J. Math. Soc. Japan
Vol. 65, No. 3 (2013) pp. 787–796
doi: 10.2969/jmsj/06530787

A simple improvement of a differentiable classification result for complete submanifolds

By Ezequiel R. Barbosa

(Received Nov. 18, 2011)

Abstract. We consider M^n, $n \geq 3$, an n-dimensional complete submanifold of a Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$. We prove that if for all point $x \in M^n$ the following inequality is satisfied

$$S \leq \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-1},$$

with strictly inequality at one point, where S and H denote the squared norm of the second fundamental form and the mean curvature of M^n respectively, then M^n is either diffeomorphic to a spherical space form or the Euclidean space $\mathbb{R}^n$. In particular, if M^n is simply connected, then M^n is either diffeomorphic to the sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.

1. Introduction and main result.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$. Here we assume that $n \geq 2$. For an arbitrary fixed point $x \in M^n$, we choose an orthonormal local frame field $\{e_1, \ldots, e_{n+p}\}$ in $\overline{M}^{n+p}$ such that $\{e_1, \ldots, e_n\}$ is tangent to M^n. Let $\{\omega_1, \ldots, \omega_{n+p}\}$ the dual frame field of $\{e_1, \ldots, e_{n+p}\}$. Let Rm and $\overline{Rm}$ be the Riemannian curvature tensors of M^n and $\overline{M}^{n+p}$ respectively, and h the second fundamental form of M^n. Then

$$Rm = \sum_{i,j,k,l=1}^{n} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$

$$\overline{Rm} = \sum_{i,j,k,l=1}^{n+p} \overline{R}_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l$$

and

2010 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C20, 53C44.
Key Words and Phrases. submanifold, second fundamental form, differentiable sphere theorem, Ricci flow, sectional curvature.

$$h = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_\alpha .$$

The Riemann curvature tensors Rm, $\overline{R}m$ and the second fundamental form h are related by the Gauss equation:

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_{\alpha=n+1}^{n+p} \left(h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha} \right). \tag{1}$$

The squared norm S of the second fundamental form and the mean curvature H of M^n are given by

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha} \right)^2$$

and

$$H = \frac{1}{n} \left| \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^{n} h_{ii}^{\alpha} e_\alpha \right| .$$

Suppose, for a moment, $(\overline{M}^{n+p}, \overline{g})$ has constant curvature c. From Gauss equation, we obtain

$$R_g = n(n-1)c + n^2 H^2 - S,$$

where R_g is the scalar curvature of (M^n, g) and g is the induced metric of $\overline{g}$. Hence, in this case, we find that

$$R_g \geq \frac{n^2 (n-2) H^2}{n-1} + (n+1)(n-2)c$$

if and only if

$$S \leq \frac{n^2 H^2}{n-1} + 2c.$$

Note that if $n = 2$, then the Gauss curvature is nonnegative if and only if $S \leq (n^2 H^2/(n-1)) + 2c$. In 1936, J. J. Stoker [**10**] proved the following theorem.

Theorem 1.1 (J. J. Stoker - 1936). *Assume that M^2 is a complete submanifold of the Euclidean space $\mathbb{R}^3$. If the Gaussian curvature K is positive, the M^2 is either diffeomorphic to the sphere $\mathbb{S}^2$ or the Euclidean space $\mathbb{R}^2$.*

In 1971, do Carmo and Lima [**5**] improved the above theorem:

Theorem 1.2 (do Carmo, Lima - 1971). *Assume that M^2 is a complete submanifold of the Euclidean space $\mathbb{R}^3$. If the Gaussian curvature K is nonnegative, and positive at one point, then M^2 is either diffeomorphic to the sphere $\mathbb{S}^2$ or the Euclidean space $\mathbb{R}^2$.*

One can find some related results about the geometry and topology of manifolds with a curvature satisfying some strictly inequality at one point, e.g., in [**8**], [**9**], [**12**]. In this paper, we want to discuss a result like Theorem 1.2 that improves a result like Theorem 1.1. Before state our main result, let we make a definition. Denote by $K_x(\pi)$ the sectional curvature of M^n for tangent 2-plane $\pi \subset T_xM^n$ at point $x \in M^n$, $\overline{K}_x(\pi)$ the sectional curvature of $\overline{M}^{n+p}$ for tangent 2-plane $\pi \subset T_x\overline{M}^{n+p}$ at point $x \in \overline{M}^{n+p}$. Set

$$\overline{K}_{\min}(x) := \min_{\pi \subset T_x\overline{M}^{n+p}} \overline{K}_x(\pi)$$

and

$$\overline{K}_{\max}(x) := \max_{\pi \subset T_x\overline{M}^{n+p}} \overline{K}_x(\pi).$$

The following theorem was proved by Hong-Wei Xu and Juan-Ru Gu [**11**] (see Theorem 1.1 in [**11**]).

Theorem 1.3 (Hong-Wei Xu and Juan-Ru Gu - 2010). *Assume that M^n is a complete submanifold and, for all point $x \in M$,*

$$S < \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-1}. \tag{2}$$

Then M^n is either diffeomorphic to a spherical space form or the Euclidean space $\mathbb{R}^n$. In particular, if M^n is simply connected, then M^n is either diffeomorphic to the sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.

The above result is interesting because when M^n is a compact submanifold of codimension zero, Theorem 1.3 reduces to the differentiable pinching theorem of Brendle and Schoen [**4**]. Another hand, L. Ni and B. Wilking, combining Theorem

3.1 and Theorem 3.2 in [**8**], provide a classification, up to diffeomorphism, of all compact Riemannian manifolds that are almost strictly (1/4)-pinched in the pointwise sense. We define almost strictly (1/4)-pinched manifold in the pointwise sense, in the following way: we say that (M, g) is almost strictly (1/4)-pinched in the pointwise sense if $0 \leq K(\pi_1) \leq 4K(\pi_2)$ for all points $x \in M$ and all two-dimensional planes π_1, $\pi_2 \subset T_xM$, and there exists a point $p \in M$ such that $K(\pi_1) < 4K(\pi_2)$ for all two-dimensional planes π_1, $\pi_2 \subset T_pM$.

Therefore it is natural to ask whether the Theorem 1.3 can also be achieved by improving the inequality (2). Our main result is the following:

THEOREM 1.4. *Assume that M^n is a complete submanifold and, for all point $x \in M^n$,*

$$S \leq \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-1},$$

with strictly inequality at one point, then M^n is either diffeomorphic to a spherical space form or the Euclidean space $\mathbb{R}^n$. In particular, if M^n is simply connected, then M^n is either diffeomorphic to the sphere $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.

Theorem 1.4 has been presented at the Workshop "Geometric Analysis", IMPA - Institut Fourier at Rio de Janeiro - Brazil, November 16–26, 2010. Note that Theorem 1.4 improves the Theorem 1.3 in the sense that inequality (2) can be assumed equality and the same conclusion of Theorem 1.3 follows.

ACKNOWLEDGMENTS. This work is supported by CNPq-Brazil.

2. Proof of Theorem 1.4.

We divide the proof in a few steps. The first one is exactly the Lemma 4.1 in [**11**]. For convenience, we will post here the proof of that lemma.

Step 1: Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$, and π a tangent 2-plane in T_xM^n at point $x \in M^n$. Choose an orthonormal two-frame $\{e_1, e_2\}$ at x such that $\pi = \operatorname{span}\{e_1, e_2\}$. Then

$$K_x(\pi) \geq \frac{1}{2}\left(2\overline{K}_{\min}(x) + \frac{n^2H^2}{n-1} - S\right) + \sum_{\alpha=n+1}^{n+p} \sum_{j>i,(i,j)\neq(1,2)}^{n} (h_{ij}^{\alpha})^2.$$

PROOF. We extend the orthonormal two-frame $\{e_1, e_2\}$ to an orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $\{e_1, \ldots, e_n\}$ are tangent to M^n. Setting $S_\alpha = \sum_{i,j=1}^n (h_{ij}^\alpha)^2$, we have

$$\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 = (n-1)\left[\sum_{i=1}^n \left(h_{ii}^\alpha\right)^2 + \sum_{i\neq j}^n \left(h_{ij}^\alpha\right)^2 + \frac{1}{n-1}\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 - S_\alpha\right]. \quad (3)$$

Note that

$$\begin{aligned}\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 &\leq (n-1)\left[\left(h_{11}^\alpha + h_{22}^\alpha\right)^2 + \sum_{i>2}^n \left(h_{ii}^\alpha\right)^2\right] \\ &= (n-1)\left[\sum_{i=1}^n \left(h_{ii}^\alpha\right)^2 + 2h_{11}^\alpha h_{22}^\alpha\right].\end{aligned}$$

This together with (3) implies

$$2h_{11}^\alpha h_{22}^\alpha \geq \sum_{i\neq j}^n \left(h_{ij}^\alpha\right)^2 + \frac{1}{n-1}\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 - S_\alpha. \quad (4)$$

From the Gauss equation (1) and (4) we get

$$\begin{aligned}K(\pi) &= \overline{R}_{1212} + \sum_{\alpha=n+1}^{n+p} \left[h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2\right] \\ &\geq \sum_{\alpha=n+1}^{n+p}\left[\sum_{j>2}^n \left(h_{1j}^\alpha\right)^2 + \sum_{j>2}^n \left(h_{2j}^\alpha\right)^2 + \sum_{j>i>2}^n \left(h_{ij}^\alpha\right)^2\right] + \frac{1}{2}\left(\frac{n^2H^2}{n-1} - S\right) + \overline{K}_{\min} \\ &= \frac{1}{2}\left(2\overline{K}_{\min} + \frac{n^2H^2}{n-1} - S\right) + \sum_{\alpha=n+1}^{n+p}\sum_{j>i,(i,j)\neq(1,2)}^n \left(h_{ij}^\alpha\right)^2. \qquad \square\end{aligned}$$

Next step, we use the same idea of the proof of Theorem 4.1 in [**11**]. We just make small changes in the statement of that theorem that are important to the step 3.

Step 2: Let M^n be an $n(\geq 4)$-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$. Then, for all point $x \in M^n$, all orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_xM^n$, and all $\lambda, \mu \in [-1, 1]$,

$$
\begin{aligned}
&R_{1313}+\lambda^2 R_{1414}+\mu^2 R_{2323}+\lambda^2\mu^2 R_{2424}-2\lambda\mu R_{1234}\\
&\quad\geq \frac{1}{2}(1+\lambda^2+\mu^2+(\lambda\mu)^2)\left(\frac{8}{3}\left(\overline{K}_{\min}-\frac{1}{4}\overline{K}_{\max}\right)+\frac{n^2H^2}{n-1}-S\right).
\end{aligned}
$$

Proof. From Berger's inequality (see [**7**]), we have

$$
\left|\overline{R}_{ijkl}\right| \leq \frac{2}{3}\left(\overline{K}_{\max}-\overline{K}_{\min}\right).
$$

Hence, from equation (1), we obtain

$$
\begin{aligned}
|R_{1234}| &= \left|\overline{R}_{1234}+\sum_{\alpha=n+1}^{n+p}\left(h_{13}^{\alpha}h_{24}^{\alpha}-h_{14}^{\alpha}h_{23}^{\alpha}\right)\right|\\
&\leq \frac{2}{3}\left(\overline{K}_{\max}-\overline{K}_{\min}\right)+\sum_{\alpha=n+1}^{n+p}\left|h_{13}^{\alpha}h_{24}^{\alpha}-h_{14}^{\alpha}h_{23}^{\alpha}\right|.
\end{aligned}
$$

This together with Step 1 implies

$$
\begin{aligned}
&R_{1313}+\lambda^2 R_{1414}+\mu^2 R_{2323}+\lambda^2\mu^2 R_{2424}-2\lambda\mu R_{1234}\\
&\geq \frac{1}{2}(1+\lambda^2+\mu^2+(\lambda\mu)^2)\left(2\overline{K}_{\min}+\frac{n^2H^2}{n-1}-S\right)\\
&\quad+\sum_{\alpha=n+1}^{n+p}\sum_{j>i,(i,j)\neq(1,3)}^{n}\left(h_{ij}^{\alpha}\right)^2+\lambda^2\sum_{\alpha=n+1}^{n+p}\sum_{j>i,(i,j)\neq(1,4)}^{n}\left(h_{ij}^{\alpha}\right)^2\\
&\quad+\mu^2\sum_{\alpha=n+1}^{n+p}\sum_{j>i,(i,j)\neq(2,3)}^{n}\left(h_{ij}^{\alpha}\right)^2+(\lambda\mu)^2\sum_{\alpha=n+1}^{n+p}\sum_{j>i,(i,j)\neq(2,4)}^{n}\left(h_{ij}^{\alpha}\right)^2\\
&\quad-2|\lambda\mu|\left[\frac{2}{3}\left(\overline{K}_{\max}-\overline{K}_{\min}\right)+\sum_{\alpha=n+1}^{n+p}\left|h_{13}^{\alpha}h_{24}^{\alpha}-h_{14}^{\alpha}h_{23}^{\alpha}\right|\right]\\
&\geq \frac{1}{2}(1+\lambda^2+\mu^2+(\lambda\mu)^2)\left(2\overline{K}_{\min}+\frac{n^2H^2}{n-1}-S\right)\\
&\quad+\sum_{\alpha=n+1}^{n+p}\left[(h_{14}^{\alpha})^2+\lambda^2(h_{13}^{\alpha})^2+\mu^2(h_{24}^{\alpha})^2+(\lambda\mu)^2(h_{23}^{\alpha})^2\right]\\
&\quad-2|\lambda\mu|\left[\frac{2}{3}\left(\overline{K}_{\max}-\overline{K}_{\min}\right)\right]-2|\lambda\mu|\sum_{\alpha=n+1}^{n+p}\left|h_{13}^{\alpha}h_{24}^{\alpha}\right|-2|\lambda\mu|\sum_{\alpha=n+1}^{n+p}\left|h_{14}^{\alpha}h_{23}^{\alpha}\right|.
\end{aligned}
$$

Note that

$$(h_{14}^\alpha)^2 + (\lambda\mu)^2(h_{23}^\alpha)^2 \geq 2|\lambda\mu||h_{14}^\alpha h_{23}^\alpha|,$$
$$\lambda^2(h_{13}^\alpha)^2 + \mu^2(h_{24}^\alpha)^2 \geq 2|\lambda\mu||h_{13}^\alpha h_{24}^\alpha|$$

and

$$-4|\lambda\mu| \geq -(1+\lambda^2+\mu^2+(\lambda\mu)^2).$$

Hence,

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}$$
$$\geq \frac{1}{2}(1+\lambda^2+\mu^2+(\lambda\mu)^2)\left(\frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-1} - S\right). \qquad \square$$

The next step is the most important here. It is the differential to improve the Theorem 1.3.

Step 3: Let M^n be an $n(\geq 4)$-dimensional compact submanifold of an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \overline{g})$. Assume that, for all point $x \in M^n$,

$$S \leq \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-1},$$

with strictly inequality at one point $p_0 \in M^n$. Then, there exists a metric on M^n such that, for all point $x \in T_xM^n$, all orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_xM^n$, and all $\lambda, \mu \in [-1, 1]$,

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} > 0.$$

Proof. It follows from Step 2 and from the strictly inequality

$$S(p_0) < \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H(p_0)^2}{n-1}$$

that for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_{p_0}M^n$, and all $\lambda, \mu \in [-1, 1]$,

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} > 0.$$

Another hand, from Step 2 and work [**4**] due to Brendle-Schoen, $(M^n, g_0) \times \mathbb{R}^2$ possesses nonnegative isotropic curvature, where g_0 denotes the induced metric of $\overline{g}$. Let $g(t)$ be the solution to the Ricci flow on M^n with initial metric g_0 and maximal interval of definition $[0, T)$. From the S. Brendle and R. Schoen's work [**4**], we have that, for all $0 \leq t < T$, $(M^n, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature. Another hand, from the S. Brendle and R. Schoen's work [**3**], given $0 < t < T$ and $\lambda, \mu \in [-1, 1]$, the set of all four-frames $\{e_1, e_2, e_3, e_4\}$ that are orthonormal with respect to $g(t)$ and satisfy

$$\begin{aligned} &R_{g(t)}(e_1, e_3, e_1, e_3) + \lambda^2 R_{g(t)}(e_1, e_4, e_1, e_4) + \mu^2 R_{g(t)}(e_2, e_3, e_2, e_3) \\ &\quad + \lambda^2\mu^2 R_{g(t)}(e_2, e_4, e_2, e_4) - 2\lambda\mu R_{g(t)}(e_1, e_2, e_3, e_4) = 0 \end{aligned}$$

is invariant under parallel transport. Hence, if $(M^n, g(t)) \times \mathbb{R}^2$, $0 < t < T$, does not have positive isotropic curvature, it follows from the invariance under parallel transport that there is a four-frame $\{e_1(t), e_2(t), e_3(t), e_4(t)\} \subset T_{p_0}M^n$ and $\lambda(t), \mu(t) \in [-1, 1]$ for which

$$\begin{aligned} &R_{g(t)}(e_1(t), e_3(t), e_1(t), e_3(t)) + \lambda(t)^2 R_{g(t)}(e_1(t), e_4(t), e_1(t), e_4(t)) \\ &\quad + \mu(t)^2 R_{g(t)}(e_2(t), e_3(t), e_2(t), e_3(t)) + \lambda(t)^2\mu(t)^2 R_{g(t)}(e_2(t), e_4(t), e_2(t), e_4(t)) \\ &\quad - 2\lambda(t)\mu(t) R_{g(t)}(e_1(t), e_2(t), e_3(t), e_4(t)) = 0. \end{aligned}$$

Hence, if for each $0 < t < T$, $(M^n, g(t)) \times \mathbb{R}^2$ does not have positive isotropic curvature, we obtain a time dependent four-frame $\{e_1(t), e_2(t), e_3(t), e_4(t)\}$ at p_0 and a family $\lambda(t), \mu(t) \subset [-1, 1]$ satisfying the equality above. We can choose a sequence of times $t_i \to 0$ as $i \to +\infty$ for which the corresponding sequence of four-frames converge to an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_{p_0}M^n$ at p_0 with respect to the metric g_0. Since $[-1, 1]$ is compact, there exists two points $\lambda_0, \mu_0 \in [-1, 1]$ such that, passing to a subsequence if necessary, $\lambda(t_i) \to \lambda_0$ and $\mu(t_i) \to \mu_0$. Thus, we find an orthonormal four-frame $\{e_1, e_2, e_3, e_4\} \subset T_{p_0}M^n$ with respect to the metric g_0 and two points $\lambda_0, \mu_0 \in [-1, 1]$ such that

$$\begin{aligned} &R_{g_0}(e_1, e_3, e_1, e_3) + \lambda_0^2 R_{g_0}(e_1, e_4, e_1, e_4) + \mu_0^2 R_{g_0}(e_2, e_3, e_2, e_3) \\ &\quad + \lambda_0^2\mu_0^2 R_{g_0}(e_2, e_4, e_2, e_4) - 2\lambda_0\mu_0 R_{g_0}(e_1, e_2, e_3, e_4) = 0. \end{aligned}$$

This is a contradiction. Therefore, there exist $t > 0$ such that $(M^n, g(t)) \times \mathbb{R}^2$ possesses positive isotropic curvature. $\square$

Final Step: It follows from Step 1 that the sectional curvatures of (M^n, g_0) is nonnegative and there exists a point $p_0 \in M^n$ such that $K_{p_0}(\pi) > 0$ for all 2-plane $\pi \subset T_{p_0}M^n$.

1. Suppose that M^n $(n \geq 3)$ is complete and non-compact. It follows from Perelman's Soul Theorem (see [**9**]) that M^n is diffeomorphic to the Euclidean space $\mathbb{R}^n$, since the sectional curvatures of (M^n, g_0) is nonnegative and there exists a point $p_0 \in M^n$ such that $K_{p_0}(\pi) > 0$ for all 2-plane $\pi \subset T_{p_0}M^n$.

2. Suppose that M^n is compact and $n = 3$. It follows from Step 1 that $\mathrm{Ric}_{g_0} \geq 0$ on M^n and there exists a point $p_0 \in M^n$ such that $\mathrm{Ric}_{g_0} > 0$ at this point, since the sectional curvatures of (M^n, g_0) is nonnegative and there exists a point $p_0 \in M^n$ such that $K_{p_0}(\pi) > 0$ for all 2-plane $\pi \subset T_{p_0}M^n$. From a theorem due to T. Aubin [**1**], we construct a Riemannian metric h on M^3 such that $\mathrm{Ric}_h > 0$ on M^3. Hence, from a classification result of compact 3-dimensional manifolds with positive Ricci curvature due to R. Hamilton [**6**] we have the manifold M^3 is diffeomorphic to a spherical space form.

3. Suppose that M^n is compact and $n \geq 4$. From Step 3, there exists a metric h_0 on M such that $(M^n, h_0) \times \mathbb{R}^2$ possesses positive isotropic curvature. From a result of Brendle and Schoen, the normalized Ricci flow

$$\frac{\partial}{\partial t}h(t) = -2\,\mathrm{Ric}_{h(t)} + \frac{2}{n} r_{h(t)} h(t)$$

where $r_{h(t)}$ denotes the mean value of the scalar curvature of $h(t)$, with initial metric h_0 exists for all time and converges to a constant curvature metric as $t \to +\infty$. Hence, M^n is diffeomorphic to a spherical space form. □

References

[1] T. Aubin, Métriques riemanniennes et courbure, J. Differential Geometry, **4** (1970), 383–424.

[2] S. Brendle, A general convergence result for the Ricci flow in higher dimensions, Duke Math. J., **145** (2008), 585–601.

[3] S. Brendle and R. Schoen, Classification of manifolds with weakly 1/4-pinched curvatures, Acta Math., **200** (2008), 1–13.

[4] S. Brendle and R. Schoen, Manifolds with 1/4-pinched curvature are space forms, J. Amer. Math. Soc., **22** (2009), 287–307.

[5] M. do Carmo and E. Lima, Immersions of manifolds with non-negative sectional curvatures, Bol. Soc. Brasil. Mat., **2** (1971), 9–22.

[6] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom., **17** (1982), 255–306.

[7] H. Karcher, A short proof of Berger's curvature tensor estimates, Proc. Amer. Math. Soc., **26** (1970), 642–644.

[8] L. Ni and B. Wilking, Manifolds with 1/4-pinched flag curvature, Geom. Funct. Anal., **20** (2010), 571–591.

[9] G. Perelman, Proof of the soul conjecture of Cheeger and Gromoll, J. Differential Geom.,

40 (1994), 209–212.
[10] J. J. Stoker, Über die Gestalt der positiv gekrümmten offenen Flächen im dreidimensionalen Raume, Compositio Math., **3** (1936), 55–88.
[11] H.-W. Xu and J.-R. Gu, An optimal differentiable sphere theorem for complete manifolds, Math. Res. Lett., **17** (2010), 1111–1124.
[12] S.-H. Zhu, On open three-manifolds of quasi-positive Ricci curvature, Proc. Amer. Math. Soc., **120** (1994), 569–572.

Ezequiel R. Barbosa
Departamento de Matemática
Universidade Federal de Minas Gerais
Caixa Postal 702
30123-970, Belo Horizonte
MG, Brazil
E-mail: ezequiel@mat.ufmg.br