2 dt + \int_0^T |f_V(t)|^2 dt + \int_0^T |\tilde{g}_V(t)|^2 dt < \epsilon^{\beta^{-2(i+1)}} \frac{1}{k_i} \right\}, \\ C_V(\epsilon) &= \left\{ \int_0^T |a_V(t)|^2 dt + \int_0^T |f_V(t)|^2 dt + \int_0^T |\tilde{g}_V(t)|^2 dt < \epsilon^{\beta^{-2i-1}} \right\}. \end{aligned}$$

Then we have the relation

$$E_i \cap E_{i+1}^c \subset \bigcup_{V\in\Upsilon_i^\delta} A_V(\epsilon) \cap B_V(\epsilon)^c \subset \bigcup_{V\in\Upsilon_i^\delta} A_V(\epsilon) \cap C_V(\epsilon)^c.$$

Set $\tilde{\epsilon} = \epsilon^{\beta^{-2i-1}}$. Then we have $\tilde{\epsilon}^\beta = \epsilon^{\beta^{-2i}}$. We can apply the inequality (7.7) in Section 7 for two sets $A(\tilde{\epsilon}) := A_V(\epsilon)$ and $B(\tilde{\epsilon}) := C_V(\epsilon)$ for any $\tilde{p} = \beta^{2i+1}h$. Then we have

$$\sup_{|x|\le N, v\in S_{d-1}} P(A_V(\epsilon)\cap C_V(\epsilon)^c) < c\tilde{\epsilon}^{\tilde{p}} \le c\epsilon^h, \quad \forall 0<\epsilon<\epsilon_1.$$

Therefore,

$$\sup_{|x|\le N, v\in S_{d-1}} P(E_i\cap (E_{i+1})^c) \le C\epsilon^h,$$

proving (4.14).

We shall next prove (4.15). Set

$$K = K(x,v) = \sum_{i=0}^{n_1}\int_0^T \sum_{V\in\Upsilon_i^\delta} |Y_V^{x,v}(t)|^2 dt. \tag{4.16}$$

If $\omega \in F = F^{x,v}$, then $K(x,v) < (n_1+1)\epsilon^{\beta^{-2n_1}}$. Therefore, we have $F^{x,v} \subset \{K(x,v) < (n_1+1)\epsilon^{\beta^{-2n_1}}\}$. On the other hand, we have

$$\begin{aligned} K(x,v) &\ge \int_0^{T_h(x)} v^T(\nabla\xi_t(x))^{-1}H(\xi_t(x),t)((\nabla\xi_t(x))^{-1})^T v dt \\ &\ge c_1\int_0^{T_h(x)} |v^T\nabla\xi_t(x)^{-1}|^2 dt \\ &\ge c_1(M_h+1)^{-2}T_h(x), \end{aligned}$$

where $T_h(x)$ is the stopping time of order h with respect to N and $k=0$. Therefore, if we take γ such that $(n_1+1)\epsilon^{\beta^{-2n_1}} = c_1(M_h+1)^{-2}\gamma$, we have $\{K(x,v) < (n_1+1)\epsilon^{\beta^{-2n_1}}\} \subset \{T_h(x)<\gamma\}$. Consequently we have

$$P(F^{x,v}) \le P(T_h(x)<\gamma) \le c_h\gamma^h = C'\epsilon^{\beta^{-2n_1}h},$$

for all $|x|\le N$, $v\in S_{d-1}$. This proves (4.15).

Step 2: We shall next prove (4.8) in the case $k=1$. Let $u=(s_1,z_1)$ and consider a semimartingale $\bar{Y}_V^{x,v}(t) := v^T(\xi_t^{-1})^*V(x,t)\circ\epsilon_u^+$. Set

$$\bar{E}_i^{x,v} = \Big\{\sum_{V\in\Upsilon_i^\delta}\int_0^T \big|\bar{Y}_V^{x,v}(t)\big|^2 dt < \epsilon^{\beta^{-2i}}\Big\}, \quad i=0,1,2,\ldots,n_1.$$

It holds

$$\bar{Y}_V^{x,v}(t) = \begin{cases} v^T(\xi_t^{-1})^*(\phi_u^{-1})^* V(x,t), & \text{if } s_1 < t, \\ v^T(\xi_t^{-1})^* V(x,t), & \text{if } s_1 > t, \end{cases}$$

and for $s_1 < t$,

$$\begin{aligned}
&(\xi_t^{-1})^*(\phi_u^{-1})^* V(x,t) \\
&\quad = (\xi_s^{-1})^*(\phi_u^{-1})^* V(x,s_1) + \int_{s_1}^t (\xi_r^{-1})^* \mathcal{L}^\delta (\phi_u^{-1})^* V(x,r)dr \\
&\qquad + \sum_j \int_{s_1}^t (\xi_r^{-1})^*(\phi_u^{-1})^*[V_j(r), V(r)](x)dW^j(r) \\
&\qquad + \int_{s_1}^t \int_{|z|\le\delta} (\xi_{r-}^{-1})^* \Big\{ (\phi_{r,z}^{-1})^*(\phi_u^{-1})^* V(x,r) - (\phi_u^{-1})^* V(x,r) \Big\} \tilde{N}(drdz) \\
&\qquad + \int_{s_1}^t \int_{|z|>\delta} (\xi_{r-}^{-1})^* \Big\{ (\phi_{r,z}^{-1})^*(\phi_u^{-1})^* V(x,r) - (\phi_u^{-1})^* V(x,r) \Big\} N(drdz).
\end{aligned}$$

Then we can show similarly as in Step 1 that for any N there exists $C > 0$ such that

$$\sup_{|x|\le N, v\in S_{d-1}} \sup_{u\in A(1)} P\big(\bar{E}_i^{x,v} \cap (\bar{E}_{i+1}^{x,v})^c\big) < C\epsilon^h, \quad i = 0,\dots,n_1-1, \tag{4.17}$$

holds for any $0 < \epsilon < \epsilon_1$.

Next, set $\bar{F}^{x,v} = \bar{E}_1^{x,v} \cap \cdots \cap \bar{E}_{n_1}^{x,v}$. We want to prove

$$\sup_{|x|\le N, v\in S_{d-1}} \sup_{u\in A(1)} P(\bar{F}^{x,v}) < C'\epsilon^{\beta^{-2n_1}h}. \tag{4.18}$$

Set $\bar{K}(x,v) = \int_0^T |\bar{Y}^{x,v}(t)|^2 dt$. Then we have

$$\begin{aligned}
\bar{K}(x,v) \ge &\int_0^{s_1\wedge T_h(x)} v^T(\nabla\xi_t(x))^{-1} H(\xi_t(x),t)((\nabla\xi_t(x))^{-1})^T v dt \\
&+ \int_{s_1\wedge T_h(x)}^{T_h(x)} v(\nabla(\phi_u\circ\xi_{t-})(x))^{-1} \\
&\qquad\qquad \times H(\phi_u\circ\xi_{t-}(x),t)((\nabla(\phi_u\circ\xi_{t-})(x))^{-1})^T v dt.
\end{aligned}$$

Two integrands of the above integrals dominates $c|v|^2, c > 0$ for $t < T_h(x)$. Then we get (4.18) similarly as in Step 1. Therefore the inequality (4.8) holds in the case $k = 1$.

We can verify (4.8) for any $k \in \mathbb{N}$ similarly. □

4.4. Nondegenerate properties.

The first assertion of Theorem 4.1 is immediate from the following Lemma.

LEMMA 4.3. *Assume Conditions* (G) *and* $(MSH)_\delta$ *for some* $0 < \delta < \delta_0$. *Then the Malliavin covariance* $\Pi(x)$ *is invertible a.s. It satisfies the inequality* (4.9) *for any* $N > 1$, $k \in \mathbb{N}$ *and* $p > 1$.

PROOF. We first consider the case where the additional Condition (D) is satisfied. For given N, k, p, choose h such that $h > 2p\beta^{2n_1}$ and apply Lemma 4.2. Then we find that the modified Malliavin's covariance $\Xi(x)$ satisfies (4.8).

Now if Condition (D) is not satisfied, we can verify (4.9) by the method of perturbation as in Section 3.3. □

We shall next consider the case where Condition (G) is not satisfied. Instead, we assume Condition $(MUH)_\delta$. The second assertion of Theorem 4.1 follows from the next lemma.

LEMMA 4.4. *Assume Condition* $(MUH)_\delta$ *for some* $0 < \delta < \delta_0$. *Then the Malliavin covariance is invertible a.s. and its inverse satisfies* (4.9) *for any* $N > 1$, $k \in \mathbb{N}$ *and* $p > 1$.

PROOF. We shall only prove the inequality (4.9) in the case $s = 0$, $t = T$ and $k = 0$, i.e., the inequality

$$\sup_{|x|<N} \sup_{v \in S_{d-1}} E[(v^T \Pi(x) v)^{-p}] < \infty. \tag{4.19}$$

We will apply a method of perturbation discussed in Section 3.3. Let $\xi^\delta_{s,t}(x)$ be the truncated Lévy flow generated by $X^\delta(x,t)$ of (3.9). Let us recall the argument in the proof of Lemma 4.2. Denote events E_i and F associated with $\xi^\delta_{s,t}(x)$ by $E_i(\delta)$ and $F(\delta)$, respectively. Instead of (4.14) we have for any $h \in \mathbb{N}$

$$\sup_{|x|\le N} \sup_{v \in S_{d-1}, \boldsymbol{u} \in A(1)^k} P\big(E_i(\delta) \circ \epsilon^+_{q''} \cap (E_{i+1}(\delta) \circ \epsilon^+_{q''})^c\big) \le C\epsilon^h. \tag{4.20}$$

Let $K^\delta(x, v)$ be the functional for $\xi_t = \xi^\delta_{0,t}(x)$ given by (4.16). Since it satisfies Condition $(MUH)_\delta$, we have the inequality

$$K^\delta(x,v) \geq c_1 \int_0^T |v^T(\nabla\xi_t^\delta)^{-1}|^2 dt, \quad a.s.,$$

where c_1 is the constant in Condition $(MUH)_\delta$. This implies

$$K^\delta(x,v)^{-1} \leq \frac{1}{c_1 T^2 |v|^2} \int_0^T |\nabla\xi_t^\delta|^2 dt \quad a.s.$$

Therefore, we have for any $h > 1$,

$$P\bigg(K^\delta(x,v)^{-1} \circ \epsilon_{q''}^+ > \frac{1}{\epsilon}\bigg) \leq \frac{1}{(c_1 T^2 |v|^2)^h} E\bigg[\bigg(\int_0^T \big|\nabla\xi_t^\delta \circ \epsilon_{q''}^+\big|^2 dt\bigg)^h\bigg]\epsilon^h,$$

in view of Chebyschev's inequality. Since

$$F(\delta) \subset \{K^\delta(x,v) < (n_1+1)\epsilon^{\beta^{-2n_1}}\}$$

holds as in the proof of Lemma 4.2, we get from the above

$$\begin{aligned} P\big(\bar{F}(\delta) \circ \epsilon_{q''}^+\big) &\leq P\big(K^\delta(x,v) \circ \epsilon_{q''}^+ < (n_1+1)\epsilon^{\beta^{-2n_1}}\big) \\ &\leq c_p' \epsilon^{\beta^{-2n_1} h}, \end{aligned} \tag{4.21}$$

for any $|x| \leq N, v \in S_{d-1}$. Two inequalities (4.20) and (4.21) imply

$$\sup_{|x|\leq N} \sup_{v \in S_{d-1}} E\big[(v^T \Xi^\delta(x) v)^{-2p} \circ \epsilon_{q''}^+\big] < \infty,$$

if $p < (1/2)\beta^{-2n_1}h$. Then we get

$$\sup_{|x|\leq N}, \sup_{v \in S_{d-1}} E\big[(v^T \Pi^\delta(x) v)^{-p} \circ \epsilon_{q''}^+\big] < \infty,$$

if $p < (1/2)\beta^{-2n_1}h$. Since $\Pi^\delta(x) \circ \epsilon_{q''}^+ = \Pi(x)$ holds a s., we get the inequality (4.19) for the above p. Finally, since we can take any $h \in \mathbb{N}$, inequality (4.19) holds for any $p > 1$. □

5. SDE with strong Hörmander condition.

We assume that the Lévy measure associated with SDE (1.1) has a mean vector in the wide sense, i.e.,

$$m := \lim_{\delta \to 0} \int_{|z|>\delta} z\nu(dz) \quad \text{exists.} \tag{5.1}$$

If the exponent α of the order condition for the Lévy measure ν is greater than 1, then its mean vector m exists and is finite. If the exponent α is less than or equal to 1, we may have $\int_{|z|>0} |z|\nu(dz) = \infty$. However if the Lévy measure ν is symmetric, the mean vector m exists and is equal to 0.

We will show that the function $b_0(x,t)$ given by (1.15) exists and it is a $\mathcal{C}_b^\infty$-function. We may rewrite the coefficient $g(x,t,z)$ for $|z| \le \delta_0$ as

$$g(x,t,z) = \sum_j z_j \partial_{z_j} g(x,t,z)|_{z=0} + r(x,t,z),$$

where $r(x,t,z)/|z|^2$ is bounded for $|z| \le \delta_0$. Then $r(x,t,z)$ is integrable for $|z| \le \delta_0$ and $\int_{|z|\le\delta_0} r(x,t,z)\nu(dz)$ is a $\mathcal{C}_b^\infty$-function. Therefore the $b_0(x,t)$ exists and is a $\mathcal{C}_b^\infty$-function.

We define a time-dependent vector field $V_0(x,t)$ by (1.16). Then it is also a $\mathcal{C}_b^\infty$-function. Equation (4.2) can be rewritten as

$$d\xi_t = V_0(\xi_t,t)dt + \sum_{j=1}^m V_j(\xi_t,t) \circ dW^j(t) + \lim_{\delta\to 0} \int_{|z|>\delta} g(\xi_{t-},t,z)N(dtdz), \tag{5.2}$$

using the Stratonovitch integral. Families of vector fields $\Sigma_k, k = 0,1,\dots$ are defined in Section 1.

A (*nonstationary*) *uniform Hörmander condition* and *strong Hörmander condition* for jump-diffusion is defined as follows.

Condition (UH). *There exists* $n_0 \in \mathbb{N}$ *and* $c_0 > 0$ *such that*

$$\sum_{k=0}^{n_0} \sum_{V \in \Sigma_k} |v^T V(x,t)|^2 \ge c_0 |v|^2, \quad \forall x,t,v. \tag{5.3}$$

Condition (SH). *There exists* $n_0 \in \mathbb{N}$ *such that the family of time-dependent vector fields* $\bigcup_{k=0}^{n_0} \Sigma_k$ *spans* $\mathbb{R}^d$ *for all* x,t.

We shall consider the relation between modified Hörmander conditions and Hörmander conditions.

Lemma 5.1. *Consider SDE* (5.2)*.*

1) *Suppose that it satisfies Condition* (SH)*. Then for any $M > 1$ there exists $0 < \delta_M < \delta_0$ such that, for any $0 < \delta < \delta_M$, $\bigcup_{k=0}^{2n_0+1} \Upsilon_k^\delta$ spans $\mathbb{R}^d$ for all $|x| \leq M, t$.*
2) *Suppose that it satisfies Condition* (UH)*. Then Condition $(MUH)_\delta$ is satisfied for any $0 < \delta < \delta_0$.*

Proof. We give the proof of the first assertion only, since the second assertion can be verified more easily. In the first step of the proof, we consider another families of time-dependent vector fields Υ_k^0, $k = 0, 1, 2, \ldots$. These are defined as (4.4), replacing $\mathcal{L}^\delta$ of (4.3) by the following $\mathcal{L}^0$:

$$\mathcal{L}^0 V(t) = V_t(t) + [V_0(t), V(t)] + \frac{1}{2}\sum_{j=1}^{m}[V_j(t), [V_j(t), V(t)]]. \tag{5.4}$$

It holds for any $n \in \mathbb{N}$,

$$\text{linear span of } \bigcup_{k=0}^{n} \Sigma_k \subset \text{linear span of } \bigcup_{k=0}^{2n+1} \Upsilon_k^0. \tag{5.5}$$

Indeed, if $V \in \Sigma_0 = \Upsilon_0$, then

$$V_t + [V_0, V] = \mathcal{L}^0 V - \frac{1}{2}\sum_{j=1}^{m}[V_j[V_j, V]] \in \text{linear span of } \bigcup_{k=0}^{3} \Upsilon_k^0.$$

Therefore, linear span of $\Sigma_0 \cup \Sigma_1 \subset$ linear span of $\bigcup_{k=0}^{3} \Upsilon_k^0$. Repeating this argument inductively, we get (5.5).

Now, suppose that Condition (SH) holds. Then $\bigcup_{k=0}^{2n_0+1} \Upsilon_k^0$ spans $\mathbb{R}^d$ for all x, t. Since $\mathcal{L}_\delta V$ converges to $\mathcal{L}_0 V$ uniformly as δ tend to 0, we find that for any $M > 1$ there exists $0 < \delta_M < \delta_0$ such that for any $0 < \delta < \delta_M$ $\bigcup_{k=0}^{2n_0+1} \Upsilon_k^\delta$ spans $\mathbb{R}^d$ for all $|x| \leq M, t$. □

Theorem 5.1. *Consider SDE* (5.2)*, whose Lévy measure has a finite mean vector.* 1) *If it satisfies Conditions* (SH) *and* (G)*, it is nondegenerate.* 2) *If it satisfies Condition* (UH)*, it is nondegenerate.*

Proof. The latter assertion of the theorem is immediate from Theorem 4.1, (2) and Lemma 5.1, (2). We will prove the first assertion, assuming Conditions (SH) and (D). Though property 2) of Lemma 5.1 is slightly weaker than Condition

$(MSH)_\delta$, we can proceed our arguments in the same way as in Theorem 4.1. Indeed, given $N > 1$, k, $h \in \mathbb{N}$, choose M_h as (3.3). There exists $\delta_{M_h} > 0$ such that, for any $0 < \delta < \delta_{M_h}$, $\bigcup_{k=0}^{2n_0+1} \Upsilon_k^\delta$ spans $\mathbb{R}^d$ for any t, $|x| \le M_h$. Therefore there exists $c_1 > 0$ such that

$$\sum_{k=0}^{n_1} \sum_{V \in \Upsilon_k^\delta} |v^T V(x,t)|^2 > c_1 |v|^2, \quad \forall |x| \le M_h, t \tag{5.6}$$

holds. Then the assertion of Lemma 4.2 is valid, i.e., the modified Malliavin's covariance $\Xi(x)$ satisfies (4.8) if $1 < p < (1/2)\beta^{-2n_1}h$. Hence the Malliavin covariance $\Pi(x)$ satisfies (4.9) if $1 < p < (1/2)\beta^{-2n_1}h$. Since we can take any $h \in \mathbb{N}$, inequality (4.9) holds for any $p > 1$. Therefore the SDE is nondegenerate.

We can relax Condition (D) to Condition (G) again by the method of perturbations. □

Finally, we shall introduce other two classes of time-dependent vector fields. Set $\Sigma_0^h = \Sigma_0$ and for $k = 1, 2, \ldots$, set

$$\Sigma_k^h = \{[V_0, V](t), [V_j, V](t), [\tilde{V}_j, V](t);\ j = 1, \ldots, m, V(t) \in \Sigma_{k-1}^h\}.$$

A *homogeneous strong Hörmander's condition* may be defined as follows.

Condition (HSH). *There exists $n_0 \in \mathbb{N}$ such that $\bigcup_{k=0}^{n_0} \Sigma_k^h$ spans $\mathbb{R}^d$ for any x, t.*

If the associated vector fields $V_0, \ldots, V_m$ of the SDE are time independent, it holds $\Sigma_k^h = \Sigma_k$. Therefore Condition (SH) and Condition (HSH) coincide each other. However, if the vector fields are time-dependent, these two conditions are different. Condition (HSH) does not imply hypoelliptic properties, in general. Instead, we will introduce a *restricted homogegenous strong Hörmander condition.* We set $\bar{\Sigma}_0^h = \Sigma_0$ and for $k = 1, 2, \ldots$, set

$$\bar{\Sigma}_k^h = \{[V_j, V](t), [\tilde{V}_j, V](t);\ j = 1, \ldots, m, V(t) \in \hat{\Sigma}_{k-1}^h\}.$$

Condition (RHSH). *There exists $n_0 \in \mathbb{N}$ such that $\bigcup_{k=0}^{n_0} \bar{\Sigma}_k^h$ spans $\mathbb{R}^d$ for any x, t.*

If Condition (RHSH) is satisfied, it satisfies Condition (SH), obviously. Therefore the associated SDE is nondegenerate. See Taniguchi [**20**].

We will give some examples which satisfy Conditiion (SH) but do not satisfy Condition (RHSH).

Example. Let $Z(t)$ be a one dimensional Lévy process represented by

$$Z(t) = c_0 t + c_1 W(t) + c_2 \int_0^t \int_{\mathbb{R}-\{0\}} zN(dsdz), \tag{5.7}$$

where $W(t)$ is a standard Brownian motion and $N(dtdz)$ is a Poisson random measure, independent of $W(t)$, whose Lévy measure ν satisfies the order condition and has a mean vector in the wide sense. We assume $|c_1| + |c_2| > 0$. Then the Lévy process $Z(t)$ is nondegenerate and its law has a C^∞-density.

1) We consider an SDE defined on $\mathbb{R}^2$ as follows.

$$\begin{cases} d\xi_t^1 = dZ(t), \\ d\xi_t^2 = tdZ(t). \end{cases} \tag{5.8}$$

Vector fields of coefficients are given by $V_0(t) = c_0(1,t)$ and $V_1(t) = c_1(1,t)$ and $\tilde{V}_1(t) = c_2(1,t)$. Suppose $c_1 \neq 0$. Then we have $dV_1/dt = c_1(0,1)$ and $[V_0(t), V_1(t)] = 0$. Then it satisfies Condition (UH). Suppose next $c_1 = 0$ and $c_2 \neq 0$. Then we have $d\tilde{V}_1(t)/dt = c_2(0,1)$. Therefore it satisfies Condition (UH). Consequently in any case the SDE is nondegenerate. This example does not satisfy Condition (HSH).

If the second coponent of the equation (5.8) is changed to $d\xi_t^2 = cdZ(t)$, where c is a constant, then the SDE is homogeneous (stationary) but its law is singular with respect to the two dimensional Lebesgue measure.

2) We next consider a homogeneous SDE with drift term such that

$$\begin{cases} d\xi_t^1 = dZ(t), \\ d\xi_t^2 = \xi_t^1 dt. \end{cases} \tag{5.9}$$

In the case where $Z(t)$ is a standard Brownian motion, it is known as Kolmogorov's example of hypoelliptic SDE. We have $V_0 = (c_0, x)$, $V_1 = (c_1, 0)$ and $\tilde{V}_1 = (c_2, 0)$. It holds $[V_1, V_0] = (0, c_1)$ and $[\tilde{V}_1, V_0] = (0, c_2)$. Therefore it satisfies Condition (UH) and it is nondegenerate.

We saw that Condition (UH) is stronger than Condition $(MUH)_\delta$. The uniform Hörmander condition is stated in terms of a drift vector fields V_0 and vector fields V_j, $\tilde{V}_j$, $j = 1, \dots, m$ only. It does not care jump coefficient $g(x,t,z)$ if $\partial_z g(x,t,z)|_{z=0} = 0$. On the other hand, the modified uniform Hörmander condition may take care of such jump coefficients. Thus Condition (UH) is truly stronger than Condition $(MUH)_\delta$. We will give an example.

EXAMPLE. 3) We consider a homogeneous SDE on $\mathbb{R}^2$ for $\xi_t = (\xi_t^1, \xi_t^2)$;

$$\begin{cases} d\xi_t^1 = dZ(t), \\ d\xi_t^2 = \xi_t^1 \int_{|z|\le 1} z^2 N(dtdz). \end{cases} \tag{5.10}$$

Note that the drift part $\xi_t^1 dt$ in equation (5.9) is replaced by a pure jump part $\xi_t^1 \int_{|z|\le 1} z^2 N(dtdz)$ in (5.10). We will show that the equation is again nondegenerate. We frirst consider the case $c_1 \neq 0$. We have $V_1 = (c_1, 0)$ and $\mathcal{L}^\delta V_1 = (0, -c_1 \int_{|z|\le\delta} z^2 \nu(dz))$. Hence V_1 and $\mathcal{L}^\delta V_1$ span $\mathbb{R}^2$ for any x, t so that the SDE satisfies $(MUH)_\delta$ for any $0 < \delta < \delta_0$. Thus it is nondegenerate. However, since $\partial_z g(x,t,z)|_{z=0} = 0$, $\Sigma_0 = \{V_1\}$ and $\Sigma_k = \{0\}$ for $k = 1, 2, \ldots$. Therefore it does not satisfy Condition (UH). The same fact can also be shown for the case $c_2 \neq 0$.

6. SDE lacking order conditions and continuous SDE.

In this section we do not assume the order condition for the Lévy measure. A typical case is that the Lévy measure is a bounded measure. The Malliavin covariance of $F \in \boldsymbol{D}_\infty^d$ is defined by

$$\hat{\Pi}^F = \int_0^T D_t F (D_t F)^T dt,$$

where D_t is the Malliavin-Shigekawa's derivative. F is called *nondegenerate* if $\hat{\Pi}^F$ are invertible a.s. and for any $p > 1$, $E[(v^T \hat{\Pi}^F v)^{-p}]$ is bounded for $v \in S_{d-1}$. We will refer an estimate of polynomial decay of the weighted characteristic function of a nondegenerate functional ([**8**, Theorem 2.1]). For any $n \in \mathbb{N}$, there exists a positive constant C_n such that the inequality

$$|E[e^{i(v,F)} G]| \le C_n (1 + |v|^2)^{-n/2} |G|_{0,n,2^{n+2}} \hat{\Theta}_n(F) \tag{6.1}$$

holds for any nondegenerate $F \in \boldsymbol{D}_\infty^d$ and $G \in \boldsymbol{D}_\infty$, where

$$\hat{\Theta}_n(F) = |F|_{0,n+1,2^{n+2}}^n \big|(v^T \hat{\Pi}^F v)^{-1}\big|_{0,n,2^{n+2}}^n. \tag{6.2}$$

Let us consider again SDE (1.1). We assume that coefficients of the SDE satisfies the same condition as in Section 2. Let $\xi_{s,t}(x)$ be the solution starting from x at time s. It is a smooth functional. Its Malliavin covariance denoted by $\hat{\Pi}(x)$ is written by

$$\hat{\Pi}(x) = \int_s^t \nabla\xi_{u,t}(x) A(\xi_{s,u}(x), u) \nabla\xi_{u,t}(x)^T du, \tag{6.3}$$

where $A(x,t) = \sigma(x,t)\sigma(x,t)^T$.

The SDE is called *nondegenerate*, if the family of solutions $\{\xi_{s,t}(x); |x| \le N\}$ is uniformly nondegenerate for any $N > 1, s < t$, i.e., the inequality

$$\sup_{|x|\le N} \sup_{v \in S_{d-1}} E[(v^T \hat{\Pi}(x) v)^{-p}] < \infty, \quad \forall p > 1 \tag{6.4}$$

holds for any $N > 1$, $s < t$. If an SDE is nondegenerate, its weighted characteristic function is of polynomial decay: For any $n \in \mathbb{N}$ and $N > 1$, there exists a positive constant $C = C_{s,t,N}$ such that

$$\sup_{|x|\le N} \big|E[e^{i(v,\xi_{s,t}(x))}G]\big| \le C(1+|v|^2)^{-n/2} |G|_{0,n,2^{n+2}}, \quad \forall v$$

holds. Then the following theorem can be verified similarly as Theorem 2.1.

Theorem 6.1. *Any nondegenerate SDE has hypoelliptic properties* I *and* II.

We showed in [**8**] that if the matrix $A(x,t)$ is uniformly positive, i.e., the inequality (1.5) holds for $C(x,t) = A(x,t)$, then the SDE is nondegenerate. Further, we can show that if $A(x,t)$ is positive definite for any x, t and Condition (G) is satisfied, the SDE is nondegenerate, similarly as in Section 3.

We next assume that coefficients of the SDE are smooth with respect to t as in Section 4. If ν does not satisfy the order condition, $\int_{|z|\le\rho} |z|^2 \nu(dz) \prec \rho^\alpha$ holds for any $\alpha \in (1,2)$. Then it has a finite mean vector m. Hence equation (1.1) is rewritten as (5.2), using time-dependent vector fields $V_0(t), V_1(t), \dots, V_m(t)$. Given $0 \le \delta < \delta_0$, let $\mathcal{L}^\delta$ be the linear map of time dependent vector fields defined by (4.3). We introduce families of time-dependent vector fields $\hat{\Upsilon}_k^\delta$, $k = 0, 1, 2, \dots$ by induction as

$$\begin{aligned} &\hat{\Upsilon}_0^\delta = \{V_1(t), \dots, V_m(t)\}, \\ &\hat{\Upsilon}_k^\delta = \big\{\mathcal{L}^\delta V(t), [V_j(t), V(t)];\ j = 1, \dots, m, V \in \hat{\Upsilon}_{k-1}^\delta\big\}. \end{aligned} \tag{6.5}$$

Then, the modified uniform Hörmander condition and the modified strong Hörmander condition given in previous sections, can also be defined in the present setting, replacing Υ_k^δ to $\hat{\Upsilon}_k^\delta$. We will denote these conditions as $(M\hat{U}H)_\delta$ and $(M\hat{S}H)_\delta$, respectively. We can show similarly as in Section 4 that the SDE is nondegenerate if either Condition $(M\hat{U}H)_\delta$ for some $0<\delta<\delta_0$ or the pair of Condition (G) and Condition $(M\hat{S}H)_\delta$ for some $0<\delta<\delta_0$ is satisfied. Here, we should use Theorem 7.2 instead of Theorem 7.1. Details are left to the reader.

Further, we set

$$\hat{\Sigma}_0 = \{V_j(t), j=1,\dots,m\},$$
$$\hat{\Sigma}_k = \big\{V_t(t) + [V_0(t),V(t)], [V_j(t),V(t)];\ j=1,\dots,m, V(t)\in\hat{\Sigma}_{k-1}\big\}.$$

Replacing Σ_k by $\hat{\Sigma}_k$ in Conditions (UH) and (SH), we define Condition $(\hat{UH})$ and Condition $(\hat{SH})$, respectively. Then similarly as in the previous section, we have:

Theorem 6.2. *Consider SDE* (4.2) *lacking the order condition. It is nondegenerate if any one of the following four conditions is satisfied.*

1) *Condition* $(M\hat{U}H)_\delta$ *for some* $0<\delta<\delta_0$.
2) *Conditions* $(M\hat{S}H)_\delta$ *for some* $0<\delta<\delta_0$ *and* (G).
3) *Condition* $(\hat{UH})$.
4) *Conditions* $(\hat{SH})$ *and* (G).

Example. 4) We give a hypoelliptic example, which satisfies Condition $(M\hat{U}H)_1$, but does not satisfy Condition $(\hat{UH})$. Consider a two dimensional SDE

$$\begin{cases} d\xi_t^1 = dZ(t), \\ d\xi_t^2 = \xi_t^1 dN(t), \end{cases} \tag{6.6}$$

where $Z(t)$ is a one dimensional Lévy process of the form (5.7) such that $c_1\neq 0$, and $N(t)$ is Poisson process independent of $Z(t)$. Note that the drift part $\xi_t^1 dt$ of (5.9) is replaced by the pure jumps $\xi_t^1 dN(t)$. Drift and diffusion vector fields are given by $V_0=(c_0,0)$ and $V_1=(c_1,0)$, respectively. It holds $\mathcal{L}^1 V_1 = (0,-c_1)$, similarly as in Example 3) in Section 5. Then it satisfies Condition $(M\hat{U}H)_1$ and hence it is nondegenerate.

6.1. Continuous SDE.

Finally we will consider a continuous SDE. We may relax Condition $(\hat{SH})$ to the following Hörmander condition.

Condition $(\hat{H})$. $\bigcup_{k=0}^\infty \hat{\Sigma}_k$ *spans* $\mathbb{R}^d$ *for all* x,t.

Theorem 6.3. *Any continuous SDE satisfying Condition* $(\hat{H})$ *is nondegenerate.*

Proof. Suppose Condition $(\hat{H})$. Then for any $N > 1$, there exists $n_1' \in \mathbb{N}$ and $c_1' > 0$ such that

$$\sum_{k=0}^{n_1'} \sum_{V \in \hat{\Upsilon}_k^\delta} |v^T V(x,t)|^2 \geq c_1' |v|^2, \quad \forall |x| \leq N+1, t.$$

Associated with the above N, let $\tau(x)$ be the stopping time such that

$$\tau(x) = \inf\{t > 0;\ |\xi_t(x)| \geq N+1\} \wedge T.$$

Then, for any $h > 1$ there exists a positive constant $C_h > 0$ such that the inequality $\sup_{|x| \leq N} P(\tau(x) < \epsilon) \leq C_h \epsilon^h$ holds for $0 < \epsilon < 1$. It is close to the inequality of stopping time $T_h(x)$ given by (3.4). Then, using the stopping time $\tau(x)$ instead of $T_h(x)$, we can proceed in the argument in the proof of Lemma 4.2. Then we obtain $\sup_{|x| \leq N} \sup_{v \in S_{d-1}} E[(v^T \hat{\Xi}(x) v)^{-2p}] < \infty$ if $p < (1/2)\beta^{-2n_1'} h$, where β is a positive constant greater than 8. Therefore, the SDE is nondegenerate. $\square$

7. Appendix: Another estimate of Norris' type.

In [**6**], we discussed Norris' type estimate for a semimartingale with jumps. The estimate was complicated, since it was intended for the direct application to canonical SDE with jumps. In this section we will give two less complicated estimates for it (Theorems 7.1 and 7.2). These two estimates are used in Sections 4 and 6, respectively for the proof of the nondegenerate properties of SDE's.

We consider a d-dimensional semimartingale $Y^\gamma(t)$, $t \in \mathbb{T}$ with parameter γ defined by

$$\begin{aligned} Y^\gamma(t) = y^\gamma &+ \int_0^t a^\gamma(s) ds + \sum_j \int_0^t f_j^\gamma(s) dW^j(s) \\ &+ \int_0^t \int_{|z| \leq \delta} g^\gamma(s,z) \tilde{N}(dsdz) + \int_0^t \int_{|z| > \delta} g^\gamma(s,z) N(dsdz), \end{aligned} \tag{7.1}$$

where $a^\gamma(s)$, $f_j^\gamma(s)$ and $g^\gamma(s,z)$ are d-dimensional left continuous predictable processes, continuous with respect to parameter $\gamma \in \Gamma$, where Γ is a compact space. We set $f^\gamma(s) = (f_1^\gamma(s), \ldots, f_m^\gamma(s))$. We assume that $g^\gamma(s,z)$ is twice continuously differentiable with respect to z and $g^\gamma(s,0) = 0$. We denote by $\partial g^\gamma(s) = (\partial_1 g^\gamma(s), \ldots, \partial_m g^\gamma(s))$, where $\partial_j g^\gamma(s)$ is the partial derivatives of $g^\gamma(s,z)$ at $z = 0$ with respect to z_j.

We assume further that the drift coefficient $a^\gamma(t)$ is a semimartingale represented by

$$a^\gamma(t) = a^\gamma + \int_0^t b^\gamma(s)ds + \sum_j \int_0^t e_j^\gamma(s)dW^j(s)$$
$$+ \int_0^t \int_{|z|\le\delta} h^\gamma(s,z)d\tilde{N} + \int_0^t \int_{|z|>\delta} h^\gamma(s,z)dN, \tag{7.2}$$

where $b^\gamma(s), e_i^\gamma(s), h^\gamma(s,z)$, $s \ge 0$ are processes which have the same properties as those of $a^\gamma(s), b^\gamma(s)$ and $g^\gamma(s,z)$ in (7.1), respectively. We set

$$\theta^\gamma = \|a^\gamma\|^2 + \|b^\gamma\|^2 + \|f^\gamma\|^2 + \|e^\gamma\|^2 + \|g^\gamma\|^2 + \|h^\gamma\|^2$$
$$+ \|\partial g^\gamma\|^2 + \|\partial h^\gamma\|^2 + \|r^\gamma/|z|^2\|^4 + \|s^\gamma/|z|^2\|^4, \tag{7.3}$$

where $r^\gamma = g^\gamma - \sum_j z_j \partial_j g^\gamma$, $s^\gamma = h^\gamma - \sum_j z_j \partial_j h^\gamma$, $\|f^\gamma\| = \sup_{0\le t\le T} |f^\gamma(t)|$ and $\|g^\gamma\| = \sup_{0\le t\le T, |z|\le\delta} |g^\gamma(t,z)|$ etc.

We will first assume that the Lévy measure ν satisfy the order condition with exponent $0<\alpha<2$. Let B_ρ, $0<\rho<\rho_0$ be the correlation matrices of the Lévy measure ν defined in Section 1. Let B be a a lower bound of B_ρ, $0<\rho<\rho_0$ and $\sqrt{B} = (\tau_{ij})$ be a square root of B. We set

$$\tilde{g}^\gamma(t) = \partial g^\gamma(t)\sqrt{B}. \tag{7.4}$$

Let β be a positive number such that $\beta > \max\{4/(2-\alpha), 8\}$. We shall consider two events for $\epsilon > 0$:

$$A^\gamma(\epsilon) = \left\{ \int_0^{T_0} |Y_{t-}^\gamma|^2 dt < \varepsilon^\beta \right\}, \tag{7.5}$$

$$B^\gamma(\epsilon) = \left\{ \int_0^{T_0} \{|a^\gamma(t)|^2 + |f^\gamma(t)|^2 + |\tilde{g}^\gamma(t)|^2\} dt < \varepsilon \right\}. \tag{7.6}$$

We show that the probability where both $A^\gamma(\epsilon)$ and $B^\gamma(\epsilon)^c$ occur simultaneously is small if ϵ is small.

THEOREM 7.1. *The order condition is assumed for the Lévy measure. Assume that* $\sup_\gamma E[(\theta^\gamma)^p] < \infty$ *holds for any* $p > 1$. *Then for any* $p > 1$, *there exists a positive constant* C_p *such that the inequality*

$$\sup_{\gamma} P(A^{\gamma}(\epsilon) \cap B^{\gamma}(\epsilon)^c) < C_p \epsilon^p \tag{7.7}$$

holds for any semimartingale $Y^{\gamma}(t)$ *represented by* (7.1) *and* (7.2) *and for any* $\epsilon > 0$.

In order to prove the above theorem, we need the following estimate due to Komatsu-Takeuchi [**5**].

KOMATSU-TAKEUCHI'S ESTIMATE ([**5**, Theorem 3]). Let υ be an arbitrary number such that $0 < \upsilon < 1/4$. There exist a positive random variable $\mathcal{E}(\lambda, \gamma)$ with $E[\mathcal{E}(\lambda, \gamma)] \leq 1$ and positive constants C, C_0, C_1, C_2 independent of λ, γ such that the inequality

$$\begin{aligned} &\lambda^4 \int_0^{T_0} |Y^{\gamma}(t)|^2 \wedge \frac{1}{\lambda^2} dt + \lambda^{-\upsilon} \log \mathcal{E}(\lambda, \gamma) + C \\ &\geq C_0 \lambda^{1-4\upsilon} \int_0^{T_0} |a^{\gamma}(t)|^2 dt + C_1 \lambda^{2-2\upsilon} \int_0^{T_0} |f^{\gamma}(t)|^2 dt \\ &\quad + C_2 \lambda^{2-2\upsilon} \int_0^{T_0} \int_{\mathbb{R}^m} |g^{\gamma}(t, z)|^2 \wedge \frac{1}{\lambda^2} dt \nu(dz) \end{aligned} \tag{7.8}$$

holds on the set $A = \{\theta^{\gamma} \leq \lambda^{2\upsilon}\}$ for all $\lambda > 1$ and Y^{γ}.

PROOF OF THEOREM 7.1. We will choose positive constants $0 < \upsilon < 1/8$ and $0 < \eta < 1$ such that

$$2 - \alpha(1 + \eta) - 2\upsilon > \frac{4}{\beta}.$$

Then choose $r > 1$ such that

$$\frac{\beta}{4} > r > \frac{1}{2 - \alpha(1 + \eta) - 2\upsilon} \vee \frac{1}{1 - 4\upsilon}.$$

We first consider the last term of (7.8). We expand $g^{\gamma}(t, z)$ around the origin as $g^{\gamma}(t, z) = \partial g^{\gamma}(t) z + r^{\gamma}(t, z)$. Note the inequality

$$(a + b)^2 \wedge \frac{1}{\lambda^2} \geq \frac{1}{2}\left(a^2 \wedge \frac{1}{\lambda^2}\right) - b^2.$$

Then it holds for any $0 < \kappa < \lambda$,

$$
\begin{aligned}
&\int_{\mathbb{R}_0^m}\left(|g^\gamma(t,z)|^2\wedge\frac{1}{\lambda^2}\right)\nu(dz)\\
&\geq\int_{|z|<\kappa/\lambda}\left(|g^\gamma(t,z)|^2\wedge\frac{1}{\lambda^2}\right)\nu(dz)\\
&\geq\frac{1}{2}\int_{|z|<\kappa/\lambda}\left(\left|\partial g^\gamma(t)\frac{z}{|z|}\right|^2\wedge\frac{1}{|z|^2\lambda^2}\right)|z|^2\nu(dz)\\
&\quad-\left(\frac{\kappa}{\lambda}\right)^2\int_{|z|<\kappa/\lambda}\left(\frac{r^\gamma(t,z)}{|z|^2}\right)^2|z|^2\nu(dz)\\
&\geq\frac{1}{2}\varphi\left(\frac{\kappa}{\lambda}\right)\int_{|z|<\kappa/\lambda}\left(\left|\partial g^\gamma(t)\frac{z}{|z|}\right|^2\wedge\frac{1}{\kappa^2}\right)\bar{\mu}_{\kappa/\lambda}(dz)-\varphi\left(\frac{\kappa}{\lambda}\right)\left(\frac{\kappa}{\lambda}\right)^2\left\|\frac{r^\gamma}{|z|^2}\right\|^2,
\end{aligned}
\tag{7.9}
$$

where $\varphi(\rho)=\int_{|z|\leq\rho}|z|^2\nu(dz)$ and $\bar{\mu}_\epsilon(dz)$ is a probability measure such that $\bar{\mu}_\epsilon(dz)=(|z|^2/\varphi(\epsilon))1_{|z|<\epsilon}\nu(dz)$.

Now set $\lambda=\varepsilon^{-r}$ and $\kappa=\varepsilon^{\eta r}$. Then $\kappa/\lambda=\varepsilon^{(1+\eta)r}$ and $\varphi(\kappa/\lambda)\geq C_4\varepsilon^{\alpha(1+\eta)r}$ by the order condition for the function φ. Since $\|r^\gamma/|z|^2\|^2<1/\kappa<\lambda^{2\upsilon}$ holds on the set $\{\theta^\gamma\leq\lambda^{2\upsilon}\}$, the last term of (7.9) dominates

$$C_4\varepsilon^{\alpha(1+\eta)r}\left|\partial g^\gamma(t)B_{\epsilon^{(1+\eta)r}}\partial g^\gamma(t)^T\right|-C_5\epsilon^{(2+\alpha)(1+\eta)r}$$

on the set $\{\theta^\gamma\leq\lambda^{2\upsilon}\}$. Note $B_{\epsilon^{(1+\eta)r}}\geq B$. Then Komatsu-Takeuchi's inequality (7.8) implies

$$
\begin{aligned}
&\varepsilon^{-4r}\int_0^{T_0}|Y^\gamma(t)|^2\wedge\varepsilon^{2r}dt+\varepsilon^{\upsilon r}\log\mathcal{E}(\varepsilon^{-r},\gamma)+C\\
&\quad\geq C_0\varepsilon^{-r(1-4\upsilon)}\int_0^{T_0}|a^\gamma(t)|^2dt+C_1\varepsilon^{-r(2-2\upsilon)}\int_0^{T_0}|f^\gamma(t)|^2dt\\
&\quad\quad+C_2C_4\varepsilon^{-r(2-2\upsilon)}\epsilon^{\alpha(1+\eta)r}\int_0^{T_0}|\partial g^\gamma(t)B\partial g^\gamma(t)^T|dt\\
&\quad\quad-C_2C_5\varepsilon^{-r(2-2\upsilon)}\epsilon^{(2+\alpha)(1+\eta)r}.
\end{aligned}
$$

Set $\rho=r\min\{(1-4\upsilon),(2-2\upsilon)-\alpha(1+\eta)\}-1$. Then it holds $\rho>0$. Since $|\partial g^\gamma(t)B\partial g^\gamma(t)^T|=|\tilde{g}^\gamma(t)|^2$, the above inequality yields

$$\varepsilon^{-4r}\int_0^{T_0}|Y_t^\gamma|^2\wedge\varepsilon^{2r}dt+\varepsilon^{\upsilon r}\log\mathcal{E}(\varepsilon^{-r},\gamma)+C$$
$$\geq C_6\varepsilon^{-(\rho+1)}\int_0^{T_0}\big\{|a^\gamma(t)|^2+|f^\gamma(t)|^2+|\tilde{g}^\gamma(t)|^2\big\}dt-C_7\epsilon^{-(\rho+1)}\epsilon^{r'} \tag{7.10}$$

on the set $\{\theta^\gamma\leq\epsilon^{-\upsilon r}\}$, where $C_6=\min\{C_0,C_1,C_2C_4\}$, $C_7=C_2C_5$ and $r'>1$.

Now we define two events by

$$E_1^\gamma=\big\{\theta^\gamma>\epsilon^{-\upsilon r}\big\},$$
$$E_2^\gamma=\big\{\theta^\gamma\leq\epsilon^{-\upsilon r}\big\}\bigcap\bigg\{\int_0^{T_0}|Y_{t-}^\gamma|^2\wedge\varepsilon^{2r}dt<\varepsilon^\beta\bigg\}$$
$$\bigcap\bigg\{\int_0^{T_0}\big(|a^\gamma(t)|^2+|f^\gamma(t)|^2+|\tilde{g}^\gamma(t)|^2\big)dt>\varepsilon\bigg\}.$$

Then it holds

$$A^\gamma(\varepsilon)\cap B^\gamma(\varepsilon)^c\subset E_1^\gamma\cup E_2^\gamma.$$

Therefore, the probability of (7.7) is dominated by $P(E_1^\gamma)+P(E_2^\gamma)$. We shall get estimates of $P(E_i^\gamma), i=1,2$. In view of our assumption of the theorem, the first one is estimated as

$$\sup_\gamma P(E_1^\gamma)\leq\varepsilon^pE\Big[(\sup_\gamma\theta^\gamma)^{p/\upsilon r}\Big]\leq c_p\varepsilon^p. \tag{7.11}$$

For the estimate of $P(E_2^\gamma)$, we remark that (7.10) implies

$$E_2^\gamma\subset\big\{\mathcal{E}(\varepsilon^{-r},\gamma)^{\varepsilon^{\upsilon r}}\geq\exp\big(-\varepsilon^{\beta-4r}+C_6\varepsilon^{-\rho}-C_7\epsilon^{-\rho}\epsilon^{r'-1}-C\big)\big\}.$$

Therefore, by Chebyschev's inequality

$$P(E_2^\gamma)\leq e^C\exp\big(\varepsilon^{\beta-4r}+C_7\epsilon^{-\rho}\epsilon^{r'-1}-C_6\varepsilon^{-\rho}\big)E\big[\mathcal{E}(\varepsilon^{-r},\gamma)^{\varepsilon^{\upsilon r}}\big].$$

We have $\varepsilon^{\beta-4r}+C_7\epsilon^{-\rho}\epsilon^{r'-1}<(C_6/2)\varepsilon^{-\rho}$ holds for $\varepsilon<\varepsilon_0$ with some $\epsilon_0>0$. Further it holds $E[\mathcal{E}(\varepsilon^{-r},\gamma)^{\varepsilon^{\upsilon r}}]\leq 1$ since $0<\varepsilon^{\upsilon r}\leq 1$ for small ϵ. Therefore for any $p>1$,

$$P(E_2^\gamma) \le e^C \exp\left(-\frac{C_6}{2}\varepsilon^{-\rho}\right) \le c_p'\varepsilon^p \tag{7.12}$$

for any $\varepsilon < \varepsilon_0$ and γ. Two inequalities (7.11) and (7.12) prove (7.7). □

Finally we will consider the case where the Lévy measure may not satisfy the order condition. Instead of θ^γ given by (7.3), we consider the following

$$\theta^\gamma = \|a^\gamma\|^2 + \|b^\gamma\|^2 + \|f^\gamma\|^2 + \|e^\gamma\|^2 + \|g^\gamma\|^2 + \|h^\gamma\|^2.$$

Let β be a positive number such that $\beta > 8$. We shall consider two events for $\epsilon > 0$:

$$A^\gamma(\epsilon) = \left\{ \int_0^{T_0} |Y^\gamma(t)|^2 dt < \varepsilon^\beta \right\}, \tag{7.13}$$

$$B^\gamma(\epsilon) = \left\{ \int_0^{T_0} \{|a^\gamma(t)|^2 + |f^\gamma(t)|^2\} dt < \varepsilon \right\}. \tag{7.14}$$

THEOREM 7.2. *Assume* $\sup_\gamma E[(\theta^\gamma)^p] < \infty$ *holds for any* $p > 1$. *Then for any* $p > 1$, *there exists a positive constant* C_p *such that the inequality*

$$\sup_\gamma P(A^\gamma(\epsilon) \cap B^\gamma(\epsilon)^c) < C_p \epsilon^p \tag{7.15}$$

holds for any semimartingale $Y^\gamma(t)$ *represented by* (7.1) *and* (7.2) *and for any* $\epsilon > 0$.

PROOF. Neglecting the last term of Komatsu-Takeuchi estimate (7.8), the inequality

$$\begin{aligned} &\lambda^4 \int_0^{T_0} |Y^\gamma(t)|^2 \wedge \frac{1}{\lambda^2} dt + \lambda^{-\upsilon} \log \mathcal{E}(\lambda, \gamma) + C \\ &\quad \ge C_0 \lambda^{1-4\upsilon} \int_0^{T_0} |a^\gamma(t)|^2 dt + C_1 \lambda^{2-2\upsilon} \int_0^{T_0} |f^\gamma(t)|^2 dt \end{aligned} \tag{7.16}$$

holds on the set $A = \{\theta^\gamma \le \lambda^{2\upsilon}\}$ for all $\lambda > 1$ and Y^γ. Using this inequality, we can get the inequality (7.15), similarly as in the proof of Theorem 7.1. □

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