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Conjugate functions on spaces of parabolic Bloch type

By Yôsuke Hishikawa, Masaharu Nishio and Masahiro Yamada

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Abstract. Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space. Let $0<\alpha\le 1$ and $m(\alpha)=\min\{1,1/(2\alpha)\}$. For $\sigma>-m(\alpha)$, the α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$ on H is the set of all solutions u of the equation $(\partial/\partial t+(-\Delta_x)^\alpha)u=0$ with finite Bloch norm $\|u\|_{\mathcal{B}_\alpha(\sigma)}$ of a weight t^σ. It is known that $\mathcal{B}_{1/2}(0)$ coincides with the classical harmonic Bloch space on H. We extend the notion of harmonic conjugate functions to functions in the α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$. We study properties of α-parabolic conjugate functions and give an application to the estimates of tangential derivative norms on $\mathcal{B}_\alpha(\sigma)$. Inversion theorems for α-parabolic conjugate functions are also given.

1. Introduction.

Let $n\ge 1$ and let H be the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H=\{X=(x,t)\in\mathbb{R}^{n+1}: x=(x_1,\dots,x_n)\in\mathbb{R}^n,\ t>0\}$. For $0<\alpha\le 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)}:=\partial_t+(-\Delta_x)^\alpha, \tag{1.1}$$

where $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, and $\Delta_x=\partial_1^2+\cdots+\partial_n^2$. A continuous function (real-valued) u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u=0$ in the sense of distributions. (For details, see Section 2.) Put $m(\alpha)=\min\{1,1/(2\alpha)\}$. For a real number $\sigma>-m(\alpha)$, let $\mathcal{B}_\alpha(\sigma)$ be the set of all $L^{(\alpha)}$-harmonic functions $u\in C^1(H)$ with the norm

$$\|u\|_{\mathcal{B}_\alpha(\sigma)}:=|u(0,1)|+\sup_{(x,t)\in H}t^\sigma\bigl\{t^{1/(2\alpha)}|\nabla_x u(x,t)|+t|\partial_t u(x,t)|\bigr\}<\infty, \tag{1.2}$$

where $\nabla_x=(\partial_1,\dots,\partial_n)$. We call $\mathcal{B}_\alpha(\sigma)$ the α-parabolic Bloch type space. Since

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$\mathcal{B}_\alpha(\sigma)$ contains constant functions, we may identify $\mathcal{B}_\alpha(\sigma)/\mathbb{R} \cong \widetilde{\mathcal{B}}_\alpha(\sigma)$, where

$$\widetilde{\mathcal{B}}_\alpha(\sigma) := \{u \in \widetilde{\mathcal{B}}_\alpha(\sigma) : u(0,1) = 0\}.$$

The α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$ is introduced and studied in [**4**]. In this paper, we continue to investigate further properties of this space. In particular, we study properties of conjugate functions on $\mathcal{B}_\alpha(\sigma)$. In [**3**], the notion of conjugate functions on α-parabolic Bergman spaces was introduced and studied (we call them α-parabolic conjugate functions). For $1 \leq p < \infty$ and $\lambda > -1$, the α-parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions on H which belong to $L^p(H, t^\lambda dV)$, where dV is the Lebesgue measure on H. In [**3**], we studied the existance of α-parabolic conjugate functions V of $u \in \boldsymbol{b}_\alpha^p(\lambda)$, and we also gave estimates of the norms of V. Furthermore, as an application, we showed that tangential derivative norms of $u \in \boldsymbol{b}_\alpha^p(\lambda)$ are comparable to the norms of u. The α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$ can be considered the appropriate limit of the α-parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ as $p \to \infty$ by [**4**, Theorem 3]. In this paper, we also introduce the notion of α-parabolic conjugate functions of $u \in \mathcal{B}_\alpha(\sigma)$, and we study properties of them. Using these results, we shall give estimates of the tangential derivative norms of $u \in \mathcal{B}_\alpha(\sigma)$. Moreover, we shall give inversion theorems, that is, for a function V, we construct a function $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ such that V is an α-parabolic conjugate function of u.

We describe the main results of this paper. We begin with recalling the definition of conjugate functions of usual harmonic functions with $(n+1)$-variables. For a harmonic function u on H, a vector-valued function $V = (v_1, \ldots, v_n)$ on H is called a harmonic conjugate of u if $v_j \in C^1(H)$ and V satisfies the following generalized Cauchy-Riemann equations:

$$\partial_t v_j = \partial_j u, \quad \partial_k v_j = \partial_j v_k, \quad 1 \leq j,\ k \leq n, \tag{1.3}$$

and

$$-\partial_t u = \sum_{j=1}^{n} \partial_j v_j \tag{1.4}$$

(see [**9**]). When $\alpha = 1/2$ and $\sigma = 0$, it is known by [**4**, Remark 3.3] that $\mathcal{B}_{1/2}(0)$ coincides with the classical harmonic Bloch space of Ramey and Yi [**8**]. In [**8**], properties of harmonic conjugates of $u \in \mathcal{B}_{1/2}(0)$ are studied and estimates of the tangential derivative norms of u are given. As in the definition of α-parabolic conjugate functions on $\boldsymbol{b}_\alpha^p(\lambda)$ of [**3**], we introduce the notion of conjugacy to $\mathcal{B}_\alpha(\sigma)$. In $\mathcal{B}_\alpha(\sigma)$, we can not obtain similar generalization to $\boldsymbol{b}_\alpha^p(\lambda)$, because the case $\alpha = 1$

is critical. However, we shall observe that the case $\alpha = 1$ is referred to the case $0 < \alpha < 1$. First, we give the definition of α-parabolic conjugate functions on $\mathcal{B}_\alpha(\sigma)$ with $0 < \alpha < 1$.

DEFINITION 1. Let $0 < \alpha < 1$ and $\sigma > -m(\alpha)$. For a function $u \in \mathcal{B}_\alpha(\sigma)$, we shall say that a vector-valued function $V = (v_1, \ldots, v_n)$ on H is *an α-parabolic conjugate function of u* if $v_j \in C^1(H)$ and V satisfies the following equations:

$$-\mathcal{D}_t v_j = \partial_j u, \quad \partial_k v_j = \partial_j v_k, \quad 1 \leq j,\ k \leq n, \tag{C.1}$$

and

$$\mathcal{D}_t^{1/\alpha - 1} u = \sum_{j=1}^{n} \partial_j v_j, \tag{C.2}$$

where $\mathcal{D}_t = -\partial_t$ and $\mathcal{D}_t^{1/\alpha-1}$ is the fractional differential operator defined in Section 2.

We remark that when $\alpha = 1/2$, Equations (C.1) and (C.2) coincide with the generalized Cauchy-Riemann equations (1.3) and (1.4). In [**3**], the previous generalization to the α-parabolic Bergman spaces was given by (C.1) and (C.2) for all $0 < \alpha \leq 1$. When $0 < \alpha < 1$, properties of α-parabolic conjugate functions are given in Theorem 1.

THEOREM 1. *Let $0 < \alpha < 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Put $\eta := 1/(2\alpha) - 1 + \sigma$. If $\eta > -1/(2\alpha)$, then there exists a unique α-parabolic conjugate function $V = (v_1, \ldots, v_n)$ of u such that $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$. Also, there exists a constant $C = C(n, \alpha, \sigma) > 0$ independent of u such that*

$$C^{-1} \|u\|_{\mathcal{B}_\alpha(\sigma)} \leq \sum_{j=1}^{n} \|v_j\|_{\mathcal{B}_\alpha(\eta)} \leq C \|u\|_{\mathcal{B}_\alpha(\sigma)}. \tag{1.5}$$

The condition of η in Theorem 1 is equivalent to $\sigma > 1 - 1/\alpha$. If $0 < \alpha \leq 1/2$, then this condition is always satisfied for all $\sigma > -m(\alpha)$. However, when $1/2 < \alpha < 1$, we do not know whether Theorem 1 can be extended to the full range $\sigma > -m(\alpha)$.

When $\alpha = 1$, we shall introduce the notion of conjugacy to $\mathcal{B}_1(\sigma)$. If $\alpha = 1$, then the condition $\eta > -1/(2\alpha)$ is equivalent to $\sigma > 0$. Thus, when $\alpha = 1$ and $\sigma > 0$, we construct a counterpart of Definition 1.

DEFINITION 2. Let $\alpha = 1$ and $\sigma > 0$. For a function $u \in \mathcal{B}_1(\sigma)$, we shall say that a vector-valued function $V = (v_1, \dots, v_n)$ on H is *a* 1*-parabolic conjugate function of* u if $v_j \in C^1(H)$ and V satisfies the following equations:

$$-\mathcal{D}_t v_j = \partial_j u, \quad \partial_k v_j = \partial_j v_k, \quad 1 \le j,\ k \le n, \tag{C.1}$$

and

$$u - \lim_{t\to\infty} u(0,t) = \sum_{j=1}^{n} \partial_j v_j. \tag{C.2$'$}$$

We note that $\lim_{t\to\infty} u(0,t) = 0$ for $u \in \boldsymbol{b}_\alpha^p(\lambda)$, so that (C.2′) can be replaced by a straightforward generalization $u = \sum_{j=1}^n \partial_j v_j$ of (C.2). However, if $u \in \mathcal{B}_\alpha(\sigma)$, then $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u(0,t)$ for all $x \in \mathbb{R}^n$ and yet the limit need not vanish (see Remark 5.6 and Example 5.7). So, we have to subtract $\lim_{t\to\infty} u(0,t)$ from u as in the left hand side of (C.2′). We give properties of α-parabolic conjugate functions on $\mathcal{B}_\alpha(\sigma)$ for $\alpha = 1$.

THEOREM 2. *Let $\sigma > 0$ and $u \in \widetilde{\mathcal{B}}_1(\sigma)$. Put $\eta := 1/(2\alpha) - 1 + \sigma$ with $\alpha = 1$. Then, there exists a unique* 1*-parabolic conjugate function $V = (v_1, \dots, v_n)$ of u such that $v_j \in \widetilde{\mathcal{B}}_1(\eta)$. Also, there exists a constant $C = C(n,\sigma) > 0$ independent of u such that*

$$C^{-1}\|u\|_{\mathcal{B}_1(\sigma)} \le \sum_{j=1}^{n} \|v_j\|_{\mathcal{B}_1(\eta)} \le C\|u\|_{\mathcal{B}_1(\sigma)}. \tag{1.6}$$

As an application of Theorems 1 and 2, we give estimates of tangential derivative norms on $\widetilde{\mathcal{B}}_\alpha(\sigma)$. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Furthermore, for a function f on H, let $\|f\|_{L^\infty} = \operatorname{ess\,sup}\{|f(x,t)| : (x,t) \in H\}$.

THEOREM 3. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Then, for each $m \in \mathbb{N}$, there exists a constant $C = C(n,\alpha,\sigma,m) > 0$ independent of u such that*

$$C^{-1}\|u\|_{\mathcal{B}_\alpha(\sigma)} \le \sum_{|\gamma|=m} \|t^{m/(2\alpha)+\sigma}\partial_x^\gamma u\|_{L^\infty} \le C\|u\|_{\mathcal{B}_\alpha(\sigma)}. \tag{1.7}$$

We describe the construction of this paper. In Section 2, the definition of $L^{(\alpha)}$-harmonic functions is presented. Furthermore, fractional calculus is important tool for analysis of α-parabolic conjugate functions. Therefore, we also recall the definition of fractional differential operators and describe fundamental results of them. In Section 3, we list basic lemmas which shall be used in later arguments. In particular, we present some properties of the fundamental solution of $L^{(\alpha)}$ and basic results concerning $\mathcal{B}_\alpha(\sigma)$-functions. In Section 4, we study α-parabolic conjugate functions on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ with $0 < \alpha \leq 1$, and give the proof of Theorem 1. In Section 5, we also prove Theorem 2 which is a counterpart of Theorem 1 to the case of $\alpha = 1$. In Section 6, we give estimates of tangential derivative norms on $\widetilde{\mathcal{B}}_\alpha(\sigma)$, that is, we give the proof of Theorem 3. In Section 7, we study inversion theorems. In Theorems 1 and 2, we construct α-parabolic conjugate functions of $\widetilde{\mathcal{B}}_\alpha(\sigma)$-functions. In this section, for a vector-valued function V, we construct a function $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ such that V is the α-parabolic conjugate function of u.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries.

In this section, we describe definitions of $L^{(\alpha)}$-harmonic functions and fractional differential operators.

Let $C_c^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H with compact support. Then, for $0 < \alpha < 1$, the convolution operator $(-\Delta_x)^\alpha$ is defined by

$$(-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} \big(\psi(x+y,t) - \psi(x,t)\big)|y|^{-n-2\alpha} dy \tag{2.1}$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$ and $\int_H u \cdot \widetilde{L}^{(\alpha)}\psi dV = 0$ for all $\psi \in C_c^\infty(H)$. By (2.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that

$$\mathrm{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S := \mathbb{R}^n \times [t_1, t_2] \tag{2.2}$$

and

$$|\widetilde{L}^{(\alpha)}\psi(x,t)| \leq C(1+|x|)^{-n-2\alpha} \quad \text{for } (x,t) \in S. \tag{2.3}$$

Hence, the condition $\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$ for all $\psi \in C_c^\infty(H)$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dxdt < \infty. \tag{2.4}$$

When $\alpha = 1$, $L^{(1)} = \partial_t - \Delta_x$ is the usual heat operator. For more details of $L^{(\alpha)}$-harmonic functions, see [**6**].

We describe the fractional differential operators. Our definition of fractional differentiation follows that of [**5**]. Let $\mathbb{R}_+ = (0,\infty)$. For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \left\{\varphi \in C(\mathbb{R}_+) : \varphi(t) = O(t^{-\kappa'})\ (t \to \infty) \quad \text{for some } \kappa' > \kappa\right\}. \tag{2.5}$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)}\int_0^\infty \tau^{\kappa-1}\varphi(\tau+t)d\tau, \quad t \in \mathbb{R}_+. \tag{2.6}$$

We put $\mathcal{FC}^0 := C(\mathbb{R}_+)$ and $\mathcal{D}_t^0\varphi := \varphi$. Moreover, let

$$\mathcal{FC}^{\kappa} := \left\{\varphi : \partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}\right\}, \tag{2.7}$$

where $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. Then, we can also define the fractional derivative $\mathcal{D}_t^\kappa\varphi$ of $\varphi \in \mathcal{FC}^\kappa$ by

$$\mathcal{D}_t^\kappa\varphi(t) := \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)}(-\partial_t)^{\lceil\kappa\rceil}\varphi(t), \quad t \in \mathbb{R}_+. \tag{2.8}$$

Clearly, when $\kappa \in \mathbb{N}_0$, the operator $\mathcal{D}_t^\kappa$ coincides with the ordinary differential operator $(-\partial_t)^\kappa$. For a real number κ, we may call both (2.6) and (2.8) the fractional derivatives of φ with order κ. And, we call $\mathcal{D}_t^\kappa$ the fractional differential operator with order κ. We use the following lemma in our later arguments.

Lemma 2.1 ([**1**, Proposition 2.1] and [**2**, Proposition 2.2]). *For real numbers $\kappa,\nu > 0$, the following statements hold.*

- (i) *If $\varphi \in \mathcal{FC}^{-\kappa}$, then $\mathcal{D}_t^{-\kappa}\varphi \in C(\mathbb{R}_+)$.*
- (ii) *If $\varphi \in \mathcal{FC}^{-\kappa-\nu}$, then $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\kappa-\nu}\varphi$.*
- (iii) *If $\partial_t^k\varphi \in \mathcal{FC}^{-\nu}$ for all integers k, $0 \le k \le \lceil\kappa\rceil - 1$ and $\partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)-\nu}$, then $\mathcal{D}_t^\kappa\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\nu}\mathcal{D}_t^\kappa\varphi = \mathcal{D}_t^{\kappa-\nu}\varphi$.*
- (iv) *If $\partial_t^{k+\lceil\nu\rceil}\varphi \in \mathcal{FC}^{-(\lceil\nu\rceil-\nu)}$ for all integers k, $0 \le k \le \lceil\kappa\rceil - 1$, $\partial_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}$ for all integers ℓ, $0 \le \ell \le \lceil\nu\rceil - 1$, and $\partial_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in$*

$\mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, *then* $\mathcal{D}_t^\kappa\mathcal{D}_t^\nu\varphi = \mathcal{D}_t^{\kappa+\nu}\varphi$.

(v) *If* $\partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-\lceil\kappa\rceil}$ *and* $\lim_{t\to\infty}\partial_t^k\varphi(t) = 0$ *for all integers* k, $0 \le k \le \lceil\kappa\rceil - 1$, *then* $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^\kappa\varphi = \varphi$.

3. Basic results.

In this section, we present several basic results, which will be used in later. We begin with recalling the definition of the fundamental solution of the operator $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\ x\cdot\xi)\ d\xi & (t>0) \\ 0 & (t \le 0), \end{cases}$$

where $x\cdot\xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi\cdot\xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H. We note that

$$W^{(\alpha)} > 0 \text{ on } H \quad \text{and} \quad \int_{\mathbb{R}^n} W^{(\alpha)}(x,t)dx = 1 \text{ for all } 0 < t < \infty. \tag{3.1}$$

Furthermore, $W^{(\alpha)} \in C^\infty(H)$. Several properties of fractional derivatives of $W^{(\alpha)}$ are presented in the following lemma.

Lemma 3.1 ([**1**, Theorem 3.1]). *Let* $0 < \alpha \le 1$ *and* $\kappa > -n/(2\alpha)$. *Let* $\gamma \in \mathbb{N}_0^n$ *be a multi-index. Then, the following statements hold.*

(i) *The derivatives* $\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)}$ *and* $\mathcal{D}_t^\kappa\partial_x^\gamma W^{(\alpha)}$ *can be defined, and the equation* $\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)} = \mathcal{D}_t^\kappa\partial_x^\gamma W^{(\alpha)}$ *holds. Furthermore, there exists a constant* $C = C(n,\alpha,\gamma,\kappa) > 0$ *such that*

$$\left|\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)}(x,t)\right| \le C(t+|x|^{2\alpha})^{-(n+|\gamma|)/(2\alpha)-\kappa}$$

for all $(x,t) \in H$.

(ii) *If a real number* ν *satisfies the condition* $\nu+\kappa > -n/(2\alpha)$, *then the derivative* $\mathcal{D}_t^\nu\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)}$ *is well defined, and*

$$\mathcal{D}_t^\nu\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)}(x,t) = \partial_x^\gamma\mathcal{D}_t^{\nu+\kappa}W^{(\alpha)}(x,t)$$

for all $(x,t) \in H$.

(iii) *The derivative* $\partial_x^\gamma\mathcal{D}_t^\kappa W^{(\alpha)}$ *is* $L^{(\alpha)}$*-harmonic on* H.

We also need the following lemma.

Lemma 3.2 ([**3**, Lemma 4.1]). *Let $0 < \alpha \leq 1$. Then,*

$$\big(\mathcal{D}_t^{1/\alpha} + \Delta_x\big) W^{(\alpha)}(x,t) = 0$$

for all $(x,t) \in H$.

For a multi-index $\gamma \in \mathbb{N}_0^n$ and a real number $\nu > -n/(2\alpha)$, in view of Lemma 3.1 (i), we define a function $\omega_\alpha^{\gamma,\nu}$ on $H \times H$ by

$$\omega_\alpha^{\gamma,\nu}(x,t;y,s) = \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s) - \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(-y,1+s)$$

for all $(x,t),(y,s) \in H$. The function $\omega_\alpha^{\gamma,\nu}$ shall be used for defining conjugate functions on $\widetilde{\mathcal{B}}_\alpha(\sigma)$. Several properties of $\omega_\alpha^{\gamma,\nu}$ are the following.

Lemma 3.3 ([**4**, Lemma 5.6]). *Let $0 < \alpha \leq 1$ and $\nu > -n/(2\alpha)$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index. Then, the following statements hold.*

(i) *For every $(x,t) \in H$, there exists a constant $C = C(n,\alpha,\gamma,\nu,x,t) > 0$ such that*

$$\big|\omega_\alpha^{\gamma,\nu}(x,t;y,s)\big| \leq C(1+s+|y|^{2\alpha})^{-(n+|\gamma|)/(2\alpha)-\nu-m(\alpha)}$$

for all $(y,s) \in H$.

(ii) *Let $\rho > -1$ and put $\eta := |\gamma|/(2\alpha) + \nu - \rho - 1$. If $\eta > -m(\alpha)$, then there exists a constant $C = C(n,\alpha,\gamma,\nu,\rho) > 0$ such that*

$$\int_H \big|\omega_\alpha^{\gamma,\nu}(x,t;y,s)\big| s^\rho dV(y,s) \leq C F_{\alpha,\eta}(x,t)$$

for all $(x,t) \in H$, where

$$F_{\alpha,\eta}(x,t) := \begin{cases} 1 + |x|^{-2\alpha\eta} + t^{-\eta} & (0 > \eta > -m(\alpha)) \\ 1 + \log(1+|x|) + |\log t| & (\eta = 0) \\ 1 + t^{-\eta} & (\eta > 0). \end{cases} \tag{3.2}$$

We also present basic properties of fractional derivatives of $\mathcal{B}_\alpha(\sigma)$-functions.

Lemma 3.4 ([**4**, Proposition 5.4]). *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $\kappa = 0$*

or $\kappa > \max\{0, -\sigma\}$*. Let* $\gamma \in \mathbb{N}_0^n$ *be a multi-index. If* $u \in \mathcal{B}_\alpha(\sigma)$*, then the following statements hold.*

(i) *The derivatives* $\partial_x^\gamma \mathcal{D}_t^\kappa u$ *and* $\mathcal{D}_t^\kappa \partial_x^\gamma u$ *can be defined, and the equation* $\partial_x^\gamma \mathcal{D}_t^\kappa u = \mathcal{D}_t^\kappa \partial_x^\gamma u$ *holds. Furthermore, if* $(\gamma, \kappa) \neq (0,0)$*, then there exists a constant* $C = C(n, \alpha, \sigma, \gamma, \kappa) > 0$ *such that*

$$\left|\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)\right| \leq C t^{-(|\gamma|/(2\alpha)+\kappa+\sigma)} \|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $(x,t) \in H$.

(ii) *If* $\nu > \max\{0, -\sigma\}$*, then*

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} u(x,t) \tag{3.3}$$

Furthermore, if $\nu < 0$*, then* (3.3) *also holds whenever* $\nu < \sigma$ *and* $\nu + \kappa > \max\{0, -\sigma\}$*.*

(iii) *The derivative* $\partial_x^\gamma \mathcal{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on H.*

In [**4**], the following result is also given, which is the reproducing formula by fractional derivatives on $\mathcal{B}_\alpha(\sigma)$.

Lemma 3.5 ([**4**, Theorem 5.7]). *Let* $0 < \alpha \leq 1$ *and* $\sigma > -m(\alpha)$*. If real numbers* $\kappa \in \mathbb{R}_+$ *and* $\nu \in \mathbb{R}$ *satisfy* $\kappa > -\sigma$ *and* $\nu > \sigma$*, then*

$$u(x,t) - u(0,1) = \frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)} \int_H \mathcal{D}_t^\kappa u(y,s) \omega_\alpha^{0,\nu}(x,t;y,s) s^{\kappa+\nu-1} dV(y,s) \tag{3.4}$$

for all $u \in \mathcal{B}_\alpha(\sigma)$ *and* $(x,t) \in H$*. Furthermore,* (3.4) *also holds for* $\nu > \max\{0, \sigma\}$ *when* $\kappa = 0$*.*

As an application of the reproducing formula, estimates of the normal derivative norms on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ are given.

Lemma 3.6 ([**4**, Theorem 5.9]). *Let* $0 < \alpha \leq 1$ *and* $\sigma > -m(\alpha)$*. Then, for every real number* $\kappa > \max\{0, -\sigma\}$*, there exists a constant* $C = C(n, \alpha, \sigma, \kappa) > 0$ *independent of u such that*

$$C^{-1} \|u\|_{\mathcal{B}_\alpha(\sigma)} \leq \|t^{\kappa+\sigma} \mathcal{D}_t^\kappa u\|_{L^\infty} \leq C \|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$.

In our later arguments, we use the following lemma frequently.

Lemma 3.7 ([**7**, Lemma 5]). *Let $\theta, c \in \mathbb{R}$. If $\theta > -1$ and $\theta - c + n/(2\alpha) + 1 < 0$, then there exists a constant $C = C(n, \alpha, \theta, c) > 0$ such that*

$$\int_H \frac{s^\theta}{(t + s + |x - y|^{2\alpha})^c} dV(y, s) = C t^{\theta - c + n/(2\alpha) + 1}$$

for all $(x, t) \in H$.

Let $\boldsymbol{b}_\alpha^\infty$ be the set of all $L^{(\alpha)}$-harmonic functions u on H with $u \in L^\infty(H, dV)$. We also present the following lemma, which is [**6**, Theorem 4.1].

Lemma 3.8 ([**6**, Theorem 4.1]). *Let $0 < \alpha \leq 1$. Then, every $u \in \boldsymbol{b}_\alpha^\infty$ satisfies the following Huygens property, that is,*

$$u(x, t + s) = \int_{\mathbb{R}^n} u(x - y, t) W^{(\alpha)}(y, s) dy = \int_{\mathbb{R}^n} u(y, t) W^{(\alpha)}(x - y, s) dy$$

holds for all $x \in \mathbb{R}^n$ and $0 < s,\ t < \infty$.

4. Conjugate functions on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ with $0 < \alpha < 1$.

In this section, we study conjugate functions of $\widetilde{\mathcal{B}}_\alpha(\sigma)$-functions with $0 < \alpha \leq 1$ and give the proof of Theorem 1. First, we consider the integral operator stated below. Let f be a function on H. For $0 < \alpha \leq 1$, $\gamma \in \mathbb{N}_0^n$, $\nu > -n/(2\alpha)$, and $\rho \in \mathbb{R}$, the integral operator $\Pi_\alpha^{\gamma,\nu,\rho}$ is defined by

$$\Pi_\alpha^{\gamma,\nu,\rho} f(x, t) := \int_H f(y, s) \omega_\alpha^{\gamma,\nu}(x, t; y, s) s^\rho dV(y, s) \tag{4.1}$$

for all $(x, t) \in H$, whenever the integral is well defined. The next proposition is important for investigation of conjugate functions.

Proposition 4.1. *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. For $\gamma \in \mathbb{N}_0^n$, $\nu > -n/(2\alpha)$, and $\rho > -1$, put $\eta := |\gamma|/(2\alpha) + \nu - \rho - 1$. If $\eta > -m(\alpha)$, then the following statements hold.*

(i) *The function $\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1} \mathcal{D}_t u)$ belongs to $\widetilde{\mathcal{B}}_\alpha(\eta)$. Furthermore, there exists a constant $C = C(n, \alpha, \sigma, \gamma, \nu, \rho) > 0$ independent of u such that*

$$\left\| \Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1} \mathcal{D}_t u) \right\|_{\mathcal{B}_\alpha(\eta)} \leq C \|u\|_{\mathcal{B}_\alpha(\sigma)}.$$

(ii) *Let $\beta \in \mathbb{N}_0^n$ and $\kappa \geq 0$ be a real number such that $\kappa > -\eta$. Then, the derivatives $\partial_x^\beta \mathcal{D}_t^\kappa(\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1}\mathcal{D}_t u))$ and $\mathcal{D}_t^\kappa \partial_x^\beta(\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1}\mathcal{D}_t u))$ can be defined, and the equation $\partial_x^\beta \mathcal{D}_t^\kappa(\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1}\mathcal{D}_t u)) = \mathcal{D}_t^\kappa \partial_x^\beta(\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1}\mathcal{D}_t u))$ holds. Furthermore, if $(\beta,\kappa) \neq (0,0)$, then there exists a constant $C = C(n,\alpha,\sigma,\gamma,\nu,\rho,\beta,\kappa) > 0$ such that*

$$\left|\partial_x^\beta \mathcal{D}_t^\kappa(\Pi_\alpha^{\gamma,\nu,\rho}(s^{\sigma+1}\mathcal{D}_t u))(x,t)\right| \leq Ct^{-(|\beta|/(2\alpha)+\kappa+\eta)}\|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $(x,t) \in H$.

Proof. (i) Put $f(y,s) = s^{\sigma+1}\mathcal{D}_t u(y,s)$ for all $(y,s) \in H$. Suppose $\eta := |\gamma|/(2\alpha) + \nu - \rho - 1 > -m(\alpha)$. Then, by (i) of Lemma 3.3 and Lemma 3.7, $\Pi_\alpha^{\gamma,\nu,\rho} f(x,t)$ is well defined for every $(x,t) \in H$. Furthermore, we show that $\Pi_\alpha^{\gamma,\nu,\rho} f \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and there exists a constant $C > 0$ independent of u such that $\|\Pi_\alpha^{\gamma,\nu,\rho} f\|_{\mathcal{B}_\alpha(\eta)} \leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}$. In fact, by (ii) of Lemma 3.3, for every $0 < t_1 < t_2 < \infty$, we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} &\left|\Pi_\alpha^{\gamma,\nu,\rho} f(x,t)\right|(1+|x|)^{-n-2\alpha}dxdt \\ &\leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} F_{\alpha,\eta}(x,t)(1+|x|)^{-n-2\alpha}dxdt < \infty,\end{aligned}$$

where $F_{\alpha,\eta}$ is the function defined in (3.2). Therefore, $\Pi_\alpha^{\gamma,\nu,\rho} f$ satisfies the condition (2.4). Thus, by the definition of $\omega_\alpha^{\gamma,\nu}(x,t;y,s)$, $\Pi_\alpha^{\gamma,\nu,\rho} f$ is $L^{(\alpha)}$-harmonic and $\Pi_\alpha^{\gamma,\nu,\rho} f(0,1) = 0$. Moreover, differentiating through the integral (4.1), we obtain by Lemma 3.1 (i) that there exists a constant $C > 0$ independent of u such that

$$\begin{aligned}\left|\partial_j \Pi_\alpha^{\gamma,\nu,\rho} f(x,t)\right| &\leq \int_H |f(y,s)|\left|\partial_{x_j}\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)\right|s^\rho dV(y,s) \\ &\leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}\int_H \frac{s^\rho}{(t+s+|x-y|^{2\alpha})^{(n+|\gamma|+1)/(2\alpha)+\nu}}dV(y,s)\end{aligned}$$

for all $(x,t) \in H$. Therefore, by the condition $\eta > -m(\alpha)$ and Lemma 3.7, we obtain

$$\left|\partial_j \Pi_\alpha^{\gamma,\nu,\rho} f(x,t)\right| \leq Ct^{-(\eta+1/(2\alpha))}\|u\|_{\mathcal{B}_\alpha(\sigma)}.$$

Similarly, we get

$$\left|\partial_t \Pi_\alpha^{\gamma,\nu,\rho} f(x,t)\right| \leq C t^{-(\eta+1)} \|u\|_{\mathcal{B}_\alpha(\sigma)}.$$

Hence, we obtain $\Pi_\alpha^{\gamma,\nu,\rho} f \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and $\|\Pi_\alpha^{\gamma,\nu,\rho} f\|_{\mathcal{B}_\alpha(\eta)} \leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}$.

The second assertion is an immediate consequence of (i) of Lemma 3.4 and (i) of Proposition 4.1. □

We also show the following proposition.

Proposition 4.2. *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Put $\eta := 1/(2\alpha) - 1 + \sigma$. If $\eta > -1/(2\alpha)$, then there exists a unique (vector-valued) function $V = (v_1, \ldots, v_n)$ such that $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and V satisfies Equation* (C.1). *Also, there exists a constant $C = C(n, \alpha, \sigma) > 0$ independent of u such that $\|v_j\|_{\mathcal{B}_\alpha(\eta)} \leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}$.*

Proof. For $1 \leq j \leq n$, we can define a function $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ by

$$\begin{aligned} v_j(x,t) &= -\frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \omega_\alpha^{\gamma(j),\sigma+1}(x,t;y,s) s^{\sigma+2} dV(y,s) \\ &= -\frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \Pi_\alpha^{\gamma(j),\sigma+1,1}(s^{\sigma+1}\mathcal{D}_t u), \end{aligned} \tag{4.2}$$

where $\gamma(j) = (\delta_{j1}, \ldots, \delta_{jn}) \in \mathbb{N}_0^n$ and $\delta_{j\ell}$ is the Kronecker δ. In fact, by the hypothesis $\sigma > -m(\alpha)$, the condition $\eta = 1/(2\alpha) - 1 + \sigma > -1$ always holds. Therefore, we have $\eta > -m(\alpha)$. Hence, (i) of Proposition 4.1 implies that $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and there exists a constant $C > 0$ independent of u such that

$$\|v_j\|_{\mathcal{B}_\alpha(\eta)} \leq C\|u\|_{\mathcal{B}_\alpha(\sigma)}. \tag{4.3}$$

We show that the functions u and $V = (v_1, \ldots, v_n)$ satisfy Equation (C.1). Differentiating through the integral (4.2), we obtain from Lemma 3.5 that

$$\begin{aligned} -\mathcal{D}_t v_j(x,t) &= \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \partial_{x_j} \mathcal{D}_t^{\sigma+2} W^{(\alpha)}(x-y, t+s) s^{\sigma+2} dV(y,s) \\ &= \partial_j u(x,t). \end{aligned}$$

Since the equation $\partial_k v_j = \partial_j v_k$ is clearly satisfied by the definition (4.2), the functions u and v_j satisfy (C.1).

To show the uniqueness, we suppose that there exists a function $U = (u_1, \ldots, u_n)$ with $u_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and that u and U satisfy Equation (C.1). Then, for each $1 \le j \le n$, by the first inequality of Lemma 3.6 with $\kappa = 1$ and Equation (C.1), we have

$$\|v_j - u_j\|_{\mathcal{B}_\alpha(\eta)} \le C\big\|t^{1+\eta}\mathcal{D}_t(v_j - u_j)\big\|_{L^\infty} = C\big\|t^{1+\eta}(\partial_j u - \partial_j u)\big\|_{L^\infty} = 0.$$

Since $\|\cdot\|_{\mathcal{B}_\alpha(\eta)}$ is the norm on $\widetilde{\mathcal{B}}_\alpha(\eta)$, we obtain $v_j(x,t) = u_j(x,t)$ for all $(x,t) \in H$. It follows that the existence of the function $V = (v_1, \ldots, v_n)$ with $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ satisfying Equation (C.1) is unique. $\square$

Now, we give the proof of Theorem 1.

Proof of Theorem 1. By Proposition 4.2, there exists a unique function $V = (v_1, \ldots, v_n)$ such that $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and V satisfies Equation (C.1). Furthermore, the second inequality of (1.5) has already obtained.

We show that the functions u and V satisfy Equation (C.2). Differentiating through the integral (4.2), we have

$$\begin{aligned}\sum_{j=1}^{n} &\partial_j v_j(x,t)\\ &= -\frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s)\Delta_x \mathcal{D}_t^{\sigma+1} W^{(\alpha)}(x-y, t+s)s^{\sigma+2}dV(y,s).\end{aligned} \tag{4.4}$$

Let $1/\alpha - 1 \notin \mathbb{N}_0$. Then, by the definitions (2.6) and (2.8), Lemma 3.5 also implies

$$\begin{aligned}\mathcal{D}_t^{1/\alpha-1}u(x,t) = &\frac{1}{\Gamma(\lceil 1/\alpha - 1\rceil - (1/\alpha - 1))}\frac{2^{\sigma+3}}{\Gamma(\sigma+3)}\int_0^\infty \tau^{\lceil 1/\alpha-1\rceil-(1/\alpha-1)-1}\\ &\times \int_H \mathcal{D}_t u(y,s)\mathcal{D}_t^{\lceil 1/\alpha-1\rceil+\sigma+2}W^{(\alpha)}(x-y, t+\tau+s)s^{\sigma+2}dV(y,s)d\tau.\end{aligned} \tag{4.5}$$

We claim that we can apply the Fubini theorem to the right-hand side of the equality (4.5). Indeed, by (i) of Lemma 3.1 and (i) of Lemma 3.4, the condition $\eta > 1/(2\alpha)$ and Lemma 3.7 imply that

$$\begin{aligned}\int_0^\infty &\tau^{\lceil 1/\alpha-1\rceil-(1/\alpha-1)-1}\\ &\times \int_H |\mathcal{D}_t u(y,s)|\big|\mathcal{D}_t^{\lceil 1/\alpha-1\rceil+\sigma+2}W^{(\alpha)}(x-y, t+\tau+s)\big|s^{\sigma+2}dV(y,s)d\tau\end{aligned}$$

$$\leq C \int_0^\infty \tau^{\lceil 1/\alpha-1\rceil-(1/\alpha-1)-1}$$
$$\times \int_H \frac{s}{(t+\tau+s+|x-y|^{2\alpha})^{n/(2\alpha)+\lceil 1/\alpha-1\rceil+\sigma+2}} dV(y,s)d\tau$$
$$\leq C \int_0^\infty \tau^{\lceil 1/\alpha-1\rceil-(1/\alpha-1)-1} \frac{1}{(t+\tau)^{\lceil 1/\alpha-1\rceil+\sigma}} d\tau < \infty.$$

Therefore, (4.5) and the Fubini theorem show that

$$\mathcal{D}_t^{1/\alpha-1}u(x,t) = \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{1/\alpha+\sigma+1} W^{(\alpha)}(x-y,t+s)s^{\sigma+2}dV(y,s).$$

Hence, by (4.4) and Lemma 3.2, the functions u and V satisfy (C.2). The proof of the case $1/\alpha-1 \in \mathbb{N}$ is easy. Thus u and V satisfy Equation (C.2).

We show the first inequality of (1.5). By the first inequality of Lemma 3.6 with $\kappa = 1/\alpha-1$ and Equation (C.2), we have

$$\begin{aligned}\|u\|_{\mathcal{B}_\alpha(\sigma)} &\leq C\left\|t^{1/\alpha-1+\sigma}\mathcal{D}_t^{1/\alpha-1}u\right\|_{L^\infty} \\ &\leq C\sum_{j=1}^n \left\|t^{1/\alpha-1+\sigma}\partial_j v_j\right\|_{L^\infty} \leq C\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)}.\end{aligned}$$

Hence, this completes the proof of Theorem 1. □

5. Conjugate functions on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ for $\alpha = 1$.

By Theorem 1, we give properties of conjugate functions on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ with $0 < \alpha < 1$. In this section, we give the proof of Theorem 2. When $\alpha = 1$, generalization of conjugacy to $\widetilde{\mathcal{B}}_1(\sigma)$ is not given by (C.1) and (C.2). For a real number $\delta > 0$ and a function u on H, let $u_\delta(x,t) = u(x,t+\delta)$ for $(x,t) \in H$. We show the following lemmas, which are only used to the case $\alpha = 1$.

Lemma 5.1. *Let $0 < \alpha \leq 1$, $\sigma > 0$, $u \in \mathcal{B}_\alpha(\sigma)$, and $(x,t) \in H$. Then, $\lim_{s\to\infty} u(x,s)$ exists. Furthermore, if $k \in \mathbb{N}$, $\delta > 0$, and c_1, $c_2 > 0$, then*

$$\begin{aligned} &u_\delta(x,t) - \lim_{s\to\infty} u(x,s) \\ &\quad = \frac{(c_1+c_2)^k}{\Gamma(k)} \int_H \mathcal{D}_t^k u_\delta(y,c_1 s) W^{(\alpha)}(x-y,t+c_2 s)s^{k-1}dV(y,s). \end{aligned} \tag{5.1}$$

Proof. First, we show that the integrand in the right-hand side of the equality (5.1) belongs to $L^1(H, dV)$. In fact, (3.1) and (i) of Lemma 3.4 imply that

$$\begin{aligned}
&\int_H \left|\mathcal{D}_t^k u_\delta(y, c_1 s) W^{(\alpha)}(x-y, t+c_2 s) s^{k-1}\right| dV(y,s) \\
&\le C \int_0^\infty (c_1 s+\delta)^{-(k+\sigma)} s^{k-1} \int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t+c_2 s) dy\ ds \\
&\le C \int_0^\infty (c_1 s+\delta)^{-(1+\sigma)} ds < \infty, \qquad (5.2)
\end{aligned}$$

because $\sigma > 0$. Thus, the Fubini theorem and Lemma 3.8 show that

$$\begin{aligned}
&\int_H \mathcal{D}_t^k u_\delta(y, c_1 s) W^{(\alpha)}(x-y, t+c_2 s) s^{k-1} dV(y,s) \\
&= \int_0^\infty \int_{\mathbb{R}^n} \mathcal{D}_t^k u_\delta(y, c_1 s) W^{(\alpha)}(x-y, t+c_2 s) dy\ s^{k-1} ds \\
&= \int_0^\infty \mathcal{D}_t^k u_\delta(x, t+(c_1+c_2)s) s^{k-1} ds. \qquad (5.3)
\end{aligned}$$

We claim that $\lim_{s\to\infty} u(x,s)$ exists and the right-hand side of (5.3) is equal to $\Gamma(k)(c_1+c_2)^{-k}(u_\delta(x,t) - \lim_{s\to\infty} u(x,s))$. In fact, if $k=1$, then (5.2) shows that the right-hand side of (5.3) with $k=1$ converges, that is, $\lim_{s\to\infty} u(x,s)$ exists and

$$\int_0^\infty \mathcal{D}_t u_\delta(x, t+(c_1+c_2)s) ds = (c_1+c_2)^{-1}\Big(u_\delta(x,t) - \lim_{s\to\infty} u(x,s)\Big).$$

For induction, we assume that the right-hand side of (5.3) is equal to $\Gamma(k)(c_1+c_2)^{-k}(u_\delta(x,t) - \lim_{s\to\infty} u(x,s))$. Then, integrating by parts, we have

$$\begin{aligned}
&\int_0^\infty \mathcal{D}_t^{k+1} u_\delta(x, t+(c_1+c_2)s) s^k ds \\
&= -(c_1+c_2)^{-1}\left[\mathcal{D}_t^k u_\delta(x, t+(c_1+c_2)s) s^k\right]_0^\infty \\
&\quad + (c_1+c_2)^{-1} k \int_0^\infty \mathcal{D}_t^k u_\delta(x, t+(c_1+c_2)s) s^{k-1} ds. \qquad (5.4)
\end{aligned}$$

By (i) of Lemma 3.4, the first term of the right-hand side of (5.4) is equal to 0.

Therefore, by assumption, we obtain the right-hand side of (5.4) is equal to $\Gamma(k+1)(c_1+c_2)^{-(k+1)}(u_\delta(x,t)-\lim_{s\to\infty} u(x,s))$. This completes the proof. $\square$

LEMMA 5.2. *Let $0<\alpha\le 1$, $\sigma>0$, $u\in\mathcal{B}_\alpha(\sigma)$, and $(x,t)\in H$. Then, the following statements hold.*

(i) *If $k,\ m\in\mathbb{N}$, $\delta>0$, and $c_1,c_2>0$, then*

$$u_\delta(x,t)-\lim_{s\to\infty} u(x,s) = \frac{(c_1+c_2)^{k+m}}{\Gamma(k+m)}\int_H \mathcal{D}_t^k u_\delta(y,c_1 s)\mathcal{D}_t^m W^{(\alpha)}(x-y,t+c_2 s)s^{k+m-1}dV(y,s). \tag{5.5}$$

(ii) *If $\kappa>0$ and $\nu>\sigma$, then*

$$u(x,t)-\lim_{s\to\infty} u(x,s) = \frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)}\int_H \mathcal{D}_t^\kappa u(y,s)\mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)s^{\kappa+\nu-1}dV(y,s). \tag{5.6}$$

PROOF. (i) Let $k,\ m\in\mathbb{N}$. Then, Lemmas 3.1, 3.4 and 3.7 imply that

$$\begin{aligned}&\int_H \left|\mathcal{D}_t^k u_\delta(y,c_1 s)\mathcal{D}_t^m W^{(\alpha)}(x-y,t+c_2 s)s^{k+m-1}\right|dV(y,s)\\ &\quad\le C\int_H \frac{(c_1 s+\delta)^{-(1+\sigma)}s^m}{(t+c_2 s+|x-y|^{2\alpha})^{n/(2\alpha)+m}}dV(y,s)<\infty.\end{aligned}$$

Hence, the integrand in the right-hand side of the equality (5.5) belongs to $L^1(H,dV)$.

We show the equality (5.5) by the induction on m. Let $m=1$ and $k\in\mathbb{N}$. Then, integrating by parts, we obtain from Lemmas 3.1, 3.4 and 5.1 that

$$\begin{aligned}&\int_H \mathcal{D}_t^k u_\delta(y,c_1 s)\mathcal{D}_t W^{(\alpha)}(x-y,t+c_2 s)s^k dV(y,s)\\ &=\int_{\mathbb{R}^n}\int_0^\infty \mathcal{D}_t^k u_\delta(y,c_1 s)\mathcal{D}_t W^{(\alpha)}(x-y,t+c_2 s)s^k ds\ dy\\ &=-\frac{1}{c_2}\int_{\mathbb{R}^n}\left[\mathcal{D}_t^k u_\delta(y,c_1 s)W^{(\alpha)}(x-y,t+c_2 s)s^k\right]_0^\infty dy\end{aligned}$$

$$
\begin{aligned}
&- \frac{c_1}{c_2}\int_{\mathbb{R}^n}\int_0^\infty \mathcal{D}_t^{k+1}u_\delta(y,c_1s)W^{(\alpha)}(x-y,t+c_2s)s^k ds\ dy \\
&+ \frac{k}{c_2}\int_{\mathbb{R}^n}\int_0^\infty \mathcal{D}_t^{k}u_\delta(y,c_1s)W^{(\alpha)}(x-y,t+c_2s)s^{k-1} ds\ dy \\
= &- \frac{c_1\Gamma(k+1)}{c_2(c_1+c_2)^{k+1}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big) \\
&+ \frac{k\Gamma(k)}{c_2(c_1+c_2)^{k}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big) \\
= &\ \frac{\Gamma(k+1)}{(c_1+c_2)^{k+1}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big).
\end{aligned}
$$

Let $m\in\mathbb{N}$ be fixed, and assume that the equality (5.5) holds for all $k\in\mathbb{N}$. Then, integrating by parts, we have from Lemmas 3.1, 3.4 and the assumption that

$$
\begin{aligned}
&\int_H \mathcal{D}_t^{k}u_\delta(y,c_1s)\mathcal{D}_t^{m+1}W^{(\alpha)}(x-y,t+c_2s)s^{k+m}dV(y,s) \\
&= -\frac{1}{c_2}\int_{\mathbb{R}^n}\left[\mathcal{D}_t^{k}u_\delta(y,c_1s)\mathcal{D}_t^{m}W^{(\alpha)}(x-y,t+c_2s)s^{k+m}\right]_0^\infty dy \\
&\quad - \frac{c_1}{c_2}\int_{\mathbb{R}^n}\int_0^\infty \mathcal{D}_t^{k+1}u_\delta(y,c_1s)\mathcal{D}_t^{m}W^{(\alpha)}(x-y,t+c_2s)s^{k+m} ds\ dy \\
&\quad + \frac{k+m}{c_2}\int_{\mathbb{R}^n}\int_0^\infty \mathcal{D}_t^{k}u_\delta(y,c_1s)\mathcal{D}_t^{m}W^{(\alpha)}(x-y,t+c_2s)s^{k+m-1} ds\ dy \\
&= -\frac{c_1\Gamma(k+m+1)}{c_2(c_1+c_2)^{k+m+1}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big) \\
&\quad + \frac{(k+m)\Gamma(k+m)}{c_2(c_1+c_2)^{k+m}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big) \\
&= \frac{\Gamma(k+m+1)}{(c_1+c_2)^{k+m+1}}\Big(u_\delta(x,t)-\lim_{s\to\infty}u(x,s)\Big).
\end{aligned}
$$

Therefore, we obtain that the equality (5.5) with $m+1$ holds for all $k\in\mathbb{N}$.

(ii) First, we claim that for each $k\in\mathbb{N}$, $m>\sigma$, and $c_1,c_2>0$,

$$
\begin{aligned}
&\frac{(c_1+c_2)^{k+m}}{\Gamma(k+m)}\int_H \mathcal{D}_t^{k}u(y,c_1s)\mathcal{D}_t^{m}W^{(\alpha)}(x-y,t+c_2s)s^{k+m-1}dV(y,s) \\
&\quad = u(x,t)-\lim_{s\to\infty}u(x,s). \qquad (5.7)
\end{aligned}
$$

Indeed, by Lemmas 3.1, 3.4 and 3.7, the equality (5.7) follows from Lemma 5.1 and the dominated convergence theorem.

Let $\kappa > 0$ and $\nu > \sigma$. Suppose that $\kappa,\ \nu \notin \mathbb{N}$. Then, the definitions (2.6) and (2.8) imply that

$$\begin{aligned}
&\int_H \mathcal{D}_t^\kappa u(y,s)\mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)s^{\kappa+\nu-1}dV(y,s) \\
&= \int_H \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1}\mathcal{D}_t^{\lceil\kappa\rceil}u(y,s+\tau_1)d\tau_1 \\
&\quad\times \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^\infty \tau_2^{\lceil\nu\rceil-\nu-1}\mathcal{D}_t^{\lceil\nu\rceil}W^{(\alpha)}(x-y,t+s+\tau_2)d\tau_2\ s^{\kappa+\nu-1}dV(y,s) \\
&= \int_H \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1}\mathcal{D}_t^{\lceil\kappa\rceil}u(y,(1+\tau_1)s)d\tau_1 \\
&\quad\times \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^\infty \tau_2^{\lceil\nu\rceil-\nu-1}\mathcal{D}_t^{\lceil\nu\rceil}W^{(\alpha)}(x-y,t+(1+\tau_2)s)d\tau_2 \\
&\qquad\qquad\times s^{\lceil\kappa\rceil+\lceil\nu\rceil-1}dV(y,s).
\end{aligned}$$

Furthermore, (i) of Lemma 3.1 and (i) of Lemma 3.4 imply that

$$\begin{aligned}
&\int_H\int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1}\left|\mathcal{D}_t^{\lceil\kappa\rceil}u(y,(1+\tau_1)s)\right|d\tau_1 \\
&\quad\times \int_0^\infty \tau_2^{\lceil\nu\rceil-\nu-1}\left|\mathcal{D}_t^{\lceil\nu\rceil}W^{(\alpha)}(x-y,t+(1+\tau_2)s)\right|d\tau_2\ s^{\lceil\kappa\rceil+\lceil\nu\rceil-1}dV(y,s) \\
&\le C\int_H\int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{((1+\tau_1)s)^{\lceil\kappa\rceil+\sigma}}d\tau_1 \\
&\quad\times \int_0^\infty \frac{\tau_2^{\lceil\nu\rceil-\nu-1}}{(t+(1+\tau_2)s+|x-y|^{2\alpha})^{n/(2\alpha)+\lceil\nu\rceil}}d\tau_2\ s^{\lceil\kappa\rceil+\lceil\nu\rceil-1}dV(y,s) \\
&= C\int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{(1+\tau_1)^{\lceil\kappa\rceil+\sigma}}d\tau_1\int_0^\infty \frac{\tau_2^{\lceil\nu\rceil-\nu-1}}{(1+\tau_2)^{-\sigma+\lceil\nu\rceil}}d\tau_2 \\
&\quad\times \int_H \frac{s^{-\sigma+\lceil\nu\rceil-1}}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\lceil\nu\rceil}}dV(y,s).
\end{aligned}$$

Since $\sigma > 0$, $\kappa > 0$, and $\nu > \sigma$, we have

$$\int_0^\infty \frac{\tau_1^{\lceil\kappa\rceil-\kappa-1}}{(1+\tau_1)^{\lceil\kappa\rceil+\sigma}}d\tau_1 < \infty$$

and

$$\int_0^\infty \frac{\tau_2^{\lceil\nu\rceil-\nu-1}}{(1+\tau_2)^{-\sigma+\lceil\nu\rceil}}d\tau_2 < \infty,$$

respectively. Moreover, by the conditions $\nu > \sigma$ and $\sigma > 0$, Lemma 3.7 implies that

$$\int_H \frac{s^{-\sigma+\lceil\nu\rceil-1}}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\lceil\nu\rceil}}dV(y,s) < \infty.$$

Hence, by the Fubini theorem, (5.7) shows that

$$\begin{aligned}
&\int_H \mathcal{D}_t^\kappa u(y,s)\mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)s^{\kappa+\nu-1}dV(y,s)\\
&= \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)\Gamma(\lceil\nu\rceil-\nu)}\int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1}\int_0^\infty \tau_2^{\lceil\nu\rceil-\nu-1}\\
&\quad\times\int_H \mathcal{D}_t^{\lceil\kappa\rceil}u(y,(1+\tau_1)s)\mathcal{D}_t^{\lceil\nu\rceil}W^{(\alpha)}(x-y,t+(1+\tau_2)s)\\
&\quad\times s^{\lceil\kappa\rceil+\lceil\kappa\rceil-1}dV(y,s)d\tau_1 d\tau_2\\
&= \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)\Gamma(\lceil\nu\rceil-\nu)}\int_0^\infty \tau_1^{\lceil\kappa\rceil-\kappa-1}\int_0^\infty \tau_2^{\lceil\nu\rceil-\nu-1}\\
&\quad\times\frac{\Gamma(\lceil\kappa\rceil+\lceil\nu\rceil)}{(2+\tau_1+\tau_2)^{\lceil\kappa\rceil+\lceil\nu\rceil}}\Big(u(x,t)-\lim_{s\to\infty}u(x,s)\Big)d\tau_1 d\tau_2\\
&= \frac{\Gamma(\kappa+\nu)}{2^{\kappa+\nu}}\Big(u(x,t)-\lim_{s\to\infty}u(x,s)\Big).
\end{aligned}$$

The proof of the case $\kappa \in \mathbb{N}$ or $\nu \in \mathbb{N}$ is parallel to that of the case $\kappa,\ \nu \notin \mathbb{N}$. (When $\kappa \in \mathbb{N}$ and $\nu \in \mathbb{N}$, the assertion follows from (5.7) directly.) $\square$

REMARK 5.3. Let $0 < \alpha \le 1$, $\sigma > 0$, and $u \in \mathcal{B}_\alpha(\sigma)$. Then, by the proof of (ii) of Lemma 5.2, when $\kappa > 0$ and $\nu > \sigma$, we have

$$\int_H \big|\mathcal{D}_t^\kappa u(y,s)\big|\big|\mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)\big|s^{\kappa+\nu-1}dV(y,s) < \infty$$

for all $(x,t) \in H$.

We give a counterpart of (C.2) for $\alpha = 1$.

Proposition 5.4. *Let $\sigma > 0$ and $u \in \widetilde{\mathcal{B}}_1(\sigma)$. Let $V = (v_1, \ldots, v_n)$ be the function obtained by Proposition* 4.2, *then u and V satisfy the following equation* (C.2$'$):

$$u - \lim_{t\to\infty} u(0,t) = \sum_{j=1}^{n} \partial_j v_j. \tag{C.2$'$}$$

Proof. As in the proof of Theorem 1, differentiating through the integral, we have from Lemma 3.2 that

$$\begin{aligned}\sum_{j=1}^{n} \partial_j v_j(x,t) &= -\frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \Delta_x \mathcal{D}_t^{\sigma+1} W^{(1)}(x-y, t+s) s^{\sigma+2} dV(y,s) \\ &= \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{\sigma+2} W^{(1)}(x-y, t+s) s^{\sigma+2} dV(y,s).\end{aligned}$$

On the other hand, Lemma 3.5 and Remark 5.3 imply that

$$\begin{aligned}u(x,t) = &\frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{\sigma+2} W^{(1)}(x-y, t+s) s^{\sigma+2} dV(y,s) \\ &- \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{\sigma+2} W^{(1)}(-y, 1+s) s^{\sigma+2} dV(y,s).\end{aligned}$$

Since $u \in \widetilde{\mathcal{B}}_1(\sigma)$, we have from Lemma 5.2 (ii) that

$$u(x,t) = \sum_{j=1}^{n} \partial_j v_j(x,t) + \lim_{s\to\infty} u(0,s).$$

Hence, u and V satisfy Equation (C.2$'$). □

Now, we give a counterpart of Theorem 1 to the case $\alpha = 1$ and complete the proof of Theorem 2.

Proof of Theorem 2. Let $\sigma > 0$, $\eta := -1/2 + \sigma$ and $u \in \widetilde{\mathcal{B}}_1(\sigma)$. Then, by Proposition 4.2, there exists a unique function $V = (v_1, \ldots, v_n)$ such that $v_j \in \widetilde{\mathcal{B}}_1(\eta)$ and V satisfies Equation (C.1). Furthermore, the second inequality of

(1.6) has already obtained. Thus, by Proposition 5.4, it suffices to show the first inequality of (1.6). By Equation (C.2′), we have

$$\begin{aligned}\|u\|_{\mathcal{B}_1(\sigma)} &\le \sup_{(x,t)\in H} t^{\sigma+1/2}|\nabla_x u(x,t)| + \sup_{(x,t)\in H} t^{\sigma+1}|\partial_t u(x,t)| \\ &\le \sum_{j=1}^{n} \sup_{(x,t)\in H} t^{\sigma+1/2}\big|\nabla_x \partial_j v_j(x,t)\big| + \sum_{j=1}^{n} \sup_{(x,t)\in H} t^{\sigma+1}|\partial_t \partial_j v_j(x,t)|.\end{aligned}$$

Since $v_j \in \widetilde{\mathcal{B}}_1(\eta)$, we obtain from (i) of Lemma 3.4 that

$$|\nabla_x \partial_j v_j(x,t)| \le C t^{-(1+\eta)} \|v_j\|_{\mathcal{B}_1(\eta)}$$

for all $(x,t) \in H$. Therefore, since $1+\eta = \sigma + 1/2$, we have

$$\sup_{(x,t)\in H} t^{\sigma+1/2}|\nabla_x \partial_j v_j(x,t)| \le C\|v_j\|_{\mathcal{B}_1(\eta)}.$$

Similarly, we get

$$\sup_{(x,t)\in H} t^{\sigma+1}|\partial_t \partial_j v_j(x,t)| \le C\|v_j\|_{\mathcal{B}_1(\eta)}.$$

Hence, the first inequality of (1.6) is obtained. □

The following lemma is (2) of [**4**, Lemma 4.2].

LEMMA 5.5 ((2) of [**4**, Lemma 4.2]). *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, $\gamma \in \mathbb{N}_0^n$, and $k \in \mathbb{N}_0$. Then, for every $(x,t) \in H$, there exists a constant $C = C(n, \alpha, \gamma, k, x, t) > 0$ such that*

$$\big|\partial_x^\gamma \mathcal{D}_t^k u(x, t+s) - \partial_x^\gamma \mathcal{D}_t^k u(0, 1+s)\big| \le C\|u\|_{\mathcal{B}_\alpha(\sigma)}(1+s)^{-|\gamma|/(2\alpha)-k-m(\alpha)-\sigma}$$

for all $u \in \mathcal{B}_\alpha(\sigma)$ and $s \ge 0$.

We give the following remark.

REMARK 5.6. By Lemmas 5.1 and 5.5, if $0 < \alpha \le 1$, $\sigma > 0$ and $u \in \mathcal{B}_\alpha(\sigma)$, then $\lim_{t\to\infty} u(x,t)$ exists and $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u(0,t)$ for all $x \in \mathbb{R}^n$.

For $0 < \alpha \le 1$ and $\sigma > 0$, we can construct a function $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ with $\lim_{t\to\infty} u(0,t) \ne 0$. We give the following example.

Example 5.7. For $0 < \alpha \le 1$ and $\sigma > 0$, we put $\kappa := -n/(2\alpha) + \sigma$. Since $\sigma > 0$, we can define the derivative $\mathcal{D}_t^\kappa W^{(\alpha)}$ by Lemma 3.1. Thus, we define an $L^{(\alpha)}$-harmonic function u on H with $u(0,1) = 0$ by

$$u(x,t) := \mathcal{D}_t^\kappa W^{(\alpha)}(x,t) - \mathcal{D}_t^\kappa W^{(\alpha)}(0,1), \quad (x,t) \in H.$$

By Lemma 3.1, we also have

$$|\partial_j u(x,t)| \le C(t + |x|^{2\alpha})^{-\sigma-1/(2\alpha)} \le Ct^{-\sigma-1/(2\alpha)}, \quad 1 \le j \le n$$

and

$$|\partial_t u(x,t)| \le C(t + |x|^{2\alpha})^{-\sigma-1} \le Ct^{-\sigma-1}$$

for all $(x,t) \in H$. Therefore, we obtain

$$\|u\|_{\mathcal{B}_\alpha(\sigma)} = \sup_{(x,t)\in H} t^\sigma \big\{ t^{1/(2\alpha)} |\nabla_x u(x,t)| + t|\partial_t u(x,t)| \big\} < \infty,$$

that is, $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Since Lemma 3.1 also implies that $|\mathcal{D}_t^\kappa W^{(\alpha)}(0,t)| \le Ct^{-\sigma}$, we have

$$\lim_{t\to\infty} u(0,t) = \lim_{t\to\infty} \mathcal{D}_t^\kappa W^{(\alpha)}(0,t) - \mathcal{D}_t^\kappa W^{(\alpha)}(0,1) = -\mathcal{D}_t^\kappa W^{(\alpha)}(0,1).$$

Furthermore, as in the proof of Lemma 4.1 of [**3**], we get

$$\mathcal{D}_t^\kappa W^{(\alpha)}(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{2\alpha\kappa} \exp(-t|\xi|^{2\alpha} + i\, x \cdot \xi)\, d\xi,$$

so we obtain $\lim_{t\to\infty} u(0,t) = -\mathcal{D}_t^\kappa W^{(\alpha)}(0,1) \ne 0$.

6. Estimates of tangential derivative norms.

In this section, we estimate tangential derivative norms on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ and give the proof of Theorem 3. We need the following lemma.

Lemma 6.1. *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Then,*

$$\big(\mathcal{D}_t^{1/\alpha} + \Delta_x\big) u(x,t) = 0$$

for all $(x,t) \in H$.

Proof. By Lemma 3.5 with $\kappa = 1$ and $\nu = \sigma + 2$, we have

$$u(x,t) = \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \omega_\alpha^{0,\sigma+2}(x,t;y,s) s^{\sigma+2} dV(y,s). \tag{6.1}$$

Differentiating through the integral (6.1), we get

$$\Delta_x u(x,t) = \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{\sigma+2} \Delta_x W^{(\alpha)}(x-y,t+s) s^{\sigma+2} dV(y,s).$$

By (ii) of Lemma 3.1 and Lemma 3.2, it suffices to show that

$$\mathcal{D}_t^{1/\alpha} u(x,t) = \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{1/\alpha} \mathcal{D}_t^{\sigma+2} W^{(\alpha)}(x-y,t+s) s^{\sigma+2} dV(y,s). \tag{6.2}$$

In fact, when $1/\alpha \in \mathbb{N}$, (6.2) immediately follows by differentiating through the integral (6.1). Let $1/\alpha \notin \mathbb{N}$. Then, (2.6) and (2.8) imply that

$$\begin{aligned}\mathcal{D}_t^{1/\alpha} u(x,t) = {} & \frac{1}{\Gamma(\lceil 1/\alpha \rceil - 1/\alpha)} \frac{2^{\sigma+3}}{\Gamma(\sigma+3)} \int_0^\infty \tau^{\lceil 1/\alpha \rceil - 1/\alpha - 1} \\ & \times \int_H \mathcal{D}_t u(y,s) \mathcal{D}_t^{\lceil 1/\alpha \rceil} \mathcal{D}_t^{\sigma+2} W^{(\alpha)}(x-y,t+\tau+s) s^{\sigma+2} dV(y,s) d\tau.\end{aligned}$$

Since $\sigma > -m(\alpha) > -1/\alpha > -\lceil 1/\alpha \rceil$, Lemmas 3.1, 3.4 and 3.7 imply that

$$\begin{aligned}&\int_0^\infty \tau^{\lceil 1/\alpha \rceil - 1/\alpha - 1} \int_H |\mathcal{D}_t u(y,s)| \left| \mathcal{D}_t^{\lceil 1/\alpha \rceil} \mathcal{D}_t^{\sigma+2} W^{(\alpha)}(x-y,t+\tau+s) \right| s^{\sigma+2} dV(y,s) d\tau \\ &\quad \le C \int_0^\infty \tau^{\lceil 1/\alpha \rceil - 1/\alpha - 1} \int_H \frac{s}{(t+\tau+s+|x-y|^{2\alpha})^{n/(2\alpha)+\sigma+2+\lceil 1/\alpha \rceil}} dV(y,s) d\tau \\ &\quad \le C \int_0^\infty \frac{\tau^{\lceil 1/\alpha \rceil - 1/\alpha - 1}}{(t+\tau)^{\sigma+\lceil 1/\alpha \rceil}} d\tau < \infty.\end{aligned}$$

Therefore, the Fubini theorem shows (6.2). □

Now, we give the proof of Theorem 3. We begin with showing the second inequality of (1.7).

Lemma 6.2. *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Then, for each $m \in \mathbb{N}$, there exists a constant $C = C(n, \alpha, \sigma, m) > 0$ independent of u such that*

$$\left\| t^{m/(2\alpha)+\sigma} \partial_x^\gamma u \right\|_{L^\infty} \leq C \|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all multi-indices $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = m$.

Proof. Let $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, and $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = m$. Then, (i) of Lemma 3.4 implies that

$$\left| \partial_x^\gamma u(x,t) \right| \leq C t^{-(|\gamma|/(2\alpha)+\sigma)} \|u\|_{\mathcal{B}_\alpha(\sigma)} = t^{-(m/(2\alpha)+\sigma)} \|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $(x,t) \in H$. Hence, the desired result immediately follows. □

We show the first inequality of (1.7) when m is an even number.

Lemma 6.3. *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Then, for each even number $m \in \mathbb{N}$, there exists a constant $C = C(n, \alpha, \sigma, m) > 0$ independent of u such that*

$$\|u\|_{\mathcal{B}_\alpha(\sigma)} \leq C \sum_{|\gamma|=m} \|t^{m/(2\alpha)+\sigma} \partial_x^\gamma u\|_{L^\infty}.$$

Proof. Let $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Since $m \in \mathbb{N}$ is even, there exists $k \in \mathbb{N}$ such that $m = 2k$. Then, by the condition $1/\alpha > m(\alpha) > -\sigma$, we have from (ii) of Lemma 3.4 and Lemma 6.1 that

$$\mathcal{D}_t^{m/(2\alpha)} u = \mathcal{D}_t^{k/\alpha} u = \left(\mathcal{D}_t^{1/\alpha}\right)^k u = (-1)^k \Delta_x^k u = (-1)^k \sum_{j_1,\ldots,j_k=1}^{n} \partial_{j_1}^2 \cdots \partial_{j_k}^2 u. \quad (6.3)$$

Therefore, Lemma 3.6 shows that

$$\begin{aligned}\|u\|_{\mathcal{B}_\alpha(\sigma)} &\leq C \left\| t^{m/(2\alpha)+\sigma} \mathcal{D}_t^{m/(2\alpha)} u \right\|_{L^\infty} \leq C \sum_{j_1,\ldots,j_k=1}^{n} \left\| t^{m/(2\alpha)+\sigma} \partial_{j_1}^2 \cdots \partial_{j_k}^2 u \right\|_{L^\infty} \\ &\leq C \sum_{|\gamma|=m} \left\| t^{m/(2\alpha)+\sigma} \partial_x^\gamma u \right\|_{L^\infty}.\end{aligned}$$

Hence, we obtain the desired inequality. □

We give the proof of Theorem 3.

PROOF OF THEOREM 3. By Lemmas 6.2 and 6.3, it suffices to show the first inequality of (1.7) when m is an odd number. Therefore, suppose that there exists $k \in \mathbb{N}_0$ such that $m = 2k+1$ and let $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$.

Put $v(x,t) = \mathcal{D}_t u(x,t) - \mathcal{D}_t u(0,1)$ and $\lambda = \sigma + 1$. Then, by (iii) of Lemma 3.4, v is $L^{(\alpha)}$-harmonic on H. Furthermore, we claim that v belongs to $\widetilde{\mathcal{B}}_\alpha(\lambda)$ and there exists a constant $C > 0$ independent of u such that

$$C^{-1}\|u\|_{\mathcal{B}_\alpha(\sigma)} \le \|v\|_{\mathcal{B}_\alpha(\lambda)} \le C\|u\|_{\mathcal{B}_\alpha(\sigma)}. \tag{6.4}$$

In fact, (i) of Lemma 3.4 shows that

$$|\partial_j v(x,t)| = \left|\partial_j \mathcal{D}_t u(x,t)\right| \le Ct^{-(1/(2\alpha)+1+\sigma)}\|u\|_{\mathcal{B}_\alpha(\sigma)} = Ct^{-\lambda}\cdot t^{-1/(2\alpha)}\|u\|_{\mathcal{B}_\alpha(\sigma)}$$

and

$$|\partial_t v(x,t)| = \left|\mathcal{D}_t^2 u(x,t)\right| \le Ct^{-(2+\sigma)}\|u\|_{\mathcal{B}_\alpha(\sigma)} = Ct^{-\lambda}\cdot t^{-1}\|u\|_{\mathcal{B}_\alpha(\sigma)}.$$

Hence, v belongs to $\widetilde{\mathcal{B}}_\alpha(\lambda)$ and $\|v\|_{\mathcal{B}_\alpha(\lambda)} \le C\|u\|_{\mathcal{B}_\alpha(\sigma)}$. Moreover, Lemma 3.6 implies that

$$\begin{aligned}
\|u\|_{\mathcal{B}_\alpha(\sigma)} &\le C\left\|t^{\sigma+2}\mathcal{D}_t^2 u\right\|_{L^\infty} = C\sup_{(x,t)\in H} t^{\sigma+1}\cdot t|\partial_t(\mathcal{D}_t u)(x,t)| \\
&\le C\sup_{(x,t)\in H} t^{\sigma+1}\left\{t^{1/(2\alpha)}|\nabla_x(\mathcal{D}_t u)(x,t)| + t|\partial_t(\mathcal{D}_t u)(x,t)|\right\} \\
&= C\sup_{(x,t)\in H} t^{\sigma+1}\left\{t^{1/(2\alpha)}|\nabla_x v(x,t)| + t|\partial_t v(x,t)|\right\} = C\|v\|_{\mathcal{B}_\alpha(\lambda)}.
\end{aligned}$$

Therefore, we obtain (6.4).

Moreover, we claim that there exists a constant $C > 0$ independent of u such that

$$\left\|t^{m/(2\alpha)+\sigma+1}\partial_x^\gamma v\right\|_{L^\infty} = \left\|t^{m/(2\alpha)+\sigma+1}\mathcal{D}_t(\partial_x^\gamma u)\right\|_{L^\infty} \le C\left\|t^{m/(2\alpha)+\sigma}\partial_x^\gamma u\right\|_{L^\infty} \tag{6.5}$$

with $|\gamma| = m$. Indeed, (iii) of Lemma 3.4 implies that $\partial_x^\gamma u$ is $L^{(\alpha)}$-harmonic on H. Furthermore, (i) of Lemma 3.4 shows that

$$\left|\partial_j(\partial_x^\gamma u)(x,t)\right| \le Ct^{-((m+1)/(2\alpha)+\sigma)}\|u\|_{\mathcal{B}_\alpha(\sigma)} = Ct^{-(m/(2\alpha)+\sigma)}\cdot t^{-1/(2\alpha)}\|u\|_{\mathcal{B}_\alpha(\sigma)}$$

and

$$\left|\partial_t(\partial_x^\gamma u)(x,t)\right| \leq Ct^{-(m/(2\alpha)+1+\sigma)}\|u\|_{\mathcal{B}_\alpha(\sigma)} = Ct^{-(m/(2\alpha)+\sigma)} \cdot t^{-1}\|u\|_{\mathcal{B}_\alpha(\sigma)}.$$

Therefore, we have $\partial_x^\gamma u \in \mathcal{B}_\alpha(\sigma')$, where $\sigma' = m/(2\alpha) + \sigma$. Hence, Lemma 3.5 implies that

$$\partial_x^\gamma u(x,t) - \partial_x^\gamma u(0,1) = \frac{2^{\sigma'+1}}{\Gamma(\sigma'+1)} \int_H \partial_x^\gamma u(y,s)\omega_\alpha^{0,\sigma'+1}(x,t;y,s)s^{\sigma'}dV(y,s).$$

Differentiating through the integral, we get

$$\begin{aligned}
\left|\partial_x^\gamma v(x,t)\right| &= \left|\mathcal{D}_t(\partial_x^\gamma u)(x,t)\right| \\
&\leq C\int_H \left|\partial_x^\gamma u(y,s)\right|\left|\mathcal{D}_t^{\sigma'+2}W^{(\alpha)}(x-y,t+s)\right|s^{\sigma'}dV(y,s) \\
&\leq C\left\|s^{m/(2\alpha)+\sigma}\partial_x^\gamma u\right\|_{L^\infty}\int_H \frac{1}{(t+s+|x-y|^{2\alpha})^{(n+m)/(2\alpha)+\sigma+2}}dV(y,s).
\end{aligned}$$

Therefore, Lemma 3.7 implies the inequality (6.5).

Now, we show the first inequality of (1.7). By (6.4) and (6.5), it suffices to show the inequality

$$\|v\|_{\mathcal{B}_\alpha(\lambda)} \leq C\sum_{|\gamma|=m}\left\|t^{m/(2\alpha)+\sigma+1}\partial_x^\gamma v\right\|_{L^\infty}. \tag{6.6}$$

Suppose $0 < \alpha < 1$. Then, by Theorem 1, there exists an α-parabolic conjugate function $V = (v_1,\ldots,v_n)$ of v such that $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$, where $\eta = 1/(2\alpha) - 1 + \lambda = 1/(2\alpha) + \sigma > 0 > -1/(2\alpha)$. Thus, by (ii) of Lemma 3.4 and Equation (C.2), we have

$$\mathcal{D}_t^{(m+1)/(2\alpha)}v = \mathcal{D}_t^{k/\alpha+1+1/\alpha-1}v = \mathcal{D}_t^{k/\alpha+1}\mathcal{D}_t^{1/\alpha-1}v = \sum_{j=1}^n \mathcal{D}_t^{k/\alpha+1}\partial_j v_j.$$

Therefore, Lemma 3.6 implies that

$$\begin{aligned}
\|v\|_{\mathcal{B}_\alpha(\lambda)} &\leq C\left\|t^{(m+1)/(2\alpha)+\lambda}\mathcal{D}_t^{(m+1)/(2\alpha)}v\right\|_{L^\infty} \\
&\leq C\sum_{j=1}^n\left\|t^{(m+1)/(2\alpha)+\lambda}\mathcal{D}_t^{k/\alpha+1}\partial_j v_j\right\|_{L^\infty}.
\end{aligned} \tag{6.7}$$

Furthermore, since (i) of Lemma 3.4 shows that

$$\begin{aligned}\left|\mathcal{D}_t^{k/\alpha+1}\partial_j v_j(x,t)\right| &\le Ct^{-(1/(2\alpha)+k/\alpha+1+\eta)}\|v_j\|_{\mathcal{B}_\alpha(\eta)}\\ &= Ct^{-((m+1)/(2\alpha)+\lambda)}\|v_j\|_{\mathcal{B}_\alpha(\eta)},\end{aligned}$$

we obtain $\|t^{(m+1)/(2\alpha)+\lambda}\mathcal{D}_t^{k/\alpha+1}\partial_j v_j\|_{L^\infty} \le C\|v_j\|_{\mathcal{B}_\alpha(\eta)}$ for all $1 \le j \le n$. Hence, (6.7) and Lemma 3.6 imply that

$$\|v\|_{\mathcal{B}_\alpha(\lambda)} \le C\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} \le C\sum_{j=1}^n \left\|t^{k/\alpha+1+\eta}\mathcal{D}_t^{k/\alpha+1}v_j\right\|_{L^\infty}.$$

Since Equation (C.1) implies that $\mathcal{D}_t v_j = -\partial_j v$, we have from (ii) of Lemma 3.4 and (6.3)

$$\begin{aligned}\|v\|_{\mathcal{B}_\alpha(\lambda)} &\le C\sum_{j=1}^n \left\|t^{k/\alpha+1+\eta}\mathcal{D}_t^{k/\alpha}\partial_j v\right\|_{L^\infty}\\ &\le C\sum_{j=1}^n \sum_{j_1,\dots,j_k=1}^n \left\|t^{k/\alpha+1+\eta}\partial_{j_1}^2\cdots\partial_{j_k}^2\partial_j v\right\|_{L^\infty}\\ &\le C\sum_{|\gamma|=m}\left\|t^{k/\alpha+1+\eta}\partial_x^\gamma v\right\|_{L^\infty} = C\sum_{|\gamma|=m}\left\|t^{m/(2\alpha)+\sigma+1}\partial_x^\gamma v\right\|_{L^\infty}.\end{aligned}$$

Hence, we obtain the inequality (6.6).

Suppose $\alpha = 1$. Then, since $\lambda = \sigma + 1 > 0$, Theorem 2 implies that there exists a function $V = (v_1,\dots,v_n)$ with $v_j \in \widetilde{\mathcal{B}}_1(\eta)$ such that v and V satisfy Equations (C.1) and (C.2′), where $\eta = 1/2 - 1 + \lambda = \sigma + 1/2$. Therefore, Equation (C.2′) implies that

$$\mathcal{D}_t^{(m+1)/2}v = \mathcal{D}_t^{k+1}v = \mathcal{D}_t^{k+1}\Big(v - \lim_{t\to\infty} v(0,t)\Big) = \sum_{j=1}^n \mathcal{D}_t^{k+1}\partial_j v_j.$$

Hence, the remaining proof of the case $\alpha = 1$ is parallel to that of the case $0 < \alpha < 1$. This completes the proof of Theorem 3. $\square$

7. Inversion theorems.

In this section, we give inversion theorems, that is, for a vector-valued function $V = (v_1, \ldots, v_n)$ with $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$, we construct a function $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ such that V is an α-parabolic conjugate function of u.

Let $0 < \alpha \leq 1$ and $\eta > -m(\alpha)$. If $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$, then by (iii) of Lemma 3.4, we can define an $L^{(\alpha)}$-harmonic function v_0 on H by

$$v_0(x,t) := \sum_{j=1}^{n} \partial_j v_j(x,t), \quad (x,t) \in H. \tag{7.1}$$

Furthermore, (i) of Lemma 3.4 implies that

$$|v_0(x,t)| \leq C t^{-(1/(2\alpha)+\eta)} \sum_{j=1}^{n} \|v_j\|_{\mathcal{B}_\alpha(\eta)} \tag{7.2}$$

for all $(x,t) \in H$, where C is independent of v_j. The function v_0 defined by (7.1) shall be used in the proof of our inversion theorems.

First, we give an inversion theorem when $0 < \alpha < 1$.

Theorem 7.1. *Let $0 < \alpha < 1$. Suppose that a vector-valued function $V = (v_1, \ldots, v_n)$ on H satisfies $v_j \in \widetilde{\mathcal{B}}_\alpha(\eta)$ and $\partial_k v_j = \partial_j v_k$ for all $1 \leq j, k \leq n$. Put $\sigma := 1 - 1/(2\alpha) + \eta$. If $\sigma > 0$ (thus, η also satisfies the condition $\eta > -m(\alpha)$), then there exists a unique function u on H such that $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ and V is an α-parabolic conjugate function of u. Also, there exists a constant $C = C(n, \alpha, \eta) > 0$ independent of V such that*

$$C^{-1} \sum_{j=1}^{n} \|v_j\|_{\mathcal{B}_\alpha(\eta)} \leq \|u\|_{\mathcal{B}_\alpha(\sigma)} \leq C \sum_{j=1}^{n} \|v_j\|_{\mathcal{B}_\alpha(\eta)}. \tag{7.3}$$

Proof. Let v_0 be the $L^{(\alpha)}$-harmonic function defined by (7.1). Put $\kappa := 1/\alpha - 1 > 0$. Then the hypothesis $\sigma > 0$ implies $1/(2\alpha) + \eta > \kappa$, that is, $v_0(x, \cdot) \in \mathcal{F}C^{-\kappa}$ for each $x \in \mathbb{R}^n$. Thus, we can define a function u_0 on H by

$$u_0(x,t) := \mathcal{D}_t^{-\kappa} v_0(x,t) = \mathcal{D}_t^{-\kappa}\left(\sum_{j=1}^{n} \partial_j v_j(x,t)\right), \quad (x,t) \in H. \tag{7.4}$$

We show that the function u_0 belongs to $\mathcal{B}_\alpha(\sigma)$. In fact, let $1 \leq k \leq n$. Then, by (i) of Lemma 3.4, v_0 belongs to $C^\infty(H)$ and

$$\int_0^\infty \tau^{\kappa-1}|\partial_k v_0(x,\tau+t)|d\tau \le C\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} \int_0^\infty \tau^{\kappa-1}(\tau+t)^{-(1/\alpha+\eta)}d\tau$$
$$= Ct^{\kappa-(1/\alpha+\eta)}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} = Ct^{-(1+\eta)}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} \tag{7.5}$$

for all $(x,t) \in H$, where C is independent of V. Therefore, the derivative $\partial_k u_0$ exists, that is,

$$\partial_k u_0(x,t) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1}\partial_k v_0(x,\tau+t)d\tau \tag{7.6}$$

and

$$|\partial_k u_0(x,t)| \le Ct^{-(1+\eta)}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} = Ct^{-\sigma}\cdot t^{-1/(2\alpha)}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)}$$

for all $(x,t) \in H$. Similarly, the derivative $\partial_t u_0$ exists and

$$|\partial_t u_0(x,t)| \le Ct^{-(2-1/(2\alpha)+\eta)}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)} = Ct^{-\sigma}\cdot t^{-1}\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)}$$

for all $(x,t) \in H$. Hence, we obtain

$$\|u_0\|_{\mathcal{B}_\alpha(\sigma)} \le C\sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)}, \tag{7.7}$$

where C is independent of V. Since v_0 belongs to $C^\infty(H)$, so dose u_0. We claim that u_0 is $L^{(\alpha)}$-harmonic on H. Indeed, let $\psi \in C_c^\infty(H)$. Then, there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ which satisfy (2.2) and (2.3). Therefore, similar calculations to (7.5) show that

$$\int_H \int_0^\infty \tau^{\kappa-1}|v_0(x,\tau+t)|d\tau \left|\widetilde{L}^{(\alpha)}\psi(x,t)\right|dV(x,t)$$
$$\le C\int_H t^{-(1-1/(2\alpha)+\eta)}\left|\widetilde{L}^{(\alpha)}\psi(x,t)\right|dV(x,t)$$
$$\le C\int_{t_1}^{t_2}\int_{\mathbb{R}^n} t^{-(1-1/(2\alpha)+\eta)}(1+|x|)^{-n-2\alpha}dxdt < \infty.$$

Thus, $\int_H |u_0 \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$, and the Fubini theorem implies that

$$\begin{aligned}\int_H & u_0(x,t) \cdot \widetilde{L}^{(\alpha)}\psi(x,t) dV(x,t) \\ &= \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \int_H v_0(x,\tau+t) \cdot \widetilde{L}^{(\alpha)}\psi(x,t) dV(x,t)\ d\tau = 0,\end{aligned}$$

because v_0 is $L^{(\alpha)}$-harmonic on H. Hence, u_0 is $L^{(\alpha)}$-harmonic on H, so $u_0 \in \mathcal{B}_\alpha(\sigma)$.

Put $u(x,t) := u_0(x,t) - u_0(0,1)$. Then, $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, and (7.7) implies

$$\|u\|_{\mathcal{B}_\alpha(\sigma)} = \|u_0\|_{\mathcal{B}_\alpha(\sigma)} \le C \sum_{j=1}^n \|v_j\|_{\mathcal{B}_\alpha(\eta)}, \tag{7.8}$$

where C is independent of v_j. We show that u and V satisfy Equations (C.1) and (C.2). By (7.6), the hypothesis $\partial_k v_j = \partial_j v_k$, and Lemma 6.1, we obtain

$$\begin{aligned}\partial_k u(x,t) = \partial_k u_0(x,t) &= \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \sum_{j=1}^n \partial_j \partial_k v_j(x,\tau+t) d\tau \\ &= \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \Delta_x v_k(x,\tau+t) d\tau \\ &= -\frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \mathcal{D}_t^{1/\alpha} v_k(x,\tau+t) d\tau \\ &= -\mathcal{D}_t^{-\kappa} \mathcal{D}_t^{1/\alpha} v_k(x,t).\end{aligned}$$

By the definition (2.8), we have

$$\mathcal{D}_t^{-\kappa} \mathcal{D}_t^{1/\alpha} v_k(x,t) = \mathcal{D}_t^{-\kappa} \mathcal{D}_t^{-\nu} \mathcal{D}_t^{\lceil 1/\alpha \rceil} v_k(x,t) = \mathcal{D}_t^{-\kappa} \mathcal{D}_t^{-\nu} \varphi(x,t), \tag{7.9}$$

where $\nu := \lceil 1/\alpha \rceil - 1/\alpha$ and $\varphi(x,t) := \mathcal{D}_t^{\lceil 1/\alpha \rceil} v_k(x,t)$. Since (i) of Lemma 3.4 implies $|\varphi(x,t)| \le C t^{-(\lceil 1/\alpha \rceil + \eta)}$ for all $(x,t) \in H$ and $\kappa + \nu = \lceil 1/\alpha \rceil - 1$, (ii) of Lemma 2.1 shows that

$$\begin{aligned}\mathcal{D}_t^{-\kappa} \mathcal{D}_t^{-\nu} \varphi(x,t) &= \mathcal{D}_t^{-\kappa-\nu} \varphi(x,t) = \mathcal{D}_t^{-(\lceil 1/\alpha \rceil - 1)} \varphi(x,t) \\ &= \mathcal{D}_t^{-(\lceil 1/\alpha \rceil - 1)} \mathcal{D}_t^{\lceil 1/\alpha \rceil - 1} \mathcal{D}_t v_k(x,t).\end{aligned} \tag{7.10}$$

Furthermore, (i) of Lemma 3.4 and (v) of Lemma 2.1 also show

$$\mathcal{D}_t^{-(\lceil 1/\alpha\rceil-1)}\mathcal{D}_t^{\lceil 1/\alpha\rceil-1}\mathcal{D}_t v_k(x,t)=\mathcal{D}_t v_k(x,t). \tag{7.11}$$

Hence, (7.9), (7.10), and (7.11) imply $\partial_k u(x,t)=-\mathcal{D}_t v_k(x,t)$, so u and V satisfy Equation (C.1). Moreover, by (i) of Lemma 3.4, (iii) of Lemma 2.1, and (7.4), we also obtain

$$\mathcal{D}_t^{1/\alpha-1}u=\mathcal{D}_t^{\kappa}u=\mathcal{D}_t^{\kappa}u_0=\mathcal{D}_t^{\kappa}\mathcal{D}_t^{-\kappa}v_0=v_0=\sum_{j=1}^{n}\partial_j v_j,$$

so u and V satisfy Equation (C.2).

We show the inequalities of (7.3). By (7.8), it suffices to show the first inequality of (7.3). Indeed, Lemma 3.6 and Equation (C.1) imply that

$$\begin{aligned}\|v_j\|_{\mathcal{B}_\alpha(\eta)}&\le C\big\|t^{1+\eta}\mathcal{D}_t v_j\big\|_{L^\infty}=C\big\|t^{1+\eta}\partial_j u\big\|_{L^\infty}\\&=C\sup_{(x,t)\in H}t^{\sigma}\cdot t^{1/(2\alpha)}|\partial_j u(x,t)|\le C\|u\|_{\mathcal{B}_\alpha(\sigma)},\end{aligned}$$

where C is independent of v_j. Therefore, we obtain the first inequality of (7.3).

Suppose that a function v on H belongs to $\widetilde{\mathcal{B}}_\alpha(\sigma)$ and V is an α-parabolic conjugate function of v. Then, Lemma 3.6 and Equation (C.2) imply that

$$\begin{aligned}\|u-v\|_{\mathcal{B}_\alpha(\sigma)}&\le C\big\|t^{1/\alpha-1+\sigma}\mathcal{D}_t^{1/\alpha-1}(u-v)\big\|_{L^\infty}\\&=C\left\|t^{1/\alpha-1+\sigma}\left(\sum_{j=1}^{n}\partial_j v_j-\sum_{j=1}^{n}\partial_j v_j\right)\right\|_{L^\infty}=0.\end{aligned}$$

Hence, we obtain $u=v$. This completes the proof. □

Next, we also give an inversion theorem when $\alpha=1$. We remark that if $\alpha=1$, then $m(1)=\min\{1,1/2\}=1/2$. Thus, the condition $\eta>-m(1)$ is equivalent to $1/2+\eta>0$.

Theorem 7.2. *Suppose that a vector-valued function $V=(v_1,\dots,v_n)$ on H satisfies $v_j\in\widetilde{\mathcal{B}}_1(\eta)$ and $\partial_k v_j=\partial_j v_k$ for all $1\le j,\ k\le n$. Put $\sigma:=1/2+\eta$. If $\sigma>0$, then there exists a unique function u on H such that $u\in\widetilde{\mathcal{B}}_1(\sigma)$ and u satisfies Equations* (C.1) *and* (C.2′)*. Also, there exists a constant $C=C(n,\eta)>0$ independent of V such that*

$$C^{-1}\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)} \le \|u\|_{\mathcal{B}_1(\sigma)} \le C\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)}. \tag{7.12}$$

Proof. Let v_0 be the $L^{(\alpha)}$-harmonic function on H define by (7.1). And put

$$u(x,t) = v_0(x,t) - v_0(0,1), \quad (x,t) \in H.$$

Then, (i) of Lemma 3.4 implies that

$$|\partial_k u(x,t)| \le Ct^{-(1+\eta)}\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)} = Ct^{-\sigma}\cdot t^{-1/2}\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)}$$

and

$$|\partial_t u(x,t)| \le Ct^{-(1/2+1+\eta)}\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)} = Ct^{-\sigma}\cdot t^{-1}\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)}$$

for all $(x,t) \in H$. Hence, we obtain $u \in \widetilde{\mathcal{B}}_1(\sigma)$ and

$$\|u\|_{\mathcal{B}_1(\sigma)} \le C\sum_{j=1}^{n}\|v_j\|_{\mathcal{B}_1(\eta)}, \tag{7.13}$$

where C is independent of V.

We show that u and V satisfy Equations (C.1) and (C.2′). By (7.1), the hypothesis $\partial_k v_j = \partial_j v_k$, and Lemma 6.1, we obtain

$$\partial_k u(x,t) = \partial_k v_0(x,t) = \sum_{j=1}^{n}\partial_j\partial_k v_j(x,t) = \Delta_x v_k(x,t) = -\mathcal{D}_t v_k(x,t),$$

so u and V satisfy Equation (C.1). Furthermore, by (7.2), we have

$$\lim_{r\to\infty} u(x,r) = \lim_{r\to\infty} v_0(x,r) - v_0(0,1) = -v_0(0,1)$$

for all $x \in \mathbb{R}^n$. Thus, by Remark 5.6, we obtain

$$u(x,t) = v_0(x,t) - v_0(0,1) = \sum_{j=1}^{n}\partial_j v_j(x,t) + \lim_{r\to\infty} u(0,r),$$

so u and V satisfy Equation (C.2′).

Moreover, Lemma 3.6 and Equation (C.1) imply that for every $j = 1, 2, \ldots, n$,

$$\|v_j\|_{\mathcal{B}_1(\eta)} \le C\|t^{1+\eta}\mathcal{D}_t v_j\|_{L^\infty} = C\|t^{1+\eta}\partial_j u\|_{L^\infty}$$
$$= C \sup_{(x,t)\in H} t^\sigma \cdot t^{1/2}|\partial_j u(x,t)| \le C\|u\|_{\mathcal{B}_1(\sigma)},$$

where C is independent of V. Therefore, by (7.13), we obtain the inequalities of (7.12).

To show the uniqueness, we suppose that a function v on H belongs to $\widetilde{\mathcal{B}}_1(\sigma)$ and v satisfies Equations (C.1) and (C.2′). Then, Equation (C.2′) implies that

$$\|u - v\|_{\mathcal{B}_1(\sigma)} = \left\| \sum_{j=1}^{n} \partial_j v_j - \sum_{j=1}^{n} \partial_j v_j \right\|_{\mathcal{B}_1(\sigma)} = 0.$$

Hence, we obtain $u = v$. This completes the proof. □

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Yôsuke Hishikawa
Department of General Education
Gifu National College of Technology
Kamimakuwa 2236-2, Motosu City
Gifu 501-0495, Japan
E-mail: yosuke-h@gifu-nct.ac.jp

Masaharu Nishio
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi 3-3-138
Osaka 558-8585, Japan
E-mail: nishio@sci.osaka-cu.ac.jp

Masahiro Yamada
Department of Mathematics
Faculty of Education
Gifu University
Yanagido 1-1
Gifu 501-1193, Japan
E-mail: yamada33@gifu-u.ac.jp