Equivalence relations
for two variable real analytic function germs

By Satoshi Koike and Adam Parusiński

Abstract. For two variable real analytic function germs we compare the blow-analytic equivalence in the sense of Kuo to other natural equivalence relations. Our main theorem states that C^1 equivalent germs are blow-analytically equivalent. This gives a negative answer to a conjecture of Kuo. In the proof we show that the Puiseux pairs of real Newton-Puiseux roots are preserved by the C^1 equivalence of function germs. The proof is achieved, being based on a combinatorial characterisation of blow-analytic equivalence, in terms of the real tree model.

We also give several examples of bi-Lipschitz equivalent germs that are not blow-analytically equivalent.

Introduction.

In this paper we compare different equivalence relations of real analytic function germs. We say that two such germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^r (*right*) *equivalent*, $r = 1, 2, \ldots, \infty$, if there is a local C^r diffeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \sigma$. We say that f and g are *topologically* or C^0 *equivalent* if σ is a homeomorphism, and *analytically* or C^ω *equivalent* if σ is an analytic isomorphism. We say that f and g are *bi-Lipschitz equivalent* if σ is a local bi-Lipschitz homeomorphism. By definition, we have the following implications:

$$C^0 \text{ eq.} \Leftarrow \text{bi-Lipschitz eq.} \Leftarrow C^1 \text{ eq.} \Leftarrow C^2 \text{ eq.}$$
$$\Leftarrow \cdots \Leftarrow C^\infty \text{ eq.} \Leftarrow C^\omega \text{ eq.} \tag{0.1}$$

By Artin's Approximation Theorem [**2**], C^∞ equivalence implies C^ω equivalence. But the other converse implications of (0.1) do not hold. Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$

be polynomial functions defined by

$$f(x,y) = (x^2 + y^2)^2, \quad g(x,y) = (x^2 + y^2)^2 + x^{r+4}$$

for $r = 1, 2, \ldots$. N. Kuiper [16] and F. Takens [26] showed that f and g are C^r equivalent, but not C^{r+1} equivalent.

In the family of germs

$$K_t(x,y) = x^4 + tx^2y^2 + y^4,$$

the phenomenon of continuous C^1 moduli appears: for $t_1, t_2 \in I$, K_{t_1} and K_{t_2} are C^1 equivalent if and only if $t_1 = t_2$, where $I = (-\infty, -6]$, $[-6, -2]$ or $[-2, \infty)$, see Example 1.3 below. On the other hand, T.-C. Kuo proved that this family is C^0 trivial over any interval not containing -2, by a C^0 trivialisation obtained by the integration of a vector field, cf. [17]. A similar phenomenon is present in Whitney's example $L_t(x,y) = xy(x-y)(x-ty)$, $t \in (0,1)$. In the homogeneous case, as that of K_t or L_t, the Kuo vector field is Lipschitz and hence the trivialisation is bi-Lipschitz. Thus Kuo's construction gives examples of bi-Lipschitz equivalent germs that are not C^1 equivalent.

As shown in [10], [11], the phenomenon of continuous moduli is present also for the bi-Lipschitz equivalence. For instance the family

$$A_t(x,y) = x^3 - 3txy^4 + 2y^6, \quad t > 0,$$

is topologically trivial and if A_{t_1} is bi-Lipschitz equivalent to A_{t_2}, $t_1, t_2 > 0$, then $t_1 = t_2$. Thus C^0 equivalent germs are not necessarily bi-Lipschitz equivalent.

0.1. Blow-analytic equivalence.

Blow-analytic equivalence of real analytic function germs was proposed by Tzee-Char Kuo [19] as a counterpart of the topological equivalence of complex analytic germs. Kuo showed in [21] the local finiteness (i.e. the absence of continuous moduli) of blow-analytic types for analytic families of isolated singularities.

We say that a homeomorphism germ $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a *blow-analytic homeomorphism* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\tilde{\mu} : (\tilde{M}, \tilde{\mu}^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (\tilde{M}, \tilde{\mu}^{-1}(0))$ so that $\sigma \circ \mu = \tilde{\mu} \circ \Phi$. Two real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exists a blow-analytic homeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \sigma$. The formal definition of real modification of [19] is somewhat technical. As we have shown in [15], if $n = 2$ then μ is a real modification if and only if it is a finite

composition of point blowings-up.

As an example consider the family K_t, $t \neq -2$. This family becomes real analytically trivial after the blowing-up of the $t-$axis, cf. Kuo [**19**]. Thus for $t < -2$, or $t > -2$ respectively, all K_t are blow-analytically equivalent. Similarly, the family A_t becomes real analytically trivial after a toric blowing-up in $x, y-$variables, cf. Fukui and Yoshinaga [**5**] or Fukui and Paunescu [**8**], and hence it is blow-analytically trivial. Thus blow-analytic equivalence does not imply neither C^r equivalence, $r \geq 1$, nor bi-Lipschitz equivalence.

Blow-analytic equivalence is stronger than C^0 equivalence. For instance, $f(x,y) = x^2 - y^3$, $g(x,y) = x^2 - y^5$ are C^0 equivalent, but not blow-analytically equivalent, though the proof of it is not immediate. This can be seen using the Fukui invariant [**6**], that we recall in Section 6 below, or it follows directly from the following theorem.

THEOREM 0.1 (see [**15**]).   *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs. Then the following conditions are equivalent*:

(1) *f and g are blow-analytically equivalent.*
(2) *f and g have isomorphic minimal resolutions.*
(3) *The real tree models of f and g are isomorphic.*

For more on the blow-analytic equivalence in the general case n-dimensional we refer the reader to recent surveys [**7**], [**9**].

0.2.   Main result of this paper.
The main result of this paper is the following.

THEOREM 0.2.   *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs and suppose that there exists a C^1 diffeomorphism germ $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ \sigma$. Then f and g are blow-analytically equivalent.*

If, moreover, σ preserves orientation, then f and g are blow-analytically equivalent by an orientation preserving blow-analytic homeomorphism.

To see how surprising this result is let us state a very special corollary of it.

COROLLARY 0.3 (cf. Proposition 4.4 below).   *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism, and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma \subset f^{-1}(0)$ and $\tilde{\gamma} \subset g^{-1}(0)$ be Newton-Puiseux roots of f and g respectively such that $\sigma(\gamma) = \tilde{\gamma}$ as set germs. Then the Puiseux characteristic pairs of γ and $\tilde{\gamma}$ coincide.*

In [**19**] Kuo expressed his belief that there is no direct relation between the

blow-analytic equivalence and C^r equivalences, $1 \leq r < \infty$, and he confirmed it at the invited address of the annual convention of the Mathematical Society of Japan, autumn 1984 ([**20**]), by asserting that blow-analytic equivalence is independent of C^r equivalences. As Theorem 0.2 shows this is not the case in the two variable case.

It is believed that there is a relation between blow-analytic and bi-Lipschitz properties, though the nature of this relation is still not clear. It is not difficult to construct examples showing that

$$\text{blow-analytic eq.} \not\Rightarrow \text{bi-Lipschitz eq},$$

as the example A_t above. In this paper we construct several examples showing that

$$\text{blow-analytic eq.} \not\Leftarrow \text{bi-Lipschitz eq}.$$

Thus, there is no direct relation between these two notions. Nevertheless, as shown in [**15**], a blow-analytic homeomorphism that gives blow-analytic equivalence between two 2-variable real analytic function germs, preserves the order of contact between *non-parameterised* real analytic arcs. Note that by the curve selection lemma, a subanalytic homeomorphism is bi-Lipschitz if and only if it preserves the order of contact between *parameterised* real analytic arcs.

For more than two variables we have another phenomenon. Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be the Briançon-Speder family defined by $f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$. Although f_0 and f_{-1} are blow-analytically equivalent, any blow-analytic homeomorphism that gives the blow-analytic equivalence between them does not preserve the order of contact between some analytic arcs contained in $f_0^{-1}(0)$, cf. [**13**].

0.3.   Comparison to the complex analytic case.

In the complex analytic case the two most interesting equivalence relations are the analytic one and the topological one. The latter is classified by numerical invariants. More precisely, for an isolated singularity $f \colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ the embedded topological type of a plane curve singularity $(f^{-1}(0), 0) \subset (\mathbb{C}^2, 0)$ is determined by the Puiseux pairs of each irreducible component and the intersection numbers of any pairs of distinct components. It can be shown, cf. [**24**], that the topological type of function germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is also completely characterised, also in the non-reduced case $f = \prod f_i^{d_i}$, by the embedded topological type of its zero set and the multiplicities d_i of its irreducible components. Thus in the complex case there is no real difference between the topological classification of embedded zero sets and the function germs.

As shown by H. King [**12**] a similar phenomenon holds in the real case. The homeomorphism class of the germ at 0 of $(\mathbf{R}^2, f^{-1}(0))$ determines the homeomorphism class of the germ of f at 0, for $f \colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ analytic with an isolated singularity in 0. But the topological equivalence is too weak in the real analytic set-up. It does not distinguish the following germs $f(x, y) = x,\ x^2 - y^3,\ x^2 - y^5$, etc.

Thus for us a real analytic counterpart of C^0 equivalence of complex analytic germs is blow-analytic equivalence, of functions and not of their zero sets. The equivalence classes of this equivalence are classified by corresponding combinatorial objects, Puiseux exponents, intersection numbers. The signs of some coefficients have to be added in the real case, see [**15**]. The results of this paper say that these invariants are preserved by C^1 equivalence but not by bi-Lipschitz equivalence.

0.4.   Organisation of this paper.

In Section 2 we construct new invariants of bi-Lipschitz and C^1 equivalences. These invariants can be nicely described in terms of the Newton polygon relative to a curve, the notion introduced in [**22**]. Roughly speaking, if $f = g \circ \sigma$ with σ bi-Lipschitz then the Newton boundaries of f relative to an arc γ coincides with the Newton boundary of g relative to $\sigma(\gamma)$. If σ is C^1 and $D\sigma(0) = \mathrm{Id}$ then, moreover, the corresponding coefficients on the Newton boundaries are identical. In Subsection 2.3 we extend the construction of Section 2 to all C^1 diffeomorphisms (we drop the assumption $D\sigma(0) = \mathrm{Id}$).

In Section 3 we show the C^1 invariance of Puiseux pairs of the Newton-Puiseux roots.

Theorem 0.2 is shown in Section 4. The proof is based on Theorem 0.1 so we recall in this section the construction of real tree model.

In Section 5 we study bi-Lipschitz and C^1 equivalences of weighted homogeneous function germs. In particular we show that for such germs C^1 and analytic equivalences coincide.

Section 6 contains examples of bi-Lipschitz equivalent and blow-analytically non-equivalent germs. The construction of such examples is not simple since such a bi-Lipschitz equivalence cannot be natural. Let us first recall the construction of invariants of bi-Lipschitz equivalence of [**10**], [**11**], [**3**]. Suppose that the generic polar curve of $f(x, y)$ has at least two branches γ_i. Fix reasonable parametrisations of these branches, either as $x = \lambda_i(y)$ or by the distance to the origin, and expand f along each such branch. Suppose that the expansions along different branches $f(\lambda_i(y), y) = a_i y^s + \cdots$ have the same leading exponent s, and that the term y^s is sufficiently big in comparison to the distance between the branches. Then the ratio of the leading coefficients a_i/a_j is a bi-Lipschitz invariant (and a continuous modulus). Our construction of bi-Lipschitz homeomorphism goes along the same lines but in the opposite direction. First we choose carefully $f(x, y),\ g(x, y)$ so

that the expansions of f, resp. g, along polar branches are compatible, so that we write down explicitly bi-Lipschitz equivalences between horn neighbourhoods of polar curves of f and g respectively. Then we show that in simple examples these equivalences can be glued together using partition of unity.

1. Preliminary observations.

We begin with elementary observations in the general n-variable case. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic function germ of the form

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots, \quad f_m \not\equiv 0,$$

where f_j denotes the j-th homogeneous form of f and $m = \mathrm{mult}_0 \, f$ is the multiplicity of f at 0. Since m can be characterised by the following property

$$\exists C, c > 0 \quad \exists r > 0 \quad \forall 0 < \rho < r \quad c \leq \sup_{0 < |x| \leq \rho} \frac{|f(x)|}{|x|^m} \leq C,$$

it is a bi-Lipschitz invariant.

PROPOSITION 1.1 ([**25**]).   *Suppose that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are bi-Lipschitz equivalent. Then $\mathrm{mult}_0 \, f = \mathrm{mult}_0 \, g$.*

For C^1 equivalence the initial homogeneous form, up to linear equivalence, is an invariant. More precisely we have the following result. We leave the proof to the reader.

PROPOSITION 1.2.   *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic function germs of the form*

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots, \quad f_m \not\equiv 0,$$
$$g(x) = g_k(x) + g_{k+1}(x) + \cdots, \quad g_k \not\equiv 0.$$

Suppose that f and g are C^1 equivalent. Then $k = m$ and f_m and g_m are linearly equivalent. In particular, if homogeneous polynomial functions are C^1 equivalent, then they are linearly equivalent.

EXAMPLE 1.3.   Let $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be a polynomial function defined by

$$f_t(x, y) = x^4 + tx^2y^2 + y^4.$$

By an elementary calculation, we can see that there are a, b, c, $d \in \mathbb{R}$ with $ad - bc \neq 0$ such that

$$(ax + by)^4 + t_1(ax + by)^2(cx + dy)^2 + (cx + dy)^4 = x^4 + t_2x^2y^2 + y^4$$

if and only if $t_1 = t_2$ or $(t_1+2)(t_2+2) = 16$. It follows from Proposition 1.2 that f_{t_1} and f_{t_2}, t_1, $t_2 \in \mathbb{R}$, are C^1 equivalent if and only if $t_1 = t_2$ or $(t_1 + 2)(t_2 + 2) = 16$.

2. Construction of bi-Lipschitz and C^1 invariants.

Let $f(x, y)$ be a real analytic two variable function germ:

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots , \tag{2.1}$$

where f_j denotes the j-th homogeneous form of f. We say that f is *mini-regular in x* if $f_m(1, 0) \neq 0$. Unless otherwise specified we shall always assume that the real analytic function germs are mini-regular in x.

By *a real analytic demi-branch* at $0 \in \mathbb{R}^2$ we mean the image $\varphi([0, \delta))$, where $\delta > 0$ and $\varphi(t) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is a real analytic arc not identically equal to zero, considered as a set-germ at the origin. By *the tangent direction* of the demi-branch we mean $\lim_{t \to 0^+} \varphi(t)/\|\varphi(t)\| \in S^1$. If the system of coordinates x, y is fixed we shall consider the demi-branches of real analytic arcs at $0 \in \mathbb{R}^2$ of the following form

$$\gamma : x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots , \quad y \geq 0,$$

where $\lambda(y)$ is a convergent fractional power series, N and $n_1 < n_2 < \cdots$ are positive integers having no common divisor, $a_i \in \mathbb{R}$. We shall call such a demi-branch *allowable* if $n_1/N \geq 1$, that is γ is transverse to the x-axis.

DEFINITION 2.1. Given an analytic function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and a real analytic demi-branch γ. We say that (x, y) is *an admissible system of local analytic coordinates for f and γ* if γ is allowable and $f(x, y)$ is mini-regular in x.

Given f and γ as above. We define *the order function of f relative to γ*, $\mathrm{ord}_\gamma f : [1, \infty) \to \mathbb{R}$ as follows. Fix $\xi \geq 1$ and expand

$$f(\lambda(y) + zy^\xi, y) = P_{f,\gamma,\xi}(z)y^{\mathrm{ord}_\gamma f(\xi)} + \cdots , \tag{2.2}$$

where the dots denote higher order terms in y and $\operatorname{ord}_\gamma f(\xi)$ is the smallest exponent with non-zero coefficient. This coefficient, $P_{f,\gamma,\xi}(z)$, is a polynomial function of z.

By *the Newton polygon of f relative to γ*, denoted by $NP_\gamma f$, we mean the Newton polygon of $f(X + \lambda(Y), Y)$, see [**22**]. Its boundary, called the *Newton boundary* and denoted by $NB_\gamma f$, is the union of compact faces of $NP_\gamma f$.

REMARK 2.2. Both the Newton boundary $NB_\gamma f$ and the order function $\operatorname{ord}_\gamma f : [1, \infty) \to \mathbb{R}$ depend only on f and on the demi-branch γ considered as a set germ at the origin. They are independent of the choice of admissible local coordinate system. This follows from Corollary 2.7. As for $P_{f,\gamma,\xi}$, it is invariant by change of admissible local coordinates, $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ if $D\sigma(0) = \operatorname{Id}$, see Corollary 2.8. For general change of admissible local coordinates see Proposition 2.12.

PROPOSITION 2.3. *The Newton boundary $NB_\gamma f$ determines the order function $\operatorname{ord}_\gamma f$ and vice versa. More precisely, let $\varphi : (0, m] \to [0, \infty]$ be the piecewise linear function whose graph $y = \varphi(x)$ is $NB_\gamma f$, then we have*

$$\varphi(x) = \max_\xi (\operatorname{ord}_\gamma f(\xi) - \xi x) \quad (\text{Legendre transform}),$$

$$\operatorname{ord}_\gamma f(\xi) = \min_x (\varphi(x) + \xi x) \quad (\text{inverse Legendre transform}).$$

PROOF. Let $f(X + \lambda(Y), Y) = \sum_{i,j} c_{ij} X^i Y^j$. Then,

$$\operatorname{ord}_\gamma f(\xi) = \min_{i,j} \{ j + i\xi ; c_{i,j} \neq 0 \} = \min_x \{ \varphi(x) + x\xi \}.$$

That shows the second formula. The order $\operatorname{ord}_\gamma f(\xi)$ can be also seen graphically from the Newton boundary by connecting all the vertices to the y-axis by the lines of slope $-\xi$. Then the lowest dot on the y-axis is $(0, \operatorname{ord}_\gamma f(\xi))$, see Figure 1.
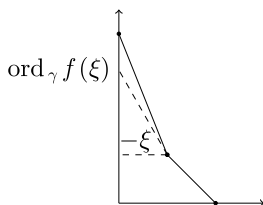

Figure 1. $NP_\gamma f$.

The first formula follows from the second one. $\qquad\square$

EXAMPLE 2.4. Let $f(x,y) = x^2 - y^3$, and let $\gamma_1 : x = y^{3/2}$ and $\gamma_2 : x = y^{3/2} + y^{5/2}$. Then

$$f(y^{3/2} + zy^\xi, y) = 2zy^{3/2+\xi} + z^2 y^{2\xi},$$

$$f(y^{3/2} + y^{5/2} + zy^\xi, y) = 2y^4 + y^5 + 2zy^{3/2+\xi} + 2zy^{5/2+\xi} + z^2 y^{2\xi}.$$

Therefore the order functions of γ_1 and γ_2 are given by

$$\operatorname{ord}_{\gamma_1} f(\xi) = \begin{cases} 2\xi & \text{for } 1 \leq \xi \leq \dfrac{3}{2} \\ \dfrac{3}{2} + \xi & \text{for } \xi \geq \dfrac{3}{2} \end{cases}$$

$$\operatorname{ord}_{\gamma_2} f(\xi) = \begin{cases} 2\xi & \text{for } 1 \leq \xi \leq \dfrac{3}{2} \\ \dfrac{3}{2} + \xi & \text{for } \dfrac{3}{2} \leq \xi \leq \dfrac{5}{2} \\ 4 & \text{for } \xi \geq \dfrac{5}{2}. \end{cases}$$
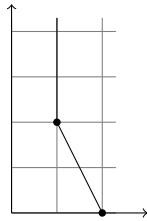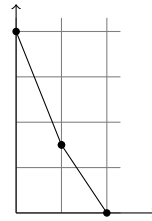

Figure 2. $NP_{\gamma_1} f$.

Figure 3. $NP_{\gamma_2} f$.

We finish the example by computing the Newton boundaries: $f(X + Y^{3/2}, Y) = X^2 + 2XY^{3/2}$ and $f(X + Y^{3/2} + Y^{5/2}, Y) = X^2 + 2XY^{3/2} + 2XY^{5/2} + 2Y^4 + Y^5$. Therefore

$$\varphi_1(x) = \begin{cases} \infty & \text{for } 0 < x < 1 \\ 3 - \dfrac{3}{2}x & \text{for } 1 \leq x \leq 2 \end{cases}, \qquad \varphi_2(x) = \begin{cases} 4 - \dfrac{5}{2}x & \text{for } 0 < x < 1 \\ 3 - \dfrac{3}{2}x & \text{for } 1 \leq x \leq 2. \end{cases}$$

Let $f(X + \lambda(Y), Y) = \sum_{i,j} c_{ij} X^i Y^j$. Then by *the initial Newton polynomial*

of f relative to γ, we mean

$$in_\gamma f = \sum_{(i,j)\in NB_\gamma(f)} c_{ij} X^i Y^j. \tag{2.3}$$

Note that $in_\gamma f$ is a fractional polynomial, $(i,j) \in \mathbb{Z} \times \mathbb{Q}$, $i \geq 0$, $j \geq 0$.
Given $\xi \geq 1$. We denote

$$\Gamma_\xi = \{(i,j) \in NB_\gamma(f); \xi i + j = \text{ord}_\gamma f(\xi)\}.$$

Then Γ_ξ is either an edge of $NB_\gamma(f)$ of slope $-\xi$ or is reduced to a vertex. Since

$$P_{f,\gamma,\xi}(z) = \sum_{(i,j)\in\Gamma_\xi} c_{ij} z^i, \tag{2.4}$$

the information given in $in_\gamma f$ is exactly equal to the information given by all $P_{f,\gamma,\xi}$. Note that Γ_ξ is reduced to a vertex if and only if $P_{f,\gamma,\xi}$ is a monomial.

2.1. Bi-Lipschitz invariants.

Using the Newton boundary and the order function we construct new invariants of bi-Lipschitz equivalence of two variable real analytic function germs.

For an allowable real analytic demi-branch $\gamma : x = \lambda(y)$ we define *the horn-neighbourhood of γ with exponent $\xi \geq 1$ and width $N > 0$* by

$$H_\xi(\gamma; N) := \{(x,y); |x - \lambda(y)| \leq N|y|^\xi, y > 0\}.$$

PROPOSITION 2.5.   *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a bi-Lipschitz homeomorphism, and let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be two real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma, \tilde{\gamma}$ are allowable real analytic demi-branches and that there exist $\xi_0 \geq 1$ and $N > 0$ such that*

$$\sigma(\gamma) \subset H_{\xi_0}(\tilde{\gamma}; N).$$

Then, for $1 \leq \xi \leq \xi_0$, $\text{ord}_\gamma f(\xi) = \text{ord}_{\tilde{\gamma}} g(\xi)$ and $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde{\gamma},\xi}$.

PROOF.   If $\xi = 1$ then $\text{ord}_\gamma f(\xi) = \deg P_{f,\gamma,\xi} = \text{mult}_0 f$ and the claim follows from bi-Lipschitz invariance of multiplicity.

Suppose that $\xi_0 > 1$ and $1 < \xi \leq \xi_0$. Let $\sigma(x,y) = (\sigma_1(x,y), \sigma_2(x,y))$, $\gamma : x = \lambda(y)$, $\tilde{\gamma} : x = \tilde{\lambda}(y)$. Then, there exists $C > 0$ such that $\tilde{y}(y) := \sigma_2(\lambda(y), y)$ satisfies

$$\frac{1}{C}y \leq \tilde{y}(y) \leq Cy.$$

LEMMA 2.6. *For any $1 < \xi \leq \xi_0$ and $M > 0$ there is $\tilde{M}_\xi$ such that*

$$\sigma(H_\xi(\gamma; M)) \subset H_\xi(\tilde{\gamma}; \tilde{M}_\xi).$$

Moreover, there is $A > 0$ independent of ξ such that $\tilde{M}_\xi$ can be chosen of the form $\tilde{M}_\xi = AM$ if $\xi < \xi_0$ and $\tilde{M}_{\xi_0} = AM + N$.

PROOF.   By the Lipschitz property, for $(x, y) \in H_\xi(\gamma; M)$ near $0 \in \mathbb{R}^2$,

$$|\sigma_2(x, y) - \tilde{y}| = |\sigma_2(x, y) - \sigma_2(\lambda(y), y)| \leq LMy^\xi \leq LMC^\xi \tilde{y}^\xi = o(\tilde{y}),$$

and

$$|\sigma_1(x, y) - \tilde{\lambda}(\tilde{y})| \leq |\sigma_1(x, y) - \sigma_1(\lambda(y), y)| + |\sigma_1(\lambda(y), y) - \tilde{\lambda}(\tilde{y})|$$
$$\leq LMC^\xi \tilde{y}^\xi + N\tilde{y}^{\xi_0}.$$

Finally, for an arbitrary $\varepsilon > 0$, there is a neighbourhood U_ϵ of $0 \in \mathbb{R}^2$ such that for $(x, y) \in H_\xi(\gamma; M) \cap U_\epsilon$,

$$|\sigma_1(x, y) - \tilde{\lambda}(\sigma_2(x, y))| \leq |\sigma_1(x, y) - \tilde{\lambda}(\tilde{y})| + |\tilde{\lambda}(\tilde{y}) - \tilde{\lambda}(\sigma_2(x, y))|$$
$$\leq LMC^\xi \tilde{y}^\xi + N\tilde{y}^{\xi_0} + (\tilde{\lambda}'(0) + \varepsilon)LMC^\xi \tilde{y}^\xi. \qquad \square$$

Since σ is bi-Lipschitz it can be shown by a similar argument that there exists N' for which $\sigma(H_\xi(\gamma; N')) \supset \tilde{\gamma}$, that is $\sigma^{-1}(\tilde{\gamma}) \subset H_{\xi_0}(\gamma; N')$. Thus the assumptions of Proposition 2.5 are symmetric with respect to f and g.

Let $x = \delta(y) = \lambda(y) + cy^\xi$, c arbitrary. On one hand, by Lemma 2.6,

$$|g(\sigma(\delta(y), y))| \leq \max\{P_{g,\tilde{\gamma},\xi}(z); |z| \leq A|c| + N + 1\} \tilde{y}^{\mathrm{ord}_{\tilde{\gamma}} g(\xi)}.$$

On the other hand

$$g(\sigma(\delta(y), y)) = f(\lambda(y) + cy^\xi, y) = P_{f,\gamma,\xi}(c)y^{\mathrm{ord}_\gamma f(\xi)} + \cdots.$$

This implies $\mathrm{ord}_\gamma f(\xi) \geq \mathrm{ord}_{\tilde{\gamma}} g(\xi)$, then by symmetry $\mathrm{ord}_\gamma f(\xi) = \mathrm{ord}_{\tilde{\gamma}} g(\xi)$, and finally $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde{\gamma},\xi}$. This ends the proof of Proposition 2.5. $\qquad \square$

Corollary 2.7. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a bi-Lipschitz homeomorphism, and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be two real analytic function germs such that $f = g \circ \sigma$. Suppose that γ, $\tilde{\gamma}$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde{\gamma}$ as set-germs at $(0,0)$. Then, for all $\xi \geq 1$, $\mathrm{ord}_\gamma f(\xi) = \mathrm{ord}_{\tilde{\gamma}} g(\xi)$ and $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde{\gamma},\xi}$. In particular, $\mathrm{NB}_\gamma f = \mathrm{NB}_{\tilde{\gamma}} g$.*

2.2. C^1 invariants.

For the C^1 equivalence we first consider the equivalence given by C^1 diffeomorphisms $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with $D\sigma(0) = \mathrm{Id}$. The general case will be treated in Section 2.3.

Proposition 2.8. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism with $D\sigma(0) = \mathrm{Id}$, and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that γ, $\tilde{\gamma}$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde{\gamma}$ as set-germs at $(0,0)$. Then for all $\xi \geq 1$*

$$P_{f,\gamma,\xi} = P_{g,\tilde{\gamma},\xi}.$$

Proof.   It is more convenient to work in a wider category and assume that f and g are convergent fractional power series of the form

$$\sum_{i,j} c_{ij} x^i y^j, \tag{2.5}$$

where $i \in \mathbb{N} \cup \{0\}$, $j \in \frac{1}{q}\mathbb{N} \cup \{0\}$, q is a positive integer, and $c_{ij} \in \mathbb{R}$. Such a series defines a function germ on $y \geq 0$. Let $\gamma : x = \lambda(y)$, $\tilde{\gamma} : x = \tilde{\lambda}(y)$. In what follows we extend λ and $\tilde{\lambda}$ to y negative by setting $\lambda(-y) = -\lambda(y)$ and $\tilde{\lambda}(-y) = -\tilde{\lambda}(y)$. Note that both λ and $\tilde{\lambda}$ are C^1. We have

$$f \circ H_1 = g \circ H_2 \ \circ (H_2^{-1} \circ \sigma \circ H_1),$$

where $H_1(x, y) = (x + \lambda(y), y)$ and $H_2(X, Y) = (X + \tilde{\lambda}(Y), Y)$. Then $\tilde{f} = f \circ H_1$, $\tilde{g} = g \circ H_2$ are fractional power series. The map $\tilde{\sigma} = H_2^{-1} \circ \sigma \circ H_1$ is C^1 and $D\tilde{\sigma}(0) = \mathrm{Id}$. The latter follows from the fact that the tangent directions at the origin to γ and $\tilde{\gamma}$ coincide. Thus by replacing f, g, σ by $\tilde{f}, \tilde{g}, \tilde{\sigma}$ we may suppose that $\lambda \equiv \tilde{\lambda} \equiv 0$, that is the image of the y-axis is the y-axis. Therefore σ is of the form

$$\sigma(x, y) = (x\varphi(x, y), y + \psi(x, y)),$$

with $\varphi(x, y), \psi(x, y)$ continuous and $\varphi(0, 0) = 1, \psi(0, 0) = 0$.

Let $g(x, y)$ be a fractional power series as in (2.5). The expansion (2.2) still holds for g and any allowable demi-branch. We use this property for the (positive) y-axis as a demi-branch that we denote below by $\underline{0}$ (since it is given by $\lambda \equiv 0$).

LEMMA 2.9.   *Let $g(x, y)$ be a fractional power series as in (2.5). Then for all $\alpha(y), \beta(y), \tau(y)$ such that $\alpha(y) = o(y)$, $\beta(y) = o(y)$, $\tau(y) = o(y)$, $\xi \geq 1$, and $z \in \mathbb{R}$ bounded*

$$g((z + \alpha(y))(1 + \tau(y))y^\xi, y + \beta(y)) = P_{g,\underline{0},\xi}(z)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}).$$

PROOF.   We have

$$g(zy^\xi, y) = P_{g,\underline{0},\xi}(z)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}).$$

More precisely $g(zy^\xi, y) - P_{g,\underline{0},\xi}(z)y^{\operatorname{ord}_{\underline{0}} g(\xi)} \to 0$ as $y \to 0$ and z is bounded. Then

$$\begin{aligned}
g((z + \alpha(y))(1 + \tau(y))y^\xi, y + \beta(y)) &= g(\tilde{z}(y + \beta(y)^\xi, y + \beta(y)) \\
&= P_{g,\underline{0},\xi}(\tilde{z})(y + \beta(y))^{\operatorname{ord}_{\underline{0}} g(\xi)} + o((y)^{\operatorname{ord}_{\underline{0}} g(\xi)}) \\
&= P_{g,\underline{0},\xi}(z)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}),
\end{aligned}$$

where $\tilde{z} = (z + \alpha(y))(1 + \tau(y))(1 + \beta(y)/y)^{-\xi}$. $\qquad\square$

To complete the proof of Proposition 2.8 we apply Lemma 2.9 to g, $\alpha(y) = \varphi(cy^\xi, y) - 1$, and $\beta(y) = \psi(cy^\xi, y)$, where $c \in \mathbb{R}$ is a constant. Then

$$\begin{aligned}
f(cy^\xi, y) = g \circ \sigma(cy^\xi, y) &= g(cy^\xi \varphi(cy^\xi, y), y + \psi(cy^\xi, y)) \\
&= P_{g,\underline{0},\xi}(c)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}).
\end{aligned}$$

Therefore, by expanding $f(cy^\xi, y) = P_{f,\underline{0},\xi}(c)y^{\operatorname{ord}_{\underline{0}} f(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} f(\xi)})$, we obtain

$$P_{f,\underline{0},\xi}(c)y^{\operatorname{ord}_{\underline{0}} f(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} f(\xi)}) = P_{g,\underline{0},\xi}(c)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}),$$

that shows $P_{f,\underline{0},\xi} = P_{g,\underline{0},\xi}$. This ends the proof of Proposition 2.8. $\qquad\square$

PROPOSITION 2.10.   *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism with $D\sigma(0) = \operatorname{Id}$, and let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma, \tilde{\gamma}$ are allowable real analytic demi-branches and that there exist $\xi_0 \geq 1, N > 0$ such that*

$$\sigma(\gamma) \subset H_{\xi_0}(\tilde{\gamma}; N).$$

Then for all $1 \leq \xi < \xi_0$,

$$P_{f,\gamma,\xi} = P_{g,\tilde{\gamma},\xi}.$$

Moreover, P_{f,γ,ξ_0} *and* $P_{g,\tilde{\gamma},\xi_0}$ *have the same degrees and their leading coefficients coincide.*

PROOF.   Let $\xi_0 > 1$. Then, since $D\sigma(0) = \mathrm{Id}$, the tangent directions at the origin to γ and $\tilde{\gamma}$ coincide. After a C^1 change of local coordinates as in the proof of Proposition 2.10, we may assume that both γ and $\tilde{\gamma}$ are equal to the (positive) y-axis. Write

$$\sigma(x,y) = (\sigma_1(0,y) + x\varphi(x,y), y + \psi(x,y)),$$

with $\varphi(x,y)$, $\psi(x,y)$ continuous and $\varphi(0,0) = 1$, $\psi(0,0) = 0$. The assumption on the image of γ gives

$$|\sigma_1(0,y)| \leq N_1|y|^{\xi_0},$$

with $N_1 > N$. Then, for $\xi < \xi_0$, $\sigma_1(0,y) = o(y^\xi)$ and

$$f(cy^\xi, y) = g \circ \sigma(cy^\xi, y) = g(cy^\xi\varphi(cy^\xi, y) + \sigma_1(0,y), y + \psi(cy^\xi, y))$$
$$= g(\tilde{c}y^\xi, y + \beta(y))$$

with $\tilde{c} = (c + \alpha(y))(1 + \tau(y))$, where $\alpha(y) = o(y)$, $\beta(y) = o(y)$. Thus the first claim follows again from Lemma 2.9.

If $\xi = \xi_0 > 1$ then the same computation shows that

$$P_{f,\underline{0},\xi}(c) \in \{P_{g,\underline{0},\xi}(z); |z - c| \leq N_1\},$$

for all c. That shows that the degrees of $P_{f,\underline{0},\xi}$ and $P_{g,\underline{0},\xi}$ and their leading coefficients coincide.

If $\xi_0 = 1$ then $P_{f,\gamma,1}$ depends only on the initial homogeneous form of f, denoted by f_m in (2.1), and the tangent direction to γ at the origin. Then $m = \deg P_{f,\gamma,1}$ and the leading coefficient of $P_{f,\gamma,1}$ is independent of the choice of γ. But the initial homogeneous forms of f and g coincide by Lemma 1.2. This completes the proof of Proposition 2.10. $\square$

By (2.4) we have the following.

COROLLARY 2.11. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism with $D\sigma(0) = \mathrm{Id}$, and let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma, \tilde{\gamma}$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde{\gamma}$ as set-germs at $(0, 0)$. Then $\mathrm{in}_\gamma f = \mathrm{in}_{\tilde{\gamma}} g$.*

2.3. Arbitrary C^1 equivalence.

If f and g are C^1 equivalent by a C^1 diffeomorphism σ, $f = g \circ \sigma$, then usually we compose f or g with a linear isomorphism and assume that $D\sigma(0) = \mathrm{Id}$. Thus to extend the results of the previous subsection to the general case it suffices to consider linear changes of coordinates, which leads to elementary computations.

PROPOSITION 2.12. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism such that $D\sigma(0)(x, y) = (ax + by, cx + dy)$ and let $f(x, y)$, $g(x, y)$ be real analytic function germs, mini-regular in x, such that $f = g \circ \sigma$. Suppose that $\gamma, \tilde{\gamma}$ are allowable real analytic demi-branches and that there exist $\xi_0 > 1$, $N > 0$ such that*

$$\sigma(\gamma) \subset H_{\xi_0}(\tilde{\gamma}; N).$$

Then, for $\xi \in (1, \xi_0)$, $P_{f, \gamma, \xi}$ and $P_{g, \tilde{\gamma}, \xi}$ are related by

$$P_{f, \gamma, \xi}(z) = (c\lambda'(0) + d)^{\mathrm{ord}_\gamma f(\xi)} P_{g, \tilde{\gamma}, \xi}\left(\frac{ad - bc}{(c\lambda'(0) + d)^{\xi+1}} z \right) \tag{2.6}$$

and $c\lambda'(0) + d$. If $\xi = 1$ then

$$P_{f, \gamma, 1}(z) = (c\lambda'(0) + d + cz)^m P_{g, \tilde{\gamma}, 1}\left(\frac{ad - bc}{c\lambda'(0) + d} \cdot \frac{z}{c\lambda'(0) + d + cz} \right).$$

PROOF. By Proposition 2.10 it suffices to consider only the case of σ linear

$$(\tilde{x}, \tilde{y}) = \sigma(x, y) = (ax + by, cx + dy), \quad \det \sigma = ad - bc \neq 0,$$

and $\tilde{\gamma} = \sigma(\gamma)$. Let $\gamma : x = \lambda(y)$, $\tilde{\gamma} : x = \tilde{\lambda}(y)$. Then

$$\lambda(y)a + by = \tilde{\lambda}(c\lambda(y) + dy), \tag{2.7}$$

$c\lambda(y) + dy$ parametrises the positive y-axis, and

$$\tilde{\lambda}'(0) = \frac{a\lambda'(0) + b}{c\lambda'(0) + d}, \qquad c\lambda'(0) + d > 0.$$

Fix $\xi > 1$. Clearly $\mathrm{ord}_\gamma f(\xi) = \mathrm{ord}_{\tilde{\gamma}} g(\xi)$. Put $Y = c(\lambda(y) + zy^\xi) + dy$. Then $Y = (c\lambda'(0) + d)y + o(y)$ and $y = (c\lambda'(0) + d)^{-1}Y + o(Y)$, and consequently

$$\begin{aligned}
f(\lambda(y) + zy^\xi, y) &= g(a(\lambda(y) + zy^\xi) + by, c(\lambda(y) + zy^\xi) + dy) \\
&= g(\tilde{\lambda}(c\lambda(y) + dy) + azy^\xi, Y) \\
&= g(\tilde{\lambda}(Y) - \tilde{\lambda}'(0)czy^\xi + azy^\xi + o(y^\xi), Y) \\
&= g\left(\tilde{\lambda}(Y) + \frac{ad - bc}{(c\lambda'(0) + d)^{\xi+1}}zY^\xi + o(Y^\xi), Y\right).
\end{aligned}$$

Hence, comparing this formula with (2.2), we get

$$P_{f,\gamma,\xi}(z)y^{\mathrm{ord}_\gamma f(\xi)} = P_{g,\tilde{\gamma},\xi}\left(\frac{ad - bc}{(c\lambda'(0) + d)^{\xi+1}}z\right)Y^{\mathrm{ord}_{\tilde{\gamma}} g(\xi)},$$

that gives (2.6). The case $\xi = 1$ is left to the reader.   □

COROLLARY 2.13.   *Given an analytic function germ* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$
and a real analytic demi-branch γ. *Then* $NB_\gamma f$ *is independent of the choice of
admissible coordinate systems. Moreover, for each edge* $\Gamma \subset NB_\gamma(f)$ *with slope
smaller than* -1, *the polynomial* $P_\Gamma(z) = \sum_{(i,j)\in\Gamma} c_{ij}z^i$ *is well-defined up to left
and right multiplications as in* (2.6).

EXAMPLE 2.14 (Compare [**10**], [**11**] for the bi-Lipschitz equivalence).   Con-
sider the family

$$A_t(x, y) = x^3 - 3txy^4 + 2y^6.$$

This family is equivalent to the family J_{10} of [**1**], where it is shown that t is
a continuous modulus for the analytic change of coordinates. For each t, A_t is
mini-regular in x, and $\partial A_t/\partial x = 3(x^2 - ty^4)$.

For $t > 0$ let us consider the Newton polygon of A_t relative to a polar curve
$\gamma_t : x = \sqrt{t}\, y^2$. Then we have

$$A_t(X + \sqrt{t}\, Y^2, Y) = X^3 + 3\sqrt{t}\, X^2 Y^2 + 2(1 - t\sqrt{t}\,)Y^6,$$

and

$$P_{\Gamma_t}(z) = z^3 + 3\sqrt{t}\, z^2 + 2(1 - t\sqrt{t}).$$

If for t, $t' \in (0, \infty)$, A_t and $A_{t'}$ are bi-Lipschitz equivalent, $A_t = A_{t'} \circ \varphi$, then, by [**10**], [**11**], the image φ of the polar curve of A_t is in a horn neighbourhood, of width > 1, of a polar curve of $A_{t'}$. Thus if A_t and $A_{t'}$ are C^1 equivalent then, by Proposition 2.12, there are α, $\beta \neq 0$ such that $P_{\Gamma_{t'}}(z) = \beta^6 P_{\Gamma_t}((\alpha/\beta^3)z)$. By an easy computation, we obtain that $\alpha^3 = \beta^2 = 1$ and $P_{\Gamma_t} \equiv P_{\Gamma_{t'}}$. Therefore A_t and $A_{t'}$ are C^1 equivalent if and only if $t = t'$.

3. $\mathbf{C^1}$ invariance of Puiseux pairs of roots.

Let $\gamma : x = \lambda(y)$ be an allowable real analytic demi-branch. The Puiseux pairs of γ are pairs of relatively prime positive integers $(n_1, d_1), \ldots, (n_q, d_q)$, $d_i > 1$ for $i = 1, \ldots, q$, $n_1/d_1 < n_2/d_1 d_2 < \cdots < n_q/(d_1 \ldots d_q)$, such that

$$\lambda(y) = \sum_{\alpha} a_{\alpha} y^{\alpha} = \sum_{j=1}^{[n_1/d_1]} a_j y^j + \sum_{j=n_1}^{[n_2/d_2]} a_{j/d_1} y^{j/d_1}$$

$$+ \sum_{j=n_2}^{[n_3/d_3]} a_{j/d_1 d_2} y^{j/d_1 d_2} + \cdots + \sum_{j=n_q}^{\infty} a_{j/d_1 d_2 \ldots d_q} y^{j/(d_1 d_2 \ldots d_q)} \quad (3.1)$$

and $a_{n_i/d_1 \ldots d_i} \neq 0$ for $i = 1, \ldots, q$, cf. e.g. [**28**]. The exponents $n_i/(d_1 \ldots d_i)$ will be called *the (Puiseux) characteristic exponents* of γ. The corresponding coefficients $A_i(\gamma) := a_{n_i/d_1 \ldots d_i}$ for $i = 1, \ldots, q$ will be called *the characteristic coefficients* of γ.

We say that γ is a *root* of $f(x, y)$ if $\gamma \subset f^{-1}(0)$. We show that if γ is a root of $f(x, y)$ then the Puiseux exponents, or equivalently the Puiseux pairs, of γ are determined by $in_{\gamma} f$.

PROPOSITION 3.1.   *Let* $\gamma : x = \lambda(y) = \sum a_{\alpha} y^{\alpha}$ *be an allowable real analytic demi-branch and let* $f(x, y)$ *be a real analytic function germ mini-regular in* x. *Let* γ *be a root of* f. *Let* ξ *be an arbitrary positive rational number. Write* $\xi = n/(d_1 \ldots d_i d)$, $\gcd(n, d) = 1$, *where* $n_1/d_1 < n_2/d_1 d_2 < \cdots < n_q/(d_1 \ldots d_i)$ *are all characteristic exponents of* γ *smaller than* ξ. *Then*

(1) *If* ξ *is a characteristic exponent of* γ *then* $NB_{\gamma}(f)$ *has an edge of slope* $-\xi$.
(2) *Suppose that* $NB_{\gamma}(f)$ *has an edge of slope* $-\xi$ *and let*

$$P_{f, \gamma, \xi} = B_0 z^s + B_1 z^{s-1} + \cdots, \quad B_0 \neq 0.$$

Then ξ is a characteristic exponent of γ if and only if $d > 1$ and $B_1 \neq 0$, and if this is the case then the characteristic coefficient $a_\xi = B_1/sB_0$.

Proof.   Consider the truncation γ_ξ of γ at ξ

$$\gamma_\xi : x = \lambda_\xi(y) = \sum_{\alpha < \xi} a_\alpha y^\alpha.$$

Denote $\mathrm{ord} = \mathrm{ord}_{\gamma_\xi} f(\xi) = \mathrm{ord}_\gamma f(\xi)$ and $P(z) = P_{f,\gamma,\xi}$, $P_0(z) = P_{f,\gamma_\xi,\xi}$. Then $P(z) = P_0(z + a_\xi)$.

Lemma 3.2.   *There exist an integer $k \geq 0$ and a polynomial $\tilde{P}_0$ such that $P_0(z) = z^k \tilde{P}_0(z^d)$.*

Proof.   Write

$$f(\lambda_\xi(y) + x, y) = \sum_{\xi i + j \geq \mathrm{ord}} c_{ij} x^i y^j = P_0(x/y^\xi) y^{\mathrm{ord}} + \sum_{\xi i + j > \mathrm{ord}} c_{ij} x^i y^j.$$

Since the Puiseux pairs of γ_ξ determine the possible denominators of the exponents of y in $f(\lambda_\xi(y) + x, y)$,

$$c_{ij} \neq 0 \Rightarrow j(d_1 \dots d_i) \in \mathbb{Z}. \tag{3.2}$$

Let $i_0 = \deg P_0$, $j_0 = \mathrm{ord} - \xi i_0$. Then $P_0(z) = c_{i_0 j_0} z^{i_0} +$ lower degree terms.

Fix (i, j) such that $\xi i + j = \mathrm{ord}$. Since $\xi(i - i_0) + (j - j_0) = 0$ and $j_0(d_1 \dots d_i) \in \mathbb{Z}$, we have

$$j(d_1 \dots d_i) \in \mathbb{Z} \Leftrightarrow \xi(i - i_0) d_1 \dots d_i \in \mathbb{Z} \Leftrightarrow (i - i_0)n/d \in \mathbb{Z} \Leftrightarrow (i - i_0) \in d\mathbb{N}.$$

Therefore, by (3.2), $P_0 = c_{i_0 j_0} z^k \hat{P}_0(z^d)$ with $\hat{P}_0$ unitary.   □

If $\gamma \subset f^{-1}(0)$ then $P(0) = 0$ and therefore P is not identically equal to a constant. By Lemma 3.2

$$P(z) = P_0(z + a_\xi) = (z + a_\xi)^k \tilde{P}_0((z + a_\xi)^d) = B_0 z^s + B_1 z^{s-1} + \cdots.$$

Therefore if $d > 1$ we may compute $a_\xi = B_1/sB_0$. If $NB_\gamma(f)$ has no edge of slope $-\xi$ then $P_{f,\gamma,\xi}$ is a monomial and if moreover $d > 1$ then $a_\xi = 0$. This shows the first part of Proposition 3.1 and the second part follows similarly.   □

COROLLARY 3.3. *Let* $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a* C^1 *diffeomorphism and let* $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be real analytic function germs such that* $f = g \circ \sigma$. *Suppose that* $\gamma \subset f^{-1}(0)$, $\tilde{\gamma} \subset g^{-1}(0)$ *are allowable real analytic demi-branches such that* $\sigma(\gamma) = \tilde{\gamma}$ *as set germs. Then the Puiseux characteristic pairs of* γ *and* $\tilde{\gamma}$ *coincide.*

Moreover, if $D\sigma(0)$ *preserves orientation then the signs of characteristic co-efficients of* γ *and* $\tilde{\gamma}$ *coincide.*

PROOF. It follows easily from Proposition 2.12. We check only the last claim. If σ preserves the orientation then $c\lambda'(0) + d > 0$ and $ad - bc > 0$. Therefore the sign of characteristic coefficients is preserved by the formula $a_\xi = B_1/sB_0$. $\square$

4. C^1 equivalent germs are blow-analytically equivalent.

In this section we show Theorem 0.2. The proof is based on the characterisation (3) of Theorem 0.1. First we recall briefly the construction of real tree model, for the details see [**15**]. The construction of [**15**] develops an earlier construction of [**23**].

Let $f(x, y)$ be a real analytic function germ. The real tree model of f is an adaptation of the Kuo-Lu tree model [**18**] of complex analytic function germs. The main differences are the following. The Newton-Puiseux roots of f

$$x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots, \tag{4.1}$$

$y \in \mathbb{C}$, are replaced by real analytic demi-branches obtained by restricting (4.1) to $y \in \mathbb{R}$, $y \geq 0$, and then truncating it at the first non-real coefficient a_i. Since we have to keep track of the exponent corresponding to this coefficient, it is replaced by a symbol c

$$x = \lambda(y) = a_1 y^{n_1/N} + \cdots + a_{i-1} y^{n_{i-1}/N} + c y^{n_i/N}, \quad y \geq 0. \tag{4.2}$$

We call (4.2) *a truncated root.* Geometrically we may think of the union over $c \in \mathbb{R}$ of all demi-branches of (4.2) that defines a set called a root horn. It is a kind of trace in the real domain of the complex root.

4.1. Real tree model $\mathbb{R}T_v(f)$ of f relative to a tangent direction v.
Fix v a unit vector of $\mathbb{R}^2$. Fix any local system of coordinates x, y such that:

- $f(x, y)$ *is mini-regular in* x;
- v *is of the form* (v_1, v_2) *with* $v_2 > 0$.

Let $x = \lambda(y)$ be a Newton-Puiseux root of f of the form (4.1). If λ is not real and a_i is the first non-real coefficient we replace this root by (4.2). Let Λ_v denote

the set of real roots and truncated roots, restricted to $y \geq 0$, that are tangent to v at the origin.

Suppose that Λ_v is non-empty. We apply the Kuo-Lu construction to Λ_v as follows. We define *the contact order of* λ_i *and* λ_j *of* Λ_v as

$$O(\lambda_i, \lambda_j) := \mathrm{ord}_0 \, (\lambda_i - \lambda_j)(y),$$

where $\mathrm{ord}_0 \, \varphi(y)$ denotes the order of vanishing of φ at the origin. Let $h \in \mathbb{Q}$. We say that λ_i, λ_j are *congruent modulo* h^+ if $O(\lambda_i, \lambda_j) > h$.

Draw a vertical line as the *main trunk* of the tree. Mark the number m_v of roots in Λ_v counted with multiplicities alongside the trunk. Let $h_1 := \min\{O(\lambda_i, \lambda_j) | 1 \leq i, j \leq m_v\}$. Then draw a bar, B_1, on top of the main trunk. Call $h(B_1) := h_1$ the *height* of B_1.

The roots of Λ_v with the coefficient a_{h_1} at y^{h_1} real are divided into equivalence classes, called *bunches*, modulo h_1^+. Each equivalence class is represented by a vertical line segment (trunk) drawn on top of B_1 in the order corresponding to the order of a_{h_1} coefficients. If a trunk consists of s roots we say it has *multiplicity* s, and mark s alongside (if $s = 1$ it is usually not marked). The other roots of Λ_v, that is those with the symbol c as the coefficient at y^{h_1}, do not produce a trunk over B_1. We say that they disappear at B_1.

Now, the same construction is repeated recursively on each trunk, getting more bars, then more trunks, etc. The height of each bar and the multiplicity of a trunk, are defined likewise. Each trunk has a unique bar on top of it. The construction terminates at the stage where either all the roots disappear (in the complex domain) at the bar or the bar has infinite height, that is on top of a trunk that contain a single, maybe multiple, real root of f.

To each bar B corresponds a unique trunk supporting it and a unique bunch of roots $A(B)$ bounded by B. In this way there is a one-to-one correspondence between trunks, bars, and bunches. We denote by m_B the multiplicity of the trunk supporting B.

Unlike in the complex case the mere information of the contact orders with the other roots does not determine the Puiseux pairs of a real root, or of the real truncation of a complex root. Therefore, whenever a bar B gives a new Puiseux pair to a root of $A = A(B)$ we mark 0 on B. If a trunk T' growing on B corresponds to the roots of A with coefficient $a_{h(B)} = 0$, resp. $a_{h(B)} < 0$, $a_{h(B)} > 0$, then we mark T' as growing at $0 \in B$, resp. to the left of 0, to the right of 0. Graphically, we mark $0 \in B$ by identifying it with the point of B that belongs to the trunk supporting B.

4.2. Real tree model of f.

The *real tree model* $\mathbb{R}T(f)$ *of* f is defined as follows.

- Draw a bar B_0 that is identified with S^1. We define $h(B_0) = 1$ and call B_0 *the ground bar*. We mark $m(B_0) := 2\,\mathrm{mult}_0\, f(x,y)$ below the ground bar.
- Grow on B_0 all non-trivial $\mathbb{R}T_v(f)$ for $v \in S^1$, keeping the clockwise order.
- Let v_1, v_2 be any two subsequent unit vectors for which $\mathbb{R}T_v(f)$ is nontrivial. Mark the sign of f in the sector between v_1 and v_2. Note that one such sign determines all the other signs between two subsequent unit vectors for which $\mathbb{R}T_v(f)$ is nontrivial (passing v changes this sign if and only if Λ_v contains an odd number of roots.)

If the leading homogeneous part f_m of f satisfies $f_m^{-1}(0) = 0$ then B_0 is the only bar of $\mathbb{R}T(f)$.

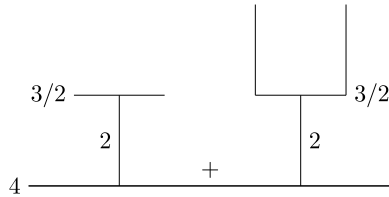We give below the real tree model of $f(x,y) = x^2 - y^3$ as example. More examples are presented in Section 6.

Figure 4.   Real tree model of $f(x,y) = x^2 - y^3$.

We call two real trees $\mathbb{R}T(f), \mathbb{R}T(g)$ *isomorphic* if there is a homeomorphism φ of their ground bars sending one tree to the other and preserving the multiplicities and heights of bars and signs of the characteristic coefficients. If, moreover, φ preserves the orientation we call the trees *orientably isomorphic*.

4.3. Horns.

Recall that for an allowable real analytic demi-branch $\gamma : x = \lambda(y)$ the horn-neighbourhood of γ with exponent $\xi \geq 1$ and width $N > 0$ is given by

$$H_\xi(\gamma; N) := \{(x,y);\ |x - \lambda(y)| \leq N|y|^\xi, y > 0\}.$$

We define *the horn-neighbourhood of* γ *of exponent* ξ as $H_\xi(\gamma; C)$ for C large and we denote it by $H_\xi(\gamma)$. *A horn is a horn-neighbourhood with exponent* $\xi > 1$.

If $\gamma_1 : x = \lambda_1(y)$, $\gamma_2 : x = \lambda_2(y)$, and $O(\lambda_1, \lambda_2) \geq \xi$ then we identify $H_\xi(\gamma_1) = H_\xi(\gamma_2)$ by meaning that for any $C_1 > 0$ there is $C_2 > 0$ such that

$$H_\xi(\gamma_1; C_1) \subset H_\xi(\gamma_2; C_2), \qquad H_\xi(\gamma_2; C_1) \subset H_\xi(\gamma_1; C_2). \tag{4.3}$$

EXAMPLE 4.1. Let B be a bar of $\mathbb{R}T_v(f)$, $h(B) > 1$. Then B defines a horn

$$H_B := \{(x, y); \ |x - \lambda(y)| \le C|y|^{h(B)}\},$$

where C is a large constant and $x = \lambda(y)$ is any root of the bunch $A(B)$.

DEFINITION 4.2. A horn that equals H_B for a bar B is called *a root horn* of f.

Let $H = H_\xi(\gamma), \gamma : x = \lambda(y)$, be a horn of exponent ξ. Let $\lambda_H(y)$ denote the truncation of λ at ξ, that is $\lambda_H(y)$ is the sum of all terms of $\lambda(y)$ of exponent $< \xi$. We define the truncated demi-branch by $\gamma_H : x = \lambda_H(y)$ and the generic demi-branch $\gamma_{H,gen} : x = \lambda_{H,gen}(y)$ by

$$\lambda_{H,gen}(y) = \lambda_H(y) + cy^\xi, \quad y \ge 0, \tag{4.4}$$

where c is a symbol signifying generic $c \in \mathbb{R}$. We define the *characteristic exponents of H* as those of $\gamma_{H,gen}$ and *the signs of characteristic coefficients of H* as those of $\gamma_{H,gen}$ (or equivalently of γ_H) that corresponds to exponents smaller than ξ. Let $\gamma' : x = \lambda'(y)$ be any allowable real analytic demi-branch contained in H. Then the order function $\text{ord}_{\gamma'} f$, defined by (2.2), restricted to $[1, \xi]$ is independent of the choice of γ' and so is the polynomial $P_{f,\gamma',\xi'}(z)$ for $\xi' < \xi$. The polynomial $P_{f,\gamma',\xi}(z)$ is independent up to a shift of variable z: if the coefficient of $\lambda'(y)$ at y^ξ is a then

$$P_{f,\gamma',\xi}(z) = P_{f,\gamma_H,\xi}(z + a).$$

PROPOSITION 4.3 ([**15**, Proposition 7.5]). *Let H be a horn of exponent ξ. Then H is a root horn for $f(x, y)$ if and only if $P_{f,\gamma_H,\xi}(z)$ has at least two distinct complex roots.*

If this is the case, $H = H_B$, then $h(B) = \xi$ and $m_B = \deg P_{f,\gamma_H,\xi_H}$.

Proposition 4.3 shows that for a root horn H of width ξ, $\deg P_{f,\gamma_H,\xi} > 1$. Moreover, for any characteristic exponent of γ_H, $\xi' < \xi$, the horn $H_\xi(\gamma_H)$ is a root horn. Indeed, if $\xi' = n_i/(d_1 \ldots d_i)$, then $\deg P_{f,\gamma_H,\xi'} = d_i$. Therefore we may extend the argument of the proof of Corollary 3.3 to the root horn case.

PROPOSITION 4.4. *Let $H = H_\xi(\gamma)$ be a horn root. Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism and let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma, \tilde{\gamma}$ are allowable real analytic demi-branches such that*

$$\sigma(\gamma) \subset H_\xi(\tilde{\gamma}; N).$$

Then the Puiseux characteristic exponents H and $\tilde{H}$ coincide.

Moreover, if $D\sigma(0)$ preserves orientation then the signs of characteristic coefficients of H and $\tilde{H}$ coincide.

4.4. Characterisation of real tree model in terms of root horns.

The real tree model $\mathbb{R}T(f)$ can be determined by the root horns and their numerical invariants, cf. [**15**, Subsection 7.3]. The root horns are ordered by inclusion and by clockwise order around the origin. Thus H_B is contained in $H_{B'}$ if and only if the bar B grows over B'. The multiplicity m_B and the height $h(B)$ are expressed in terms of invariants of the horn H_B by the formulae of Proposition 4.3.

Let $\gamma : x = \lambda(y)$ be a root of $A = A(B)$. Then the Puiseux characteristic exponents of γ that are $< h(B)$ and the corresponding signs of characteristic coefficients are those of $\gamma_{H_B,gen}$ (or, equivalently, of γ_{H_B}). If $\tilde{A} = A(\tilde{B})$ be a sub-bunch of A containing γ then the invariants of $H_{\tilde{B}}$ determine whether γ takes a new Puiseux pair at $h(B)$ and, if this is the case, the sign of the characteristic coefficient at $h(B)$.

4.5. Proof of Theorem 0.2.

By Propositions 2.5 and 4.3 the image of a root horn H_B is a root horn $H_{\widehat{B}}$. Thus obtained one-to-one correspondence $B \leftrightarrow \widehat{B}$ preserves the multiplicities and the heights of the bars. The Puiseux characteristic exponents and the corresponding signs of Puiseux coefficients are also preserved as follows from 4.4. If σ preserves the orientation then it preserves the clockwise order of root horns and hence the clockwise order on the trees. Therefore the real tree models of f and g are isomorphic and Theorem 0.2 follows from Theorem 0.1. $\qquad\square$

5. Bi-Lipschitz and C^1 equivalences of weighted homogeneous function germs.

Let $f \in \mathbb{R}[x, y]$ be a weighted homogeneous polynomial with respect to the weights $q, p \in \mathbb{N}$, $1 \le p \le q$, $(p, q) = 1$, and of weighted degree $d > 0$. We may write

$$f(x, y) = Cy^l \left(x^{d'/q} + \sum_{qi+pj=d'} a_{ij} x^i y^j \right), \qquad (5.1)$$

where C is a non-zero constant, $d' = d - pl$, and $\xi = q/p$. We distinguish three cases:

(A) homogeneous: $p = q = 1$;
(B) $1 = p < q$;
(C) $1 < p < q$.

In each of these three cases we call the following polynomials *monomial-like*

(Am) $A(ax + by)^k (cx + dy)^l$, $ad - bc \neq 0$,
(Bm) $A(x + by^q)^k y^l$,
(Cm) $Ax^k y^l$.

The name comes from the fact that they are equivalent by an obvious change of variables to the monomial $\pm x^k y^l$.

Write

$$f(x, y) = Cy^l \left(\prod_i (x^p - c_i y^q)^{m_i} \right) f_0(x, y), \qquad (5.2)$$

where $f_0 \equiv 1$ or has an isolated zero at the origin. Thus the zero set of f is either reduced to the origin or consists of finitely many irreducible real analytic curve germs. Each of these curves consists of two demi-branches that we call roots of f. To each such root γ we associate its tangent direction (a half-line at the origin) and its multiplicity as a root, denoted by m_γ. For instance, if γ is given by $x = c_i y^\xi, y \geq 0, \xi > 1$, then $x = 0$, $y \geq 0$ is its tangent direction and $m_\gamma = m_i$.

The following lemma is an easy consequence of Proposition 1.1.

LEMMA 5.1.   *Let* $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a bi-Lipschitz homeomorphism, and let* $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be real analytic function germs such that* $f = g \circ \sigma$. *Then* $\mathrm{mult}_0 f = \mathrm{mult}_0 g$.

Suppose moreover that $\gamma_i, i = 1, 2$, *respectively* $\tilde{\gamma}_i, i = 1, 2$, *are roots of* f, *respectively of* g, *such that* $\sigma(\gamma_i) = \tilde{\gamma}_i$. *Then* $m_{\gamma_i} = m_{\tilde{\gamma}_i}$ *and the tangent directions of* γ_1 *and* γ_2, *coincide if and only if they coincide for* $\tilde{\gamma}_1$ *and* $\tilde{\gamma}_2$.

COROLLARY 5.2.   *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *and* $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be two germs of weighted homogeneous polynomials. Suppose that* f *and* g *are bi-Lipschitz equivalent. Then, if* f *is monomial-like then so is* g *and* f *and* g *are analytically equivalent.*

PROOF.   If f is monomial-like then it satisfies the following property: f has 2 or 4 roots (demi-branches) with distinct tangent directions, and the sum of multiplicities roots of f is equal to $2 \mathrm{mult}_0 f$. By Lemma 5.1 this property is invariant by bi-Lipschitz homeomorphisms. It is easy to check that none of the not monomial-like weighted homogeneous polynomials satisfies this property.   □

The main results of this section are the two propositions below. Their proof will be based on Propositions 2.5 and 2.12.

PROPOSITION 5.3. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *and* $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be two germs of weighted homogeneous polynomials. Suppose that* f *and* g *are bi-Lipschitz equivalent and not monomial-like. Then* f *and* g *have the same weights and the same weighted degree.*

Moreover, if these common weights q, p *satisfy* $q/p > 1$ *and we write* f *and* g *in the form* (5.1), *then the factor* y *appears with the same exponent for* f *and for* g.

PROPOSITION 5.4. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *and* $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be two germs of weighted homogeneous polynomials. Suppose that* f *and* g *are* C^1 *equivalent and not monomial-like. Then*

(1) *If* f *is of type* (A) *then so is* g, *and* f *and* g *are equivalent by a linear change of coordinates.*
(2) *If* f *is of type* (B) *then so is* g, *and there exist* $c_1 \neq 0$, $c_2 \neq 0$, *and* b *such that*

$$f(x, y) = g(c_1 x - by^q, c_2 y).$$

(3) *If* f *is of type* (C) *then so is* g, *and there exist* $c_1 \neq 0$, $c_2 \neq 0$ *such that*

$$f(x, y) = g(c_1 x, c_2 y).$$

Thus Propositions 5.3 and 5.4 imply that for weighted homogeneous germs C^1 and analytic equivalences coincide.

Before we begin the proofs of Propositions 5.3 and 5.4 we need some preparation. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be the germ of a not monomial-like weighted homogeneous polynomial as in (5.1) and let γ be a root of f. We list all possibilities for the Newton boundary $NB_\gamma f$ in an admissible system of coordinates. Note that x, y is admissible if and only if f is not divisible by y. Otherwise we have to perform a generic linear change of coordinates. Then f need not be weighted homogeneous in the new system of coordinates.

We denote $m = \mathrm{mult}_0 f$. There are three distinct possibilities for the Newton boundary $NB_\gamma f$:

(i) If f is homogeneous then $NB_\gamma f$ has two vertices at $(m, 0)$, $(m_\gamma, m - m_\gamma)$, and hence one nontrivial compact edge of slope -1. This is also the Newton boundary for a non-homogeneous f and the root γ being a demi-branch of $y = 0$ in an admissible system of coordinates (that is different from x, y in this case).

(ii) If f is not homogeneous and γ is not a demi-branch of $y = 0$, then we have two possiblities:

    (a) If $l = 0$ then $NB_\gamma f$ has one nontrivial edge of slope $-\xi$ and vertices $(m, 0)$, $(m_\gamma, \xi(m - m_\gamma))$.

    (b) If $l \neq 0$ then $NB_\gamma f$ has two nontrivial edges: Γ_1 of slope -1 and vertices $(m, 0)$, $(m - l, l)$, and Γ_2 of slope $-\xi$ and vertices $(m - l, l)$, $(m_\gamma, \xi(m - l - m_\gamma) + l)$.
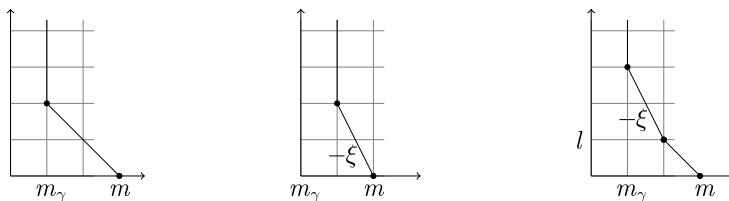

Figure 5.   Cases: (i), (ii, a), (ii, b).

PROOF OF PROPOSITION 5.3.   If f has a root γ such that $NB_\gamma f$ contains an edge of slope $-\xi < -1$ then Proposition 5.3 follows easily from the computation of the Newton boundary, see Case (ii) above. Indeed, $\xi = q/p$ with q and p coprime, so the weights can be computed from $NB_\gamma f$. If $(a, b) \in NB_\gamma f$ belongs to the edge of slope $-\xi$ then $d = qa + pb$. Moreover, if f and g are bi-Lipschitz equivalent $f = g \circ \sigma$ and $\tilde\gamma = \sigma(\gamma)$ then, by Corollary 2.7, $NB_\gamma f = NB_{\tilde\gamma} g$, so the weights and the weighted degree of f and g coincide. If the exponent l of (5.1) for f is nonzero, then $NB_\gamma f$ has two edges and we conclude by a similar argument that the exponent l of (5.1) for g is nonzero. By comparing the Newton polygons of f and g relative to the roots demi-branches of $y = 0$, Case (i) above, we conclude that these exponents coincide.

Thus to complete the proof it suffices to consider the case where for every root of f and g, the boundary of the Newton polygon contains only one edge and the slope of this edge is -1. This includes the following cases. Firstly f may have no roots, that is the origin is an isolated zero of f, and then the same holds for g. Secondly f may have two roots, and then so does g. If f has more than two roots then, in the cases that remains, it has to be homogeneous, and then so is g. This case is easy. Thus in what follows we will suppose that f and g have at most two roots. We shall use the Łojasiewicz exponents.

Take a real analytic arc $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$

$$\varphi(t) = (a_1 t^{n_1} + a_2 t^{n_2} + \cdots, \ b_1 t^{m_1} + b_2 t^{m_2} + \cdots).$$

Let $N = \min\{n_1, m_1\}$. Let us define $\ell_f(\varphi) \in \mathbb{Q}^+$ by

$$\|f(\varphi(t))\| \sim \|\varphi(t)\|^{\ell_f(\varphi)} \sim |t|^{N\ell_f(\varphi)}, \qquad (5.3)$$

where $A \sim B$ means that A/B lies between two positive constants.

If f has no roots then there exist constants $C, c > 0$ such that

$$c\|(x,y)\|^{d/p} \le \|f(x,y)\| \le C\|(x,y)\|^{d/q}, \qquad (5.4)$$

and the exponents are realised as $\ell(\varphi)$ for φ parameterising $x = 0$ and $y = 0$. Clearly these exponents are bi-Lipschitz invariants, and hence so are p, q and d.

Suppose that $l > 0$ and that f vanishes only on $y = 0$. We have a similar inequality in the complement of any small horn neighbourhood of the roots $\{(x,y); |y| \le \varepsilon|x|\}$

$$c\|(x,y)\|^{l+d'/p} \le \|f(x,y)\| \le C\|(x,y)\|^{l+d'/q}. \qquad (5.5)$$

Thus again, the exponents $d'/p + l$ and $d'/q + l$ are bi-Lipschitz invariants and so is l as the multiplicity of a root. This ends the proof of Proposition 5.3. $\square$

PROOF OF PROPOSITION 5.4. The homogeneous case $p = q = 1$ follows from Proposition 1.2.

We define *the associated one variable polynomial* $P(z) = f(z, 1)$. Then

$$f(x,y) = Cy^l\left(x^{d'/q} + \sum_{qi+pj=d'} a_{ij}x^i y^j\right) = y^{d/p}P(x/y^\xi). \qquad (5.6)$$

Consider the case when f has a root γ such that $NB_\gamma f$ contains an edge of slope $-\xi < -1$. We suppose that γ is given by $x = ay^\xi, y \ge 0$, where a is a root of $P(z)$. By Proposition 2.5, $NB_\gamma f = NB_{\tilde\gamma}g$, where $\tilde\gamma = \sigma(\gamma)$. Replacing $g(x,y)$ by $g(-x,-y)$, if necessary, we may suppose that $\tilde\gamma : x = \tilde a y^\xi, y \ge 0$. Let $\tilde P(z) = g(z,1)$. Then $\tilde P(\tilde a) = 0$. Since σ is C^1, by Proposition 2.12, $P_{f,\gamma,\xi} = P(z-a)$ and $P_{g,\tilde\gamma,\xi} = \tilde P(z - \tilde a)$ coincide up to left and right multiplications. Multiplying x by a positive constant, if necessary, we may suppose that

$$P(z-a) = \tilde P(\alpha(z - \tilde a)). \qquad (5.7)$$

For $p = 1$ this gives $f(x,y) = g(c_1 x - by^\xi, c_2 y)$ (taking into account of the changes we have made already) and ends the proof of (2).

If $p > 1$ then

$$P(z) = z^l Q(z^p), \tilde{P}(z) = z^l \tilde{Q}(z^p).$$

and therefore the arithmetic mean of complex roots of P, and the one of the roots of $\tilde{P}$, equals 0. By (5.7), if z is a complex root of P then $\alpha(z + a - \tilde{a})$ is a root of $\tilde{P}$. Thus by comparing both arithmetic means we get $a = \tilde{a}$. Consequently, $P(z - a) = \tilde{P}(\alpha(z - a))$ or, by replacing $z - a$ by z, $P(z) = \tilde{P}(\alpha z)$, and hence we may conclude finally that

$$f(x, y) = g(c_1 x, c_2 y).$$

This ends the proof of Proposition 5 in this case.

In the remaining cases, when f has no or two roots and $\xi > 1$, we replace roots of f by polar roots, that is by roots of $\partial f / \partial x = 0$. We consider only those polar roots that are not included in $y = 0$. By assumption, $P(z) = f(z, 1)$ has no real root, and therefore P' must have one. If $P'(a) = 0$ then

$$\frac{\partial f}{\partial x}(ay^\xi, y) \equiv 0,$$

and the curve $\gamma_a : x = ay^\xi$, $y \geq 0$, is a polar root of f. The Newton boundary $NB_{\gamma_a} f$ has an edge of slope $-\xi$. The main point is to show that if f and g are bi-Lipschitz equivalent, $f = g \circ \sigma$, then $\sigma(\gamma_a)$ is included in a horn neighbourhood of width bigger than ξ of a polar root of g. Then we may use Proposition 2.10.

Consider the germ at the origin of

$$V^\delta(f) = \{(x, y) \in \mathbb{R}^2; \delta r \| \operatorname{grad} f(x, y) \| \leq |f(x, y)|\}, \tag{5.8}$$

where $r = \|(x, y)\|$ and $\delta > 0$. If δ is sufficiently small then each polar root γ_a is in $V^\delta(f)$. Indeed, if $P(a) \neq 0$ and $P'(a) = 0$ then

$$\| \operatorname{grad} f(ay^\xi, y) \| = \left| \frac{\partial f}{\partial y}(ay^\xi, y) \right|$$

$$= (d/p)|P(a)y^{d/p-1} + \cdots| \sim r^{-1}(d/p)|f(ay^\xi, y)|.$$

Conversely, if a real analytic demi-branch

$$\gamma : x = \lambda(y) = a_\xi y^\xi + \sum_{i > N\xi} a_{i/N} y^{i/N}, \quad y \geq 0,$$

is contained in $V^\delta(f)$, then $P'(a_\xi) = 0$. Indeed, otherwise

$$\| \operatorname{grad} f(\lambda(y), y) \| \sim \left| \frac{\partial f}{\partial x}(\lambda(y), y) \right| \sim |P'(a) y^{d/q-1}| \sim r^{d/q - d/p - 1} |f(a y^\xi, y)|.$$

Moreover, if $P'(a_\xi) = 0$ then γ is contained in a horn neighbourhood $H_\mu(\gamma_{a_\xi}, M)$, with $\mu > \xi$.

By (3.1) of [**11**], if f and g are bi-Lipschitz equivalent then $\sigma(V^\delta(f)) \subset V^{\delta'}(g)$, where δ'' can be given in terms of δ and the Lipschitz constant of L. Consequently, for a polar root γ_a, $\sigma(\gamma_a)$ is included in a horn neighbourhood of width bigger than ξ of a polar root of g, and we may assume that this root is of the form $\gamma_{\tilde{a}} : x = \tilde{a} y^\xi$, $y \geq 0$.

By Proposition 2.10, and multiplying x by a positive constant, if necessary, we may suppose that

$$P'(z - a) = \tilde{P}'(\alpha(z - \tilde{a})).$$

Then we may proceed as in the previous case. $\qquad\square$

6. Bi-Lipschitz equivalence does not imply blow-analytic equivalence.

In this section we present examples of bi-Lipschitz equivalent real analytic function germs that are not blow-analytically equivalent. In order to distinguish different blow-analytic types we use either the real tree model of [**15**] or the Fukui invariants. Recall the definition of Fukui invariants of blow-analytic equivalence [**6**]. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ. Set

$$A(f) := \{\operatorname{ord}(f(\gamma(t))) \in \mathbb{N} \cup \{\infty\}; \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)\ C^\omega\}.$$

Let $\lambda : U \to \mathbb{R}^n$ be an analytic arc with $\lambda(0) = 0$, where U denotes a neighbourhood of $0 \in \mathbb{R}$. We call λ *nonnegative* (resp. *nonpositive*) *for* f if $(f \circ \lambda)(t) \geq 0$ (resp. ≤ 0) in a positive half neighbourhood $[0, \delta) \subset U$. Then we set

$$A_+(f) := \{\operatorname{ord}(f \circ \lambda); \lambda \text{ is a nonnegative arc for } f\},$$

$$A_-(f) := \{\operatorname{ord}(f \circ \lambda); \lambda \text{ is a nonpositive arc for } f\}.$$

Fukui proved that $A(f)$, $A_+(f)$ and $A_-(f)$ are blow-analytic invariants. Namely, if analytic functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$, $A_+(f) = A_+(g)$ and $A_-(f) = A_-(g)$. We call $A(f)$, $A_\pm(f)$ *the Fukui invariant*, *the Fukui invariants with sign*, respectively. Apart from the Fukui invariants, motivic type invariants, *zeta functions*, are also known cf. [**14**], [**4**].

6.1.   Example.

$$f(x,y) = x(x^3 - y^5), \quad g(x,y) = x(x^3 + y^5).$$

By [**15**], f and g are not blow-analytically equivalent by an orientation preserving blow-analytic homeomorphism.
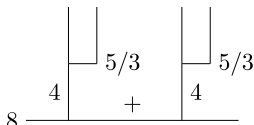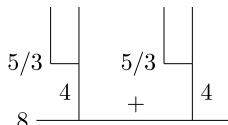
Figure 6. $\mathbb{R}T(f)$.   Figure 7. $\mathbb{R}T(g)$.

We construct below an orientation preserving bi-Lipschitz homeomorphism $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ \sigma$. The construction uses the fact that f and g are weighted homogeneous with weights 5 and 3. Write

$$f(x,y) = x(x^3 - y^5) = y^{20/3} P\left(\frac{x}{y^{5/3}}\right), \quad P(z) = z^4 - z,$$

$$g(x,y) = x(x^3 + y^5) = y^{20/3} Q\left(\frac{x}{y^{5/3}}\right), \quad Q(z) = z^4 + z.$$

Proposition 6.1.   *There exists a unique increasing real analytic diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $P = Q \circ \varphi$. Moreover, for this φ, φ' and $\varphi - z\varphi'$ are globally bounded and $\varphi(z)/z \to 1$ as $z \to \infty$.*

Proof. P and Q have unique critical points: $z_0 = \sqrt[3]{1/4}, P'(z_0) = 0$, $\tilde{z}_0 = -z_0, Q'(\tilde{z}_0) = 0$. Therefore $\varphi : (-\infty, z_0] \to (-\infty, \tilde{z}_0]$, defined as $Q^{-1} \circ P$, is continuous and analytic on $(-\infty, z_0)$. Similarly for $\varphi : [z_0, \infty) \to [\tilde{z}_0, \infty)$. Thus $\varphi : \mathbb{R} \to \mathbb{R}$ is well-defined and continuous. In a neighbourhood of z_0, that is a non-degenerate critical point, P is analytically equivalent to $z^2 + P(z_0)$. Similarly Q near $\tilde{z}_0$ is analytically equivalent to $z^2 + Q(\tilde{z}_0)$. Finally, since $P(z_0) = Q(\tilde{z}_0)$, P near z_0 is analytically equivalent to Q near $\tilde{z}_0$.

Let $w = 1/z$. Consider real analytic function germs

$$p(w) := (P(w^{-1}))^{-1} : (\mathbb{R}, 0) \to (\mathbb{R}, 0), \quad p(w) = w^4 + \cdots,$$

$$q(w) := (Q(w^{-1}))^{-1} : (\mathbb{R}, 0) \to (\mathbb{R}, 0), \quad q(w) = w^4 + \cdots.$$

Then $p = q \circ \psi$ with $\psi(w) = w + \cdots$. Since $\varphi(z) = (\psi(z^{-1}))^{-1}$, the last claim of proposition can be verified easily. $\square$

COROLLARY 6.2. $\quad \sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, *defined by*

$$\sigma(x, y) = \begin{cases} \left(y^{5/3} \varphi\left(\dfrac{x}{y^{5/3}} \right), y \right) & \text{if } y \neq 0, \\ (x, 0) & \text{if } y = 0, \end{cases}$$

is bi-Lipschitz and $f = g \circ \sigma$.

PROOF. We only check that σ is Lipschitz. This follows from the fact that the partial derivatives of σ are bounded

$$\partial\sigma/\partial x = (\varphi'(z), 0), \quad \partial\sigma/\partial y = (5/3 y^{2/3}(\varphi(z) - z\varphi'(z)), 1).$$

where $z = x/y^{5/3}$. $\square$

6.2. Example.

$$f(x, y) = x(x^3 - y^5)(x^3 + y^5), \quad g(x, y) = x(x^3 - ay^5)(x^3 - by^5),$$

where $0 < a < b$ are constants. The real trees of f and g are not equivalent, see below, and hence by [**15**], f and g are not blow-analytically equivalent.
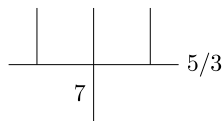

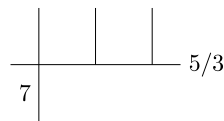Figure 8. $\mathbb{R}T_{(0,1)}(f)$.

Figure 9. $\mathbb{R}T_{(0,1)}(g)$.

Note that the Fukui invariants and the zeta functions of f and g coincide, see cf. Example 1.4 in [**15**]. We show below that for a choice of a and b, f and g are bi-Lipschitz equivalent. Write

$$f(x, y) = y^{35/3} P\left(\dfrac{x}{y^{5/3}} \right), \quad P(z) = z(z^3 - 1)(z^3 + 1),$$

$$g(x,y) = y^{35/3} Q\left(\frac{x}{y^{5/3}}\right), \quad Q(z) = z(z^3 - a)(z^3 - b).$$

The polynomial P has two non-degenerate critical points $-1 < z_1 < 0$, $z_2 = -z_1$ and $P(z_1) > 0$, $P(z_2) = -P(z_1) < 0$. The polynomial Q has also two non-degenerate critical points $0 < \tilde{z}_1 < \sqrt[3]{a} < \tilde{z}_2 < \sqrt[3]{b}$ and $Q(\tilde{z}_1) > 0$, $Q(\tilde{z}_2) < 0$. Indeed, the discriminant of $Q'(z) = 7z^6 - 4(a+b)z^3 + ab$ with respect to z^3 equals $\Delta = 4(4a^2 + 4b^2 + ab) > 0$. This also shows that these critical points $\tilde{z}_1(a,b)$, $\tilde{z}_2(a,b)$ depend smoothly on a, b.

LEMMA 6.3.   *There exist a, b, $0 < a < b$, such that $Q(\tilde{z}_1(a,b)) = P(z_1)$, $Q(\tilde{z}_2(a,b)) = P(z_2)$.*

PROOF.   Fix $b > 0$. If $a \to 0$ then $Q(\tilde{z}_1) \to 0$ and $Q(\tilde{z}_2) \to const < 0$. If $a \to b$ then $Q(\tilde{z}_1) \to const > 0$ and $Q(\tilde{z}_2) \to 0$. Therefore there is an $a(b)$ such that $Q(\tilde{z}_1(a(b), b)) = -Q(\tilde{z}_2(a(b), b))$.

Write $Q_{a,b}$ instead of Q to emphasise that Q depends on a and b. If $\alpha > 0$ then $Q_{a,b}(\alpha z) = \alpha^7 Q_{a/\alpha^3, b/\alpha^3}(z)$. Thus, there is $\alpha > 0$ such that the critical values of $Q_{a(b)/\alpha^3, b/\alpha^3}$ are precisely $P(z_1), P(z_2)$. This shows the lemma.   □

Then, for a and b satisfying Lemma 6.3, the construction of bi-Lipschitz homeomorphism σ such that $f = g \circ \sigma$ is similar to that of Example 6.1.

6.3.   Example.

$$\begin{aligned} f(x,y) &= x(x^3 - y^5)((x^3 - y^5)^3 - y^{17}) \\ g(x,y) &= x(x^3 + ay^5)(x^3 - y^7)(x^6 + by^{10}), \end{aligned} \qquad (6.1)$$

where $a > 0, b > 0$ are real constants. As we show below, for a choice of a and b, f and g are bi-Lipschitz equivalent. They have different real tree models, see below, so they are not blow-analytically equivalent. Moreover, in contrast to the previous two examples, f and g have different Fukui invariants.
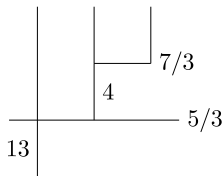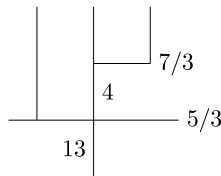

Figure 10. $\mathbb{R}T_{(0,1)}(f)$.

Figure 11. $\mathbb{R}T_{(0,1)}(g)$.

PROPOSITION 6.4. *Let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be polynomial functions defined by* (6.1). *Then*

$$A(f) = \{13, 22, 23, 24, \dots\} \cup \{\infty\}, \quad A(g) = \{13, 23, 25, 26, \dots\} \cup \{\infty\}.$$

Thus f and g are not blow-analytically equivalent.

PROOF. Let us express an analytic arc at $(0, 0) \in \mathbb{R}^2$, $\lambda(t) = (\lambda_1(t), \lambda_2(t))$, as follows:

$$\lambda_1(t) = c_1 t + c_2 t^2 + \cdots, \quad \lambda_2(t) = d_1 t + d_2 t^2 + \cdots.$$

To compute $A(f)$, we consider $f(\lambda(t))$;

$$f(\lambda(t)) = (c_1 t + c_2 t^2 + \cdots)(c_1^3 t^3 + \cdots - d_1^5 t^5 - \cdots)$$
$$\times \big((c_1^3 t^3 + \cdots - d_1^5 t^5 - \cdots)^3 - d_1^{17} t^{17} - \cdots\big).$$

In case $c_1 \neq 0$, we have $\mathrm{ord}(f \circ \lambda) = 13$. In case $c_1 = 0$, we have $\mathrm{ord}(f \circ \lambda) \geq 22$. For any $s = 2, 3, \dots, 20 + s$ is attained by the arc $\lambda(t) = (t^s, t)$. Therefore we have

$$A(f) = \{13, 22, 23, 24, \dots\} \cup \{\infty\}.$$

We next compute $A(g)$. Then

$$g(\lambda(t)) = (c_1 t + c_2 t^2 + \cdots)(c_1^3 t^3 + \cdots + a d_1^5 t^5 + \cdots)$$
$$\times (c_1 t^3 + \cdots - d_1^7 t^7 - \cdots)(c_1^6 t^6 + \cdots + b d_1^{10} t^{10} + \cdots).$$

In case $c_1 \neq 0$, we have $\mathrm{ord}(f \circ \lambda) = 13$. In case $c_1 = 0$, $c_2 \neq 0$ and $d_1 \neq 0$, we have $\mathrm{ord}(f \circ \lambda) = 23$. In case $c_1 = c_2 = 0$ or $c_1 = d_1 = 0$, we have $\mathrm{ord}(g \circ \lambda) \geq 25$. For any $s = 3, 4, \dots, 22 + s$ is attained by the arc $\lambda(t) = (t^s, t)$. Therefore we have

$$A(g) = \{13, 23, 25, 26, \dots\} \cup \{\infty\}. \qquad \square$$

Next we compute the polar roots of f and g. The polynomials f and g are not weighted homogeneous and therefore it is useful to apply the method of Newton polygon as in [**22**], [**10**].

The one variable polynomial associated to the leading weighted homogeneous part of f with respect to the weights 5 and 3 equals $P_1(z) = z(z^3 - 1)^4$. Besides a multiple root $z = 1$, P_1 has a unique non-denegenerate critical point a_1, $0 < a_1 < 1$,

which gives rise to a polar curve

$$\gamma_1 : x = \lambda_1(y) = a_1 y^{5/3} + \cdots, \quad f(\lambda_1(y), y) = A_1 y^{21(2/3)} + O(y^{23(2/3)})$$

where $A_1 = P_1(a_1)$. The Newton polygon of f relative to $\gamma : x = y^{5/3}$ has two edges: one of slope $-5/3$ and one of slope $-7/3$. The one variable polynomial associated to the latter is $P_2(z) := P_{f,\gamma,7/3}(z) = 3^4 z^4 - 3z$. The unique non-degenerate critical point a_2 of P_2 gives rise to a polar curve

$$\gamma_2 : x = \lambda_2(y) = y^{5/3} + a_2 y^{7/3} + \cdots, \quad f(\lambda_2(y), y) = A_2 y^{24(1/3)} + O(y^{25})$$

where $A_2 = P_2(a_2)$. There are no more real polar roots of f.

The one variable polynomial associated to the leading weighted homogeneous part of g equals $Q_1(z) = z^4(z^3 + a)(z^6 + b)$. If $10^2 a^2 - (7 \cdot 39)b < 0$ then $Q_1'(z) = 13z^{12} + 10az^9 + 7bz^6 + 4abz^3$ has a single simple non-zero real root. Indeed, let $S(t) = 13t^3 + 10at^2 + 7bt + 4ab$. Then $S'(t) = 39t^2 + 20at + 7b$ and the discriminant of $S'(t)$ is $\Delta = 4(10^2 a^2 - (7 \cdot 39)b)$. Therefore, if we suppose that

$$a > 0, \ b > 0, \ 10^2 a^2 - (7 \cdot 39)b < 0, \tag{6.2}$$

then $S(t)$ has a single simple root, that shows our claim on Q_1'. Let $\tilde{a}_1$ denote this critical point of Q_1, $\tilde{a}_1 < 0$. Then there exists a polar curve of g

$$\tilde{\gamma}_1 : x = \tilde{\lambda}_1(y) = \tilde{a}_1 y^{5/3} + \cdots, \quad g(\tilde{\lambda}_1(y), y) = \tilde{A}_1 y^{21(2/3)} + O(y^{23(2/3)}).$$

The one variable polynomial associated to the face of the Newton polygon of g of slope $-7/3$ is $Q_2(z) = z^4 - z$. It has a single non-degenerate critical point $\tilde{a}_2$ that gives a polar curve

$$\tilde{\gamma}_2 : x = \tilde{\lambda}_2(y) = \tilde{a}_2 y^{7/3} + \cdots, \quad g(\tilde{\lambda}_2(y), y) = \tilde{A}_2 y^{24(1/3)} + O(y^{26(1/3)})$$

where $\tilde{A}_2 = Q_2(\tilde{a}_2)$. One checks easily that $\tilde{A}_2 = A_2$. There are no more real polar roots of g.

LEMMA 6.5. *There are constants a, b satisfying (6.2) for which $\tilde{A}_1 = A_1$.*

PROOF. Denote by $\tilde{a}_1(a, b)$ the unique non-zero critical point of Q_1 thus emphasising that it depends on a, b. Note that $\tilde{a}_1(a, b)$ is between the two roots of Q_1, $-\sqrt[3]{a} < \tilde{a}_1(a, b) < 0$. For b fixed $Q_1(\tilde{a}_1(a, b)) \to 0$ as $a \to 0$. Fix a and let $b \to \infty$. Then $Q_1(-(1/2)\sqrt[3]{a}) \to \infty$ and hence $Q_1(\tilde{a}_1(a, b)) \to \infty$. Thus there

exist a, b for which $Q_1(\tilde{a}_1(a, b)) = A_1$. $\qquad\square$

Next for f, and then for g, we introduce a new system of local coordinates $(\tilde{x}, \tilde{y}) = H(x, y)$ in which f has particularly simple form near the polar curves. We shall use partitions of unity that we always assume to be either C^∞ or semi-algebraic of class C^k, $2 \leq k < \infty$. Fix such a partition of unity $\varphi_0, \varphi_1, \varphi_2$ on $\mathbb{R}$ such that

(i) supp φ_1 is a small neighbourhood of a_1 and $\varphi_1 \equiv 1$ in a neighbourhood of a_1.

(ii) supp φ_2 is a small neighbourhood of 1 and $\varphi_2 \equiv 1$ in a neighbourhood of 1.

Then $\varphi_0 = 1 - \varphi_1 - \varphi_2$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a (C^∞ or semialgebraic and C^k, $k \geq 2$) diffeomorphism such that

(i) $\psi(a_1) = 0$ and $\psi(z) = z - a_1$ for z near a_1.

(ii) $\psi(1) = 1$ and $\psi(z) = z$ for z near 1.

(iii) $\psi(z) = z$ for $|z|$ large.

Finally, for each polar curve γ_i, $i = 1, 2$, separately, we reparametrise λ_i by replacing y by an invertible fractional power series $\tilde{y}_i(y)$ so that

$$f(\lambda_1(y(\tilde{y}_1)), y(\tilde{y}_1)) = A_1 \tilde{y}_1^{21(2/3)}, \quad \tilde{y}_1 = y + O(y^3), \tag{6.3}$$

$$f(\lambda_2(y(\tilde{y}_2)), y(\tilde{y}_2)) = A_2 \tilde{y}_2^{24(1/3)}, \quad \tilde{y}_2 = y + O(y^{5/3}). \tag{6.4}$$

Denote $\xi = 5/3$ and $z = x/y^\xi$ for short. We set

$$H(x, y) = \left(y^\xi \psi(z) - \sum_{i=0}^{2} \delta_i(y)\varphi_i(z), \ \sum_{i=0}^{2} \tilde{y}_i(y)\varphi_i(z) \right) \quad \text{if } y \neq 0,$$

$$H(x, 0) = (x, 0),$$

where $\tilde{y}_0(y) = y$, $\delta_0 \equiv 0$, and

$$\delta_1(y) = y^\xi \psi(\lambda_1(y)/y^\xi), \quad \delta_2(y) = y^\xi \psi(\lambda_2(y)/y^\xi) - \tilde{y}_2^\xi.$$

Then $\delta_1(y)$ and $\delta_2(y)$ are fractional power series in y and $\delta_i(y) = o(y^\xi)$, $i = 1, 2$. The image of polar curves γ_1, γ_2 by H is given by

$$H(\lambda_1(y), y) = (0, \tilde{y}_1(y)), \quad H(\lambda_2(y), y) = (\tilde{y}_2^\xi, \tilde{y}_2(y)).$$

Denote $\tilde{f}(\tilde{x}, \tilde{y}) = f \circ H^{-1}(\tilde{x}, \tilde{y})$.

Lemma 6.6.

(1) H is a bi-Lipschitz local homeomorphism.
(2) $D^2 H = O(y^{-\xi})$, $D^2 H^{-1} = O(\tilde{y}^{-\xi})$.
(3) In horn neighbourhoods of γ_1 and γ_2 with exponent ξ, H is given by respectively by

$$H(x, y) = (x - \lambda_1(y), \tilde{y}_1(y)), \quad H(x, y) = (x - \lambda_2(y) + \tilde{y}_2^{\xi}(y), \tilde{y}_2(y)).$$

(4) For C large and $|x| \geq C|y|^{\xi}$, $H(x, y) = (x, y)$.
(5) $\{\partial \tilde{f}/\partial \tilde{x} = 0\} = H(\{\partial f/\partial x = 0\}) = H(\gamma_1) \cup H(\gamma_2)$ and $H(\gamma_1) = \{\tilde{x} = 0\}$, $H(\gamma_2) = \{\tilde{x} = \tilde{y}^{\xi}\}$.

Proof. (3) and (4) are given by construction.

We show that the partial derivatives of H are bounded. By (4) we may assume z bounded. We use that $\sum_{i=0}^{2} \tilde{y}_i(y)\varphi_i(z) = y + \sum_{i=0}^{2}(\tilde{y}_i(y) - y)\varphi_i(z)$.

$$\partial H/\partial x = \left(\psi' - \sum_i y^{-\xi}\delta_i\varphi_i', \sum_i y^{-\xi}(\tilde{y}_i - y)\varphi_i' \right) = (\psi' + o(1), o(1)),$$

where $o(1) \to 0$ as $y \to 0$.

$$\partial H_1/\partial y = \xi y^{\xi-1}\left(\psi - z\psi' + \sum_i zy^{-\xi}\delta_i\varphi_i' \right) - \sum_i \delta_i'\varphi_i = O(y^{\xi-1}),$$

$$\partial H_2/\partial y = 1 + \sum_i (\tilde{y}_i - y)'\varphi_i - \xi zy^{-1}\sum_i (\tilde{y}_i - y)\varphi_i' = 1 + o(1).$$

Thus H is Lipschitz, $H^{-1}(0) = 0$, and H is a covering over the complement of the origin. Moreover $H^{-1}(0, x) = (0, x)$. Therefore H is a local homeomorphism. The formulae for the partial derivatives also show that the the inverse of Jacobian matrix of H has bounded entries. Thus H^{-1} is also Lipschitz. A similar computation gives (2).

By (3) and (4), (5) is obvious in horn neighbourhoods of γ_1 and γ_2 and for $|z| \geq C$. It suffices to show that in the complement of these sets $\partial \tilde{f}/\partial \tilde{x}$ does not vanish. Firstly, on this set, $\partial f/\partial x \sim y^{20}$ and $\partial f/\partial y = O(y^{20(2/3)})$. Moreover, there is a constant $c > 0$ such that $c \leq \partial x/\partial \tilde{x} \leq c^{-1}$. Therefore

$$\frac{\partial \tilde{f}}{\partial \tilde{x}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \tilde{x}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \tilde{x}} \sim y^{20} \sim \tilde{y}^{20}. \qquad \square$$

We apply the same procedure to $g(x, y)$ and obtain a bi-Lipschitz homeomorphism $\tilde{H}$ so that $\tilde{H}$ and $\tilde{g}(\tilde{x}, \tilde{y})$ satisfy the statement of Lemma 6.6. In what follows we shall drop the "tilda" notation for variables and consider $\tilde{f}$ and $\tilde{g}$ as functions of (x, y). We show that the homotopy

$$F(x, y, t) = t\tilde{g}(x, y) + (1 - t)\tilde{f}(x, y)$$

is bi-Lipschitz trivial and can be trivialised by the vector field

$$v(x, y, t) = \frac{\partial}{\partial t} - \frac{\partial F/\partial t}{\partial F/\partial x} \frac{\partial}{\partial x}, \quad v(x, y, t) = \frac{\partial}{\partial t} \text{ if } \partial F/\partial x = 0. \qquad (6.5)$$

Thus to complete the proof of bi-Lipschitz equivalence of f and g it suffices to show:

LEMMA 6.7. *The vector field $v(x, y, t)$ of* (6.5) *is Lipschitz.*

PROOF. The polar curves of $\tilde{f}$ and $\tilde{g}$ coincide:

$$\{\partial \tilde{f}/\partial \tilde{x} = 0\} = \{\partial \tilde{g}/\partial x = 0\} = \{x = 0\} \cup \{x = y^{\xi}\}. \qquad (6.6)$$

As we shall show also $\{\partial \tilde{F}/\partial \tilde{x} = 0\} = \{x = 0\} \cup \{x = y^{\xi}\}$.

We proceed separately in each of the horn neighbourhood with exponent ξ of the polar curves (6.6), for $|x| \geq C|y|^{\xi}$, C large, and in the complement of these three sets.

Suppose $|x| \leq \varepsilon |y|^{\xi}$, $\varepsilon > 0$ and small. In the variables $z = x/y^{\xi}, y$

$$F(z, y, t) = [tP_1(z - a_1) + (1 - t)Q_1(z)]y^{21(2/3)} + R_0(z, y) + R_1(z, y)t,$$

where $R_0, R_1 \in \mathbb{R}\{z, y^{1/q}\}$, for an odd positive integer q, with exponents bigger than $21(2/3)$ in y. By (6.3),

$$\partial F/\partial x = zy^{20}u(z, y, t), \quad \frac{\partial F}{\partial t} = \tilde{g} - \tilde{f} = z^2 y^{21(2/3)}\eta(z, y),$$

and $u(0, 0, t) \neq 0$ for $t \in [0, 1]$. Hence

$$\frac{\partial F/\partial t}{\partial F/\partial x} = zy^{\xi}h(z,y,t) = xh(z,y,t).$$

Thus $(\partial F/\partial t)/(\partial F/\partial x)$ is Lipschitz because the partial derivatives of $xh(z,y,t)$ are bounded:

$$\frac{\partial}{\partial x}(xh) = h + \frac{x}{y^{\xi}}\frac{\partial h}{\partial z},$$

$$\frac{\partial}{\partial y}(xh) = x\frac{\partial h}{\partial z}\frac{\partial z}{\partial y} + x\frac{\partial h}{\partial y} = -\xi\frac{x}{y^{\xi}}\frac{x}{y}\frac{\partial h}{\partial z} + x\frac{\partial h}{\partial y},$$

$$\frac{\partial}{\partial t}(xh) = x\frac{\partial h}{\partial t}.$$

A similar argument works for a horn neighbourhood of $x = y^{\xi}$.

Suppose now that x/y^{ξ} is bounded and that we are not in horn neighbourhoods of the polar curves. By Lemma 6.6 one can verify easily that on this set

$$\tilde{g} - \tilde{f} = O(y^{20+\xi}), \quad D(\tilde{g} - \tilde{f}) = O(y^{20}),$$

$$\partial F/\partial x \sim y^{20}, \quad D(\partial F/\partial x) = O(y^{20-\xi}).$$

Now a direct computation shows that the partial derivatives of $(\partial F/\partial t)/(\partial F/\partial x)$ are bounded.

If $|x| \geq C|y|^{\xi}$, C large, then by (4) of Lemma 6.6, $\tilde{f} = f$ and $\tilde{g} = g$. Then in variables $x, w = y^{\xi}/x$

$$F(x,w,t) = x^m(1 + u(x,w,t)),$$

where $m = 13$ and $u(0,0,t) = 0$ for $t \in [0,1]$. Then

$$\partial F/\partial x = x^{m-1}(m + v(x,w,t)), \quad \frac{\partial F}{\partial t} = x^m\eta(x,w),$$

where $v(0,0,t) = 0$ for $t \in [0.1]$. Hence

$$\frac{\partial F/\partial t}{\partial F/\partial x} = xh(x,w,t).$$

Then, an elementary computation shows that the partial derivatives of $xh(x,w,t)$ are bounded. $\qquad\square$

References

[1] V. I. Arnol'd, A. N. Varchenko and S. M. Guseĭn-Zade, Singularities of differentiable mappings, **1**, Birkhäuser, Moscow, 1982.

[2] M. Artin, On the solutions of analytic equations, Invent. Math., **5** (1968), 277–291.

[3] A. C. G. Fernandes and M. A. S. Ruas, Bi-Lipschitz determinacy of quasihomogeneous germs, Glasg. Math. J., **46** (2004), 77–82.

[4] G. Fichou, Motivic invariants of arc-symmetric sets and blow-Nash equivalence, Compos. Math., **141** (2005), 655–688.

[5] T. Fukui and E. Yoshinaga, The modified analytic trivialization of family of real analytic functions, Invent. Math., **82** (1985), 467–477.

[6] T. Fukui, Seeking invariants for blow-analytic equivalence, Compositio Math., **105** (1997), 95–108.

[7] T. Fukui, S. Koike and T.-C. Kuo, Blow-analytic equisingularities, properties, problems and progress, In: Real Analytic and Algebraic Singularities, Nagoya, Sapporo, Hachioji, 1996, (eds. T. Fukuda, T. Fukui, S. Izumiya and S. Koike), Pitman Res. Notes Math. Ser., **381**, Longman, Harlow, 1998, pp. 8–29.

[8] T. Fukui and L. Paunescu, Modified analytic trivialization for weighted homogeneous function-germs, J. Math. Soc. Japan, **52** (2000), 433–446.

[9] T. Fukui and L. Paunescu, On blow-analytic equivalence, In: Arc Spaces and Additive Invariants in Real Algebraic Geometry, Proceedings of Winter School "Real algebraic and Analytic Geometry and Motivic Integration", Aussois 2003, Panor. Synthèses, **24**, 2008, Soc. Math. France, 2007, pp. 87–125.

[10] J.-P. Henry and A. Parusiński, Existence of moduli for bi-Lipschitz equivalence of analytic functions, Compositio Math., **136** (2003), 217–235.

[11] J.-P. Henry and A. Parusiński, Invariants of bi-Lipschitz equivalence of real analytic functions, Banach Center Publ., **65** (2004), 67–75.

[12] Henry C. King, Real analytic germs and their varieties at isolated singularities, Invent. Math., **37** (1976), 193–199.

[13] S. Koike, On strong C^0-equivalence of real analytic functions, J. Math. Soc. Japan, **45** (1993), 313–320.

[14] S. Koike and A. Parusiński, Motivic-type invariants of blow-analytic equivalence, Ann. Inst. Fourier (Grenoble), **53** (2003), 2061–2104.

[15] S. Koike and A. Parusiński, Blow-analytic equivalence of two variable real analytic function germs, J. Algebraic Geom., **19** (2010), 439–472.

[16] N. H. Kuiper, C^1-equivalence of functions near isolated critical points, In: Symposium on Infinite-Dimensional Topology, Baton Rouge, 1967, (ed. R. D. Anderson), Ann. of Math. Stud., **69**, Princeton Univ. Press, 1972, pp. 199–218.

[17] T.-C. Kuo, On C^0-sufficiency of jets of potential functions, Topology, **8** (1969), 167–171.

[18] T.-C. Kuo and Y. C. Lu, On analytic function germs of two complex variables, Topology, **16** (1977), 299–310.

[19] T.-C. Kuo, The modified analytic trivialization of singularities, J. Math. Soc. Japan, **32** (1980), 605–614.

[20] T.-C. Kuo, Blow-analytic equisingularities, Invited Address at Annual Convention of the Mathematical Society of Japan, Autumn 1984, Tokyo University.

[21] T.-C. Kuo, On classification of real singularities, Invent. Math., **82** (1985), 257–262.

[22] T.-C. Kuo and A. Parusiński, Newton polygon relative to an arc, In: Real and Complex Singularities (São Carlos, 1998), Chapman & Hall/CRC Res. Notes Math, **412**, Chapman & Hall/CRC, Boca, Raton, FL, 2000, pp. 76–93.

[23] K. Kurdyka and L. Paunescu, Arc-analytic roots of analytic functions are Lipschitz, Proc. Amer. Math. Soc., **132** (2004), 1693–1702.

[24] A. Parusiński, A criterion for topological equivalence of two variable complex analytic function germs, Proc. Japan Acad. Ser. A Math. Sci., **84** (2008), 147–150.

[25] J.-J. Risler and D. Trotman, Bi-Lipschitz invariance of the multiplicity, Bull. London Math. Soc., **29** (1997), 200–204.

[26] F. Takens, A note on sufficiency of jets, Invent. Math., **13** (1971), 225–231.

[27] R. J. Walker, Algebraic Curves, Springer-Verlag, New York, Heidelberg, Berlin, 1978.

[28] C. T. C. Wall, Singular Points of Plane Curves, London Math. Soc. Stud. Texts, **63**, Cambridge University Press, 2004.

[29] O. Zariski, On the topology of algebroid singularities, Amer. J. Math., **54** (1932), 453–465.

Satoshi Koike

Department of Mathematics
Hyogo University of Teacher Education
942-1 Shimokume, Kato
Hyogo 673-1494, Japan
E-mail: koike@hyogo-u.ac.jp

Adam Parusiński

Laboratoire J. A. Dieudonné U.M.R. C.N.R.S. N 6621
Université de Nice Sophia-Antipolis
Parc Valrose 06108 Nice Cedex 02, France
E-mail: adam.parusinski@unice.fr