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On a bound of λ and the vanishing of μ of $\mathbb{Z}_p$-extensions of an imaginary quadratic field

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Abstract. Let p be an odd prime number. To ask the behavior of λ- and μ-invariants is a basic problem in Iwasawa theory of $\mathbb{Z}_p$-extensions. Sands showed that if p does not divide the class number of an imaginary quadratic field k and if the λ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of k is 2, then μ-invariants vanish for all $\mathbb{Z}_p$-extensions of k, and λ-invariants are less than or equal to 2 for $\mathbb{Z}_p$-extensions of k in which all primes above p are totally ramified. In this article, we show results similar to Sands' results without the assumption that p does not divide the class number of k. When μ-invariants vanish, we also give an explicit upper bound of λ-invariants of all $\mathbb{Z}_p$-extensions.

1. Introduction.

Let $k/\mathbb{Q}$ be a finite extension, h_k the class number of k and p a prime number. In this article, all algebraic extensions of $\mathbb{Q}$ are assumed to be contained in a fixed algebraic closure of $\mathbb{Q}$. Let k_∞/k be a $\mathbb{Z}_p$-extension and k_n its n-th layer, that is, the unique intermediate field of k_∞/k such that $[k_n : k] = p^n$, here we let $\mathbb{Z}_p$ the ring of p-adic integers. By Iwasawa's class number formula, there are non-negative integers $\lambda(k_\infty/k)$, $\mu(k_\infty/k)$ and an integer $\nu(k_\infty/k)$ depending only on k_∞/k such that the p-exponent of h_{k_n} is described as

$$\lambda(k_\infty/k)n + \mu(k_\infty/k)p^n + \nu(k_\infty/k)$$

for all sufficiently large n. These invariants are called the Iwasawa λ-, μ- and ν-invariant. Especially, the invariants λ and μ are important, these are structure invariants of ideal class groups as Galois modules. Then the following problem has been considered.

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Problem. For a fixed finite extension $k/\mathbb{Q}$ and a prime number p, how do invariants $\lambda(k_\infty/k)$ and $\mu(k_\infty/k)$ behave as k_∞ runs $\mathbb{Z}_p$-extensions of k?

Some studies on the above problem for imaginary quadratic fields have been done by several authors, for example, Bloom–Gerth [**2**], Sands [**7**] and Ozaki [**6**], and so on. Let k be an imaginary quadratic field. Then there is a unique $\mathbb{Z}_p^2$-extension $\widetilde{k}$ of k. Hence there exist infinitely many $\mathbb{Z}_p$-extensions of k. Typical examples of $\mathbb{Z}_p$-extensions are:

- The cyclotomic $\mathbb{Z}_p$-extension k_∞^c.
- The anti-cyclotomic $\mathbb{Z}_p$-extension k_∞^a when p is an odd prime number.
- Suppose that p splits in k, that is, $p = \mathfrak{p}\mathfrak{p}'$. Then there are the $\mathfrak{p}$- and the $\mathfrak{p}'$-ramified $\mathbb{Z}_p$-extensions N_∞ and N'_∞.

When p is an odd prime number, the $\mathbb{Z}_p$-extensions k_∞^c and k_∞^a are Galois extensions over $\mathbb{Q}$, and if $k_\infty/\mathbb{Q}$ is a Galois extension then $k_\infty = k_\infty^c$ or k_∞^a. Note that $k_\infty^c/\mathbb{Q}$ is abelian and that $k_\infty^a/\mathbb{Q}$ is non-abelian.

We show here completely determined cases, Sands' and Ozaki's results for our problem.

Theorem A (Completely determined cases). *Let p be an odd prime number and k an imaginary quadratic field.*

(1) *Suppose that p does not split in k and that $\lambda(k_\infty^c/k) = 0$. Then $\lambda(k_\infty/k) = \mu(k_\infty/k) = \nu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞.*
(2) *Suppose that p splits in k and that $\lambda(k_\infty^c/k) = 1$. Then, $\lambda(N_\infty/k) = \lambda(N'_\infty/k) = 0$, $\lambda(k_\infty/k) = 1$ for each $\mathbb{Z}_p$-extension k_∞ with $k_\infty \neq N_\infty, N'_\infty$, and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞.*

Sands [**7**] stated a part of Theorem A. We will prove Theorem A in the last section. However, there are no contributions by the author. Theorem A is shown by combining arguments which are already known.

Theorem B (Sands [**7**]). *Let p be an odd prime number and k an imaginary quadratic field in which p splits. Suppose that $p \nmid h_k$ and that $\lambda(k_\infty^c/k) = 2$. Then, $\lambda(k_\infty/k) \leq 2$ for each $\mathbb{Z}_p$-extension k_∞ with $k_\infty \cap N_\infty = k_\infty \cap N'_\infty = k$, and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞.*

Theorem C (Ozaki [**6**]). *Let p be an odd prime number and k an imaginary quadratic field in which p splits. Suppose that $p \nmid h_k$. Then $\lambda(k_\infty/k) = 1$ and $\mu(k_\infty/k) = 0$ for all but finite k_∞.*

In this article, we show results similar to Theorem B without the condition that $p \nmid h_k$.

THEOREM 1. *Let p be an odd prime number and k an imaginary quadratic field.*

(1) *Suppose that p splits in k and that $\lambda(k_\infty^c/k) = 2$. Then, $\lambda(k_\infty/k) \leq 2$ for each $\mathbb{Z}_p$-extension k_∞ such that $k_\infty \cap k_\infty^a = k$ and that $k_\infty \neq N_\infty$, N'_∞.*
(2) *Suppose that p does not split in k and that $\lambda(k_\infty^c/k) = 1$. Then, $\lambda(k_\infty/k) \leq 1$ for each $\mathbb{Z}_p$-extension k_∞ such that $k_\infty \cap k_\infty^a = k$.*

Here we give some remarks.

(1) By Bloom–Gerth's result [**2**], under the assumption on $\lambda(k_\infty^c/k)$ in Theorem 1, it is known that $\mu(k_\infty/k) = 0$ for each k_∞ except for k_∞^a, which will be explained lator.
(2) The proof of Theorem 1 is very similar to a method used in Bloom [**1**]. By using the action of the complex conjugation, we can obtain a detailed conclusion.

As a corollary to Theorem 1 and results which had already been obtained by several authors, we can give a partial answer to our problem.

COROLLARY. *Let p be an odd prime number and k an imaginary quadratic field in which p splits. Suppose that $p \nmid h_k$ and that $\lambda(k_\infty^c/k) = 2$.*

(1) *For all $\mathbb{Z}_p$-extensions k_∞, $\mu(k_\infty/k) = 0$.*
(2) $\lambda(N_\infty/k) = \lambda(N'_\infty/k) = 0$.
(3) $\lambda(k_\infty/k) = 1$ *for all but finite k_∞.*
(4) *For finite exceptional $\mathbb{Z}_p$-extensions k_∞ in (3) with $k_\infty \neq N_\infty, N'_\infty$, $\lambda(k_\infty/k) = 2$.*

In particular, $\lambda(k_\infty/k) \leq 2$ for all $\mathbb{Z}_p$-extensions k_∞.

The assertion (1) is a part of Theorem B. Let N_n be the unique intermediate subfield of N_∞/k with $[N_n : k] = p^n$ for each non-negative integer n. Since N_∞/k is totally ramified at $\mathfrak{p}$ and $p \nmid h_k$, we have $p \nmid h_{N_n}$. This shows (2). The assertion (3) is a special case of Theorem C. Suppose that $k_\infty \neq N_\infty, N'_\infty$. If $k_\infty \cap k_\infty^a \supsetneq k$, then $k_\infty \cap N_\infty = k_\infty \cap N'_\infty = k$ since $p \nmid h_k$. By Theorem B, $\lambda(k_\infty/k) \leq 2$. If $k_\infty \cap k_\infty^a = k$, then $\lambda(k_\infty/k) \leq 2$ by Theorem 1. This shows (4).

Next we show a result which concern an upper bound of λ and the vanishing of μ. If $p \nmid h_k$ and $\lambda(k_\infty^c/k) = 2$, then we already know $\mu(k_\infty/k) = 0$ and $\lambda(k_\infty/k) \leq 2$ for all $\mathbb{Z}_p$-extensions k_∞ from the above corollary. We then deal with the case where $p \mid h_k$.

THEOREM 2. *Let p be an odd prime number and k an imaginary quadratic field in which p splits. Suppose the following conditions*:

(1) $\lambda(k_\infty^c/k) = 2$.
(2) *The p-Hilbert class field L_k of k is contained in $\widetilde{k}$.*
(3) $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = p$, *where we denote by $\mathfrak{D}$ the decomposition group in* $\mathrm{Gal}(\widetilde{k}/k)$ *of a prime lying above p.*

Then $\lambda(k_\infty/k) \leq p$ and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞.

In fact, we will show a somewhat more general statement including the case where p does not split in k. One will see that $\lambda(k_\infty/k) \leq p$ is the best possible bound if $p \mid h_k$. We show some examples.

- Let $p = 3$. Let $k = \mathbb{Q}(\sqrt{-461})$ or $\mathbb{Q}(\sqrt{-743})$, then the prime 3 splits in k. We can check that $3 \mid h_k$, $\lambda(k_\infty^c/k) = 2$, $L_k \subseteq \widetilde{k}$ and $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = 3$. Hence $\lambda(k_\infty/k) \leq 3$ and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_3$-extensions k_∞.
- Let $p = 5$ and $k = \mathbb{Q}(\sqrt{-1214})$, then 5 splits in k. We can check that $5 \mid h_k$, $\lambda(k_\infty^c/k) = 2$, $L_k \subseteq \widetilde{k}$ and $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = 5$. Hence $\lambda(k_\infty/k) \leq 5$ and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_5$-extensions k_∞.

2. Preliminaries.

This section consists of notations and affirmations of fundamental properties of Iwasawa modules. In what follows, let p and k be an odd prime number and an imaginary quadratic field respectively. As mentioned in Section 1, there is a unique $\mathbb{Z}_p^2$-extension $\widetilde{k}$ of k. Note that all $\mathbb{Z}_p$-extensions of k are contained in $\widetilde{k}$. Note also that all primes of k lying above p are ramified in k_∞/k (not necessary totally ramified) except for $k_\infty = N_\infty$ or N'_∞. Let L_k/k be the maximal unramified abelian pro-p extension, which is also called the p-Hilbert class field. Let K/k be a $\mathbb{Z}_p$-extension or the $\mathbb{Z}_p^2$-extension and X_K the Galois group $\mathrm{Gal}(L_K/K)$ of the maximal unramified abelian pro-p extension L_K/K. When $K = \widetilde{k}$ we put $X = X_{\widetilde{k}}$. The Galois group $\mathrm{Gal}(K/k)$ acts on X_K in the manner $g(x) = \overline{g}x\overline{g}^{-1}$, where we let $g \in \mathrm{Gal}(K/k)$, $x \in X_K$ and $\overline{g}$ a lift of g to $\mathrm{Gal}(L_K/k)$. Then the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$ acts on X_K, and it is known that X_K is a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$-module. For $K = \widetilde{k}$, we set a more precise notation. We choose a basis of $\mathrm{Gal}(\widetilde{k}/k)$ as follows. Since the cyclotomic $\mathbb{Z}_p$-extension k_∞^c and the anti-cyclotomic $\mathbb{Z}_p$-extension k_∞^a are disjoint over k, we know that $\widetilde{k} = k_\infty^c k_\infty^a$, and hence $\mathrm{Gal}(\widetilde{k}/k)$ is a direct product of $\mathrm{Gal}(\widetilde{k}/k_\infty^c)$ and $\mathrm{Gal}(\widetilde{k}/k_\infty^a)$. Let σ and τ be topological generators of $\mathrm{Gal}(\widetilde{k}/k_\infty^c)$ and $\mathrm{Gal}(\widetilde{k}/k_\infty^a)$ respectively. Put $\langle J \rangle = \mathrm{Gal}(k/\mathbb{Q})$. Then J acts on $\mathrm{Gal}(\widetilde{k}/k)$ since $\widetilde{k}/\mathbb{Q}$ is a Galois extension. The action of J on $\mathrm{Gal}(\widetilde{k}/k)$ is given by $J(x) = \overline{J}x\overline{J}^{-1}$ for $x \in \mathrm{Gal}(\widetilde{k}/k)$, here $\overline{J} \in \mathrm{Gal}(\widetilde{k}/\mathbb{Q})$ is a lift of J. Since $k_\infty^c/\mathbb{Q}$ is abelian and $k_\infty^a/\mathbb{Q}$ is non-abelian, one sees that $J(\sigma) = \sigma^{-1}$ and $J(\tau) = \tau$. We then fix an isomorphism between the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/k)]]$ and the formal power series ring $\Lambda = \mathbb{Z}_p[[S,T]]$ in two variables given by $\sigma \leftrightarrow 1+S$ and $\tau \leftrightarrow 1+T$. So we regard X a Λ-module. Note that Λ is a complete noetherian local integral domain with the maximal ideal (S,T,p). We also use the power series rings $\mathbb{Z}_p[[S]]$ and $\mathbb{Z}_p[[T]]$ in one variable as a sub- or a quotient ring of Λ. For a commutative ring A, denote by $A^\times$ the unit group of A. Note that $\Lambda^\times = \Lambda - (S,T,p)$ and $\mathbb{Z}_p[[S]]^\times = \mathbb{Z}_p[[S]] - (S,p)$. Let M be a finitely generated torsion $\mathbb{Z}_p[[S]]$-module. By the structure theorem of $\mathbb{Z}_p[[S]]$-modules, M is pseudo-isomorphic to a module of the form $\bigoplus_{i=1}^r \mathbb{Z}_p[[S]]/\mathfrak{q}^{m_i}$, where r and m_i $(1 \le i \le r)$ are non-negative integers, and $\mathfrak{q}_i$s are prime ideals of $\mathbb{Z}_p[[S]]$ of height 1. Then the ideal

$$\mathrm{char}_{\mathbb{Z}_p[[S]]}(M) = \prod_{i=1}^{r} \mathfrak{q}^{m_i}$$

is called the characteristic ideal of M.

For a profinite group H and a profinite H-module M, let M_H be the H-coinvariant module of M, namely, $M_H = M/\overline{\sum_{h\in H}(h-1)M}$. If $H = \overline{\langle h\rangle}$, then $M_H = M/(h-1)M$. Let k_∞ be a $\mathbb{Z}_p$-extension and $\overline{\langle \sigma^\alpha\tau^\beta\rangle}$ the corresponding subgroup of $\mathrm{Gal}(\widetilde{k}/k)$ to k_∞, where $(\alpha,\beta) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$. Since $\sigma^\alpha\tau^\beta$ corresponds to $(1+S)^\alpha(1+T)^\beta$, we have

$$X_{\mathrm{Gal}(\widetilde{k}/k_\infty)} = X/(\sigma^\alpha\tau^\beta - 1)X = X/((1+S)^\alpha(1+T)^\beta - 1)X.$$

In this article, we use frequently such coinvariant modules, so we put $Y_{k_\infty} = X_{\mathrm{Gal}(\widetilde{k}/k_\infty)}$ for $\mathbb{Z}_p$-extensions k_∞.

LEMMA 2.1. *Let F_∞/F be a $\mathbb{Z}_p$-extension of a number field F.*

(1) $\lambda(F_\infty/F) = \mathrm{rank}_{\mathbb{Z}_p}(X_{F_\infty})$.
(2) $\mu(F_\infty/F) = 0$ *if and only if X_{F_∞} is finitely generated over $\mathbb{Z}_p$.*
(3) *Let $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, here $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$. Then $\lambda(F_\infty/F) = \lambda(g(F_\infty)/g(F))$ and $\mu(F_\infty/F) = \mu(g(F_\infty)/g(F))$.*

PROOF. For (1) and (2), see sections 13–2 and –3 of [**8**]. Let F_n be the n-th layer of F_∞/F for each non-negative integer n. Then $g(F_n)$ is the n-th layer of a $\mathbb{Z}_p$-extension $g(F_\infty)/g(F)$, and $h_{F_n} = h_{g(F_n)}$. By Iwasawa's class number formula, we have

$$\begin{aligned}&\lambda(F_\infty/F)n + \mu(F_\infty/F)p^n + \nu(F_\infty/F)\\ &\quad = \lambda(g(F_\infty)/g(F))n + \mu(g(F_\infty)/g(F))p^n + \nu(g(F_\infty)/g(F))\end{aligned}$$

for all sufficiently large n. Since $\lim_{n\to\infty} n/p^n = 0$, we have

$$\mu(F_\infty/F) = \mu(g(F_\infty)/g(F)).$$

Similarly, it follows that $\lambda(F_\infty/F) = \lambda(g(F_\infty)/g(F))$. □

Lemma 2.2. *Let p be an odd prime number and k an imaginary quadratic field. Then $L_k \cap \widetilde{k}$ is contained in k_∞^a.*

Proof. Let Cl_k be the ideal class group of k. Then, by class field theory, the Artin map induces an isomorphism $Cl_k \otimes \mathbb{Z}_p \simeq \mathrm{Gal}(L_k/k)$, in particular, this isomorphism and the action of the complex conjugation J are compatible. Since $h_{\mathbb{Q}} = 1$, J acts as inverse on $Cl_k \otimes \mathbb{Z}_p$, and hence J also acts as inverse on $\mathrm{Gal}(L_k/k)$. Thus $L_k \cap \widetilde{k}/\mathbb{Q}$ is a Galois extension and J acts as inverse on $\mathrm{Gal}(L_k \cap \widetilde{k}/k)$. This shows that the image from $\mathrm{Gal}(\widetilde{k}/k_\infty^a)$ to $\mathrm{Gal}(L_k \cap \widetilde{k}/k)$ with respect to the restriction map is trivial. Hence $L_k \cap \widetilde{k}$ is fixed by $\mathrm{Gal}(\widetilde{k}/k_\infty^a)$, and therefore $L_k \cap \widetilde{k}$ is contained in k_∞^a. □

3. Proof of Theorem 1.

First we show an explicit relation between X and X_{k_∞}.

Lemma 3.1 (See for example Lemma 1 of Ozaki [**6**]). *Suppose one of the following two conditions.*

(1) *The prime p splits in k and $k_\infty \neq N_\infty,\ N'_\infty$.*
(2) *The prime p does not split in k and k_∞/k is totally ramified at the prime lying above p.*

Then there is an exact sequence

$$0 \longrightarrow Y_{k_\infty} \longrightarrow X_{k_\infty} \longrightarrow \mathrm{Gal}(\widetilde{k} \cap L_{k_\infty}/k_\infty) \longrightarrow 0$$

of $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-modules. Here, $\mathrm{Gal}(\widetilde{k} \cap L_{k_\infty}/k_\infty)$ is isomorphic to $\mathbb{Z}_p$ if p splits in k since $\widetilde{k} \subseteq L_{k_\infty}$, and is finite cyclic otherwise.

Remark. The cyclotomic $\mathbb{Z}_p$-extension k_∞^c satisfies the condition of Lemma 3.1. If p does not split in k and if $k_\infty \cap k_\infty^a = k$, then k_∞/k is totally ramified. Indeed, let k_1 be the 1-st layer of k_∞/k. if k_1/k is unramified at prime lying above p, then is unramified at all primes of k. Hence k_1 is contained in L_k. Therefore $k_1 \subseteq k_\infty^a$ by Lemma 2.2.

From Lemma 3.1, we have

$$\lambda(k_\infty/k) = \mathrm{rank}_{\mathbb{Z}_p}(X_{k_\infty}) = \begin{cases} \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty}) + 1 & \text{if } p \text{ splits in } k, \\ \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty}) & \text{otherwise} \end{cases}$$

for suitable $\mathbb{Z}_p$-extensions.

LEMMA 3.2. *Suppose that $\lambda(k_\infty^c/k) = 2$ if p splits in k, and $\lambda(k_\infty^c/k) = 1$ otherwise. Then there are a power series $f(S) \in \mathbb{Z}_p[[S]]$ and a surjective morphism $\Lambda/(T - f(S)) \to X$ of Λ-modules.*

PROOF. By Lemma 3.1, there is the following exact sequence

$$0 \longrightarrow Y_{k_\infty^c} \longrightarrow X_{k_\infty^c} \longrightarrow \mathrm{Gal}\left(L_{k_\infty^c} \cap \widetilde{k}/k_\infty^c\right) \longrightarrow 0$$

of $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty^c/k)]]$-modules. From the fact that $X_{k_\infty^c}$ is a free $\mathbb{Z}_p$-module of rank $\lambda(k_\infty^c/k)$ (see for example corollary 13.29 of [**8**]), we find that $Y_{k_\infty^c} = X/SX \simeq \mathbb{Z}_p$. By topological version of Nakayama's lemma, there is $x \in X$ such that $X = \mathbb{Z}_p[[S]]x$. Then there is a power series $f(S) \in \mathbb{Z}_p[[S]]$ such that $Tx = f(S)x$, and $(T - f(S))X = 0$. Therefore, there is a surjective morphism

$$\Lambda/(T - f(S)) \to X,\ F(S,T) \mapsto F(S,T)x$$

of Λ-modules. □

Note that the uniqueness of a power series $f(S)$ is unknown, but we fix one $f(S)$. The uniqueness of $f(S)$ is related to so called Greenberg's generalized conjecture. The properties of $f(S)$ are also not known almost. However, we can show at least that $S \nmid f(S)$. Indeed, there is a surjective morphism $\Lambda/(S, T - f(S)) \to Y_{k_\infty^c}$. If $S \mid f(S)$ then $\mathrm{Gal}(k_\infty^c/k)$ acts on $Y_{k_\infty^c}$ trivially. But it is known that $\mathrm{Gal}(k_\infty^c/k)$ acts on $Y_{k_\infty^c}$ non-trivially, see for example Lemma 5 of Ozaki [**6**]. Therefore, S does not divide $f(S)$. By the p-adic version of Weierstrass preparation theorem, there are a non-negative integer m, a distinguished polynomial $g(S) \in \mathbb{Z}_p[S]$ and a unit power series $U(S) \in \mathbb{Z}_p[[S]]^\times$ such that $f(S) = p^m g(S)U(S)$. Here a polynomial $\varphi(S)$ with coefficients in $\mathbb{Z}_p$ is called distinguished polynomial if $\varphi(S)$ is monic and $\varphi(S) \equiv S^{\deg \varphi(S)} \bmod p$.

Let k_∞/k be a $\mathbb{Z}_p$-extension. Then there is a pair $(\alpha, \beta) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ such that $k_\infty = \widetilde{k}^{\langle \sigma^\alpha \tau^\beta \rangle}$. Suppose that k_∞ satisfies the assumption of Lemma 3.1. Then by Lemma 3.2, we have an exact sequence

$$\Lambda/((1+S)^{\alpha}(1+T)^{\beta}-1, T-f(S)) \longrightarrow X_{k_\infty} \longrightarrow \mathrm{Gal}\left(L_{k_\infty} \cap \widetilde{k}/k_\infty\right) \longrightarrow 0.$$

Put

$$I_{\alpha,\beta} = ((1+S)^{\alpha}(1+T)^{\beta}-1, T-f(S), p).$$

If $I_{\alpha,\beta} = (S, T, p)$, then

$$\Lambda/I_{\alpha,\beta} \simeq \mathbb{Z}/p,\ F(S,T) \bmod I_{\alpha,\beta} \mapsto F(0,0) \bmod p.$$

This leads the assertion of Theorem 1 by Lemma 2.1. We analyze when $I_{\alpha,\beta} = (S, T, p)$.

Lemma 3.3. *If $p \nmid \alpha$ and $p \nmid \alpha + \beta U(0)$, then $I_{\alpha,\beta} = (S, T, p)$, here $U(S)$ is a unit power series associated to $f(S)$.*

Proof. Recall $f(S) = p^m g(S)U(S)$. We prove by splitting into 2 cases.

(i) Suppose that $m \geq 1$. Suppose also that $\alpha = p^n\alpha'$ for some non-negative integer n and $\alpha' \in \mathbb{Z}_p$. Then

$$\begin{aligned} I_{\alpha,\beta} &= ((1+S)^{\alpha}(1+T)^{\beta}-1, T-p^m g(S)U(S), p) \\ &= ((1+S^{p^n})^{\alpha'}(1+T)^{\beta}-1, T, p) \\ &= \left(S^{p^n}\left(\sum_{k=1}^{\infty}\binom{\alpha'}{k}S^{p^n(k-1)}\right), T, p\right) \\ &\subseteq (S^{p^n}, T, p). \end{aligned}$$

Also, if $p \nmid \alpha$ then $n = 0$ and $\sum_{k=1}^{\infty}\binom{\alpha}{k}S^{k-1}$ is a unit of $\mathbb{Z}_p[[S]]$. Hence, in this case, $I_{\alpha,\beta} = (S, T, p)$ if and only if $p \nmid \alpha$.

(ii) Suppose that $m = 0$. Then $f(S) = g(S)U(S)$. Let $d \geq 1$ be the degree of a distinguished polynomial $g(S)$. Note that $g(S) \equiv S^d \bmod p$. Then

$$\begin{aligned} I_{\alpha,\beta} &= ((1+S)^{\alpha}(1+T)^{\beta}-1, T-S^dU(S), p) \\ &= ((1+S)^{\alpha}(1+S^dU(S))^{\beta}-1, T-S^dU(S), p) \\ &= \left(\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}S^{k+(n-k)d}U(S)^{n-k}, T-S^dU(S), p\right) \end{aligned}$$

$$= \left(S \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} S^{nd-k(d-1)-1} U(S)^{n-k}, T - S^d U(S), p \right).$$

Put $h(S) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} S^{nd-k(d-1)-1} U(S)^{n-k}$. Since

$$nd - k(d-1) - 1 \geq nd - n(d-1) - 1 = n - 1,$$

we have

$$\begin{aligned} h(0) &= \sum_{k=0}^{1} \binom{\alpha}{k} \binom{\beta}{1-k} \left[S^{d-k(d-1)-1} \right]_{S=0} U(0)^{1-k} \\ &= \begin{cases} \alpha + \beta U(0) & \text{if } d = 1, \\ \alpha & \text{if } d \geq 2. \end{cases} \end{aligned}$$

Suppose that $p \nmid \alpha$ and $p \nmid \alpha + \beta U(0)$. Then $h(S)$ is a unit power series of $\mathbb{Z}_p[[S]]$. Therefore,

$$I_{\alpha,\beta} = (Sh(S), T - S^d U(S), p) = (S, T - S^d U(S), p) = (S, T, p).$$

This completes the proof of Lemma 3.3. $\square$

Recall that $k_\infty = \widetilde{k}^{\,\overline{\langle \sigma^\alpha \tau^\beta \rangle}}$ with $(\alpha, \beta) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$.

Lemma 3.4. *$p \nmid \alpha$ if and only if $k_\infty \cap k_\infty^a = k$.*

Proof. Let k_1^a be the 1-st layer of k_∞^a/k. Then $k_\infty \cap k_\infty^a = k$ if and only if $k_1^a \not\subseteq k_\infty$ since k_∞/k is a $\mathbb{Z}_p$-extension. By the choices of σ and τ, k_1^a is fixed by τ and σ^p, whence $\mathrm{Gal}(\widetilde{k}/k_1^a) = \overline{\langle \sigma^p \rangle} \oplus \overline{\langle \tau \rangle}$. This shows that $k_1^a \not\subseteq k_\infty$ if and only if $p \nmid \alpha$. $\square$

Suppose that $p \nmid \alpha$, hence $k_\infty \cap k_\infty^a = k$. When p splits in k, suppose further that $k_\infty \neq N_\infty$, N'_∞. Assume that $p \mid \beta$. Then $\alpha + \beta U(0) \equiv \alpha \not\equiv 0 \bmod p$, and hence $I_{\alpha,\beta} = (S, T, p)$ by Lemma 3.3. Assume that $p \nmid \beta$. If $\alpha + \beta U(0) \not\equiv 0 \bmod p$, then $I_{\alpha,\beta} = (S, T, p)$ by Lemma 3.3. Suppose that $\alpha + \beta U(0) \equiv 0 \bmod p$. Since $p \nmid \alpha\beta U(0)$ and p is an odd prime number, we find that $-\alpha + \beta U(0) \not\equiv 0 \bmod p$. Recall $\langle J \rangle = \mathrm{Gal}(k/\mathbb{Q})$ and let $\overline{J} \in \mathrm{Gal}(\widetilde{k}/\mathbb{Q})$ be a lift of J. Then

$$\begin{aligned}\overline{J}(k_\infty) &= \overline{J}\big(\widetilde{k}^{\,\overline{\langle\sigma^\alpha\tau^\beta\rangle}}\big)\\ &= \widetilde{k}^{\,\overline{J}\,\overline{\langle\sigma^\alpha\tau^\beta\rangle}\,\overline{J}^{-1}}\\ &= \widetilde{k}^{\,\overline{\langle\sigma^{-\alpha}\tau^\beta\rangle}}.\end{aligned}$$

From the congruence $-\alpha + \beta U(0) \not\equiv 0 \bmod p$ and Lemma 3.3, we know that $\lambda(\overline{J}(k_\infty)/k) \le 2$ if p splits in k and $\lambda(\overline{J}(k_\infty)/k) \le 1$ otherwise. Note that $\overline{J}(k_\infty) \ne N_\infty, N'_\infty$ since $\overline{J}(N_\infty) = N'_\infty$. From Lemma 2.1 (3), we conclude that

$$\lambda(k_\infty/k) = \lambda(\overline{J}(k_\infty)/k) \le \begin{cases} 2 & \text{if } p \text{ splits in } k,\\ 1 & \text{otherwise.}\end{cases}$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2.

We show in this section the following.

Theorem 4.1. *Let $\mathfrak{D} \subseteq \mathrm{Gal}(\widetilde{k}/k)$ be the decomposition group of a prime lying above p. Suppose that $L_k \subseteq \widetilde{k}$, and that one of the following two conditions* **(S)** *or* **(NS)** *holds.*

(S) *p splits in k, $\lambda(k_\infty^{\mathrm{c}}/k) = 2$ and $\mathfrak{D}$ is normal in $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$.*
(NS) *$p \ge 5$, p does not split in k and $\lambda(k_\infty^{\mathrm{c}}/k) = 1$.*

Then $\lambda(k_\infty/k) \le [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}]$ and $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞.

Here we give some remarks.
(1) We can show that if p does not split in k and $L_k \subseteq \widetilde{k}$, then $\lambda(k_\infty/k) = \mu(k_\infty/k) = 0$ for each k_∞ with $L_k \subseteq k_\infty$ independent with the value $\lambda(k_\infty^{\mathrm{c}}/k)$. To explain this, we need the following formula (see Lemma 4.1 of Chapter 13 in [**4**]): Let n be a positive integer and K/F a cyclic extension of degree n. Let $e(K/F)$ be the product of the ramification indeces in K/F for all primes (finite and infinite) of F. Let Cl_K be the ideal class group of K and E_F the unit group of F. Then we have

$$\# Cl_K^{\mathrm{Gal}(K/F)} = \frac{e(K/F)h_F}{[K:F][E_F : E_F \cap (N_{K/F}K^\times)]},$$

here we let $Cl_K^{\mathrm{Gal}(K/F)} = \{a \in Cl_K \mid g(a) = a \text{ for all } g \in \mathrm{Gal}(K/F)\}$. Assume that p does not split in k and that $L_k \subseteq \widetilde{k}$. First, let $p = 3$ and $k = \mathbb{Q}(\sqrt{-3})$.

Since $3 \nmid h_k$, for each $\mathbb{Z}_3$-extension k_∞/k, k_∞/k is totally ramified at the prime lying above 3. Then we have $(X_{k_\infty})_{\mathrm{Gal}(k_\infty/k)} \simeq Cl_k \otimes \mathbb{Z}_3 = 0$, and so $X_{k_\infty} = 0$ by Nakayama's lemma. Hence $\lambda(k_\infty/k) = \mu(k_\infty/k) = 0$. Next, suppose that $p \geq 5$, or, $p = 3$ and $k \neq \mathbb{Q}(\sqrt{-3})$. Let k_∞/k be a $\mathbb{Z}_p$-extension which contains L_k. Choose a positive integer n with $L_k \subseteq k_n$. Since k has only one prime lying above p and k_n/k is unramified outside primes lying above p, one sees that $e(k_n/k) = [k_n : k]/[L_k : k]$. Because E_k is finite and k has no primitive p-th roots of unity in this case, p does not divide $[E_k : E_k \cap (N_{k_n/k}k_n^\times)]$. Hence from the above formula, we have

$$\#(Cl_{k_n} \otimes \mathbb{Z}_p)^{\mathrm{Gal}(k_n/k)} = \frac{([k_n : k]/[L_k : k])[L_k : k]}{[k_n : k]} = 1.$$

This implies that $Cl_{k_n} \otimes \mathbb{Z}_p = 0$ for all sufficiently large n, and hence $X_{k_\infty} = 0$. Therefore, $\lambda(k_\infty/k) = \mu(k_\infty/k) = 0$. Specifically, we have $\mu(k_\infty^a/k) = 0$. Suppose further that $\lambda(k_\infty^c/k) = 1$. By Bloom–Gerth's result [**2**], the number of $\mathbb{Z}_p$-extensions k_∞ with $\mu(k_\infty/k) > 0$ is at most $\lambda(k_\infty^c/k) = 1$ since p does not split in k. Suppose that $\mu(k_\infty/k) > 0$. Then it also holds that $\mu(\overline{J}(k_\infty)/k) > 0$. It follows that $\overline{J}(k_\infty) = k_\infty$, and this implies that $k_\infty/\mathbb{Q}$ is a Galois extension. Hence $k_\infty = k_\infty^c$ or k_∞^a. But we already know $\mu(k_\infty^c/k) = 0$, and we have proved $\mu(k_\infty^a/k) = 0$ here. Thus, $\mu(k_\infty/k) = 0$ for all $\mathbb{Z}_p$-extensions k_∞. Hence, for the vanishing of μ-invariants, there is nothing new when p does not split in k. In particular, if $\lambda(k_\infty^c/k) = 1$, $L_k \subseteq \widetilde{k}$ and $[L_k : k] = p$, then $\mu(k_\infty/k) = 0$ and $\lambda(k_\infty/k) = 0$ for each k_∞ with $k_\infty \cap k_\infty^a \neq k$ since $L_k \cap k_\infty = k_\infty^a \cap k_\infty$ from Lemma 2.2.
(2) Suppose the assumptions of Theorem 2. If further $p \mid h_k$, then the conditions of Theorem 4.1 (S) are satisfied. To check this, it suffices to show only that if $L_k \subseteq \widetilde{k}$, $p \mid h_k$ and $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = p$, then $\mathfrak{D}$ is normal in $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. Let F be the fixed field of $\mathfrak{D}$. Then $k \subseteq F \subsetneq \widetilde{k}$ and $[F : k] = p$. Let k_1^a be the 1-st layer of k_∞^a/k. Since $p \mid h_k$ and $L_k \subseteq \widetilde{k}$, k_1^a/k is unramified by Lemma 2.2. Assume that $F \neq k_1^a$. Then Fk_1^a/k is the composite of 1-st layers of all $\mathbb{Z}_p$-extensions of k and is unramified at a prime lying above p. This contradicts to the fact that k_∞^c/k is totally ramified at all primes lying above p since $(Fk_1^a) \cap k_\infty^c \neq k$. Hence $F = k_1^a$. Since $k_1^a/\mathbb{Q}$ is a Galois extension, $\mathfrak{D}$ is normal in $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. When $p \nmid h_k$, as mentioned in the above of Theorem 2, we already have a stricter result (see corollary of Theorem 1.)

From here we start to prove Theorem 4.1. As discussed in the previous section, since $\lambda(k_\infty^c/k) = 2$ if p splits in k, and $\lambda(k_\infty^c/k) = 1$ otherwise, there are a power series $f(S) = p^m g(S)U(S)$ in $\mathbb{Z}_p[[S]]$ and a surjective morphism

$$\Lambda/(T-f(S)) \to X.$$

Proposition 4.1. $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = \# \mathbb{Z}_p/f(0)\mathbb{Z}_p$.

Proof. By isomorphisms

$$\begin{aligned} \Lambda/(S) \ &\simeq \ \mathbb{Z}_p[[T]] \ \simeq \qquad \mathbb{Z}_p[[\mathrm{Gal}(k_\infty^c/k)]], \\ F(S,T) &\mapsto F(0,T) \mapsto F(0, \tau \,\mathrm{Gal}(\widetilde{k}/k_\infty^c) - 1), \end{aligned}$$

we identify these rings. Recall that $Y_{k_\infty^c} \simeq \mathbb{Z}_p$. Since

$$\begin{aligned} \Lambda/(S, T-f(S)) &= \Lambda/(S, T-f(0)) \\ &\simeq \mathbb{Z}_p[[T]]/(T-f(0)) \\ &\simeq \mathbb{Z}_p \end{aligned}$$

as $\mathbb{Z}_p$-module, one sees that

$$\begin{aligned} \Lambda/(S, T-f(S)) &\simeq \mathbb{Z}_p[[T]]/(T-f(0)) \\ &\simeq Y_{k_\infty^c} \end{aligned}$$

as $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty^c/k)]]$-modules. Applying Lemma 3.1 for k_∞^c, there is the following exact sequence

$$0 \longrightarrow \mathbb{Z}_p[[T]]/(T-f(0)) \longrightarrow X_{k_\infty^c} \longrightarrow \mathrm{Gal}\left(\widetilde{k} \cap L_{k_\infty^c}/k_\infty^c\right) \longrightarrow 0$$

of $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty^c/k)]]$-modules. Suppose the conditions **(S)**. Then $\mathrm{Gal}(\widetilde{k} \cap L_{k_\infty^c}/k_\infty^c) = \mathrm{Gal}(\widetilde{k}/k_\infty^c)$. Since $f(0) \neq 0$ as mentioned the above, it follows that

$$(\mathbb{Z}_p[[T]]/(T-f(0)))^{\mathrm{Gal}(k_\infty^c/k)} = 0,$$

here we let $M^{\mathrm{Gal}(k_\infty^c/k)}$ the invariant submodule of a $\mathrm{Gal}(k_\infty^c/k)$-module M. Also, since $\widetilde{k}/k$ is abelian, it follows that

$$\mathrm{Gal}\left(\widetilde{k}/k_\infty^c\right)^{\mathrm{Gal}(k_\infty^c/k)} = \mathrm{Gal}\left(\widetilde{k}/k_\infty^c\right)$$

and

$$\mathrm{Gal}\left(\widetilde{k}/k_\infty^c\right)_{\mathrm{Gal}(k_\infty^c/k)} \simeq \mathrm{Gal}\left(\widetilde{k}/k_\infty^c\right).$$

Hence we have an exact sequence

$$0 \longrightarrow X_{k_\infty^c}^{\mathrm{Gal}(k_\infty^c/k)} \longrightarrow \mathrm{Gal}(\widetilde{k}/k_\infty^c)$$

$$\longrightarrow \mathbb{Z}_p/f(0)\mathbb{Z}_p \longrightarrow (X_{k_\infty^c})_{\mathrm{Gal}(k_\infty^c/k)} \longrightarrow \mathrm{Gal}(\widetilde{k}/k_\infty^c) \longrightarrow 0$$

of $\mathbb{Z}_p$-modules since $\mathbb{Z}_p[[T]]/(T, T-f(0)) \simeq \mathbb{Z}_p/f(0)\mathbb{Z}_p$. By Lemma 4.1 of Okano [**5**], we know that $X_{k_\infty^c}^{\mathrm{Gal}(k_\infty^c/k)} = D_{k_\infty^c}$, which is the decomposition group in $X_{k_\infty^c} = \mathrm{Gal}(L_{k_\infty^c}/k_\infty^c)$ of a prime lying above p. Let M_k/k be the maximal pro-p abelian extension unramified outside all primes lying above p and L the fixed field of $L_{k_\infty^c}$ by $TX_{k_\infty^c}$. We claim that $\widetilde{k} = M_k = L$. By class field theory, see for example Theorem 13.4 and Corollary 13.6 of [**8**], there is an isomorphism

$$\mathrm{Tor}_{\mathbb{Z}_p} \mathrm{Gal}(M_k/k) \simeq \mathrm{Gal}(L_k/L_k \cap \widetilde{k})$$

of finite abelian groups, where $\mathrm{Tor}_{\mathbb{Z}_p} \mathrm{Gal}(M_k/k)$ is the $\mathbb{Z}_p$-torsion submodule of $\mathrm{Gal}(M_k/k)$. By our assumption that $L_k \subseteq \widetilde{k}$, it follows that

$$\mathrm{Tor}_{\mathbb{Z}_p} \mathrm{Gal}(M_k/k) \simeq \mathrm{Gal}(L_k/L_k \cap \widetilde{k}) = \mathrm{Gal}(L_k/L_k) = 0.$$

This implies that $M_k = \widetilde{k}$. It follows from the fact that M_k/k_∞^c is unramified that $M_k \subseteq L$. Since L/k is abelian and unramified outside all primes lying above p, we have $L \subseteq M_k$. Therefore, $L = M_k = \widetilde{k}$. This shows that $(X_{k_\infty^c})_{\mathrm{Gal}(k_\infty^c/k)} \simeq \mathrm{Gal}(\widetilde{k}/k_\infty^c)$. Hence we obtain the following exact sequence

$$0 \longrightarrow D_{k_\infty^c} \longrightarrow \mathrm{Gal}(\widetilde{k}/k_\infty^c) \longrightarrow \mathbb{Z}_p/f(0)\mathbb{Z}_p \longrightarrow 0$$

of $\mathbb{Z}_p$-modules. Note that

$$\mathrm{Image}\big(D_{k_\infty^c} \to \mathrm{Gal}(\widetilde{k}/k_\infty^c)\big) = \mathfrak{D} \cap \mathrm{Gal}\left(\widetilde{k}/k_\infty^c\right)$$

since $D_{k_\infty^c}$ is not depending on the choice of a prime lying above p. Since k_∞^c/k is totally ramified at all primes lying above p, by combining the above arguments, we have

$$\begin{aligned}[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] &= \#\,\mathrm{Gal}(\widetilde{k}/k)/\mathfrak{D} \\ &= \#\,\mathrm{Gal}(\widetilde{k}/k^c_\infty)\mathfrak{D}/\mathfrak{D} \\ &= \#\,\mathrm{Gal}(\widetilde{k}/k^c_\infty)/\mathfrak{D} \cap \mathrm{Gal}(\widetilde{k}/k^c_\infty) \\ &= \#\mathrm{Coker}\big(D_{k^c_\infty} \to \mathrm{Gal}(\widetilde{k}/k^c_\infty)\big) \\ &= \#\mathbb{Z}_p/f(0)\mathbb{Z}_p.\end{aligned}$$

Suppose the conditions **(NS)**. Recall $X_{k^c_\infty}$ is isomorphic to $\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules. Since $Y_{k^c_\infty} \simeq \mathbb{Z}[[T]]/(T - f(0))$, it follows that

$$(X_{k^c_\infty})_{\mathrm{Gal}(k^c_\infty/k)} \simeq \mathbb{Z}_p/f(0)\mathbb{Z}_p.$$

Since k^c_∞ has the unique prime lying above p, we also have

$$(X_{k^c_\infty})_{\mathrm{Gal}(k^c_\infty/k)} \simeq \mathrm{Gal}(L_k/k).$$

By the condition that $p \geq 5$, we have $M_k = \widetilde{k}$ since the completion at the prime lying above p has no primitive p-th root of unity. It follows that the fixed field of $\widetilde{k}$ by $\mathfrak{D}$ is L_k by class field theory because the order of the ideal class containing the prime above is prime to p. Therefore, we have

$$\begin{aligned}[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] &= \#\,\mathrm{Gal}(L_k/k) \\ &= \#(X_{k^c_\infty})_{\mathrm{Gal}(k^c_\infty/k)} \\ &= \#\mathbb{Z}_p/f(0)\mathbb{Z}_p.\end{aligned}$$

This completes the proof. □

Let $p^{n_0} = [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}]$ and put $\nu_{n_0}(S) = ((1+S)^{p^{n_0}} - 1)/S$.

Proposition 4.2. *$f(S) = \nu_{n_0}(S)U(S)$.*

Proof. For each non-negative integer n, denote by k^a_n the n-th layer of k^a_∞. Since $\mathfrak{D}$ is normal in $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$, the fixed field of $\mathfrak{D}$ is a Galois extension over $\mathbb{Q}$, and is unramified over k. This shows that the fixed field is $k^a_{n_0}$ by Lemma 2.2. Let $\widetilde{k^a_{n_0}}$ be the composite of all $\mathbb{Z}_p$-extensions of $k^a_{n_0}$. Then it is known that $\mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_{n_0}) \simeq \mathbb{Z}_p^{p^{n_0}+1}$, see [**3**] and Section 5–5 of [**8**]. We show $\nu_{n_0}(S) \mid f(S)$.

Suppose the condition **(S)**. Let $\mathfrak{I}_{n_0} \subseteq \mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_{n_0})$ be the inertia subgroup of a prime of $k^a_{n_0}$ lying above p. Since the prime number p splits completely in $k^a_{n_0}/\mathbb{Q}$, we have $\mathfrak{I}_{n_0} \simeq \mathbb{Z}_p$. Also, since $k^a_\infty/\mathbb{Q}$ is a Galois extension, all primes of k^a_∞ are ramified in k^a_∞/k. This shows that $\mathfrak{I}_{n_0} \cap \mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_\infty) = 1$, and hence $\widetilde{k^a_{n_0}}/k^a_\infty$ is unramified at all primes of k^a_∞ because $\widetilde{k^a_{n_0}}/k^a_{n_0}$ is unramified outside the all primes lying above p. Consider the natural surjective morphism

$$X_{k^a_\infty} \to \mathrm{Gal}\left(\widetilde{k^a_{n_0}}/k^a_\infty\right) \simeq \mathbb{Z}_p^{p^{n_0}}.$$

Since $\widetilde{k^a_{n_0}}$ contains $\widetilde{k} = M_k$, we have

$$\mathrm{Gal}\left(\widetilde{k^a_{n_0}}/k^a_\infty\right)_{\mathrm{Gal}(k^a_\infty/k)} \simeq \mathrm{Gal}\left(\widetilde{k}/k^a_\infty\right).$$

By isomorphisms

$$\begin{aligned} \Lambda/(T) \ &\simeq \mathbb{Z}_p[[S]] \simeq \quad \mathbb{Z}_p\big[\big[\,\mathrm{Gal}(k^a_\infty/k)\big]\big], \\ F(S,T) &\mapsto F(S,0) \mapsto F\big(\sigma\,\mathrm{Gal}(\widetilde{k}/k^a_\infty) - 1, 0\big), \end{aligned}$$

we identify these rings. Since $\widetilde{k^a_{n_0}}/k^a_{n_0}$ is abelian, $\sigma^{p^{n_0}}\,\mathrm{Gal}(\widetilde{k}/k^a_\infty) = (1+S)^{p^{n_0}}$ acts on $\mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_\infty)$ trivially. Since also $\mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_\infty) \simeq \mathbb{Z}_p^{p^{n_0}}$ as $\mathbb{Z}_p$-modules, we have

$$\mathrm{Gal}\left(\widetilde{k^a_{n_0}}/k^a_\infty\right) \simeq \mathbb{Z}_p[[S]]/((1+S)^{p^{n_0}} - 1).$$

Recall the characteristic ideal $\mathrm{char}_{\mathbb{Z}_p[[S]]}(M)$ of a finitely generated torsion $\mathbb{Z}_p[[S]]$-module M. The above isomorphism and the surjective morphism $X_{k^a_\infty} \to \mathrm{Gal}(\widetilde{k^a_{n_0}}/k^a_\infty)$ implies that

$$\mathrm{char}_{\mathbb{Z}_p[[S]]}(X_{k^a_\infty}) \subseteq ((1+S)^{p^{n_0}} - 1).$$

Also, from the exact sequence

$$0 \longrightarrow Y_{k^a_\infty} \longrightarrow X_{k^a_\infty} \longrightarrow \mathrm{Gal}(\widetilde{k}/k^a_\infty) \longrightarrow 0,$$

we have

$$\begin{aligned}\mathrm{char}_{\mathbb{Z}_p[[S]]}(X_{k^a_\infty}) &= \mathrm{char}_{\mathbb{Z}_p[[S]]}\big(\mathrm{Gal}(\widetilde{k}/k^a_\infty)\big)\mathrm{char}_{\mathbb{Z}_p[[S]]}(Y_{k^a_\infty}) \\ &= S\mathrm{char}_{\mathbb{Z}_p[[S]]}(Y_{k^a_\infty}) \\ &\subseteq ((1+S)^{p^{n_0}}-1) \\ &= S(\nu_{n_0}(S)).\end{aligned}$$

Since S and $\nu_{n_0}(S)$ are relatively prime, we have $\mathrm{char}_{\mathbb{Z}_p[[S]]}(Y_{k^a_\infty}) \subseteq (\nu_{n_0}(S))$. Finally, from the surjective morphism

$$\mathbb{Z}_p[[S]]/(f(S)) \longrightarrow Y_{k^a_\infty},$$

we have $(f(S)) \subseteq (\nu_{n_0}(S))$ and hence $\nu_{n_0}(S)$ divides $f(S)$.

Suppose the condition **(NS)**. Let $\mathfrak{I}_0$ and $\mathfrak{I}_{n_0}$ be the inertia subgroups in $\widetilde{k}/k$ and $\widetilde{k^a_{n_0}}/k^a_{n_0}$ of a prime of k and $k^a_{n_0}$ lying above p, respectively. Since $k^a_{n_0}/k$ is unramified, we have $\mathfrak{I}_0 \subseteq \mathrm{Gal}(\widetilde{k}/k^a_{n_0})$ and $\mathfrak{I}_0$ is the inertia subgroup in $\widetilde{k}/k^a_{n_0}$. Also, since there is only one prime of k lying above p, $\mathfrak{I}_0$ is isomorphic to $\mathbb{Z}_p^2$. Note that $\mathfrak{I}_{n_0}$ maps to $\mathfrak{I}_0$ surjectively. Let $\mathfrak{p}_{n_0}$ be a prime of $k^a_{n_0}$ lying above p such that $\mathfrak{I}_{n_0}$ is the inertia subgroup of $\mathfrak{p}_{n_0}$ in $\widetilde{k^a_{n_0}}/k^a_{n_0}$. Let U_{n_0} be the local principal unit group at $\mathfrak{p}_{n_0}$. Since p does not split in k and the all primes lying above p decomposed completely in k_{n_0}/k, we find that $U_{\mathfrak{p}_{n_0}} \simeq \mathbb{Z}_p^2$. By class field theory, there is a surjective map $U_{\mathfrak{p}_{n_0}} \to \mathfrak{I}_{n_0}$. Hence we find that $\mathfrak{I}_{n_0} \simeq \mathbb{Z}_p^2$ and therefore $\mathfrak{I}_{n_0} \simeq \mathfrak{I}_0$. This shows that $\mathfrak{I}_{n_0}$ maps to $\mathrm{Gal}(\widetilde{k}/k)$ injectively, and hence $\mathfrak{I}_{n_0} \cap \mathrm{Gal}(\widetilde{k^a_{n_0}}/\widetilde{k}) = 1$. Thus $\widetilde{k^a_{n_0}}/\widetilde{k}$ is an abelian unramified extension. Let L/k^a_∞ be the maximal abelian subextension of $L_{\widetilde{k}}/k^a_\infty$, we then have $\mathrm{Gal}(L/\widetilde{k}) = Y_{k^a_\infty}$. Since $\widetilde{k^a_{n_0}}/k^a_\infty$ is abelian and $\widetilde{k^a_{n_0}} \subseteq L_{\widetilde{k}}$, we have $\widetilde{k^a_{n_0}} \subseteq L$. From a surjective morphism

$$\mathrm{Gal}(L/\widetilde{k}) = Y_{k^a_\infty} \longrightarrow \mathrm{Gal}\big(\widetilde{k^a_{n_0}}/\widetilde{k}\big),$$

it follows that

$$\mathrm{char}_{\mathbb{Z}_p[[S]]}(Y_{k^a_\infty}) \subseteq \mathrm{char}_{\mathbb{Z}_p[[S]]}\big(\mathrm{Gal}(\widetilde{k^a_{n_0}}/\widetilde{k})\big).$$

By doing the same argument to the case **(S)**, we have

$$\mathrm{Gal}\big(\widetilde{k^a_{n_0}}/k^a_\infty\big) \simeq \mathbb{Z}_p[[S]]/((1+S)^{p^{n_0}}-1)$$

since $p \geq 5$ and $M_k = \widetilde{k}$. Thus $\mathrm{char}_{\mathbb{Z}_p[[S]]}(\mathrm{Gal}(\widetilde{k^a_{n_0}}/\widetilde{k})) = (\nu_{n_0}(S))$, and hence

$\mathrm{char}_{\mathbb{Z}_p[[S]]}(Y_{k_\infty^a}) \subseteq (\nu_{n_0}(S))$. Therefore we also have $\nu_{n_0}(S) \mid f(S)$.

Rewrite $f(S) = p^m \nu_{n_0}(S) g(S) U(S)$ with a distinguished polynomial $g(S)$. Note that $\nu_{n_0}(0) = p^{n_0}$. Then we have

$$\begin{aligned} p^{n_0} &= \left[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}\right] \\ &= \#\mathbb{Z}_p/f(0)\mathbb{Z}_p \\ &= \#\mathbb{Z}_p/p^m \cdot p^{n_0} \cdot g(0)\mathbb{Z}_p. \end{aligned}$$

Hence $m = 0$ and $p \nmid g(0)$, and therefore $f(S) = \nu_{n_0}(S)U(S)$. □

We finish the proof of Theorem 4.1. Suppose the condition **(S)**. If $k_\infty \neq N_\infty, N'_\infty$ then

$$\lambda(k_\infty/k) = \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty}) + 1$$

by Lemma 3.1. Suppose the condition **(NS)**. Let k_∞ be a $\mathbb{Z}_p$-extension and L/k_∞ the maximal abelian subextension of $L_{\widetilde{k}}/k_\infty$. Then $L_{k_\infty}\widetilde{k}$ is contained in L. Hence there is an exact sequence

$$Y_{k_\infty} \longrightarrow X_{k_\infty} \longrightarrow \mathrm{Gal}\left(\widetilde{k} \cap L_{k_\infty}/k_\infty\right) \longrightarrow 0$$

of $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-modules. Since $\mathfrak{D}$ is equal to the inertia subgroup in $\mathrm{Gal}(\widetilde{k}/k)$ and $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}] = [L_k : k] < \infty$, we find that

$$\left[\widetilde{k} \cap L_{k_\infty} : k_\infty\right] = \left[\mathrm{Gal}(\widetilde{k}/k_\infty) : \mathfrak{D} \cap \mathrm{Gal}(\widetilde{k}/k_\infty)\right] < \infty.$$

Therefore we have

$$\lambda(k_\infty/k) \le \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty})$$

for all $\mathbb{Z}_p$-extensions k_∞.

Let $k_\infty = k_\infty^a$ and suppose the condition **(S)**. Note that $k_\infty^a = \widetilde{k}^{\langle\tau\rangle}$. Then, by Proposition 4.2,

$$I_{0,1} = \left(T, T - S^{p^{n_0}-1}U(S), p\right) = \left(S^{p^{n_0}-1}, T, p\right)$$

and $\Lambda/I_{0,1} \simeq (\mathbb{Z}/p)^{p^{n_0}-1}$. This implies $\mu(k_\infty^a/k) = 0$ and

$$\lambda(k_\infty^a/k) = \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty^a}) + 1 \le p^{n_0}$$

from Lemma 2.1. Suppose the condition (NS). Then $\lambda(k_\infty^a/k) = \mu(k_\infty^a/k) = 0$ as mentioned in the above. In particular, $\mu(k_\infty/k) = 0$ for all k_∞ by Bloom–Gerth [**2**] (In fact, we also can show $\mu = 0$ by our argument.)

Assume that $k_\infty \cap k_\infty^a = k$. Then

$$\lambda(k_\infty/k) \le \begin{cases} 2 & \textbf{(S)}, \\ 1 & \textbf{(NS)} \end{cases}$$

by Theorem 1. Thus $\lambda(k_\infty/k) \le 2 \le p^{n_0}$.

Suppose the condition **(S)** and let $k_\infty = N_\infty$. Since $L_k \subseteq N_\infty$, we have $\lambda(N_\infty/k) = \mu(N_\infty/k) = 0$ by the formula stated in the remark (1) of Theorem 4.1.

Let k_∞ be a $\mathbb{Z}_p$-extension such that $k_\infty \cap k_\infty^a \ne k$, $k_\infty \ne k_\infty^a$, and that $k_\infty \ne N_\infty, N'_\infty$ if p splits in k. Choose $(\alpha, \beta) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ so that $k_\infty = \widetilde{k}^{\,\overline{\langle \sigma^\alpha \tau^\beta \rangle}}$. Then $p \mid \alpha$, so put $\alpha = p^s \alpha'$ for $s \in \mathbb{Z}_{\ge 1}$ and $\alpha' \in \mathbb{Z}_p^\times$. We calculate $I_{\alpha,\beta}$.

$$\begin{aligned} I_{\alpha,\beta} &= \big((1+S)^\alpha (1+T)^\beta - 1, T - S^{p^{n_0}-1}U(S), p\big) \\ &= \big((1+S^{p^s})^{\alpha'} (1+S^{p^{n_0}-1}U(S))^\beta - 1, T - S^{p^{n_0}-1}U(S), p\big) \\ &= \left(\left(\sum_{k=0}^{\infty} \binom{\alpha'}{k} S^{kp^s}\right)\left(\sum_{l=0}^{\infty} \binom{\beta}{l} S^{l(p^{n_0}-1)} U(S)^l\right) - 1, T - S^{p^{n_0}-1}U(S), p\right) \\ &= \left(\sum_{n=1}^{\infty}\sum_{k=0}^{n} \binom{\alpha'}{k}\binom{\beta}{n-k} S^{kp^s+(n-k)(p^{n_0}-1)} U(S)^{n-k}, T - S^{p^{n_0}-1}U(S), p\right). \end{aligned}$$

First suppose that $p^{n_0} - 1 < p^s$. Note that

$$kp^s + (n-k)(p^{n_0}-1) = n(p^{n_0}-1) + k(p^s - (p^{n_0}-1)) \ge n(p^{n_0}-1).$$

Thus $\sum_{n=1}^{\infty}\sum_{k=0}^{n} \binom{\alpha'}{k}\binom{\beta}{n-k} S^{kp^s+(n-k)(p^{n_0}-1)} U(S)^{n-k}$ is divided by $S^{p^{n_0}-1}$. Put

$$h_0(S) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} \binom{\alpha'}{k}\binom{\beta}{n-k} S^{kp^s+(n-k)(p^{n_0}-1)-(p^{n_0}-1)} U(S)^{n-k}.$$

Since $kp^s + (n-k)(p^{n_0}-1) - (p^{n_0}-1) \ge (n-1)(p^{n_0}-1)$, we have

$$\begin{aligned} h_0(0) &= \sum_{k=0}^{1} \binom{\alpha'}{k} \binom{\beta}{1-k} \left[S^{kp^s+(1-k)(p^{n_0}-1)-(p^{n_0}-1)}\right]_{S=0} U(0)^{1-k} \\ &= \binom{\alpha'}{0}\binom{\beta}{1} U(0) = \beta U(0) \in \mathbb{Z}_p^{\times}. \end{aligned}$$

This shows that

$$\begin{aligned} I_{\alpha,\beta} &= \left(S^{p^{n_0}-1} h_0(S), T - S^{p^{n_0}-1} U(S), p\right) \\ &= \left(S^{p^{n_0}-1}, T, p\right), \end{aligned}$$

and hence

$$\Lambda / I_{\alpha,\beta} \simeq (\mathbb{Z}/p)^{p^{n_0}-1}.$$

Therefore $\lambda(k_\infty/k) \le p^{n_0}$.

Next suppose that $p^{n_0} - 1 > p^s$. Since

$$\begin{aligned} kp^s + (n-k)(p^{n_0}-1) &= n(p^{n_0}-1) + k(p^s - (p^{n_0}-1)) \\ &\ge n(p^{n_0}-1) + n(p^s - (p^{n_0}-1)) \\ &= np^s, \end{aligned}$$

$\sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{\alpha'}{k}\binom{\beta}{n-k} S^{kp^s+(n-k)(p^{n_0}-1)} U(S)^{n-k}$ is divided by S^{p^s}. Put

$$h_1(S) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{\alpha'}{k}\binom{\beta}{n-k} S^{kp^s+(n-k)(p^{n_0}-1)-p^s} U(S)^{n-k}.$$

Since $kp^s + (n-k)(p^{n_0}-1) - p^s \ge (n-1)p^s$, we have

$$\begin{aligned} h_1(0) &= \sum_{k=0}^{1} \binom{\alpha'}{k} \binom{\beta}{1-k} \left[S^{kp^s+(1-k)(p^{n_0}-1)-p^s}\right]_{S=0} U(0)^{1-k} \\ &= \binom{\alpha'}{1}\binom{\beta}{0} = \alpha' \in \mathbb{Z}_p^{\times}. \end{aligned}$$

This shows that

$$
\begin{aligned}
I_{\alpha,\beta} &= \left(S^{p^s} h_1(S), T - S^{p^{n_0}-1}U(S), p\right) \\
&= (S^{p^s}, T, p),
\end{aligned}
$$

and hence

$$
\Lambda / I_{\alpha,\beta} \simeq (\mathbb{Z}/p)^{p^s}.
$$

Therefore $\lambda(k_\infty/k) \leq p^s + 1 \leq p^{n_0}$. This completes the proof of Theorem 4.1. $\square$

As an application to Proposition 4.2, we can obtain the following results.

Theorem 4.2. *Under the condition* **(S)**, $X_{k_\infty^a} \simeq \mathbb{Z}_p[[S]]/((1+S)^{p^{n_0}} - 1)$ *as* $\mathbb{Z}_p[[S]]$*-modules.*

Proof. Recall a surjective morphism $X_{k_\infty^a} \to \mathrm{Gal}(\widetilde{k_{n_0}^a}/k_{n_0}^a) \simeq \mathbb{Z}_p^{p^{n_0}}$. It follows that $p^{n_0} \leq \mathrm{rank}_{\mathbb{Z}_p}(X_{k_\infty^a}) = \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty^a}) + 1$ and hence we have $p^{n_0} - 1 \leq \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty^a})$. Recall also a surjective morphism $\mathbb{Z}_p[[S]]/(\nu_{n_0}(S)) \to Y_{k_\infty^a}$. Since

$$
\begin{aligned}
p^{n_0} - 1 &= \mathrm{rank}_{\mathbb{Z}_p}(\mathbb{Z}_p[[S]]/(\nu_{n_0}(S))) \\
&\geq \mathrm{rank}_{\mathbb{Z}_p}(Y_{k_\infty^a}) \\
&\geq p^{n_0} - 1,
\end{aligned}
$$

we have $\mathbb{Z}_p[[S]]/(\nu_{n_0}(S)) \simeq Y_{k_\infty^a} \simeq \mathbb{Z}_p^{p^{n_0}-1}$, and hence $X_{k_\infty^a} \simeq \mathbb{Z}_p^{p^{n_0}}$. Therefore we have $L_{k_\infty^a} = \widetilde{k_{n_0}^a}$ and

$$
X_{k_\infty^a} \simeq \mathrm{Gal}\left(\widetilde{k_{n_0}^a}/k_\infty^a\right) \simeq \mathbb{Z}_p[[S]]/((1+S)^{p^{n_0}} - 1).
$$

This completes the proof. $\square$

This isomorphism says that k_∞^a has only trivially known unramified abelian pro-p extensions.

Corollary 4.1. $(T - \nu_{n_0}(S)U(S))X = 0.$ $\square$

Remark. As mentioned in the below of Lemma 3.2, the uniqueness of $f(S)$ is unknown. Under the assumption of Theorem 4.1, we conclude that the uniqueness of $U(S)$ is unknown, namely, if a power series $F(S) \in \mathbb{Z}_p[[S]]$ satisfies $Tx = F(S)x$, then $F(S) = \nu_{n_0}(S)(U(S) + G(S))$ with $G(S) \in (S, p)$.

5. Some Discussions.

We give a proof of Theorem A here as mentioned in Section 1. Suppose that p does not split in k and $\lambda(k_\infty^c/k) = 0$. Since k_∞^c has the unique prime lying above p and k_∞^c/k is totally ramified at the prime lying above p, $(X_{k_\infty^c})_{\mathrm{Gal}(k_\infty^c/k)} \simeq \mathrm{Gal}(L_k/k)$. It is known that $X_{k_\infty^c}$ is a finitely generated free $\mathbb{Z}_p$-module of rank $\lambda(k_\infty^c/k)$. Hence $\mathrm{Gal}(L_k/k) = 0$ since $\lambda(k_\infty^c/k) = 0$. Thus $(X_{k_\infty})_{\mathrm{Gal}(k_\infty/k)} \simeq \mathrm{Gal}(L_k/k) = 0$, and therefore $X_{k_\infty} = 0$ for each k_∞. This shows $\lambda(k_\infty/k) = \mu(k_\infty/k) = \nu(k_\infty/k) = 0$ for each k_∞.

Suppose that p splits in k and $\lambda(k_\infty^c/k) = 1$. Then by Lemma 3.1, $Y_{k_\infty^c} = 0$ and hence $X = 0$. This shows that $X_{k_\infty} = \mathrm{Gal}(\widetilde{k}/k_\infty) \simeq \mathbb{Z}_p$ for each k_∞ with $k_\infty \neq N_\infty, N'_\infty$, and therefore $\lambda(k_\infty/k) = 1$ and $\mu(k_\infty/k) = 0$. Next we show that $\lambda(N_\infty/k) = \lambda(N'_\infty/k) = \mu(N_\infty/k) = \mu(N'_\infty/k) = 0$. Since $X = 0$, one sees that $\mathrm{Gal}(L_{N_\infty}\widetilde{k}/\widetilde{k}) = 0$. Since also $\widetilde{k}/N_\infty$ is ramified at primes lying above $\mathfrak{p}'$, $\mathrm{Gal}(\widetilde{k} \cap L_{N_\infty}/N_\infty)$ is finite. From the exact sequence

$$0 \to \mathrm{Gal}\left(L_{N_\infty}\widetilde{k}/\widetilde{k}\right) \to X_{N_\infty} \to \mathrm{Gal}\left(L_{N_\infty} \cap \widetilde{k}/N_\infty\right) \to 0,$$

we conclude that X_{N_∞} is finite. Therefore, $\lambda(N_\infty/k) = \mu(N_\infty/k) = 0$. By the same argument, we also have $\lambda(N'_\infty/k) = \mu(N'_\infty/k) = 0$. This completes the proof of Theorem A.

On the proof of Theorem 1, when p does not split in k, we do not use individualities of imaginary quadratic fields, it was needed that k has the complex conjugation J as an automorphism (i.e. k is a CM-field), $X_{k_\infty^c} \simeq \mathbb{Z}_p$ and that k_∞^c has only one prime lying above p. Hence we can obtain a more general result.

PROPOSITION 5.1. *Let p be an odd prime number, k a CM-field and k^+ the maximal totally real subfield of k. Suppose that k_∞^c has the unique prime lying above p, $X_{k_\infty^c} \simeq \mathbb{Z}_p$ and that k_∞^c/k is totally ramified at the prime above p. Let k_∞^a/k be an anti-cyclotomic $\mathbb{Z}_p$-extension of k, namely, k_∞^a/k^+ is a Galois extension such that* $\mathrm{Gal}(k_\infty^a/k^+)$ *is non-abelian. Put $K = k_\infty^c k_\infty^a$. Then $\lambda(k_\infty/k) \leq 1$ for each $\mathbb{Z}_p$-extension k_∞ such that $k_\infty \subseteq K$ and that $k_\infty \cap k_\infty^a = k$.*

For example, let $p = 37, 59$ or 67. Then the p-th cyclotomic field $k = \mathbb{Q}(\mu_p)$ satisfies the assumption of Proposition 5.1.

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