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Littlewood-Paley theory for variable exponent Lebesgue spaces and Gagliardo-Nirenberg inequality for Riesz potentials

By Yoshihiro Mizuta, Eiichi Nakai, Yoshihiro Sawano and Tetsu Shimomura

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Abstract. Our aim in this paper is to prove the Gagliardo-Nirenberg inequality for Riesz potentials of functions in variable exponent Lebesgue spaces, which are called Musielak-Orlicz spaces with respect to $\Phi(x,t) = t^{p(x)}(\log(c_0+t))^{q(x)}$ for $t > 0$ and $x \in \mathbb{R}^n$, via the Littlewood-Paley decomposition.

1. Introduction.

The goal of this paper is to investigate the inequality of Gagliardo-Nirenberg type, where we place ourselves in the setting of the n-dimensional Euclidean space $\mathbb{R}^n$. The Gagliardo-Nirenberg inequality is the one of the form

$$\|(-\Delta)^{\theta\alpha} f\|_X \le C\|f\|_Y^{1-\theta}\|(-\Delta)^{\alpha} f\|_Z^{\theta} \quad (0<\alpha,\ 0<\theta<1), \tag{1.1}$$

where X, Y, Z are all Banach spaces of measurable functions.

Here we are concerned with various function spaces; X, Y, Z can be various function spaces. As a model case we take up variable exponent spaces. Variable exponent spaces have been studied in many articles over the past decades. To describe variable exponent spaces employed in the present paper, we introduce

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some notations. Let $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ and $q(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be bounded functions, which are called variable exponents in the variable Lebesgue setting. Set

$$\Phi_{p(\cdot),q(\cdot)}(x, t) = t^{p(x)}(\log(c_0 + t))^{q(x)};$$

here $c_0 > e$ is chosen so large that the condition, which we shall call (Φ_1), is fulfilled:

$$(\Phi_1) \quad \Phi_{p(\cdot),q(\cdot)}(x, \cdot) \text{ is convex on } [0, \infty) \text{ for every } x \in \mathbb{R}^n.$$

Define the $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$-norm by

$$\|f\|_{L^{p(\cdot),q(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

and denote by $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ the space of all measurable functions f on $\mathbb{R}^n$ for which the norm $\|f\|_{L^{p(\cdot),q(\cdot)}}$ is finite. Note that $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is a Musielak-Orlicz space [**31**]. In case $q(\cdot) \equiv 0$, $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is denoted by $L^{p(\cdot)}(\mathbb{R}^n)$ for simplicity. The notation $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which seems unfamiliar for non-specialists, is used especially for the purpose of stressing that $p(\cdot)$ and $q(\cdot)$ are functions on $x \in \mathbb{R}^n$. When $p(\cdot)$ and $q(\cdot)$ are constant functions, then as usual we omit $(\cdot)$ and we write $L^{p,q}(\mathbb{R}^n)$.

We shall illustrate the Littlewood-Paley theory is very useful when we obtain an inequality of the form (1.1). The Littlewood-Paley theory is one of the most powerful tools in harmonic analysis. Roughly speaking, this is a technique of transforming functions into good ones in order to measure the norms. Here and below, we use the notation $A \lesssim B$ to indicate that there exists a constant C independent of functions such that $A \le CB$. If we need to emphasize that C depends on parameters $\alpha, \beta, \ldots$, then we write $\lesssim_{\alpha,\beta,\ldots}$ instead of $\lesssim$. The notation $A \sim B$ means that $A \lesssim B \lesssim A$.

Write $B(r) \equiv \{x \in \mathbb{R}^n : |x| \le r\}$ for $r > 0$. Also, for a function f, $\mathcal{F}f$ denotes the Fourier transform of f, that is,

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-\sqrt{-1}x\xi}\, dx$$

and $\mathcal{F}^{-1}F$ denotes its inverse, that is,

$$\mathcal{F}^{-1}F(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} F(\xi)e^{\sqrt{-1}x\xi}\, d\xi.$$

The first result in the present paper, which extends what is known for classical $L^p(\mathbb{R}^n)$ spaces, can be stated as follows:

THEOREM 1.1. *Suppose that the functions* $p(\cdot):\mathbb{R}^n\to(0,\infty)$ *satisfies* (p1), (p2) *and* (p3) *and that* $q(\cdot):\mathbb{R}^n\to\mathbb{R}$ *satisfies* (q1) *and* (q2), *where these conditions are*;

(p1) $1<p_-=\inf_{x\in\mathbb{R}^n}p(x)\le\sup_{x\in\mathbb{R}^n}p(x)=p_+<\infty$;

(p2) $|p(x)-p(y)|\le\dfrac{C_{\log}}{\log(e+1/|x-y|)}$, *whenever* $x\in\mathbb{R}^n$ *and* $y\in\mathbb{R}^n$;

(p3) $|p(x)-p(y)|\le\dfrac{C_{\log}}{\log(e+|x|)}$ *whenever* $|y|\ge|x|/2$;

(q1) $-\infty<q_-=\inf_{x\in\mathbb{R}^n}q(x)\le\sup_{x\in\mathbb{R}^n}q(x)=q_+<\infty$;

(q2) $|q(x)-q(y)|\le\dfrac{C_{\log\log}}{\log(e+\log(e+1/|x-y|))}$ *whenever* $x\in\mathbb{R}^n$ *and* $y\in\mathbb{R}^n$.

In the above $C_{\log}$ *and* $C_{\log\log}$ *are positive constants independent of* x *and* y.

Assume also that a non-negative function $\varphi\in\mathcal{S}(\mathbb{R}^n)$ *has its support on* $B(8)\setminus B(1)$, *that is,*

$$\operatorname{supp}(\varphi)\subset B(8)\setminus B(1) \tag{1.2}$$

and that there exists a non-negative $\widetilde{\varphi}\in\mathcal{S}(\mathbb{R}^n)$ *such that*

$$\operatorname{supp}(\widetilde{\varphi})\subset B(8)\setminus B(1),\quad \sum_{j=-\infty}^{\infty}\varphi(2^{-j}\xi)\widetilde{\varphi}(2^{-j}\xi)\equiv\chi_{\mathbb{R}^n\setminus\{0\}}(\xi)\quad(\xi\in\mathbb{R}^n). \tag{1.3}$$

Then we have the following equality and norm equivalence: *For all* $f\in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *we have*

$$f=c_n\sum_{j=-\infty}^{\infty}2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot)*2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot)*f\quad in\quad \mathcal{S}'(\mathbb{R}^n), \tag{1.4}$$

$$\left\|\left(\sum_{j=-\infty}^{\infty}2^{2jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot)*f|^2\right)^{1/2}\right\|_{L^{p(\cdot),q(\cdot)}}\sim\|f\|_{L^{p(\cdot),q(\cdot)}}. \tag{1.5}$$

This theorem can be located as an extension of the classical Littlewood-Paley theory to the variable Lebesgue spaces.

The conditions (p1), (p2), (p3), (q1) and (q2) appear throughout in the present paper to describe the conditions of variable exponents. Here clarifying remarks on $p(\cdot)$ and $q(\cdot)$ may be in order.

Remark 1.2.

1. The idea of employing $p(\cdot)$ and $q(\cdot)$ dates back to the paper by Cruz-Uribe and Fiorenza [**3**].
2. Condition (p3) implies that $p_\infty = \lim_{|x|\to\infty} p(x)$ exists and that

$$|p(x) - p_\infty| \le \frac{C_{\log}}{\log(e+|x|)} \text{ for all } x \in \mathbb{R}^n.$$

3. For later use it is convenient to see from (Φ_1) that

$$(\Phi_2) \quad t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t) \text{ is nondecreasing on } (0,\infty) \text{ for fixed } x \in \mathbb{R}^n.$$

The first thrust to investigate variable exponent spaces is to apply it to the partial differential equations by Diening and Růžička [**9**] with $q(\cdot) \equiv 0$. For a survey see [**8**], [**17**], [**37**]. These investigations have been concerned both with the spaces themselves, e.g. [**4**], [**13**], [**16**], [**18**], [**20**], [**29**], and with related differential equations [**2**], [**6**], [**11**], and with applications [**1**], [**36**].

One of the reasons why we are fascinated to consider the function $q(\cdot)$ is that this function can be used to describe the maximal operator control in very subtle settings. We denote by $B(x,r)$ the open ball centered at x and of radius r. For a locally integrable function f on $\mathbb{R}^n$, we consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

For the fundamental properties of maximal functions, see Duoandikotxea [**10**] and Stein [**42**]. It is known as Stein's theorem that there exists a universal constant $C > 0$ such that

$$\int_B Mf(x)\, dx \le C\|f\|_{L^{1,1}}$$

for all functions f supported on a ball B with radius less than 1. So in our setting it is very important to introduce the second function $q(\cdot)$.

Based upon the Littlewood-Paley characterization, we obtain inequalities of the form (1.1). First we rewrite $(-\Delta)^{\alpha/2}$ in terms of the Riesz potential. We define the Riesz potential $R_\alpha f$ of order α by

$$R_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy$$

for $0 < \alpha < n$ and a locally integrable function f on $\mathbb{R}^n$. By taking the Fourier transform, we notice that $(-\Delta)^{-\alpha/2}$ is a constant multiple of R_α for $0 < \alpha < n$. Here we assume that $R_\alpha|f| \not\equiv \infty$, which is equivalent to

$$\int_{\mathbb{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty$$

(see [**28**]). With the terminology fixed, we specify a Gagliardo-Nirenberg inequality for Riesz potentials of functions in $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ obtained in this paper. Namely, we prove the inequality

$$\|R_{\theta\alpha} f\|_{L^{p(\cdot),q(\cdot)}} \le C\|f\|^{1-\theta}_{L^{p_1(\cdot),q_1(\cdot)}} \|R_\alpha f\|^{\theta}_{L^{p_2(\cdot),q_2(\cdot)}}, \tag{1.6}$$

with $0 < \alpha < n, \theta \in (0,1)$ and a certain relation of the variable exponents. We refer the readers to the works by Nirenberg [**33**], [**34**] and Gagliardo [**14**] for $f \in C_0^\infty(\mathbb{R}^n)$ in the constant $L^p(\mathbb{R}^n)$ case. Recently, in the short paper [**21**], Kopaliani and Chelidze have proved the inequality for Sobolev functions in the variable $L^{p(\cdot)}(\mathbb{R}^n)$ case. For related results, see also Stein [**41**] and Zang-Fu [**44**]. To obtain the inequality (1.6), we use the Littlewood-Paley theory for the function space $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. The spirit is close to the one in [**39**].

For fundamental properties of these spaces, see, for example, Kováčik and Rákosník [**23**] and the authors [**30**].

Remark 1.3. The idea of using φ and $\tilde{\varphi}$ can be found in [**12**]. For the sake of convenience for readers, we give an example of φ and $\tilde{\varphi}$. Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be taken so that it is $\mathbb{R}$-valued and that

$$\operatorname{supp}(\Psi) \subset B(8) \setminus B(1), \quad \Psi \equiv 1 \text{ on } B(4) \setminus B(2).$$

We set

$$\varphi(\xi) = \tilde{\varphi}(\xi) = \Psi(\xi)\left(\sum_{j=-\infty}^{\infty} \Psi(2^{-j}\xi)^2\right)^{-1/2}.$$

Then φ and $\tilde{\varphi}$ satisfies (1.3). Another example is

$$\varphi(\xi)=\Psi(\xi), \quad \tilde{\varphi}(\xi)=\Psi(\xi)\Big(\sum_{j=-\infty}^{\infty}\Psi(2^{-j}\xi)^2\Big)^{-1}. \tag{1.7}$$

Now we formulate the Gagliardo-Nirenberg inequality for function spaces $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 1.4. *Suppose that the functions* $p(\cdot),p_1(\cdot),p_2(\cdot):\mathbb{R}^n\to(1,\infty)$ *satisfy* (p1), (p2) *and* (p3) *and that the functions* $q(\cdot),q_1(\cdot),q_2(\cdot):\mathbb{R}^n\to\mathbb{R}$ *satisfy* (q1) *and* (q2). *Assume that these functions are related by*

$$\frac{1}{p(x)}=\frac{1-\theta}{p_1(x)}+\frac{\theta}{p_2(x)}, \quad \frac{q(x)}{p(x)}=(1-\theta)\frac{q_1(x)}{p_1(x)}+\theta\frac{q_2(x)}{p_2(x)} \tag{1.8}$$

for some $\theta\in(0,1)$. *Then*

$$\|R_{\theta\alpha}f\|_{L^{p(\cdot),q(\cdot)}}\lesssim\|f\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta}\|R_\alpha|f|\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta} \tag{1.9}$$

for all $f\in L^{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

Remark 1.5. If $m\in(0,n)$ is a positive integer and λ is a multi-index with length m, then the operator

$$T:f\longrightarrow D^\lambda(R_mf)=(D^\lambda R_m)*f$$

defines a singular integral operator. This implies that R_mf belongs to the Sobolev space $W^{m,p(\cdot)}(\mathbb{R}^n)$ when $f\in L^{p(\cdot)}(\mathbb{R}^n)$ has compact support and $1<p_-\le p_+<n/\alpha$, with the aid of [**4**, Corollary 2.5].

In [**21**] Kopaliani and Chelidze dealt the case when $q(\cdot)\equiv 0$ by using an inequality due to Maz'ya and Shaposhnikova [**26**]. Indeed, Kopaliani and Chelidze [**21**] used inequalities of the form

$$|\nabla^kf(x)|\le C\big(M[|\nabla^lf|](x)\big)^{(m-k)/(m-l)}\mathcal{D}_{p,m}f(x)^{(k-l)/(m-l)},$$

where, denoting by $[m]$ the integer part of a non-integer $m>0$, we defined

$$\mathcal{D}_{p,m}f(x)=\bigg(\int_{\mathbb{R}^n}\frac{|\nabla_{[m]}u(x)-\nabla_{[m]}u(y)|^p}{|x-y|^{n+p[m]}}\,dy\bigg)^{1/p}.$$

This technique can be also used for our spaces.

However, this technique has a series of disadvantages. First, in the actual paper [**21**], the case when k and l are integers is covered. This aspect can be overcome somehow by reexamining the proof of their result and the result due to Maz'ya and Shaposhnikova [**26**]. Indeed, Maz'ya and Shaposhnikova used

$$|R_z f(x)| \le CM[R_\zeta f](x)^{\mathrm{Re}(z)/\mathrm{Re}(\zeta)} Mf(x)^{1-\mathrm{Re}(z)/\mathrm{Re}(\zeta)} \tag{1.10}$$

for all $z, \zeta \in \{w \in \mathbb{C} : 0 < \mathrm{Re}(w) < n\}$. It seems that a careful observation yields a result of Theorem 1.4. However, their method is no more applicable for other function spaces such as Hardy spaces. As an example of function spaces beyond the reach of (1.10), we take up variable exponent Hardy spaces. Assume that an exponent $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ satisfies (p2), (p3) and (p1)$_-$ described below

(p1)$_-$ $0 < p_- = \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) = p_+ < \infty$.

Then the definition of $L^{p(\cdot)}(\mathbb{R}^n)$ is still available as a linear space and the only change is that $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies only when $p_- \ge 1$. We set $\psi_t(x) = t^{-n}\exp(-|x|^2/t^2)$ for $t > 0$ and $x \in \mathbb{R}^n$. Then we define the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{H^{p(\cdot)}} := \left\| \sup_{t>0} |\psi_t * f(\cdot)| \right\|_{L^{p(\cdot)}} < \infty.$$

Proposition 1.6 ([**32**, Theorem 5.7]). *Keep to the same notations for φ and $\widetilde{\varphi}$ as Theorem* 1.1. *Assume that an exponent $p(\cdot)$ satisfies* (p1)$_-$, (p2) *and* (p3). *Then we have the following equality and norm equivalence*: *For all $f \in H^{p(\cdot)}(\mathbb{R}^n)$ we have*

$$f = c_n \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}} \sim \|f\|_{H^{p(\cdot)}}.$$

Using Proposition 1.6, we can prove the following proposition, which is beyond the reach of (1.10) because of the maximal operator.

Theorem 1.7. *Let $p(\cdot), p_1(\cdot), p_2(\cdot) : \mathbb{R}^n \to (0, \infty)$ satisfy* (p1)$_-$, (p2) *and* (p3). *Assume that these functions are related by* (1.8). *Let $0 < \theta < 1$. Then, for $f \in H^{p_1(\cdot)}(\mathbb{R}^n)$ such that $0 \notin \mathrm{supp}(\mathcal{F}f)$*,

$$\|R_{\theta\alpha}f\|_{H^{p(\cdot)}} \lesssim \|f\|_{H^{p_1(\cdot)}}^{1-\theta}\|R_\alpha f\|_{H^{p_2(\cdot)}}^{\theta}.$$

The method we shall use in the present paper can enlarge the class of functions as we will illustrate in Section 4. Another advantage of our method is that we can use a simple pointwise estimate (4.6). As an evidence, our method can be also applied to the stochastic analysis. This aspect will be described in Section 5.

The behavior of the constant C in (1.1) is investigated precisely by Kozono, Wadade and Nagayasu in the case when X, Y are Lebesgue spaces and Z is a Lebesgue space, Besov space, or the BMO space [**22**], [**43**]. For example, in [**35**], when $1 < p \le n/(n-2)$, Pino and Dolbeault established

$$\|w\|_{p+1} \le \left(\frac{y(p-1)^2}{2\pi n}\right)^{\theta/2}\left(\frac{2y}{2y+n}\right)^{(1-\theta)/2p}\left(\frac{\Gamma(n/2+1+y)}{\Gamma(1+y)}\right)^{\theta/n}\|\nabla w\|_2^{\theta}\|w\|_{2p}^{1-\theta},$$

where

$$\theta = \frac{n(p-1)}{(1+p)(n-np+2p)},$$

and this can be used for the minimizing problem

$$\text{minimize}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right)dx : u \in H^1(\mathbb{R}^n),\ \|u\|_{2p} = m\right\}$$

with $m > 0$.

The structure of the rest of this paper is as follows. We will obtain a result on boundedness of singular integral operators in Section 2 to prove the Littlewood-Paley theory for $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in Section 3. Then we give a Gagliardo-Nirenberg inequality for Riesz potentials of functions in $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in Section 4. Inequalities related to Theorem 1.4 can be found in Section 5.

2. Boundedness of the maximal operators and the singular integral operators.

In this section we obtain a boundedness result for the maximal operators and that for the singular integral operators of convolution type.

2.1. Boundedness of the maximal operators.

Let $p_- > 1$. We use the duality formula. Here and below we write $\log^a b = (\log b)^a$ for simplicity. Observe that

$$\sup_{t>0}\left(st - t^{p(x)}\log^{q(x)}(c_0+t)\right) \simeq s^{p(x)/(p(x)-1)}\log^{-q(x)/(p(x)-1)}(c_0+s),$$

whenever $p_- > 1$. For more details of this calculation, we refer to [**40**, Claim 4.4]. Therefore, if we write $\overline{p}(x) \equiv p(x)/(p(x)-1)$ and $\overline{q}(x) \equiv -q(x)/(p(x)-1)$, then we have

$$L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)' \approx L^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n) \tag{2.1}$$

and the pair $(\overline{p}(\cdot),\overline{q}(\cdot))$ still fulfills the condition of the same type as $(p(\cdot),q(\cdot))$.

We now invoke one of the fundamental properties used in this paper:

Proposition 2.1 ([**10**, Proposition 2.7], [**42**, page 63]). *Let $\phi:\mathbb{R}^n \to \mathbb{R}$ be a function, which is positive, radial, decreasing and integrable. Then*

$$\sup_{t>0}|\phi_t * f(x)| \le \|\phi\|_{L^1} Mf(x)$$

for all locally integrable functions f, where we defined $\phi_t = t^{-n}\phi(\cdot/t)$ for $t>0$. In particular, if ϕ is a measurable function such that $|\phi(x)| \lesssim (1+|x|)^{-n-1}$, then

$$\sup_{t>0}|\phi_t * f(x)| \lesssim Mf(x).$$

We know the following result concerning the boundedness of the Hardy-Littlewood maximal function M in $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which is an extension of Diening [**7**] and Cruz-Uribe, Fiorenza and Neugebauer [**5**] when $q \equiv 0$.

Lemma 2.2 ([**25**, Proposition 2.2]). *Suppose that the functions $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot):\mathbb{R}^n \to (1,\infty)$ satisfy* (p1), (p2) *and* (p3) *and that the functions $q(\cdot)$, $q_1(\cdot)$, $q_2(\cdot):\mathbb{R}^n \to \mathbb{R}$ satisfy* (q1) *and* (q2). *Then the estimate*

$$\|Mf\|_{L^{p(\cdot),q(\cdot)}} \lesssim \|f\|_{L^{p(\cdot),q(\cdot)}}$$

holds for all $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

2.2. Boundedness of the singular integral operators.

Here we develop a theory of Calderón-Zygmund in our variable exponent setting. However, our aim is very modest; the goal is to obtain the Littlewood-Paley characterization. Therefore, we assume that the integral kernel k belongs to $\mathcal{S}(\mathbb{R}^n)$ to avoid the problem of convergence of $k * f$ (see (2.3) below). Let us set

$$A(k) \equiv \sup_{\xi \in \mathbb{R}^n} |\mathcal{F}k(\xi)|, \quad B(k) \equiv \sup_{x \in \mathbb{R}^n} |x|^n |k(x)|,$$
$$C(k) \equiv \sup_{x \in \mathbb{R}^n} |x|^{n+1} |\nabla k(x)|. \tag{2.2}$$

In our variable exponent setting, we use the sum space $L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$. We denote by $L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$ the set of all functions that can be written as the sum of an $L^{p_1}(\mathbb{R}^n)$ function and an $L^{p_2}(\mathbb{R}^n)$ function. Here the norm is given by

$$\|f\|_{L^{p_1}+L^{p_2}} \equiv \inf \left\{ \|f_1\|_{L^{p_1}} + \|f_2\|_{L^{p_2}} : f = f_1 + f_2,\ f_1 \in L^{p_1}(\mathbb{R}^n),\ f_2 \in L^{p_2}(\mathbb{R}^n) \right\}.$$

Let us now prove the boundedness of the singular integral operators. Here we shall truncate the kernel because, if we do that, it is still sufficient for our purpose.

Theorem 2.3. *Suppose that the functions* $p(\cdot), p_1(\cdot), p_2(\cdot) : \mathbb{R}^n \to (1, \infty)$ *satisfy* (p1), (p2) *and* (p3) *and that the functions* $q(\cdot), q_1(\cdot), q_2(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *satisfy* (q1) *and* (q2). *Let* $k \in \mathcal{S}(\mathbb{R}^n)$ *and let* $A(k)$, $B(k)$ *and* $C(k)$ *be defined by* (2.2). *Then*

$$\|k * f\|_{L^{p(\cdot),q(\cdot)}} \lesssim_{A(k),B(k),C(k)} \|f\|_{L^{p(\cdot),q(\cdot)}}. \tag{2.3}$$

Here the implicit positive constant is dependent on $A(k)$, $B(k)$ *and* $C(k)$ *but it is independent of* $\|k\|_{L^1}$.

According to the Calderón-Zygmund theory [**10**], we have

$$\|k * f\|_{L^p} \lesssim_{A(k),B(k),C(k)} \|f\|_{L^p} \quad (1 < p < \infty),$$

where $\|\cdot\|_{L^p}$ is the $L^p(\mathbb{R}^n)$-norm.

We will need the sharp maximal operator control to prove Theorem 2.3. Denote by $\mathcal{Q}(x)$ the set of all compact cubes whose edges are parallel to the coordinate axes and which contain a point x. Given a function $g : \mathbb{R}^n \to [0, \infty]$, we denote by $g^* : [0, \infty) \to [0, \infty)$ its decreasing rearrangement. The sharp maximal operator we use in the present paper is given by

$$M^\sharp_\lambda f(x) = \sup_{Q \in \mathcal{Q}(x)} \inf_{c \in \mathbb{C}} (|f - c|\chi_Q)^*(\lambda |Q|) \quad (0 < \lambda < 1).$$

Here note the following result:

Proposition 2.4 ([**19**], [**24**]). *Let* $\lambda_n = 10^{-n}$.

1. *For any measurable function* f *with* $|\{|f| > \lambda\}| < \infty$ *for all* $\lambda > 0$, *and for any weight* w,

$$\int_{\mathbb{R}^n} |f(x)|w(x)\,dx \lesssim_n \int_{\mathbb{R}^n} M^{\sharp}_{\lambda_n} f(x) Mw(x)\,dx. \tag{2.4}$$

2. *Let* $k \in \mathcal{S}$ *and let* $A(k)$, $B(k)$ *and* $C(k)$ *be defined by* (2.2). *Then, for any* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M^{\sharp}_{\lambda_n}[k * f](x) \lesssim_{n,A(k),B(k),C(k)} Mf(x). \tag{2.5}$$

Now we turn to the proof of Theorem 2.3.

Proof of Theorem 2.3. Let us set $P^{\dagger}_{(+)} = p_+ + 1$ and $P^{\dagger}_{(-)} = (p_- + 1)/2$. If $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then it follows immediately from the definition of the norm that $f \in L^{P^{\dagger}_{(-)}}(\mathbb{R}^n) + L^{P^{\dagger}_{(+)}}(\mathbb{R}^n)$. Therefore, we conclude $k * f \in L^{P^{\dagger}_{(-)}}(\mathbb{R}^n) + L^{P^{\dagger}_{(+)}}(\mathbb{R}^n)$. Hence

$$|\{|k * f| > \lambda\}| < \infty$$

for all $\lambda > 0$, which is necessary to use (2.4). Let us denote by $L^{\overline{p}(\cdot),\overline{q}(\cdot)}_1(\mathbb{R}^n)$ the closed unit ball in $L^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)$. By the duality (2.1), (2.4) and (2.5), we have

$$\begin{aligned} \|k * f\|_{L^{p(\cdot),q(\cdot)}} &\sim \sup_{g \in L^{\overline{p}(\cdot),\overline{q}(\cdot)}_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |k * f(x)| g(x)\,dx \\ &\lesssim \sup_{g \in L^{\overline{p}(\cdot),\overline{q}(\cdot)}_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} M^{\sharp}_{10^{-n}}[k * f](x) Mg(x)\,dx \\ &\lesssim_{A(k),B(k),C(k)} \sup_{g \in L^{\overline{p}(\cdot),\overline{q}(\cdot)}_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} Mf(x) Mg(x)\,dx. \end{aligned} \tag{2.6}$$

If we invoke the boundedness of M on $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $L^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)$ (Lemma 2.2), then we have

$$\sup_{g \in L_1^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)} \int_{\mathbb{R}^n} Mf(x)Mg(x)\,dx$$
$$\lesssim \sup_{g \in L_1^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)} \|Mf\|_{L^{p(\cdot),q(\cdot)}} \|Mg\|_{L^{\overline{p}(\cdot),\overline{q}(\cdot)}} \lesssim \|f\|_{L^{p(\cdot),q(\cdot)}}. \tag{2.7}$$

Thus, if we combine (2.6) and (2.7), then we obtain (2.3). □

3. Littlewood-Paley theory for variable exponent Lebesgue spaces.

The passage of Theorem 2.3 to $\ell^2(\mathbb{Z})$-valued spaces can be achieved easily by replacing $|\cdot|$ with the $\ell^2(\mathbb{Z})$ norm. Our result reads as follows:

THEOREM 3.1. *Assume that an exponent* $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ *satisfies* (p1)$_-$, (p2) *and* (p3) *and that an exponent* $q(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *satisfies* (q1) *and* (q2). *Suppose that* $\{k_{ij}\}_{i,j\in\mathbb{Z}}$ *is a given collection with the following properties*:

1. $k_{ij} \equiv 0$ *if* $|i|+|j|$ *is sufficiently large.*
2. *The following quantities are finite*:

$$A(K) \equiv \sup_{\xi \in \mathbb{R}^n} \|\mathcal{F}K(\xi)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})},$$
$$B(K) \equiv \sup_{x \in \mathbb{R}^n} |x|^n \|K(x)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})},$$
$$C(K) \equiv \sup_{l=1,2,\dots,n} \sup_{x \in \mathbb{R}^n} |x|^{n+1} \|\partial_l K(x)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})},$$

where $K(x)$ *and* $\partial_l K(x)$ $(l = 1,2,\dots,n)$ *denote the multiplication operator on* $\ell^2(\mathbb{Z})$ *given by the matrix* $\{k_{ij}(x)\}_{i,j\in\mathbb{Z}}$ *and* $\{\partial_l k_{ij}(x)\}_{i,j\in\mathbb{Z}}$ *respectively.*

Then we have

$$\left\| \left(\sum_{i=-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} k_{ij} * f_j \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}$$
$$\lesssim_{A(K),B(K),C(K)} \left\| \left(\sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}, \tag{3.1}$$

where the implicit constant in $\lesssim_{A(K),B(K),C(K)}$ *does not depend on* $\|k_{i,j}\|_{L^1}$.

PROOF. Just reexamine the proof of Theorem 2.3 by replacing $|\cdot|$ with the $\ell^2(\mathbb{Z})$-norm. □

As an application of Theorem 3.1, we obtain the Littlewood-Paley type characterization (Theorem 1.1 in Section 1).

Proof of Theorem 1.1. Let us set $P^\dagger_{(+)} = p_+ + 1$ and $P^\dagger_{(-)} = (p_- + 1)/2$ as before. Note that $1 < P^\dagger_{(-)} \le P^\dagger_{(+)} < \infty$. Once we accept that Theorem 1.1 is true when $p(\cdot)$ is a constant function, since $L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{P^\dagger_{(-)}}(\mathbb{R}^n) + L^{P^\dagger_{(+)}}(\mathbb{R}^n)$, the convergence of (1.4) is readily obtained. Indeed, assume that $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then we have $f \in L^{P^\dagger_{(-)}}(\mathbb{R}^n) + L^{P^\dagger_{(+)}}(\mathbb{R}^n)$. Consequently, f can be written as $f = f_{(-)} + f_{(+)}$, where $f_{(+)} \in L^{P^\dagger_{(-)}}(\mathbb{R}^n)$ and $f_{(-)} \in L^{P^\dagger_{(+)}}(\mathbb{R}^n)$. According to the classical Littlewood Paley theory, we have

$$f_{(\pm)} = c_n \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f_{(\pm)} \quad \text{in} \quad L^{P^\dagger_{(\pm)}}(\mathbb{R}^n) \tag{3.2}$$

and hence

$$f_{(\pm)} = c_n \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f_{(\pm)} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

since a standard argument shows that $L^{P^\dagger_{(\pm)}}(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$. Consequently (3.2) yields the convergence of (1.4).

Let us concentrate on proving (1.5). We first let

$$k^N_{ij} \equiv \begin{cases} 2^{in}(\mathcal{F}^{-1}\varphi)(2^i\cdot) & j = 0 \text{ and } |i| \le N, \\ 0 & \text{otherwise}, \end{cases} \qquad \tilde{f}_j \equiv \begin{cases} f & j = 0, \\ 0 & \text{otherwise}, \end{cases}$$

in (3.1). Denote by K^N the multiplication operator corresponding to $\{k^N_{ij}\}_{i,j\in\mathbb{Z}}$.

By the monotone convergence theorem we have

$$\begin{aligned} &\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\ &\quad = \lim_{N\to\infty} \left\| \left(\sum_{j=-N}^{N} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}. \end{aligned}$$

Let us verify

$$\sup_{N\in\mathbb{N}} \big(A(K^N) + B(K^N) + C(K^N)\big) < \infty. \tag{3.3}$$

Since there are at most two non-zero terms in the summand below, we have

$$\begin{aligned}\sup_{N\in\mathbb{N}} A(K^N) &= \sup_{N\in\mathbb{N},\,\xi\in\mathbb{R}^n} \|\mathcal{F}K^N(\xi)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})} \\ &\le \sup_{\xi\in\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} |\varphi(2^{-j}\xi)|^2\right)^{1/2} < \infty.\end{aligned}$$

Next, let us estimate $C(K^N)$. Let us first write it out in full:

$$C(K^N) = \sup_{l=1,2,\dots,n} \sup_{x\in\mathbb{R}^n} |x|^{n+1} \|\partial_l K^N(x)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})}.$$

Again by virtue of the fact that $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$, we find

$$\begin{aligned}\sup_{N\in\mathbb{N}} C(K^N) &\le \sup_{x\in\mathbb{R}^n} |x|^{n+1} \left(\sum_{j=-\infty}^{\infty} |\nabla[2^{jn}(\mathcal{F}^{-1}\varphi)(2^j x)]|^2\right)^{1/2} \\ &\lesssim \sup_{x\in\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} \frac{|2^j x|^{2(n+1)}}{(1+|2^j x|)^{2n+3}}\right)^{1/2}.\end{aligned}$$

Observe that the function

$$F : x \in \mathbb{R}^n \mapsto \sup_{x\in\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} \frac{|2^j x|^{2(n+1)}}{(1+|2^j x|)^{2n+3}}\right)^{1/2} \in \mathbb{R}$$

satisfies $F(2x) = F(x)$. Hence it follows that

$$\sup_{N\in\mathbb{N}} C(K^N) \lesssim \sup_{1\le|x|\le 2} F(x) \sim \left(\sum_{j=-\infty}^{\infty} \frac{2^{2j(n+1)}}{(1+2^j)^{2n+3}}\right)^{1/2} \sim 1. \tag{3.4}$$

Finally, let us estimate $B(K^N)$, which is similar to the estimate of $C(K^N)$. By definition we have

$$B(K^N) = \sup_{x\in\mathbb{R}^n} |x|^n \|K^N(x)\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})} = \sup_{x\in\mathbb{R}^n} |x|^n \left(\sum_{j=-\infty}^{\infty} |2^{jn}(\mathcal{F}^{-1}\varphi)(2^j x)|^2\right)^{1/2}.$$

Since $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $|(\mathcal{F}^{-1}\varphi)(2^j x)| \lesssim (1+|2^j x|)^{-n-1}$ for all $x \in \mathbb{R}^n$. Hence, we obtain

$$\sup_{N\in\mathbb{N}} B(K^N) \lesssim \sup_{x\in\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} \frac{|2^j x|^{2n}}{(1+|2^j x|)^{2n+2}} \right)^{1/2} \sim \left(\sum_{j=-\infty}^{\infty} \frac{2^{2jn}}{(1+2^j)^{2n+2}} \right)^{1/2} \lesssim 1.$$

Therefore, (3.3) is proved and we have

$$\begin{aligned}
&\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
&= \lim_{N\to\infty} \left\| \left(\sum_{i=-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} k_{ij}^N * \tilde{f}_j \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
&\lesssim \left\| \left(\sum_{j=-\infty}^{\infty} |\tilde{f}_j|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
&= \|f\|_{L^{p(\cdot),q(\cdot)}}
\end{aligned}$$

by virtue of Theorem 3.1. Meanwhile if we substitute

$$k_{ij} = \delta_{i0} 2^{jn} (\mathcal{F}^{-1}\widetilde{\varphi})(2^j x), \quad f_j = 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \quad (i, j \in \mathbb{Z})$$

in (3.1), then we have

$$\|f\|_{L^{p(\cdot),q(\cdot)}} \lesssim \left\| \sum_{j=-\infty}^{\infty} 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * 2^{jn} (\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}}$$

again by virtue of Theorem 3.1. We calculate the right-hand side carefully to have

$$\begin{aligned}
\|f\|_{L^{p(\cdot),q(\cdot)}} &\lesssim \left\| \left(\sum_{i=-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} k_{ij} * f_j \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
&= \left\| \left(\sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
&= \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}
\end{aligned}$$

Hence we obtain the desired result. □

Here is an alternative proof of Theorem 1.1.

Another proof of Theorem 1.1. Let us first establish that

$$\mathrm{I} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \lesssim \|f\|_{L^{p(\cdot),q(\cdot)}} \tag{3.5}$$

and hence

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\tilde{\varphi})(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \lesssim \|f\|_{L^{p(\cdot),q(\cdot)}}. \tag{3.6}$$

Once (3.5) is proven, then (3.6) follows immediately. Indeed, we deduce that

$$\operatorname{supp}(\widetilde{\varphi}(-\cdot)), \quad \operatorname{supp}(\varphi(-\cdot)) \subset B(8) \setminus B(1)$$

and that

$$\sum_{j=-\infty}^{\infty} \widetilde{\varphi}(-2^{-j}\xi)\varphi(-2^{-j}\xi) \equiv \chi_{\mathbb{R}^n \setminus \{0\}}(\xi) \quad (\xi \in \mathbb{R}^n).$$

from (1.2) and (1.3). Therefore by replacing φ with $\widetilde{\varphi}(-\cdot)$ and $\widetilde{\varphi}$ with $\varphi(-\cdot)$, we have only to establish (3.5).

To this end, let us take a Rademacher sequence $\{r_j(t)\}_{j\in\mathbb{Z}} \subset L^2([0,1))$. Recall that, for every $v \in (0,\infty)$, then we have

$$\left(\int_0^1 \left| \sum_{j\in\mathbb{Z}} a_j r_j(t) \right|^v dt \right)^{1/v} \sim_v \left(\sum_{j\in\mathbb{Z}} |a_j|^2 \right)^{1/2} \tag{3.7}$$

if $\{a_j\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{Z})$. For details we refer to [**15**].

If we use (3.7) with $v = 1$, then we have

$$\mathrm{I} \sim \left\| \int_0^1 \left| \sum_{j=-\infty}^{\infty} r_j(t) 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \right| dt \right\|_{L^{p(\cdot),q(\cdot)}}.$$

By the Minkowski inequality, we obtain

$$\mathrm{I} \lesssim \int_0^1 \left\| \sum_{j=-\infty}^{\infty} r_j(t) 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}} dt.$$

By the Fatou theorem, we obtain

$$\mathrm{I} \le \liminf_{N \to \infty} \int_0^1 \left\| \sum_{j=-N}^{N} r_j(t) 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}} dt. \tag{3.8}$$

Let us set

$$k_N(x,t) \equiv \sum_{j=-N}^{N} r_j(t) 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j x).$$

Then we have

$$\mathcal{F}[k_N(\cdot,t)](\xi) = \sum_{j=-N}^{N} r_j(t) \varphi(2^{-j}\xi)$$

and hence

$$A(k_N(\cdot,t)) < \infty$$

by virtue of the support condition. Since $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$B(k_N(\cdot,t)) \le \sup_{x \in \mathbb{R}^n} \sum_{j=-N}^{N} |2^j x|^n |(\mathcal{F}^{-1}\varphi)(2^j x)|.$$

In a way similar to (3.4), we deduce

$$B(k_N(\cdot,t)) \lesssim \sup_{x \in \mathbb{R}^n} \sum_{j=-\infty}^{\infty} \frac{|2^j x|^n}{(1+|2^j x|)^{n+1}} \sim \sum_{j=-\infty}^{\infty} \frac{2^{jn}}{(1+2^j)^{n+1}} \sim 1.$$

Likewise we calculate by using

$$C(k_N(\cdot,t)) \le \sup_{x \in \mathbb{R}^n} \sum_{j=-N}^{N} |2^j x|^{n+1} |(\mathcal{F}^{-1}\varphi)(2^j x)|.$$

The consequence is

$$C(k_N(\cdot,t)) \lesssim \sup_{x\in\mathbb{R}^n} \sum_{j=-\infty}^{\infty} \frac{|2^j x|^{n+1}}{(1+|2^j x|)^{n+2}} \sim \sum_{j=-\infty}^{\infty} \frac{2^{j(n+1)}}{(1+2^j)^{n+2}} \sim 1.$$

Therefore, if we invoke Theorem 2.3, then we have

$$\int_0^1 \left\| \sum_{j=-\infty}^{\infty} r_j(t) 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}} dt \lesssim \|f\|_{L^{p(\cdot),q(\cdot)}}. \tag{3.9}$$

If we insert (3.9) to (3.8), we obtain (3.5).

For the reverse inequality, we use the duality (2.1). More precisely, we proceed as follows: Let $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$ whose $L^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)$ -norm is 1. Then we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = c_n \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\tilde{\varphi})(2^j\cdot) * f(x)g(x)\,dx \tag{3.10}$$

because we have (1.4). By the Fubini theorem we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)g(x)\,dx \\
&= c_n \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j(x-y)) 2^{jn}(\mathcal{F}^{-1}\tilde{\varphi})(2^j\cdot) * f(y)\,dy \right) g(x)\,dx \\
&= c_n \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j(x-y)) g(x)\,dx \right) 2^{jn}(\mathcal{F}^{-1}\tilde{\varphi})(2^j\cdot) * f(y)\,dy \\
&= c_n \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f(x) \cdot 2^{jn}(\mathcal{F}^{-1}\tilde{\varphi})(-2^j\cdot) * g(x)\,dx.
\end{aligned}$$

Now we use the Schwarz inequality and the duality (2.1). The result is

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| \lesssim & \left\| \left(\sum_{j=-\infty}^{\infty} |2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\
& \times \left\| \left(\sum_{j=-\infty}^{\infty} |2^{jn}(\mathcal{F}^{-1}\tilde{\varphi})(-2^j\cdot) * g|^2 \right)^{1/2} \right\|_{L^{\overline{p}(\cdot),\overline{q}(\cdot)}}.
\end{aligned}$$

If we use (3.5), then we have

$$\left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| \lesssim \left\| \left(\sum_{j=-\infty}^{\infty} |2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \|g\|_{L^{\overline{p}(\cdot),\overline{q}(\cdot)}}.$$

Now that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^{\overline{p}(\cdot),\overline{q}(\cdot)}(\mathbb{R}^n)$, the reverse inequality is proved. □

About Theorem 1.1, we have the following variant.

THEOREM 3.2. *Let $p(\cdot), q(\cdot), \varphi$ and $\tilde{\varphi}$ satisfy the same condition as Theorem 1.1. If $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $0 \notin \operatorname{supp}(\mathcal{F}f)$ and*

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} < \infty, \tag{3.11}$$

then $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and f satisfies (1.4) *and* (1.5).

PROOF. The heart of the matter is to verify that $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Once this is achieved, then (1.4) and (1.5) follow automatically from Theorem 1.1. Since $0 \notin \operatorname{supp}(\mathcal{F}f)$, we have

$$f = c_n \sum_{j=-J}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \tag{3.12}$$

for some large $J \in \mathbb{N}$. Let $K \in \mathbb{N}$ be fixed. Notice that by virtue of the triangle inequality, Proposition 2.1, Lemma 2.2 and (3.11), we have

$$\begin{aligned}
&\left\| \sum_{j=-J}^{K} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}} \\
&\quad \le \sum_{j=-J}^{K} \left\| 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}} \\
&\quad \lesssim \sum_{j=-J}^{K} \left\| M[2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f] \right\|_{L^{p(\cdot),q(\cdot)}} \\
&\quad \lesssim \sum_{j=-J}^{K} \left\| 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}}
\end{aligned}$$

$$\leq (J+K+1)\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} < \infty.$$

Consequently, Theorem 1.1 is applicable to $\sum_{j=-J}^{K} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j \cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * f$ for each $K \in \mathbb{N}$. Applying Theorem 1.1, we obtain

$$\sup_{K\in\mathbb{N}} \left\| \sum_{j=-J}^{K} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j \cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * f \right\|_{L^{p(\cdot),q(\cdot)}}$$
$$\lesssim \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}.$$

Now we shall make use of the Banach-Alaoglu theorem to conclude that the sequence

$$\left\{ \sum_{j=-J}^{K} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j \cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * f \right\}_{K=1}^{\infty}$$

is a subsequence convergent in the weak topology of $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which is reflexive. Denote by $g \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ the limit above. Then by (3.12) we obtain $f = g \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and this is exactly what we wanted to prove. □

4. Gagliardo-Nirenberg inequality.

4.1. Hölder inequality for variable exponent Lebesgue spaces.

First we obtain the Hölder inequality for the variable Lebesgue spaces.

We use the following inequality of Young type:

Lemma 4.1. *Let $a, b \geq 0$ and M, q, q_1, q_2, u satisfy*

$$M > 0,\ q, q_1, q_2 \in [-M, M], \quad 1 < u < \infty$$

and

$$q = \frac{q_1}{u} + \frac{q_2}{u'}, \tag{4.1}$$

where u' denotes the Hölder conjugate of u, that is, $1/u + 1/u' = 1$. Then we have

$$ab \log^q(e+ab) \lesssim_M a^u \log^{q_1}(e+a) + b^{u'} \log^{q_2}(e+b).$$

Here the implicit constant does not depend on a and b.

PROOF. By symmetry we can assume that $a \geq b \geq 0$. If $0 \leq b \leq a \leq 1$, then the result follows from the inequality

$$ab \leq a^u + b^{u'},$$

which is a consequence of Young's inequality.

If $a \geq 1 \geq b$ and $u > 1$, we have

$$ab \log^q(e+ab) \lesssim a^u \log^{q_1}(e+a) \leq a^u \log^{q_1}(e+a) + b^{u'} \log^{q_2}(e+b).$$

Next suppose that $a \geq b \geq 1$. If $ab \log^q(e+a) \leq a^u \log^{q_1}(e+a)$, then there is nothing to prove. Let us suppose instead that $ab \log^q(e+a) > a^u \log^{q_1}(e+a)$. Namely we are now assuming that $a^{u-1} \log^{q_1-q}(e+a) < b \leq a$. Observe that $a^{u-1} \log^{q_1-q}(e+a) < b$ implies

$$a < Cb^{1/(u-1)} \log^{-(q_1-q)/(u-1)}(e+b) \tag{4.2}$$

for some constant $C > 1$. To see (4.2), we just consider the inverse of the function $f(a) = a^{u-1} \log^{q_1-q}(e+a)$. As a consequence, by using (4.1), we have

$$ab \log^q(e+ab) \lesssim b^{u/(u-1)} \log^{q-(q_1-q)/(u-1)}(e+b) = b^{u/(u-1)} \log^{q_2}(e+b).$$

The proof is therefore complete. □

THEOREM 4.2. *Let $p_1, p_2 : \mathbb{R}^n \to (0,\infty)$ satisfy* (p1)$_-$, (p2) *and* (p3) *and let $q_1, q_2 : \mathbb{R}^n \to \mathbb{R}$ satisfy* (q1), (q2). *Define $p(x)$ and $q(x)$ by*

$$\frac{1}{p(x)} = \frac{1-\theta}{p_1(x)} + \frac{\theta}{p_2(x)}, \quad \frac{q(x)}{p(x)} = (1-\theta)\frac{q_1(x)}{p_1(x)} + \theta\frac{q_2(x)}{p_2(x)}$$

for some $\theta \in (0,1)$. Then we have

$$\|f^{1-\theta} g^\theta\|_{L^{p(\cdot),q(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \|g\|_{L^{p_2(\cdot),q_2(\cdot)}}^\theta \tag{4.3}$$

for all positive measurable functions f and g.

Proof. By normalization we can assume that

$$\|f\|_{L^{p_1(\cdot),q(\cdot)}} = \|g\|_{L^{p_2(\cdot),q(\cdot)}} = 1.$$

If we let $u = p_1(x)/p(x)(1-\theta)$ and $a = |f(x)|^{p(x)(1-\theta)}$ and $b = |g(x)|^{p(x)\theta}$, then we have by virtue of Lemma 4.1

$$\begin{aligned}
&|f(x)|^{p(x)(1-\theta)}|g(x)|^{p(x)\theta}\log^{q(x)}(e+|f(x)|^{p(x)(1-\theta)}|g(x)|^{p(x)\theta})\\
&\quad\lesssim |f(x)|^{p_1(x)}\log^{q_1(x)}(e+|f(x)|^{p(x)(1-\theta)}) + |g(x)|^{p_2(x)}\log^{q_2(x)}(e+|g(x)|^{p(x)\theta})\\
&\quad\lesssim |f(x)|^{p_1(x)}\log^{q_1(x)}(e+|f(x)|) + |g(x)|^{p_2(x)}\log^{q_2(x)}(e+|g(x)|).
\end{aligned}$$

If we integrate this inequality over $\mathbb{R}^n$, then we have

$$\int_{\mathbb{R}^n}|f(x)|^{p(x)(1-\theta)}|g(x)|^{p(x)\theta}\log^{q(x)}(e+|f(x)|^{p(x)(1-\theta)}|g(x)|^{p(x)\theta})\,dx \lesssim 1.$$

This is the desired result. □

4.2. Proof of Theorems 1.4 and 1.7.

Now let us go back to the proof of Theorem 1.4. Maintain the same notation as Theorem 1.1. Let us set

$$f_j(x) = \begin{cases} f(x) & |x| \le j,\ |f(x)| \le j, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \ldots$. We assume $R_\alpha|f| \in L^{p_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$. Once we assume $R_\alpha|f| \in L^{p_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}(1+|y|)^{\alpha-n}|f(y)|\,dy < \infty$$

and hence

$$\int_{\mathbb{R}^n}(1+|y|)^{\theta\alpha-n}|f(y)|\,dy < \infty.$$

Consequently, by the Lebesgue convergence theorem, we have

$$\lim_{j\to\infty} R_\alpha f_j(x) = R_\alpha f(x), \quad \lim_{j\to\infty} R_{\theta\alpha} f_j(x) = R_{\theta\alpha} f(x).$$

By the triangle inequality we have $|R_\alpha f_j(x)| \le R_\alpha|f|(x)$. Therefore, by the Fatou lemma and the Lebesgue convergence theorem, we can assume that $f \in L^\infty_{\rm comp}(\mathbb{R}^n)$.

Let $u \in (n/(n-\alpha), \infty)$. Assuming $f \in L^\infty_{\rm comp}(\mathbb{R}^n)$, we have

$$R_{\theta\alpha} f = c_n R_{\theta\alpha}\left[\lim_{L\to\infty}\sum_{j=-L}^{L} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f\right] \quad \text{in} \quad L^u(\mathbb{R}^n),$$

since

$$f = c_n \lim_{L\to\infty}\sum_{j=-L}^{L} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f \quad \text{in} \quad L^{nu/(n+\alpha u)}(\mathbb{R}^n).$$

Again if we pass to subsequence $\{L_k\}_{k=1}^\infty$, we can assume

$$R_{\theta\alpha} f(x) = c_n R_{\theta\alpha}\left[\lim_{k\to\infty}\sum_{j=-L_k}^{L_k} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f\right](x)$$

for a.e. $x \in \mathbb{R}^n$.

Then we have

$$\begin{aligned}
&\|R_{\theta\alpha} f\|_{L^{p(\cdot),q(\cdot)}} \\
&\quad \le \liminf_{k\to\infty}\left\|R_{\theta\alpha}\left[\sum_{j=-L_k}^{L_k} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f\right]\right\|_{L^{p(\cdot),q(\cdot)}} \\
&\quad \lesssim \sup_{k\in\mathbb{N}}\left\|\left(\sum_{l=-\infty}^{\infty}\left|\sum_{j=-L_k}^{L_k} 2^{(l+2j)n}(\mathcal{F}^{-1}\varphi)(2^l\cdot) * (\mathcal{F}^{-1}\varphi)(2^j\cdot)\right.\right.\right. \\
&\qquad\qquad\qquad \left.\left.\left. * (\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * R_{\theta\alpha} f\right|^2\right)^{1/2}\right\|_{L^{p(\cdot),q(\cdot)}}
\end{aligned}$$

by Theorem 3.2. Notice that

$$\begin{aligned}
&\sum_{j=-L_k}^{L_k} 2^{(l+2j)n}(\mathcal{F}^{-1}\varphi)(2^l\cdot) * (\mathcal{F}^{-1}\varphi)(2^j\cdot) * (\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * R_{\theta\alpha} f \\
&\qquad\qquad \sim 2^{ln}(\mathcal{F}^{-1}\varphi)(2^l\cdot) * R_{\theta\alpha} f
\end{aligned}$$

when $|l| \le L_k - 6$ and that, by virtue of Proposition 2.1,

$$\begin{aligned}\Bigg| \sum_{j=-L_k}^{L_k} 2^{(l+2j)n} (\mathcal{F}^{-1}\varphi)(2^l \cdot) * (\mathcal{F}^{-1}\varphi)(2^j \cdot) * (\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * R_{\theta\alpha} f(x) \Bigg| \\ \lesssim M[2^{ln}(\mathcal{F}^{-1}\varphi)(2^l \cdot) * R_{\theta\alpha} f](x)\end{aligned}$$

for $L_k - 6 \le |l| \le L_k + 6$. Observe also that

$$\sum_{j=-L_k}^{L_k} 2^{(l+2j)n} (\mathcal{F}^{-1}\varphi)(2^l \cdot) * (\mathcal{F}^{-1}\varphi)(2^j \cdot) * (\mathcal{F}^{-1}\widetilde{\varphi})(2^j \cdot) * R_{\theta\alpha} f = 0$$

when $|l| > L_k + 6$.

Putting these observations and Lemma 2.2 together, we obtain

$$\begin{aligned}\|R_{\theta\alpha} f\|_{L^{p(\cdot),q(\cdot)}} &\lesssim \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi)(2^j \cdot) * R_{\theta\alpha} f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}} \\ &= \left\| \left(\sum_{j=-\infty}^{\infty} |2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * R_{\theta\alpha} f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}.\end{aligned}$$

Let $\gamma(\beta)$ be the constant such that

$$\mathcal{F} R_\beta f(\xi) = \gamma(\beta) |\xi|^{-\beta} \mathcal{F} f(\xi) \tag{4.4}$$

(see for example [**45**, p. 64]). If we write

$$\varphi_\kappa(\xi) = \begin{cases} 0 & \xi = 0, \\ \varphi(\xi)|\xi|^{-\kappa} & \text{otherwise} \end{cases} \tag{4.5}$$

for $\kappa \in \mathbb{R}$, then, from (4.4) and the Fourier transform, we have

$$2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * R_{\theta\alpha} f = c_n \gamma(\theta\alpha) 2^{j(n-\theta\alpha)} (\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j \cdot) * f,$$

so that

$$\|R_{\theta\alpha} f\|_{L^{p(\cdot),q(\cdot)}} \sim \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\theta\alpha)} |(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j \cdot) * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot),q(\cdot)}}.$$

The Hölder inequality and the equality

$$\left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2} = \left(\sum_{j=-\infty}^{\infty} |2^{j(n-\theta\alpha)}(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2}$$

yield the pointwise estimate; we have

$$\begin{aligned}
&\left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2} \\
&\le \left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{(1-\theta)/2} \\
&\quad \times \left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{\theta/2} \\
&= \frac{1}{\gamma(\alpha)^\theta}\left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{(1-\theta)/2} \\
&\quad \times \left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{(\theta-1)\alpha})(2^j\cdot)*R_\alpha f|^2\right)^{\theta/2}. \qquad (4.6)
\end{aligned}$$

If we invoke (4.3), then we have

$$\begin{aligned}
&\left\|\left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2}\right\|_{L^{p(\cdot),q(\cdot)}} \\
&\lesssim \left\|\left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2}\right\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \\
&\quad \times \left\|\left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{(\theta-1)\alpha})(2^j\cdot)*R_\alpha f|^2\right)^{1/2}\right\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta}.
\end{aligned}$$

Note that

$$\left\|\left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f|^2\right)^{1/2}\right\|_{L^{p_1(\cdot),q_1(\cdot)}} \sim \|f\|_{L^{p_1(\cdot),q_1(\cdot)}}$$

and that

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn} |(\mathcal{F}^{-1}\varphi_{(\theta-1)\alpha})(2^j \cdot) * R_\alpha f|^2 \right)^{1/2} \right\|_{L^{p_2(\cdot),q_2(\cdot)}} \sim \|R_\alpha f\|_{L^{p_2(\cdot),q_2(\cdot)}}$$

again by virtue of Theorem 1.1; remark that $\varphi_{\theta\alpha}$ and $\varphi_{(\theta-1)\alpha}$ satisfy the assumption (1.2) and (1.3) of Theorems 1.1 and 3.2, since φ does. If we combine the observations above, we obtain (3.11) and then (1.9).

The proof of Theorem 1.7 is similar to the above by using Proposition 1.6, Theorem 4.2 and (4.6).

Remark 4.3. The inequality

$$\|R_{\theta\alpha} f\|_{L^{p(\cdot),q(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \|R_\alpha f\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta}$$

hold for all $f \in L^{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ provided one of the following conditions is fulfilled.

1. $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$.
2. $R_\alpha |f| \in L^{p_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$.
3. The origin 0 is not contained in the support of $\mathcal{F}f$.

The proof is just a matter of reexamination of the above proof. In particular, when $\mathcal{F}f$ does not have 0 as its support, Theorem 3.2 is directly applicable and the same argument works.

4.3. Extension of Theorem 1.4.

We reexamine the fundamental inequality (Theorem 1.4) and we obtain the following extension. We define

$$R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k \cdot) * f] := \gamma(\alpha)\mathcal{F}^{-1}[|\xi|^{-\alpha}\mathcal{F}[(\mathcal{F}^{-1}\varphi)(2^k \cdot) * f]]$$

so that $R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k \cdot) * f]$ makes sense for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 4.4. *Let $p, p_1, p_2 : \mathbb{R}^n \to (1, \infty)$ satisfy* (p1), (p2) *and* (p3) *and let $q, q_1, q_2 : \mathbb{R}^n \to \mathbb{R}$ satisfy* (q1) *and* (q2). *Assume that these functions are related by*

$$\frac{1}{p(x)} = \frac{1-\theta}{p_1(x)} + \frac{\theta}{p_2(x)}, \quad \frac{q(x)}{p(x)} = (1-\theta)\frac{q_1(x)}{p_1(x)} + \theta\frac{q_2(x)}{p_2(x)}$$

for some $\theta \in (0, 1)$. Define $\varphi_{\theta\alpha}$ by (4.5). *Then, for $f \in L^{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\begin{aligned}
&\left\| \sum_{j=-\infty}^{\infty} 2^{j(n-\theta\alpha)} |(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot) * f| \right\|_{L^{p(\cdot),q(\cdot)}} \\
&\lesssim \left\| \sup_{k\in\mathbb{Z}} (2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot) * f|) \right\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \\
&\quad \times \left\| \sup_{k\in\mathbb{Z}} (2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot) * f]|) \right\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta}.
\end{aligned}$$

For a proof of Theorem 4.4, we note the following lemma.

Lemma 4.5. *Let $A, B > 0$ and $\kappa_1, \kappa_2 > 0$. Then*

$$\sum_{j=-\infty}^{\infty} \min(2^{-j\kappa_1}A, 2^{j\kappa_2}B) \lesssim A^{\kappa_2/(\kappa_1+\kappa_2)} B^{\kappa_1/(\kappa_1+\kappa_2)}.$$

Proof. We calculate directly:

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \min(2^{-j\kappa_1}A, 2^{j\kappa_2}B) &= \sum_{j<\frac{1}{\kappa_1+\kappa_2}\log_2\frac{A}{B}} 2^{j\kappa_2}B + \sum_{j\geq\frac{1}{\kappa_1+\kappa_2}\log_2\frac{A}{B}} 2^{-j\kappa_1}A \\
&\lesssim \left(\frac{A}{B}\right)^{\kappa_2/(\kappa_1+\kappa_2)} B = A^{\kappa_2/(\kappa_1+\kappa_2)} B^{\kappa_1/(\kappa_1+\kappa_2)}.
\end{aligned}$$

So the result follows. □

Proof of Theorem 4.4. Recall that φ satisfies (1.2), that is, $\operatorname{supp}(\varphi) \subset B(8) \setminus B(1)$. Also, $\varphi_{\theta\alpha}$ is given by (4.5) with $\kappa = \theta\alpha$. Therefore, if we choose $\tau \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ so that $\operatorname{supp}(\tau) \subset B(16) \setminus B(1/2)$ and that $\tau \equiv 1$ on $B(8) \setminus B(1)$, then we have

$$\varphi_{\theta\alpha}(\xi) = \tau(\xi)\varphi_{\theta\alpha}(\xi) = \tau(\xi)|\xi|^{-\theta\alpha}\varphi(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

With this in mind, let us denote $\tau_\kappa(\xi) = \tau(\xi)|\xi|^{-\kappa}$ for $\xi \in \mathbb{R}^n\setminus\{0\}$ and $\kappa > 0$. If we define $\tau_\kappa(0) = 0$, then it follows from the support condition that $\tau_{\theta\alpha} \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. As a result, we obtain

$$2^{jn}(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot) * f = c_n 2^{2jn}(\mathcal{F}^{-1}\tau_{\theta\alpha})(2^j\cdot) * (\mathcal{F}^{-1}\varphi)(2^j\cdot) * f$$

and $\mathcal{F}^{-1}\tau_{\theta\alpha} \in \mathcal{S}(\mathbb{R}^n)$. Consequently, if we invoke Proposition 2.1, then we have

$$2^{jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f(x)| \lesssim M[2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot)*f](x).$$

In particular,

$$2^{jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f(x)| \lesssim M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f|)\right](x). \tag{4.7}$$

Likewise, we have an equality

$$2^{jn}(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f = c_{n,\alpha}2^{2jn+j\alpha}(\mathcal{F}^{-1}\tau_{(\theta-1)\alpha})(2^j\cdot)*(\mathcal{F}^{-1}\varphi)(2^j\cdot)*R_\alpha f,$$

which yields

$$2^{jn}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f(x)| \lesssim 2^{j\alpha}M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f]|)\right](x) \tag{4.8}$$

by Proposition 2.1 again. We replace (4.6) with the following pointwise estimate. By virtue of (4.7) and (4.8), we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} 2^{j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f| \\
&\lesssim \sum_{j=-\infty}^{\infty} \min\left\{2^{-j\theta\alpha}M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f|)\right],\right. \\
&\qquad\qquad \left. 2^{j(1-\theta)\alpha}M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f]|)\right]\right\} \\
&\lesssim \left(M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f|)\right]\right)^{1-\theta}\left(M\left[\sup_{k\in\mathbb{Z}}(2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f]|)\right]\right)^{\theta}.
\end{aligned}$$

Here for the last inequality, we used Lemma 4.5. By this inequality, (4.3) and Lemma 2.2, we can go through the same argument as Theorem 1.4. □

To compare our results, it may be of use to observe the following:

Lemma 4.6. *Let $0<\alpha<n$. Define a Banach space $X_\alpha(\mathbb{R}^n)$ of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-functions by the norm*

$$\|f\|_{X_\alpha} = \int_{\mathbb{R}^n}(1+|y|)^{\alpha-n}|f(y)|dy.$$

1. *The space $X_\alpha(\mathbb{R}^n)$ is continuous embedded into $\mathcal{S}'(\mathbb{R}^n)$. More precisely,*

$$\int_{\mathbb{R}^n} |f(x)u(x)|\,dx \lesssim \left(\sup_{x\in\mathbb{R}^n}(1+|x|)^{-\alpha+n}|u(x)|\right)\|f\|_{X_\alpha}$$

for all $f \in X_\alpha(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$.

2. *The integral $R_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy$ also defines an element in $\mathcal{S}'(\mathbb{R}^n)$. More precisely,*

$$\int_{\mathbb{R}^n} |u(x)R_\alpha f(x)|\,dx \lesssim \left(\sup_{x\in\mathbb{R}^n}(1+|x|)^{n+1}|u(x)|\right)\|f\|_{X_\alpha}$$

for all $f \in X_\alpha(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} u(x)R_\alpha f(x)\,dx = \int_{\mathbb{R}^n} R_\alpha u(x)f(x)\,dx$$

for all $f \in X_\alpha(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$.

The proof of this lemma is based upon the observation that

$$\begin{aligned} |R_\alpha u(x)| &\le \left(\sup_{z\in\mathbb{R}^n}(1+|z|)^{n+1}|u(z)|\right)\int_{\mathbb{R}^n} \frac{|x-y|^{\alpha-n}}{(1+|y|)^{n+1}}\,dy \\ &\lesssim \left(\sup_{z\in\mathbb{R}^n}(1+|z|)^{n+1}|u(z)|\right)(1+|x|)^{\alpha-n}. \end{aligned}$$

Remark 4.7. We claim that Theorem 4.4 covers Theorem 1.4. As before, a passage to the limit allows us to consider $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ under the condition of Theorem 1.4.

If necessary, we can choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in (1.2) so that

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) \equiv \chi_{\mathbb{R}^n\setminus\{0\}}(\xi) \quad (\xi \in \mathbb{R}^n).$$

Note that $R_\alpha f$ is an $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ function and that Proposition 2.1 yields pointwise estimates

$$\sup_{k\in\mathbb{Z}}(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot) * f|) \lesssim Mf$$

$$\sup_{k\in\mathbb{Z}}(2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f]|) = \sup_{k\in\mathbb{Z}}(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot)*R_\alpha f|) \lesssim M[R_\alpha f].$$

By Lemma 2.2 we have

$$\left\|\sup_{k\in\mathbb{Z}}\left(2^{kn}|(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f|\right)\right\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \lesssim \|f\|_{L^{p_1(\cdot),q_1(\cdot)}}^{1-\theta} \tag{4.9}$$

$$\left\|\sup_{k\in\mathbb{Z}}\left(2^{kn}|R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k\cdot)*f]|\right)\right\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta} \lesssim \|R_\alpha f\|_{L^{p_2(\cdot),q_2(\cdot)}}^{\theta}. \tag{4.10}$$

Observe that

$$f(x) = \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot)*f(x)$$

in the topology of $L^{1+\varepsilon}(\mathbb{R}^n)$, where $0<\varepsilon<(n/\theta\alpha)-1$. Note that $R_{\theta\alpha}$ is a bounded linear operator from $L^{1+\varepsilon}(\mathbb{R}^n)$ to $L^{n(1+\varepsilon)/(n-\theta\alpha(1+\varepsilon))}(\mathbb{R}^n)$, since

$$\frac{1}{1+\varepsilon}-\frac{\theta\alpha}{n}=\frac{n-\theta\alpha(1+\varepsilon)}{n(1+\varepsilon)}.$$

Hence, it follows that

$$R_{\theta\alpha}f(x) = \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot)*R_{\theta\alpha}f(x)$$

in the topology of $L^{n(1+\varepsilon)/(n-\theta\alpha(1+\varepsilon))}(\mathbb{R}^n)$. Also, by the triangle inequality and the fact above, we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f(x)| &\geq \left|\sum_{j=-\infty}^{\infty} 2^{j(n-\theta\alpha)}(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot)*f(x)\right| \\ &\gtrsim \left|\sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot)*R_{\theta\alpha}f(x)\right| \\ &= |R_{\theta\alpha}f(x)|.\end{aligned}$$

Consequently, we obtain

$$\sum_{j=-\infty}^{\infty} 2^{j(n-\theta\alpha)}|(\mathcal{F}^{-1}\varphi_{\theta\alpha})(2^j\cdot) * f(x)| \gtrsim |R_{\theta\alpha}f(x)|. \tag{4.11}$$

From (4.9)–(4.11), we see that Theorem 4.4 covers Theorem 1.4.

The following example illustrates that Theorem 4.4 actually extends Theorem 1.4.

EXAMPLE 4.8. Let Ψ, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be taken so that they are $\mathbb{R}$-valued and that

$$\begin{aligned} &\mathrm{supp}(\Psi) \subset B\left(\frac{41}{10}\right)\Big\backslash B\left(\frac{19}{10}\right), \quad &&\Psi \equiv 1 \text{ on } B(4) \setminus B(2),\\ &\mathrm{supp}(\Phi) \subset B\left(\frac{1}{10}\right), &&\Phi \equiv 1 \text{ on } B\left(\frac{1}{20}\right). \end{aligned}$$

We set $\varphi(\xi) := \Psi(\xi)$. Note that φ is satisfies the requirement of Theorem 1.1 by virtue of (1.7).

The function given by

$$f = \sum_{j=1}^{\infty} \mathcal{F}^{-1}[\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \ldots, 0))]$$

is a typical example to which Theorem 4.4 is applicable but Theorem 1.4 is not. Strictly speaking, we need to truncate f so that we have $f \in L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. However, we omit the details of this tedious and routine argument.

To see this, we first note that

$$\begin{aligned} \mathrm{supp}(\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \ldots, 0))) &\subset B((3\cdot 2^{j-1}, 0, 0, \ldots, 0), 10^{-1})\\ &\subset B(3\cdot 2^{j-1} + 10^{-1}) \backslash B(3\cdot 2^{j-1} - 10^{-1})\\ &\subset B\left(\frac{31}{10}\cdot 2^{j-1}\right)\Big\backslash B\left(\frac{29}{10}\cdot 2^{j-1}\right) \end{aligned}$$

for all $j \in \mathbb{N}$.

Let $k \in \mathbb{Z}$. Observe that the relation

$$\mathrm{supp}(\Psi(2^{-k}\cdot)) \subset B\left(\frac{41}{10}\cdot 2^k\right) \Big\backslash B\left(\frac{19}{10}\cdot 2^k\right)$$

implies

$$\Psi(2^{-k+1}\cdot)\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0)) \equiv 0$$

if $j \neq k$. Meanwhile if $j = k$, the relation

$$\Psi(2^{-j}\cdot) \equiv 1 \text{ on } B(4\cdot 2^j) \setminus B(2\cdot 2^j)$$

implies

$$\Psi(2^{-k+1}\cdot)\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0)) = \Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0)).$$

Consequently

$$\Psi(2^{-k+1}\cdot)\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0)) = \delta_{jk}\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0)) \quad (j \in \mathbb{N},\ k \in \mathbb{Z}),$$

which yields

$$2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f = \begin{cases} \mathcal{F}^{-1}[\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0))] & (j \geq 1), \\ 0 & (j \leq 0) \end{cases}$$

and

$$2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * R_\alpha f \sim \begin{cases} \mathcal{F}^{-1}[|\cdot|^{-\alpha}\Phi(\cdot - (3\cdot 2^{j-1}, 0, 0, \dots, 0))] & (j \geq 1), \\ 0 & (j \leq 0). \end{cases}$$

Consequently, for $j \geq 1$, we obtain

$$\sup_{j\in\mathbb{Z}} 2^{jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f| = |\mathcal{F}^{-1}\Phi| \tag{4.12}$$

$$\sup_{j\in\mathbb{Z}} 2^{jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot) * R_\alpha f(x)| \lesssim (1+|x|)^{-n-1} \tag{4.13}$$

$$\sum_{j\in\mathbb{Z}} |2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 = \chi_{\{\mathcal{F}^{-1}\Phi\neq 0\}} \times \infty. \tag{4.14}$$

Indeed, (4.13) is verified by the following:

$$\int_{\mathbb{R}^n} \left| \frac{\partial^{m_1}}{\partial {x_1}^{m_1}} \frac{\partial^{m_2}}{\partial {x_1}^{m_2}} \cdots \frac{\partial^{m_n}}{\partial {x_1}^{m_n}} \Phi(\xi - (3 \cdot 2^{j-1}, 0, \dots, 0)) |\xi|^{-\alpha} \right| d\xi_1 \, d\xi_2 \cdots d\xi_n \lesssim 1$$

for all $j \geq 1$ and all multiindices $(m_1, m_2, \dots, m_n)$. Since $\mathcal{F}^{-1}\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\{\mathcal{F}^{-1}\Phi \neq 0\}$ is a set of positive measure, we have

$$\left\| \sup_{k \in \mathbb{Z}} (2^{kn} |(\mathcal{F}^{-1}\varphi)(2^k \cdot) * f|) \right\|_{L^{p_1(\cdot), q_1(\cdot)}}^{1-\theta} \left\| \sup_{k \in \mathbb{Z}} (2^{kn} |R_\alpha[(\mathcal{F}^{-1}\varphi)(2^k \cdot) * f]|) \right\|_{L^{p_2(\cdot), q_2(\cdot)}}^{\theta} < \infty,$$

while

$$\|f\|_{L^{p_1(\cdot), q_1(\cdot)}}^{1-\theta} \|R_\alpha f\|_{L^{p_2(\cdot), q_2(\cdot)}}^{\theta} = \infty.$$

This example shows that f is beyond the reach of Theorem 1.4 but that Theorem 4.4 can control such f.

As a corollary of Theorem 4.4, we have the boundedness of the maximal operator associated to the Gagliardo-Nirenberg inequality.

COROLLARY 4.9. *Under the same condition as Theorem* 1.4, *one has*

$$\left\| \sup_{-\infty < K \leq L < \infty} \left| \sum_{j=K}^{L} R_{\theta\alpha} \left[2^{jn} (\mathcal{F}^{-1}\varphi)(2^j \cdot) * f \right] \right| \right\|_{L^{p(\cdot), q(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot), q_1(\cdot)}}^{1-\theta} \|R_\alpha f\|_{L^{p_2(\cdot), q_2(\cdot)}}^{\theta}.$$

5. Related estimates.

5.1. Related estimates for Morrey spaces and Hardy-Morrey spaces.

Suppose that the parameters p, q, r, s satisfy

$$0 < q \leq p < \infty, \quad 0 < r \leq \infty, \quad s \in \mathbb{R}.$$

Define the Morrey norm of a measurable function f by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{x \in \mathbb{R}^n, \, r > 0} r^{(n/p)-(n/q)} \left(\int_{B(x,r)} |f(y)|^q \, dy \right)^{1/q}.$$

The parameter p seems to reflect the global integrability while q describes the local integrability.

Proposition 5.1 ([**27**, Proposition 4.1]). *Keep to the same notations for the functions φ and $\widetilde{\varphi}$ as Theorem* 1.1. *Assume that $1 < q \le p < \infty$. Then we have the following equality and norm equivalence*: *For all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ we have*

$$f = c_n \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f, \quad in \quad \mathcal{S}'(\mathbb{R}^n),$$

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{\mathcal{M}^p_q} \sim \|f\|_{\mathcal{M}^p_q}.$$

We set $\psi_t(x) = t^{-n}\exp(-|x|^2/t^2)$ for $t > 0$ and $x \in \mathbb{R}^n$ as before. Then we define the Hardy-Morrey space $H\mathcal{M}^p_q(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{H\mathcal{M}^p_q} := \left\| \sup_{t>0} |\psi_t * f(\cdot)| \right\|_{\mathcal{M}^p_q} < \infty.$$

Proposition 5.2 ([**38**, Theorem 4.2]). *Keep to the same notations for the functions φ and $\widetilde{\varphi}$ as Theorem* 1.1. *Assume that $0 < q \le p < \infty$. Then we have the following equality and norm equivalence*: *For all $f \in H\mathcal{M}^p_q(\mathbb{R}^n)$ we have*

$$f = c_n \sum_{j=-\infty}^{\infty} 2^{jn}(\mathcal{F}^{-1}\varphi)(2^j\cdot) * 2^{jn}(\mathcal{F}^{-1}\widetilde{\varphi})(2^j\cdot) * f, \quad in \quad \mathcal{S}'(\mathbb{R}^n),$$

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{2jn}|(\mathcal{F}^{-1}\varphi)(2^j\cdot) * f|^2 \right)^{1/2} \right\|_{\mathcal{M}^p_q} \sim \|f\|_{H\mathcal{M}^p_q}.$$

The Hölder inequality for Morrey spaces is as follows: The proof is straightforward by the Hölder inequality for Lebesgue spaces.

Proposition 5.3. *Let $0 < q_1 \le p_1 < \infty$, $0 < q_2 \le p_2 < \infty$ and let $0 < q \le p < \infty$. Assume that these parameters are related by*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

for some $\theta \in (0,1)$. Then,

$$\|f^{1-\theta}g^{\theta}\|_{\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{\mathcal{M}^{p_1}_{q_1}} \|g\|^{\theta}_{\mathcal{M}^{p_2}_{q_2}}$$

for all positive measurable functions f and g.

The following theorem can be proven in the same way as Theorems 1.4 or 1.7 by using Propositions 5.1, 5.2 and 5.3.

THEOREM 5.4. *Let $0 < q_1 \le p_1 < \infty$, $0 < q_2 \le p_2 < \infty$ and let $0 < q \le p < \infty$. Assume that these parameters are related by*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

for some $\theta \in (0,1)$.

1. *Assume $q_1, q_2, q > 1$. Then, for $f \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$ satisfying $R_\alpha|f| \not\equiv \infty$,*

$$\|R_{\theta\alpha} f\|_{\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{\mathcal{M}^{p_1}_{q_1}} \|R_\alpha f\|^{\theta}_{\mathcal{M}^{p_2}_{q_2}}.$$

2. *For $f \in H\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$ such that $0 \notin \mathrm{supp}(\mathcal{F}f)$,*

$$\|R_{\theta\alpha} f\|_{H\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{H\mathcal{M}^{p_1}_{q_1}} \|R_\alpha f\|^{\theta}_{H\mathcal{M}^{p_2}_{q_2}}.$$

5.2. Related estimates for Stochastic process.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of sub σ-fields. Assume that $\mathcal{F}_1$ contains all null sets in $\mathcal{F}$ and that $\mathcal{F} = \sigma\big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\big)$. Let $1 < p < \infty$. According to the well-known Burkholder-Gundy-Davis inequality, we have

$$\|X\|_{L^p(\Omega)} \sim \|E[X \mid \mathcal{F}_1]\|_{L^p(\Omega)} + \left\| \left(\sum_{j=1}^{\infty} |E[X \mid \mathcal{F}_{j+1}] - E[X \mid \mathcal{F}_j]|^2 \right)^{1/2} \right\|_{L^p(\Omega)}$$

for all $X \in L^p(\Omega)$, where $E[X|\mathcal{G}]$ denotes the conditional expectation of a σ-field $\mathcal{G}$.

In analogy with Theorem 1.1, we have the following result.

PROPOSITION 5.5. *For $\theta \in (0,1)$ and $s \in \mathbb{R}$ we have*

$$\left\| E[X \mid \mathcal{F}_1] + \sum_{j=1}^{\infty} 2^{j\theta s}(E[X \mid \mathcal{F}_{j+1}] - E[X \mid \mathcal{F}_j]) \right\|_{L^p(\Omega)}$$

$$\lesssim \|X\|_{L^p(\Omega)}^{1-\theta} \left\| E[X \mid \mathcal{F}_1] + \sum_{j=1}^{\infty} 2^{js}(E[X \mid \mathcal{F}_{j+1}] - E[X \mid \mathcal{F}_j]) \right\|_{L^p(\Omega)}^{\theta}$$

for all $X \in L^p(\Omega)$ *such that* $E[X \mid \mathcal{F}_1] + \sum_{j=1}^{\infty} 2^{js}(E[X \mid \mathcal{F}_{j+1}] - E[X \mid \mathcal{F}_j])$ *converges in the topology of* $L^p(\Omega)$.

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Yoshihiro MIZUTA
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8521, Japan

Current address:
Department of Mechanical Systems Engineering
Hiroshima Institute of Technology
2-1-1 Miyake Saeki-ku
Hiroshima 731-5193, Japan
E-mail: yomizuta@hiroshima-u.ac.jp

Eiichi NAKAI
Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: enakai@cc.osaka-kyoiku.ac.jp

Current address:
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310-8512, Japan
E-mail: enakai@mx.ibaraki.ac.jp

Yoshihiro SAWANO
Department of Mathematics
Kyoto University
Kyoto, 606-8502, Japan
E-mail: yosihiro@math.kyoto-u.ac.jp

Current address:
Department of Mathematics
and Information Science
Tokyo Metropolitan University
1-1 minami-Ohsawa, Hachioji
Tokyo 192-0397, Japan
E-mail: ysawano@tmu.ac.jp

Tetsu SHIMOMURA
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: tshimo@hiroshima-u.ac.jp