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Optimal decay rate of the energy for wave equations with critical potential

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Abstract. We study the long time behavior of solutions of the wave equation with a variable damping term $V(x)u_t$ in the case of critical decay $V(x) \geq V_0(1+|x|^2)^{-1/2}$ (see condition (A) below). The solutions manifest a new threshold effect with respect to the size of the coefficient V_0: for $1 < V_0 < N$ the energy decay rate is exactly t^{-V_0}, while for $V_0 \geq N$ the energy decay rate coincides with the decay rate of the corresponding parabolic problem.

1. Introduction.

We consider the Cauchy problem for the linear wave equation with a critical potential $V(x)$ in $\boldsymbol{R}^N$ $(N \geq 1)$:

$$u_{tt}(t,x) - \Delta u(t,x) + V(x)u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^N, \tag{1.2}$$

where (u_0, u_1) are compactly supported initial data in the energy space:

$$u_0 \in H^1(\boldsymbol{R}^N), \quad u_1 \in L^2(\boldsymbol{R}^N), \quad \operatorname{supp} u_i \subset B(R_0) := \{x \in \boldsymbol{R}^N : |x| < R_0\}, \quad (i = 0, 1),$$

and the potential $V \in L^\infty(\boldsymbol{R}^N) \cap C(\boldsymbol{R}^N)$ satisfies

$$\text{(A)} \qquad V_0(1+|x|^2)^{-1/2} \leq V(x) \leq V_1(1+|x|^2)^{-1/2} \ \text{ for } \ V_0,\ V_1 > 0.$$

Denote $X_1(0,T) := C([0,T); H^1(\boldsymbol{R}^N)) \cap C^1([0,T); L^2(\boldsymbol{R}^N))$ for $T \in (0,\infty]$.

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It is well-known that, under the above assumptions on the initial data, problem (1.1)–(1.2) has a unique solution $u \in X_1(0, +\infty)$ satisfying

$$E_u(t) + \int_0^t \int_{\boldsymbol{R}^N} V(x) u_t^2(s,x) dx ds = E_u(0), \quad t \geq 0, \tag{1.3}$$

where

$$E_u(t) := \frac{1}{2} \int_{\boldsymbol{R}^N} (u_t^2 + |\nabla u|^2) dx$$

is the total energy of u. Hence $E_u(t)$ is a non-increasing function of t. An important question is whether the energy decays as $t \to \infty$, and if so, what is the decay rate? The main objective of this paper is to find the exact decay rate of the energy $E_u(t)$ as $t \to \infty$.

1.1. Prior results.

In the case of constant potential $V(x) = V_0 > 0$, Matsumura [**9**] established the estimate $E_u(t) = O(t^{-n/2-1})$ as $t \to \infty$ by using Fourier analysis. The generalization of this estimate to the case of variable potential $V(x)$ was far from straightforward and, correspondingly, the initial decay results were dimension independent. Matsumura [**10**], Mochizuki-Nakazawa [**14**] and Uesaka [**28**] discussed the energy decay rate of problem (1.1)–(1.2) in the case $V(x) = V(t,x) \geq V_0(1 + t + |x|)^{-1}$. Their results tell us that $E_u(t) = O(t^{-1})$ as $t \to +\infty$. Mochizuki [**13**] observed the hyperbolic structure of equation (1.1) by showing non-decay results for $E_u(t)$ in the case $V(t,x) \leq V_0(1 + |x|)^{-1-\alpha}$ with $\alpha > 0$, i.e., the case of supercritical potential. Rauch-Taylor [**22**] showed non-decay results for potentials $V(x)$ with compact support.

Recently, Todorova-Yordanov [**26**] treated the x-dependent potential $V(t,x) \equiv V_0(1+|x|)^{-\alpha}$ with $\alpha \in [0,1)$, i.e., the case of subcritical potential. They derived almost optimal decay rates of the total energy $E_u(t)$. Their results also showed the diffusive structure of equation (1.1) in the subcritical case. In [**26**], the condition $\alpha < 1$ is essential (see also Ikehata [**5**]) and the results cannot be applied directly to the critical case $\alpha = 1$. So far the critical case $\alpha = 1$ has been far from well-understood.

The nonlinear version of equation (1.1) with the critical potential $V(t,x) \equiv V_0(1 + |x|)^{-1}$ $(\alpha = 1)$ is considered in Ikehata-Inoue [**6**]. Let us mention also the very recent work of Ikehata-Todorova-Yordanov [**7**] where the critical exponent problem is studied for the semilinear equation

$$u_{tt}(t,x) - \Delta u(t,x) + \frac{V_0}{(1+|x|)^\alpha} u_t(t,x) = |u(t,x)|^p, \quad \alpha \in [0,1). \tag{1.4}$$

These authors found that the critical exponent is $p_c = 1+2/(N-\alpha)$. Namely, (1.4) will admit global solutions with small initial data for $p > p_c$, while all solutions of (1.4) with positive in average initial data will blow-up in finite time for $1 < p \leq p_c$.

The case of t-dependent potentials $V(t,x) \equiv V_0(1+t)^{-1}$ is studied extensively by Wirth [**29**], [**30**], [**31**] (see also [**10**], [**28**] and Reissig [**23**]) using the Fourier transform method. These results show the following optimal decay for t-dependent potentials:

$$E_u(t) = O(t^{-\min\{V_0,2\}}), \quad t \to +\infty.$$

Unfortunately, the Fourier transform is not very effective in the case of x-dependent potentials.

1.2. Main results.

The purpose of this paper is to determine the exact decay rate of $E_u(t)$ associated to the Cauchy problem (1.1)–(1.2) in the case of potentials $V(x)$ with critical decay, i.e., condition (A). This case is extremely delicate. The solution manifests a new threshold effect with respect to the size of coefficient V_0 in condition (A). Here is our main result.

THEOREM 1.1. *Let $N \geq 3$ and $V(x)$ satisfy condition* (A). *For the solution u of the Cauchy problem* (1.1)–(1.2), *we have the following*:

(i) *if $1 < V_0 < N$, then*

$$E_u(t) = O(t^{-V_0}), \quad t \to +\infty, \tag{1.5}$$

(ii) *if $N \leq V_0$, then*

$$E_u(t) = O(t^{-N+\delta}), \quad t \to +\infty, \tag{1.6}$$

for any $\delta > 0$.

REMARK 1.2. The decay estimates (1.6) for large $V_0 \geq N$ continuously agree with those in [**26**] for $\alpha \to 1^-$. These results also apply to the exterior mixed problem for equation (1.1) with potentials $V(x) \equiv V_0|x|^{-1}$, where the exterior domain $\Omega \subset \boldsymbol{R}^N$ satisfies $0 \notin \bar{\Omega}$.

We obtain similar results in low dimensions $N = 1, 2$. Next theorem presents the estimates in the two-dimensional case. We will discuss the one-dimensional case later.

THEOREM 1.3. *Let $N = 2$ and $V(x)$ satisfy condition* (A). *For the unique solution $u \in X_1(0, +\infty)$ to* (1.1)–(1.2), *one has the following*:

(i) *in the case when $1 < V_0 \leq 2$,*

$$E_u(t) = O(t^{-V_0+\delta}), \quad t \to +\infty, \tag{1.7}$$

for any $\delta > 0$, and

(ii) *in the case when $2 < V_0$,*

$$E_u(t) = O(t^{-2}), \quad t \to +\infty. \tag{1.8}$$

A recent paper of Nishihara [**20**] shows that $\phi(t, x) = A(1 + t)^{-(N-1)} \times e^{-V_0(|x|/(1+t))}$ is the exact solution to the heat equation (see Remark 1.2)

$$\frac{V_0}{|x|}\phi_t - \Delta\phi = 0, \quad x \in \Omega, \quad t > 0.$$

One can easily check that

$$\|\phi_t(t, \cdot)\|^2 = \|\nabla\phi(t, \cdot)\|^2 = O(t^{-N}), \quad t \to +\infty.$$

If we compare this result with the decay results in Theorem 1.1 for large $V_0 \geq N$, we can say that equation (1.1) has a diffusive structure as $t \to +\infty$. When the coefficient V_0 in condition (A) is less than the threshold N, namely $1 < V_0 < N$, the diffusive structure of equation (1.1) is destroyed and the exact energy decay is t^{-V_0} (see (1.5) together with Theorem 1.6).

The proof of Theorem 1.1 relies on three different ideas. First we observe that equation (1.1) is approximately invariant under scaling $(t, x) \mapsto (\lambda t, \lambda x)$ at large $|x|$. This points towards solutions with power-type asymptotic behavior and suggests L^2 estimates with powers of $|x|$ as weights along the lines of Todorova-Yordanov [**26**]. The diffusive structure of equation (1.1) with large $V_0 \gg 1$ is the second key observation used as follows: for large V_0, the estimate $E_u(t) \leq C(1 + t)^{-d}$ implies $E_{u_t}(t) \leq C(1 + t)^{-d-2}$. Namely, the gain in the decay rate from the first-order energy $E_u(t)$ to the second-order energy $E_{u_t}(t)$ is t^{-2}. Such a gain is much stronger than the one found by Nakao [**16**]. The relationship between energies of different orders is studied in Radu-Todorova-Yordanov [**21**] for equation

(1.1) with subcritical potentials $V(x) \sim V_0(1+|x|)^{-\alpha}$, $\alpha < 1$. We note that the condition $V_0 \gg 1$ is not needed in order to reach a t^{-2} gain in the decay rate when the potential is subcritical. Here we are able to show the same gain in the decay rates from $E_u(t)$ to $E_{u_t}(t)$ when the potential $V(x)$ is critical. A similar phenomenon is observed for the heat equation

$$\partial_t v - \Delta v = 0, \quad v(0,x) = f(x).$$

In fact, the following $L^q - L^2$ decay estimates are well known:

$$\begin{aligned}\left\|\partial_t^k \nabla v(t,\cdot)\right\|^2 &\le C_k t^{-N(1/q-1/2)-2k-1}\|f\|_{L^q}^2,\\ \left\|\partial_t^{k+1} \nabla v(t,\cdot)\right\|^2 &\le C_{k+1} t^{-N(1/q-1/2)-2(k+1)-1}\|f\|_{L^q}^2,\end{aligned}$$

for all $t \gg 1$ and $k \ge 0$. Thus, each t-derivative increases the energy decay by t^{-2}. The third important idea comes from Morawetz [**15**] (see also its modified version in Ikehata [**4**] for $N = 2$). In the case $N \ge 3$ we introduce the auxiliary function

$$\chi(t,x) = \int_0^t u(s,x)ds + h(x),$$

where $u(t,x)$ is the solution of problem (1.1)–(1.2). The correction $h(x)$ is the unique solution of the Poisson equation

$$\Delta h = V(x)u_0 + u_1,$$

which decays fast at infinity (see (2.1) below). Further we derive (see Proposition 2.2 below) that the energy of this auxiliary function χ has the following decay rate

$$E_\chi(t) = O(t^{-(N-2)}), \quad t \to +\infty,$$

for large V_0. Since $E_u(t) = E_{\chi_t}(t)$, the energy of u is actually the second energy of χ. Using the gain t^{-2} in the decay rate from the first energy $E_\chi(t)$ to the second energy $E_{\chi_t}(t) = E_u(t)$, we finally derive that $E_u(t) = O(t^{-N})$ for $V_0 \ge N$.

If we impose convenient restrictions on the initial data we can get faster decay rates for the energy of (1.1)–(1.2). We state these results only in the one-dimensional case.

Theorem 1.4. *Let $N = 1$ and assume condition* (A) *on $V(x)$. If the initial data $[u_0, u_1] \in H^1(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ further satisfy*

$$\text{(H1)} \qquad \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))dx = 0,$$

then the unique solution $u \in X_1(0, +\infty)$ to (1.1)–(1.2) *satisfies*

(i) *in the case $2 < V_0$,*

$$E_u(t) = O(t^{-2}), \quad (t \to +\infty), \tag{1.9}$$

and

(ii) *in the case $2 \geq V_0 > 1$,*

$$E_u(t) = O(t^{-V_0+\delta}), \quad t \to +\infty, \tag{1.10}$$

for any small $\delta > 0$.

The decay rate in the case $1 \geq V_0 > 0$ is already known from Matsumura [**10**], Mochizuki-Nakazawa [**14**], Uesaka [**28**].

Proposition 1.5. *Let $N \geq 1$ and $V(x)$ satisfy condition* (A)*. For the solution $u \in X_1(0, +\infty)$ to* (1.1)–(1.2) *we have*

(i) *in the case $1 < V_0$,*

$$E_u(t) = O(t^{-1}), \quad t \to +\infty, \tag{1.11}$$

and

(ii) *in the case $0 < V_0 \leq 1$,*

$$E_u(t) = O(t^{-V_0+\eta}), \quad t \to +\infty, \tag{1.12}$$

for any small $\eta > 0$.

Finally, we can address the question about the exactness of decay rates in Theorem 1.1. When $0 < V_0 < N$, these rates are actually exact, since the next theorem shows that the energy cannot decay faster than t^{-V_0}.

Theorem 1.6. *Let $V \in C(\boldsymbol{R}^N)$ be radial, and $V(x) \sim V_0/|x|$ as $|x| \to +\infty$. Then there exist non-trivial initial data $[u_0, u_1] \in H^1(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N)$ such that* *the solution* $u \in X_1(0, +\infty)$ *to problem* (1.1)–(1.2) *satisfies*

$$E_u(t) \geq Ct^{-V_0}, \quad t \gg 1, \tag{1.13}$$

with some constant $C > 0$.

Remark 1.7. For large $V_0 \geq N$, the decay estimate (1.6) coincides with the decay estimate of the corresponding parabolic problem (up to small losses $\delta > 0$). Therefore, based on this heuristic argument, we expect that the decay rate in (1.6) is almost exact even for large $V_0 \geq N$.

We conclude with a few standard definitions and notations used throughout this paper.

$$(p, q) = \int_{\boldsymbol{R}^N} p(x)q(x)dx, \quad p, q \in L^2(\boldsymbol{R}^N),$$

$$\|p\| := \left\{ \int_{\boldsymbol{R}^N} |p(x)|^2 dx \right\}^{1/2}, \quad p \in L^2(\boldsymbol{R}^N),$$

$$\langle x \rangle := \sqrt{1 + |x|^2}, \quad |x| := \left(\sum_{i=1}^{N} x_i^2 \right)^{1/2}, \quad x := (x_1, x_2, \ldots, x_N) \in \boldsymbol{R}^N,$$

$$f \in BC^2(\boldsymbol{R}^N) \text{ iff } f \in C^2(\boldsymbol{R}^N) \text{ and } |f|, |\nabla f|, \text{ and } |\partial^2 f / \partial x_i \partial x_j| \text{ are bounded on } \boldsymbol{R}^N.$$

The paper is organized as follows. In section 2 we prove Theorem 1.1 by dividing the proof into several lemmas. Sections 3 and 4 are devoted to the proofs of Theorem 1.3 and Theorem 1.4, respectively. In section 5 we address the question about the optimality of decay rates. We present several weighted $L^2 - L^2$ estimates for the auxiliary elliptic problem in the Appendix.

2. High-dimensional case $N \geq 3$.

First we shall consider the Poisson equation with a restriction on the behavior at infinity:

$$\begin{cases} \Delta H(x) = f(x), & x \in \boldsymbol{R}^N, \\ H(x) = O(|x|^{-(N-2)}), & |x| \to +\infty. \end{cases} \tag{2.1}$$

It is well-known [**12**] that the unique solution $H \in H^2_{loc}(\boldsymbol{R}^N)$ to problem (2.1) is given by the Newton potential

$$H(x) = \frac{-1}{(N-2)|S^{N-1}|} \int_{\boldsymbol{R}^N} |x-y|^{-(N-2)} f(y)dy,$$

where $|S^{N-1}|$ is the area of the $(N-1)$-dimensional unit sphere, provided that $f \in L^2(\boldsymbol{R}^N)$ and $f(x) = 0$ for $|x| > R_0 > 0$. In this case, one can show the following weighted $L^2 - L^2$ estimates.

Lemma 2.1. *Let $N \geq 3$. Assume that $f \in L^2(\boldsymbol{R}^N)$ has a compact support* $\operatorname{supp} f \subset B(R_0)$. *Then the (unique) solution $H \in H^2_{loc}(\boldsymbol{R}^N)$ of the problem* (2.1) *satisfies the following estimates*

(1) *If $N-2 > s > 0$, then*

$$\int_{\boldsymbol{R}^N} \langle x \rangle^s |\nabla H(x)|^2 dx \leq \frac{4(N-2+s)}{(N-2-s)^2} \int_{\boldsymbol{R}^N} \langle x \rangle^{s+2} f(x)^2 dx,$$

(2) *If $N-2 > s > 0$, then*

$$\int_{\boldsymbol{R}^N} \langle x \rangle^{s-2} |H(x)|^2 dx \leq \frac{16(N-2+s)}{(N-2)(N-2-s)^2} \int_{\boldsymbol{R}^N} \langle x \rangle^{s+2} f(x)^2 dx.$$

In the Appendix we present the proof of Lemma 2.1.

Now we introduce the auxiliary function

$$\chi(t,x) := \int_0^t u(s,x)ds + h(x), \tag{2.2}$$

where u is the solution to (1.1)–(1.2) and the perturbation $h(x)$ is the unique smooth solution to the Poisson problem (2.1) with $f(x) := V(x)u_0(x) + u_1(x)$. The function $h(x)$ is smoother than the data $u_0(x)$ and $u_1(x)$, but $h(x)$ does not decay sufficiently fast as $|x| \to +\infty$. In fact, since $h(x)$ is a solution of (2.1), we have $h(x) = O(|x|^{-(N-2)})$ and $|\nabla h(x)| = O(|x|^{-(N-1)})$ as $|x| \to +\infty$ for smooth initial data. Lemma 2.1 implies the same behavior in L^2 sense. The decay of $h(x)$ transfers to decay of $\chi(x)$ through definition (2.2) and the finite propagation speed (FSP) of problem (1.1)–(1.2):

$$|\chi(t,x)| = O(|x|^{-(N-2)}), \quad |\nabla \chi(t,x)| = O(|x|^{-(N-1)}), \quad |x| \to +\infty, \tag{2.3}$$

for sufficiently smooth $u_0(x)$ and $u_1(x)$. It is easy to see that the auxiliary function

$\chi(t,x)$ is a solution of the following Cauchy problem:

$$\chi_{tt}(t,x) - \Delta\chi(t,x) + V(x)\chi_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^N, \tag{2.4}$$

$$\chi(0,x) = h(x), \quad \chi_t(0,x) = u_0(x), \quad x \in \boldsymbol{R}^N. \tag{2.5}$$

Clearly $\chi(t,x)$ is smoother than $u(t,x)$ and has both first and second order energy. Moreover the second energy $E_{\chi_t}(t) = E_u(t)$. The only shortcoming of $\chi(t,x)$ is that it does not decay fast enough as $|x| \to \infty$. This leads to some problems which will be overcome in the sequel.

We start with the following Proposition.

Proposition 2.2. *Let $N \geq 3$ and $V_0 \geq N-2$. Then it follows that*

$$E_\chi(t) = O(t^{-(N-2-\delta)}), \quad t \to +\infty,$$

for any $\delta > 0$.

The idea to proceed further comes from [**26**]. Indeed, we set

$$v(t,x) := \chi(t,x)w(x)^{-1}, \tag{2.6}$$

where $w(x) = \langle x \rangle^{-m}$ with $m \in \boldsymbol{R}$ to be chosen in the sequel. Then it can be shown as in [**26**] that $v(t,x)$ satisfies the following equation

$$v_{tt} - \Delta v + V(x)v_t - 2(w^{-1}\nabla w)\cdot\nabla v + w^{-1}(-\Delta w)v = 0, \quad (t,x) \in (0,\infty)\times\boldsymbol{R}^N. \tag{2.7}$$

Furthermore, we set

$$w_1(x) = k\langle x \rangle^{1-m} = k\langle x \rangle w(x),$$

where $k > 0$ is another parameter to be chosen in the sequel. As in [**26**, Proposition 2.1] we multiply equation (2.7) by $w_1 v_t + wv$ and integrate by parts over R^N to derive the following weigthed energy identity.

Lemma 2.3. *Let $N \geq 3$ and $m < N-2$. Then*

$$\frac{d}{dt}E(v_t, \nabla v, v)(t) + F(v_t, \nabla v, v)(t) = 0, \tag{2.8}$$

where the weighted energy

$$E(v_t, \nabla v, v)(t)$$
$$= \frac{1}{2}\int_{\boldsymbol{R}^N} \left[w_1(v_t^2 + |\nabla v|^2) + 2wv_tv + (w^{-1}w_1(-\Delta w) + V(x)w)v^2\right]dx \quad (2.9)$$

and

$$F(v_t, \nabla v, v)(t) = \int_{\boldsymbol{R}^N} (Vw_1 - w)v_t^2 dx + \int_{\boldsymbol{R}^N} v_t \left(\nabla w_1 - 2w_1 \frac{\nabla w}{w}\right) \cdot \nabla v dx$$
$$+ \int_{\boldsymbol{R}^N} w|\nabla v|^2 dx - \frac{1}{2}\int_{\boldsymbol{R}^N} (\Delta w)v^2 dx. \quad (2.10)$$

Outline of proof. Here we calculate only the integrals coming from the integration by parts whose convergence is not evident. Let us assume that u_0 and u_1 are smooth. Using the decay (2.3) of the auxiliary function χ and the Gauss divergence theorem on $B(\rho)$ with a large $\rho \gg 1$ we obtain

$$\int_{B(\rho)} \nabla \cdot (w(x)v_t(t,x)\nabla v(t,x))dx = \int_{B(\rho)} \nabla \cdot (\chi_t(t,x)\nabla v(t,x))dx$$
$$= \int_{B(\rho)} \nabla \cdot (u(t,x)\nabla v(t,x))dx = \int_{|\sigma|=\rho} u(t,\sigma)\nabla v(t,\sigma) \cdot \boldsymbol{n}(\sigma)d\sigma = 0, \quad \rho \gg 1,$$
$$\int_{B(\rho)} \nabla \cdot (w(x)v(t,x)\nabla v(t,x))dx = \int_{B(\rho)} \nabla \cdot (\chi(t,x)\nabla v(t,x))dx$$
$$= \int_{|\sigma|=\rho} \chi(t,\sigma)\nabla v(t,\sigma) \cdot \boldsymbol{n}(\sigma)d\sigma$$
$$= \int_{|\sigma|=\rho} h(\sigma)(w^{-1}\nabla h(\sigma) - w^{-2}h(\sigma)\nabla w) \cdot \boldsymbol{n}(\sigma)d\sigma = O(\rho^{m-(N-2)}), \quad \rho \to +\infty,$$

and

$$\int_{B(\rho)} \nabla \cdot (\nabla w(x)v(t,x)^2)dx = \int_{|\sigma|=\rho} \nabla w(\sigma) \cdot \boldsymbol{n}(\sigma)v(t,\sigma)^2 d\sigma$$
$$= \int_{|\sigma|=\rho} w^{-2}\nabla w(\sigma) \cdot \boldsymbol{n}(\sigma)h(\sigma)^2 d\sigma = O(\rho^{m-N}), \quad \rho \to +\infty.$$

In the above we used the FSP of solutions $u(t,x)$ to problem (1.1)–(1.2) and the decay (2.3) of $h(x)$ and $|\nabla h(x)|$ as $|x| \to +\infty$. The proof for general u_0 and u_1 follows from the above proof and standard approximation argument. $\square$

We will prove that the weighted energy $E(v_t, \nabla v, v)(t)$ is a non-increasing function of t for conveniently chosen weights. The parameters m and k in the weights $w(x)$ and $w_1(x)$ will be chosen so that the following conditions will hold:

1. The initial weighted energy $E(v_t, \nabla v, v)(0)$ is finite (see the restriction on m coming from Lemma 2.4).
2. The inequality $F(v_t, \nabla v, v)(t) \geq 0$ holds (see Lemma 2.6 below).

These together with Lemma 2.3 give us that the weighted energy $E(v_t, \nabla v, v)(t)$ is a non-increasing function of t.

The next result finds the restriction on m which assures that $E_0 < \infty$.

LEMMA 2.4. *The initial weighted energy*

$$E(v_t, \nabla v, v)(0) = E_0$$

is finite, provided $N - 3 > m > -1$.

PROOF. From the definition of v, it is easy to estimate

$$\begin{aligned} E_0 \leq C \int_{\boldsymbol{R}^N} \langle x \rangle^{m+1} (|u_0(x)|^2 + |\nabla h(x)|^2) dx \\ + C \int_{\boldsymbol{R}^N} \langle x \rangle^{m-1} |h(x)|^2 dx + C \int_{\boldsymbol{R}^N} \langle x \rangle^m |h(x)||u_0(x)| dx. \end{aligned}$$

Applying Hölder inequality, we get

$$\begin{aligned} &\int_{\boldsymbol{R}^N} \langle x \rangle^m |h(x)||u_0(x)| dx \\ &\quad \leq \left(\int_{\boldsymbol{R}^N} \langle x \rangle^{m-1} |h(x)|^2 dx \right)^{1/2} \left(\int_{\boldsymbol{R}^N} \langle x \rangle^{m+1} |u_0(x)|^2 dx \right)^{1/2} \\ &\quad \leq \frac{1}{2} \int_{\boldsymbol{R}^N} \langle x \rangle^{m-1} |h(x)|^2 dx + \frac{1}{2} \int_{\boldsymbol{R}^N} \langle x \rangle^{m+1} |u_0(x)|^2 dx. \end{aligned}$$

It is clear that $E_0 < +\infty$ holds if

$$I_{1,h} := \int_{\boldsymbol{R}^N} \langle x \rangle^{m+1} |\nabla h(x)|^2 dx,$$

$$I_{2,h} := \int_{\boldsymbol{R}^N} \langle x \rangle^{m-1} |h(x)|^2 dx$$

are finite, since

$$E_0 \leq C\left(I_{1,h} + I_{2,h} + \int_{\boldsymbol{R}^N} \langle x\rangle^{m+1} |u_0(x)|^2 dx \right),$$

with some generous constant $C > 0$. Applying Lemma 2.1 with $s = m+1$ and using that $N-2 > m+1 > 0$, we get the following bounds:

$$I_{1,h} \leq \frac{4(N+m-1)}{(N-m-3)^2} \int_{\boldsymbol{R}^N} \langle x\rangle^{m+3} (V(x)u_0(x) + u_1(x))^2 dx < +\infty, \tag{2.11}$$

$$I_{2,h} \leq \frac{16(N+m-1)}{(N-2)(N-m-3)^2} \int_{\boldsymbol{R}^N} \langle x\rangle^{m+3} (V(x)u_0(x) + u_1(x))^2 dx < +\infty. \tag{2.12}$$

Thus, we have

$$E_0 \leq C(N, m, k, V_1) \int_{\boldsymbol{R}^N} \langle x\rangle^{m+3} (|u_0(x)| + |u_1(x)|)^2 dx < +\infty,$$

with another constant $C = C(N, m, k, V_1) > 0$ provided that $N - 3 > m > -1$. □

Throughout the rest of this section, we consider the case $V_0 > 1$ in condition (A). Our next goal is to find admissible values of m and k for the weights $w(x)$ and $w_1(x)$. We prove some auxiliary estimates which lead to the non-negativity of $F(v_t, \nabla v, v)(t)$ in (2.10). The first three terms there form a quadratic form which is non-negative in all cases. However, the last term in (2.10), involving Δw as a weight, has a sign only when $N \geq 4$. When $N = 3$, Δw changes sign at small $|x|$. This fact requires a more delicate inequality in the case $N = 3$ and leads to different choices for k in the two cases $N \geq 4$ and $N = 3$.

LEMMA 2.5. *Let $N \geq 3$, $V_0 > 1$ and $m = \min\{N-3, V_0 - 1\} - \delta$, where δ is a small positive number.*

Case 1: *If $N \geq 4$, there are numbers $k_\pm > 0$ such that for all $k \in [k_-, k_+]$ the following estimates hold*:

(i)

$$\Delta w \leq 0, \tag{2.13}$$

(ii)

$$4w(V(x)w_1 - w) \geq \left| \nabla w_1 - 2w_1 \frac{\nabla w}{w} \right|^2. \tag{2.14}$$

Case 2: *If $N = 3$, there are numbers $k_\pm > 0$ such that for all $k \in [k_-, k_+]$ the following estimate holds with small $\delta > 0$:*

$$(4-\sqrt{\delta})w(V(x)w_1 - w) \geq \left|\nabla w_1 - 2w_1\frac{\nabla w}{w}\right|^2. \tag{2.15}$$

Proof.
Case 1 (i): Set $r = |x|$. We calculate

$$\Delta w = m(1+r^2)^{-1-(m/2)}\left\{(2+m)\frac{r^2}{(1+r^2)} - N\right\}.$$

In order to get $\Delta w \leq 0$, it suffices to have

$$\frac{r^2}{1+r^2}(2+m) \leq N. \tag{2.16}$$

Inequality (2.16) holds, since

$$m \leq N-2. \tag{2.17}$$

Case 1 (ii): Simple computations show that

$$4w(V(x)w_1 - w) \geq 4\langle x\rangle^{-2m}(V_0 k - 1) \tag{2.18}$$

and

$$\left|\nabla w_1 - 2w_1\frac{\nabla w}{w}\right|^2 = k^2(1+m)^2 r^2(1+r^2)^{-(m+1)}. \tag{2.19}$$

Based on the above inequalities (2.18) and (2.19), condition (2.14) follows from

$$4(V_0 k - 1) \geq \frac{r^2}{1+r^2}k^2(1+m)^2. \tag{2.20}$$

This can be verified using the relations

$$\frac{r^2}{1+r^2}k^2(1+m)^2 \leq k^2(1+m)^2 \leq 4(V_0 k - 1),$$

and

$$(1+m)^2k^2 - 4V_0k + 4 \le 0. \tag{2.21}$$

If we set

$$k_\pm = \frac{2V_0 \pm 2\sqrt{V_0^2-(m+1)^2}}{(m+1)^2} > 0,$$

then (2.21) and (2.20) hold for all $k \in [k_-, k_+]$. At this step it suffices to take any

$$m \le V_0 - 1. \tag{2.22}$$

Case 2: Using the same arguments as in Case 1 (ii), we need to show that

$$\frac{r^2}{1+r^2}k^2(1+m)^2 \le k^2(1+m)^2 \le (4-\sqrt{\delta})(V_0k-1),$$

or

$$(1-\delta)^2k^2 - (4-\sqrt{\delta})V_0k + (4-\sqrt{\delta}) \le 0. \tag{2.23}$$

Here we set

$$k_\pm := \frac{(4-\sqrt{\delta})V_0 \pm \sqrt{(4-\sqrt{\delta})^2V_0^2 - 4(4-\sqrt{\delta})(1-\delta)^2}}{2(1-\delta)^2} > 0.$$

Then estimate (2.23) holds with a small $\delta > 0$ for all $k \in [k_-, k_+]$. Of course we need the expression under the square root to be positive, i.e. $(4-\sqrt{\delta})V_0^2 > 4(1-\delta)^2$ for small $\delta > 0$. This property is a consequence of

$$V_0 > 2\frac{(1-\delta)}{\sqrt{4-\sqrt{\delta}}}. \tag{2.24}$$

The above inequality holds for small $\delta > 0$ since

$$\lim_{\delta\to+0} 2\frac{(1-\delta)}{\sqrt{4-\sqrt{\delta}}} = 1 \quad \text{and} \quad \mathrm{V}_0 > 1.$$

The proof of (2.15) is completed. □

Let us now choose the parameters m and k in the weights $w(x)$ and $w_1(x)$:

$$m = \min\{N-3, V_0-1\} - \delta \tag{2.25}$$

and

$$k = \begin{cases} \frac{2V_0+2\sqrt{V_0^2-(m+1)^2}}{(m+1)^2}, & N \geq 4, \\ \frac{(4-\sqrt{\delta})V_0+\sqrt{(4-\sqrt{\delta})^2V_0^2-4(4-\sqrt{\delta})(1-\delta)^2}}{2(1-\delta)^2}, & N = 3, \end{cases} \tag{2.26}$$

where δ is a small positive number that may be decreased in the sequel. We use these definitions of m and k in the proof of next Lemmas.

Lemma 2.6. *For m and k chosen above, the functional*

$$\begin{aligned} F(v_t, \nabla v, v)(t) = &\int_{\boldsymbol{R}^N} (Vw_1 - w)v_t^2 dx + \int_{\boldsymbol{R}^N} v_t\left(\nabla w_1 - 2w_1\frac{\nabla w}{w}\right)\cdot \nabla v dx \\ &+ \int_{\boldsymbol{R}^N} w|\nabla v|^2 dx - \frac{1}{2}\int_{\boldsymbol{R}^N} (\Delta w)v^2 dx \end{aligned} \tag{2.27}$$

is nonnegative at all $t \geq 0$.

Proof.

Case 1: $N \geq 4$. The nonnegativity of $F(v_t, \nabla v, v)(t)$ is a trivial consequence from estimates (2.13) and (2.14).

Case 2: $N = 3$. The proof is more involved. We can not rely anymore on the sign of Δw. Instead we apply a combination of the weighted Hardy inequality and estimate (2.15) to close the circle. Simple calculation shows that

$$\int_{\boldsymbol{R}^N} \Delta w(x)v(t,x)^2 dx \leq 3\delta \int_{\boldsymbol{R}^N} \langle x\rangle^{\delta-2} v(t,x)^2 dx.$$

On the other hand, by Hardy's inequality (cf. [**2**, Lemma 1.21]) we have

$$\int_{\boldsymbol{R}^N} \langle x\rangle^{\delta-2} v(t,x)^2 dx \leq 4\int_{\boldsymbol{R}^N} \langle x\rangle^{\delta} |\nabla v(t,x)|^2 dx.$$

Respectively, we get

$$\int_{\boldsymbol{R}^N} \Delta w(x) v(t,x)^2 dx \leq 12\delta \int_{\boldsymbol{R}^N} \langle x \rangle^\delta |\nabla v(t,x)|^2 dx.$$

This gives the lower bound

$$F(v_t, \nabla v, v)(t) \geq \int_{\boldsymbol{R}^N} (Vw_1 - w) v_t^2 dx + \int_{\boldsymbol{R}^N} v_t \left(\nabla w_1 - 2w_1 \frac{\nabla w}{w} \right) \cdot \nabla v dx + (1 - 6\delta) \int_{\boldsymbol{R}^N} w |\nabla v|^2 dx. \tag{2.28}$$

By Hölder inequality we also have

$$-v_t \left(\nabla w_1 - 2w_1 \frac{\nabla w}{w} \right) \cdot \nabla v \leq \frac{1}{2\varepsilon w} v_t^2 \left| \nabla w_1 - 2w_1 \frac{\nabla w}{w} \right|^2 + \frac{\varepsilon}{2} |\nabla v|^2 w,$$

where $\varepsilon > 0$ will be chosen later. Using (2.28) and Lemma 2.5 with a small $\delta > 0$, we get a better lower bound:

$$\begin{aligned} F(t, v_t, \nabla v, v)(t) \geq & \int_{\boldsymbol{R}^N} \frac{v_t^2}{(4 - \sqrt{\delta}) w} \left| \nabla w_1 - 2w_1 \frac{\nabla w}{w} \right|^2 dx \\ & - \int_{\boldsymbol{R}^N} \frac{1}{2\varepsilon w(x)} \left| \nabla w_1 - 2w_1 \frac{\nabla w}{w} \right|^2 v_t^2 dx \\ & + \left(1 - 6\delta - \frac{\varepsilon}{2} \right) \int_{\boldsymbol{R}^N} w(x) |\nabla v(t,x)|^2 dx. \end{aligned}$$

Thus,

$$\begin{aligned} F(t, v_t, \nabla v, v)(t) \geq & \left(\frac{1}{4 - \sqrt{\delta}} - \frac{1}{2\varepsilon} \right) \int_{\boldsymbol{R}^N} \frac{v_t^2}{w(x)} \left| \nabla w_1 - 2w_1 \frac{\nabla w}{w} \right|^2 dx \\ & + \left(1 - \frac{\varepsilon}{2} - 6\delta \right) \int_{\boldsymbol{R}^N} w(x) |\nabla v(t,x)|^2 dx \end{aligned} \tag{2.29}$$

is non-negative whenever $\varepsilon > 0$ and $\delta > 0$ are such that

$$\frac{1}{4 - \sqrt{\delta}} - \frac{1}{2\varepsilon} > 0 \quad \text{and} \quad \left(1 - \frac{\varepsilon}{2} - 6\delta \right) > 0. \tag{2.30}$$

The above inequalities are equivalent to

$$2 - \frac{\sqrt{\delta}}{2} < \varepsilon < 2 - 12\delta. \tag{2.31}$$

This completes the proof since the relation

$$0 < 2 - \frac{\sqrt{\delta}}{2} < 2 - 12\delta \tag{2.32}$$

is satisfied for sufficiently small $\delta > 0$. □

The above lemma and the weighted energy inequality (2.8), integrated over $[0, t]$, imply

$$E(v_t, \nabla v, v)(t) \leq E(v_t, \nabla v, v)(0) = E_0. \tag{2.33}$$

Hence the weighted energy of v is a non-increasing function of t. Now we are in position to get an important weighted estimate for χ.

Lemma 2.7. *Let χ be defined in* (2.2). *The following estimate holds for* $V_0 \geq N - 2$ *:*

$$\int_{\boldsymbol{R}^N} \langle x\rangle^{m+1}(\chi_t^2 + |\nabla\chi|^2)dx + V_0 \int_{\boldsymbol{R}^N} \langle x\rangle^{m-1}\chi^2 dx \leq CE_0.$$

Proof. First we prove the two auxiliary estimates (2.34) and (2.36).

Case 1: $N \geq 4$. By means of Case 1 (i) in Lemma 2.5 and (2.17), we have

$$\int_{\boldsymbol{R}^N} \left[k\langle x\rangle^{1-m}(v_t^2 + |\nabla v|^2) + 2\langle x\rangle^{-m} v_t v + V_0 \langle x\rangle^{-m-1} v^2\right] dx \leq 2E_0.$$

Since

$$-2\langle x\rangle^{-m} v_t v \leq \frac{3}{4} V_0 v^2 \langle x\rangle^{-m-1} + \frac{4}{3V_0} v_t^2 \langle x\rangle^{1-m},$$

one also has

$$\int_{\boldsymbol{R}^N} \left[\left(k - \frac{4}{3V_0}\right)\langle x\rangle^{1-m} v_t^2 + k\langle x\rangle^{1-m}|\nabla v|^2 + \left(V_0 - \frac{3V_0}{4}\right)\langle x\rangle^{-m-1} v^2\right] dx \leq 2E_0.$$

We take k and m from definitions (2.25) and (2.26), respectively, with a small $\delta > 0$. Thus we get the inequality

$$\int_{\boldsymbol{R}^N}\left\{\frac{2}{3V_0}\langle x\rangle^{1-m}v_t^2+\frac{2}{V_0}\langle x\rangle^{1-m}|\nabla v|^2+\frac{V_0}{4}\langle x\rangle^{-1-m}v^2\right\}dx\le CE_0, \tag{2.34}$$

with some constant $C>0$. Here, we used the fact that

$$k\ge\frac{2}{V_0}. \tag{2.35}$$

Indeed,

$$k\ge\frac{2V_0}{(m+1)^2}\ge\frac{2}{V_0}.$$

Case 2: $N=3$. Applying similar arguments, we can prove that

$$\int_{\boldsymbol{R}^N}\left\{\frac{2}{3V_0}\langle x\rangle^{1+\delta}v_t^2+\frac{2}{V_0}\langle x\rangle^{1+\delta}|\nabla v|^2+\left(\frac{V_0}{4}-3k\delta\right)\langle x\rangle^{-1+\delta}v^2\right\}dx\le CE_0. \tag{2.36}$$

Let us give some details. Set

$$J(t):=\frac{1}{2}\int_{\boldsymbol{R}^N}w_1(x)|\nabla v(t,x)|^2dx-\frac{1}{2}\int_{\boldsymbol{R}^N}w(x)^{-1}w_1(x)\Delta w(x)v(t,x)^2dx$$

and take a small $\delta>0$. It follows from simple computations that

$$J(t)\ge\frac{k}{2}\int_{\boldsymbol{R}^N}\langle x\rangle^{1+\delta}|\nabla v(t,x)|^2dx-\frac{3k\delta}{2}\int_{\boldsymbol{R}^N}\langle x\rangle^{\delta-1}v(t,x)^2dx.$$

Now the definition of $E(v_t,\nabla v,v)(t)$ implies

$$\begin{aligned}2E(v_t,\nabla v,v)(t)&=k\int_{\boldsymbol{R}^N}\langle x\rangle^{1+\delta}v_t^2dx+2J(t)+2\int_{\boldsymbol{R}^N}wv_tv\,dx+\int_{\boldsymbol{R}^N}V(x)wv^2dx\\&\ge k\int_{\boldsymbol{R}^N}\langle x\rangle^{1+\delta}(v_t^2+|\nabla v|^2)dx+2\int_{\boldsymbol{R}^N}wv_tv\,dx\\&\quad+(V_0-3k\delta)\int_{\boldsymbol{R}^N}\langle x\rangle^{-1+\delta}v^2dx.\end{aligned}$$

One has the desired estimate using the fact that

$$k\ge\frac{(4-\sqrt{\delta})V_0}{2(1-\delta)^2}\ge\frac{2}{V_0}, \tag{2.37}$$

for $\delta > 0$ small enough. Notice that condition (2.37) is weaker than (2.24) since $V_0 > 1$.

We can now complete the proof of Lemma 2.7 relying on estimates (2.34) and (2.36). The elementary inequality

$$\frac{1}{1+\varepsilon}|f|^2 - \frac{1}{\varepsilon}|g|^2 \le |f-g|^2, \quad (\varepsilon > 0),$$

together with the definition of $v(t,x)$ leads to

$$\begin{aligned} |\nabla v|^2 &= |w^{-1}\nabla\chi - w^{-2}\chi\nabla w|^2 \\ &\ge \frac{1}{1+\varepsilon}w^{-2}|\nabla\chi|^2 - \frac{1}{\varepsilon}w^{-4}\chi^2|\nabla w|^2, \quad (\varepsilon > 0). \end{aligned} \tag{2.38}$$

Thus, from (2.34) and (2.38) we get

$$\begin{aligned} 2E_0 &\ge \frac{2}{3V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{1-m}w^{-2}\chi_t^2dx + \frac{V_0}{4}\int_{\boldsymbol{R}^N}\langle x\rangle^{-1-m}w^{-2}\chi^2dx \\ &\quad + \frac{2}{V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{1-m}\left\{\frac{1}{1+\varepsilon}w^{-2}|\nabla\chi|^2 - \frac{1}{\varepsilon}w^{-4}\chi^2|\nabla w|^2\right\}dx \\ &\ge \frac{2}{3V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{1-m}w^{-2}\chi_t^2dx + \frac{2}{(1+\varepsilon)V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{1-m}w^{-2}|\nabla\chi|^2dx \\ &\quad + \int_{\boldsymbol{R}^N}\left\{\frac{V_0}{4}\langle x\rangle^{-1-m}w^{-2} - \frac{2}{V_0\varepsilon}w^{-4}|\nabla w|^2\langle x\rangle^{1-m}\right\}\chi^2dx. \end{aligned}$$

Since $w = \langle x\rangle^{-m}$, one has $\nabla w = -m\langle x\rangle^{-m-2}x$, so that we see

$$\begin{aligned} 2E_0 &\ge \frac{2}{3V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{m+1}\chi_t^2dx + \frac{2}{(1+\varepsilon)V_0}\int_{\boldsymbol{R}^N}\langle x\rangle^{m+1}|\nabla\chi|^2dx \\ &\quad + \int_{\boldsymbol{R}^N}\left\{\frac{V_0}{4}\langle x\rangle^{m-1} + \frac{2m^2}{\varepsilon V_0}\langle x\rangle^{m-3}(-|x|^2)\right\}\chi^2dx \\ &\ge C_1\int_{\boldsymbol{R}^N}\langle x\rangle^{m+1}(\chi_t^2 + |\nabla\chi|^2)dx + \int_{\boldsymbol{R}^N}\left\{\frac{V_0}{4}\langle x\rangle^{m-1} - \frac{2m^2}{\varepsilon V_0}\langle x\rangle^{m-1}\right\}\chi^2dx \\ &\ge C_1\int_{\boldsymbol{R}^N}\langle x\rangle^{m+1}(\chi_t^2 + |\nabla\chi|^2)dx + C_2\int_{\boldsymbol{R}^N}V_0\langle x\rangle^{m-1}\chi^2dx, \end{aligned} \tag{2.39}$$

where $\varepsilon > 0$ is a large number defined by

$$\frac{V_0}{4} > \frac{2m^2}{\varepsilon V_0}, \tag{2.40}$$

and in that case we have just set

$$C_1 = \min\left\{\frac{2}{3V_0}, \frac{2}{(1+\varepsilon)V_0}\right\}, \quad C_2 = \frac{1}{2}\left\{\frac{1}{4} - \frac{2m^2}{\varepsilon V_0^2}\right\}.$$

This implies the desired estimate. □

Next we show that the weights $\langle x\rangle^m$ and t^m are equivalent. We begin with the following simple identity.

Lemma 2.8. *Let $N \geq 3$. Then χ satisfies*

$$\frac{d}{dt}\int_{\boldsymbol{R}^N}\left(\chi_t^2 + |\nabla\chi|^2 + W\chi_t\chi + \frac{1}{2}V(x)W\chi^2\right)dx$$
$$+ \int_{\boldsymbol{R}^N}\left\{(2V(x) - W(x))\chi_t^2 + W(x)|\nabla\chi|^2 - \frac{1}{2}\Delta W\chi^2\right\}dx = 0,$$

where $W \in BC^2(\boldsymbol{R}^N)$ is a function satisfying

$$|\nabla W(x)| = O(|x|^{-2}), \quad |x| \to +\infty.$$

Proof. Multiplying both sides of (2.4) by $\chi_t + 1/2W(x)\chi$, applying the divergence theorem and rearranging the terms, we get

$$\frac{d}{dt}\int_{\boldsymbol{R}^N}(\chi_t^2 + |\nabla\chi|^2 + \chi_t\chi W)dx - \int_{\boldsymbol{R}^N}\chi_t^2 W dx + 2\int_{\boldsymbol{R}^N}V(x)\chi_t^2 dx$$
$$+ \int_{\boldsymbol{R}^N}(\nabla W\cdot\nabla\chi)\chi dx + \int_{\boldsymbol{R}^N}W|\nabla\chi|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{R}^N}VW\chi^2 dx = 0. \tag{2.41}$$

In the above we use $(\Delta\chi)W\chi = \nabla\cdot(W\chi\nabla\chi) - (\nabla W\cdot\nabla\chi)\chi - W|\nabla\chi|^2$. Based on the decay property (2.3) of χ, the boundary integral vanishes for large $|x|$

$$\left|\int_{B(\rho)}\nabla\cdot(W(x)\chi(t,x)\nabla\chi(t,x))dx\right| = O(\rho^{2-N}), \quad \rho \to +\infty.$$

It is easy to see that

$$(\nabla W \cdot \nabla \chi)\chi = \frac{1}{2}\nabla \cdot (\chi^2 \nabla W) - \frac{1}{2}\chi^2 \Delta W,$$

and the boundary integral also vanishes based on (2.3)

$$\left| \int_{B(\rho)} \nabla \cdot (\chi(t,x)^2 \nabla W(x)) dx \right| = O(\rho^{1-N}), \quad \rho \to +\infty.$$

This completes the proof. □

The key results for transferring decay estimates at large $|x|$ into decay estimates at large t are the following two lemmas.

LEMMA 2.9. *Let $N \geq 3$. Then there exists a constant $C > 0$ depending only on V_0 and the Hardy-Sobolev constant such that one has*

$$\int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (\chi_t(s,x)^2 + |\nabla \chi(s,x)|^2) dx ds \leq C \int_{\boldsymbol{R}^N} (\chi_t(t,x)^2 + |\nabla \chi(t,x)|^2) dx,$$

for $t \geq 0$.

PROOF. Set

$$G(t) := \int_{\boldsymbol{R}^N} \left(\chi_t^2 + |\nabla \chi|^2 + W \chi_t \chi + \frac{1}{2} V W \chi^2 \right) dx.$$

Choose

$$W(x) = \frac{V_0}{2} \langle x \rangle^{-1}.$$

Then, it follows from Lemma 2.8 that

$$G(T) + \int_t^T \int_{\boldsymbol{R}^N} \left\{ (2V - W)\chi_t^2 + W|\nabla \chi|^2 - \frac{1}{2}(\Delta W)\chi^2 \right\} dx ds = G(t), \qquad 0 \leq t < T. \quad (2.42)$$

Since

$$2V(x) - W(x) \geq \frac{3V_0}{2} \langle x \rangle^{-1},$$

and

$$-\Delta W(x) = \frac{V_0}{2}(1+|x|^2)^{-3/2}\frac{N+(N-3)|x|^2}{1+|x|^2} > 0,$$

we see that

$$G(T) + \int_t^T \int_{\boldsymbol{R}^N} \left(\frac{3}{2}V_0\langle x\rangle^{-1}\chi_t^2 + \frac{V_0}{2}\langle x\rangle^{-1}|\nabla\chi|^2\right)dxds \le G(t). \tag{2.43}$$

From

$$|W\chi_t\chi| \le \frac{\eta}{2}\chi_t^2 + \frac{1}{2\eta}W^2\chi^2 \le \frac{\eta}{2}\chi_t^2 + \frac{1}{2\eta}WV\chi^2, \tag{2.44}$$

with $\eta = 3/2$, it follows that

$$G(t) \ge \int_{\boldsymbol{R}^N} \left(\frac{1}{4}\chi_t^2 + |\nabla\chi|^2 + \frac{1}{6}VW\chi^2\right)dx > 0. \tag{2.45}$$

Furthermore, by using (2.44) with $\eta = 1$ one has

$$\begin{aligned} G(t) &\le \int_{\boldsymbol{R}^N} \left(\frac{3}{2}\chi_t^2 + |\nabla\chi|^2 + WV\chi^2\right)dx \\ &\le \frac{3}{2}\int_{\boldsymbol{R}^N} (\chi_t^2 + |\nabla\chi|^2)dx + C_0\int_{\boldsymbol{R}^N} \frac{\chi^2}{1+|x|^2}dx, \end{aligned}$$

where we have also used that

$$V(x)W(x) \le \frac{C_0}{1+|x|^2},$$

with some constant $C_0 > 0$. Applying the Hardy-Sobolev inequality ($N \ge 3$) one has

$$G(t) \le \frac{3}{2}\int_{\boldsymbol{R}^N} (\chi_t^2 + |\nabla\chi|^2)dx + C\int_{\boldsymbol{R}^N} |\nabla\chi|^2dx. \tag{2.46}$$

Therefore, (2.43), (2.45) and (2.46) imply

$$\int_t^T \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dxds \le C\int_{\boldsymbol{R}^N} (\chi_t^2 + |\nabla\chi|^2)dx,$$

which is the desired inequality. □

LEMMA 2.10. *Let $N \geq 3$. The following relations hold.*
(a) $\lim_{t\to+\infty} E_\chi(t) = 0$,
(b) $E_\chi(t) \leq V_1 \int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t(s,x)^2 + |\nabla\chi(s,x)|^2)dxds.$

PROOF. First, from the Hölder inequality with exponents $(m+2)$ and $((m+2)/(m+1))$ we get

$$E_\chi(t) \leq C\Big(\int_{\boldsymbol{R}^N} \langle x\rangle^{m+1}(\chi_t^2 + |\nabla\chi|^2)dx\Big)^{1/(m+2)} \times \Big(\int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dx\Big)^{(m+1)/(m+2)}.$$

By using Lemma 2.7 we have

$$E_\chi(t) \leq C\{CE_0\}^{1/(m+2)}\Big(\int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dx\Big)^{(m+1)/(m+2)}.$$

On the other hand, it follows from Lemma 2.9 with $t = 0$ that

$$\int_0^\infty \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dxds \leq C\int_{\boldsymbol{R}^N} (u_0^2 + |\nabla h|^2)dx < +\infty.$$

This shows

$$\liminf_{t\to+\infty} \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dx = 0.$$

Thus, the dissipation of the energy $E_\chi(t)$ leads to

$$\lim_{t\to+\infty} E_\chi(t) \leq CE_0^{1/(m+2)}\Big\{\liminf_{t\to+\infty} \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dx\Big\}^{(m+1)/(m+2)} = 0,$$

which implies (a). Note that $m + 1 > 0$.

Once we have obtained (a), the proof of (b) follows from (2.4) (see (1.3))

$$E_\chi(T) + \int_t^T \int_{\boldsymbol{R}^N} V(x)\chi_t^2 dxds = E_\chi(t),$$

for $0 \le t \le T$. Letting $T \to +\infty$ (a) and the assumption (A) on the potential $V(x)$ lead to

$$\begin{aligned} E_\chi(t) &\le V_1 \int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x \rangle^{-1} \chi_t^2 dxds \\ &\le V_1 \int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (\chi_t^2 + |\nabla \chi|^2) dxds, \end{aligned}$$

which proves (b). □

Under these preparations we can prove Proposition 2.2 based on Lemmas 2.7, 2.9 and 2.10.

Proof of Proposition 2.2. Let $N \ge 3$ and and m is as in Lemma 2.5, namely

$$m = \min\{N-3, V_0 - 1\} - \delta$$

with small $\delta > 0$. First, as in the proof of Lemma 2.10 from the Hölder inequality we get

$$\begin{aligned} E_\chi(t) \le C &\Big(\int_{\boldsymbol{R}^N} \langle x \rangle^{m+1} (\chi_t^2 + |\nabla \chi|^2) dx \Big)^{1/(m+2)} \\ &\times \Big(\int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (\chi_t^2 + |\nabla \chi|^2) dx \Big)^{(m+1)/(m+2)}, \end{aligned}$$

where both exponents $(m+2)$ and $((m+2)/(m+1)) > 1$ are greater than 1. Set

$$I(t) := \int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (\chi_t^2 + |\nabla \chi|^2) dxds.$$

Then, it follows from Lemmas 2.7 and 2.9 that

$$I(t) \le CE_\chi(t) \le CE_0^{1/(m+2)} \Big(\int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (\chi_t^2 + |\nabla \chi|^2) dx \Big)^{(m+1)/(m+2)}, \quad (2.47)$$

where the constant $C > 0$ depends on V_0, V_1, N, m, k. Since

$$-\frac{d}{dt}I(t) = \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(\chi_t^2 + |\nabla\chi|^2)dx > 0,$$

we can rewrite (2.47) in terms of $I(t)$ and get

$$I(t) \le CE_0^{1/(m+2)}(-I'(t))^{(m+1)/(m+2)}. \tag{2.48}$$

The estimate (2.48) implies the differential inequality:

$$E_{00} \cdot I'(t) \le -CI(t)^{(m+2)/(m+1)}, \quad I(0) < +\infty,$$

where (see the proof of Lemma 2.7)

$$0 < E_{00} := \left\{ \int_{\boldsymbol{R}^N} \langle x\rangle^{m+3}(|u_0(x)| + |u_1(x)|)^2 dx \right\}^{1/(m+1)} < +\infty.$$

By solving this inequality we get the estimate:

$$I(t) \le CE_{00}^{m+1} \cdot t^{-m-1}.$$

The Proposition 2.2 follows from the above estimate and Lemma 2.10. □

Based on Proposition 2.2, we can prove **(ii)** of Theorem 1.1. The proof of this part manifests the parabolic asymptotic profile of the equation (1.1) with large V_0. For this we shall prepare the following Proposition. This proposition is derived by slightly modifying the result obtained by Radu-Todorova-Yordanov [**21**] which has been derived in the case of subcritical potential $V(x) \approx V_0(1 + |x|)^{-\alpha}$ with $\alpha \in [0, 1)$.

PROPOSITION 2.11. *Let $N \ge 3$ and $V_0 > \mu + 2$ ($\mu > -1$). There exist constants C_{i0} ($i = 0, 1$), such that*

$$\int_0^t (R + s)^{\mu+1}E_u(s)ds \le C_{00} + C_{10}\int_0^t (R + s)^{\mu-1}E_\chi(s)ds,$$

where $u \in X_1(0, +\infty)$ is the weak solution of (1.1)–(1.2), $\chi = \chi(t, x)$ is the function defined by (2.2), and $R > 0$ is a large number satisfying $R > R_0 + 1$.

Let us postpone the proof for a while and go back to the proof of Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii) [in the case $V_0 \geq N$]. Since $V_0 \geq N-2$, we use Proposition 2.2 to obtain

$$E_\chi(t) \leq C(1+t)^{-(N-2-\delta)}.$$

Then by Proposition 2.11 we have

$$\begin{aligned}\int_0^t (R+s)^{\mu+1} E_u(s)ds &\leq C_0 + C_{1,R}\int_0^t (1+s)^{\mu-1}(1+s)^{-(N-2-\delta)}ds \\ &\leq C_0 + C_{1,R}\int_0^t (1+s)^{\mu-N+1+\delta}ds \\ &\leq C_0 + C_{1,R}\frac{1}{N-2-\mu-\delta},\end{aligned}$$

where μ is such that

$$-1 < \mu < N-2-\delta, \quad \mu+2 < V_0. \tag{2.49}$$

Thus, the energy of u satisfies

$$\int_0^t (R+s)^{\mu+1} E_u(s)ds \leq C_0 + C_{1,R,N,V_0},$$

with some generous constant $C_{1,R,N,V_0} > 0$. From the monotonicity of the energy $E_u(t)$, we get

$$\int_0^t (R+s)^{\mu+1} ds \cdot E_u(t) \leq C_0 + C_{1,R,N,V_0}.$$

Thus we have

$$\frac{(R+t)^{\mu+2}}{\mu+2} E_u(t) \leq C_0 + C_{1,R,N,V_0} + \frac{R^{\mu+2}}{\mu+2} E_u(0)$$

which implies

$$E_u(t) = O(t^{-(\mu+2)}), \quad t \to +\infty.$$

Since $V_0 \geq N$, it will be enough to choose $\mu = N - 2 - 2\delta$. Resetting $\delta := 2\delta$, we have the decay estimate in Theorem 1.1 (ii). $\square$

Next we prove Theorem 1.1 (i) in the case $1 \leq V_0 < N$. The proof is divided into two parts:
(Step 1): The case $N - 1 \leq V_0 < N$,
(Step 2): The case $1 \leq V_0 < N - 1$.

Proof of (Step 1). Let $v(t,x) = u(t,x)w(x)^{-1}$ in (2.6), where $u \in X_1(0,+\infty)$ is the solution to the original problem (1.1)–(1.2) and the weights are again

$$w(x) = \langle x \rangle^{-m}, \quad w_1(x) = k\langle x \rangle^{1-m}.$$

Compared with $\chi(t,x)$, the solution $u(t,x)$ can stand much higher powers m. The restriction on the weights for χ ($m < N - 3$) comes from the slow decay of χ as $|x| \to \infty$ (see Lemmas 2.3 and 2.4). For u the FSP applies. Therefore, we can afford much stronger weights and get a weighted energy identity for u similar to (2.8) without the condition $m < N - 2$.

Taking $m := V_0 - 1$ (see (2.22)) and $k := k_+ = 2/V_0$ (see (2.35)) we can show, as in Lemma 2.6, that the first three terms of $F(v_t, \nabla v, v)(t)$ form a nonnegative-definite quadratic form

$$\int_{\boldsymbol{R}^N} (Vw_1 - w)v_t^2 dx + \int_{\boldsymbol{R}^N} v_t \left(\nabla w_1 - 2w_1 \frac{\nabla w}{w}\right) \cdot \nabla v dx + \int_{\boldsymbol{R}^N} w|\nabla v|^2 dx \geq 0.$$

Since we can not rely anymore on the positivity of the Laplacian term we rewrite (2.8) as

$$\frac{d}{dt} E(v_t, \nabla v, v)(t) \leq \frac{1}{2} \int_{\boldsymbol{R}^N} (\Delta w(x)) v(t,x)^2 dx dt. \tag{2.50}$$

After integrating (2.50) over $[0,t]$ we get

$$E(v_t, \nabla v, v)(t) \leq E_0 + \frac{1}{2} \sup_{0 \leq t < +\infty} \int_0^t \int_{\boldsymbol{R}^N} \Delta w(x) v(t,x)^2 dx dt =: E_0^*.$$

We will prove in the sequel the following boundedness

$$\sup_{0 \leq t < +\infty} \int_0^t \int_{\boldsymbol{R}^N} \Delta w(x) v(t,x)^2 dx dt < +\infty, \tag{2.51}$$

which leads to $E_0^* < \infty$. By using the representation formula (2.9) for $E(v_t, \nabla v, v)(t)$ we get

$$\frac{1}{2}\int_{\boldsymbol{R}^N} \left[w_1(v_t^2 + |\nabla v|^2) + 2wv_tv + V(x)wv^2\right]dx$$
$$\leq \frac{1}{2}\sup_{0\leq t<+\infty}\int_{\boldsymbol{R}^N} w^{-1}w_1(\Delta w)v^2dx + E_0^* =: E_0^{**}.$$

We will prove also that $E_0^{**} < +\infty$ which corresponds to Lemma 2.7 in the case $N-1 \leq V_0 < N$ with $\chi(t,x)$ replaced by $u(t,x)$. Further, by using the same arguments as in the proofs of Lemmas 2.7, 2.8, 2.9, 2.10 and Proposition 2.2 with $\chi(t,x)$ replaced by $u(t,x)$ we can derive

$$I_u(t) := \int_t^\infty \int_{\boldsymbol{R}^N} \langle x\rangle^{-1}(u_t^2 + |\nabla u|^2)dxds$$
$$\leq C(E_0^{**})^{1/(m+2)}(-I_u'(t))^{(m+1)/(m+2)},$$

so that (see (2.48))

$$E_u(t) \leq Ct^{-(m+1)} = Ct^{-V_0}. \tag{2.52}$$

Let us now prove (2.51) and

$$\sup_{0\leq t<+\infty}\int_{\boldsymbol{R}^N} w^{-1}w_1(\Delta w)v^2dx < +\infty. \tag{2.53}$$

Indeed, we have

$$\int_0^t \int_{\boldsymbol{R}^N} \Delta w(x)v(t,x)^2dxdt$$
$$= m\int_0^t \int_{\boldsymbol{R}^N} \langle x\rangle^{-m-2}\left\{(2+m)\frac{r^2}{1+r^2} - N\right\}\langle x\rangle^{2m}u^2dxdt$$
$$\leq (V_0-1)(V_0+1-N)\int_0^{+\infty}\int_{\boldsymbol{R}^N} \langle x\rangle^{V_0-3}u^2dxdt =: (V_0-1)(V_0+1-N)I_0.$$

At this stage we impose the assumptions $V_0 \geq N-1$. To estimate I_0, we use the FSP of the solution $u(t,x)$ and get

$$
\begin{aligned}
I_0 &\le \frac{C}{V_0}\int_0^{+\infty}\int_{\boldsymbol{R}^N} V(x)u(t,x)^2(1+|x|)^{V_0-2}dxdt, \quad (V_0 \ge N-1 \ge 2)\\
&\le \frac{C}{V_0}\int_0^{+\infty}\int_{\boldsymbol{R}^N} (R+t)^{V_0-2}V(x)u(t,x)^2dxdt \quad (1+R_0<R)\\
&= \frac{C}{V_0}\int_0^{+\infty}\int_{\boldsymbol{R}^N} (R+t)^{V_0-2}V(x)\chi_t(t,x)^2dxdt.
\end{aligned}
$$

In the above

$$
\chi(t,x) = \int_0^t u(s,x)ds + h(x),
$$

where $h(x)$ is the unique solution of the Poisson problem (2.1) with $f(x) = V(x)u_0(x) + u_1(x)$. Thus we have

$$
\begin{aligned}
\int_{\boldsymbol{R}^N} w^{-1}w_1(\Delta w)v^2dx &= km\int_{\boldsymbol{R}^N}\langle x\rangle^{m-1}\Big\{(2+m)\frac{r^2}{1+r^2}-N\Big\}u^2dx\\
&\le \frac{2}{V_0}(V_0-1)(V_0+1-N)\int_{\boldsymbol{R}^N}\langle x\rangle^{V_0-2}u^2dx\\
&\le \frac{2}{V_0}(V_0-1)(V_0+1-N)\int_{\boldsymbol{R}^N}(1+|x|)^{V_0-2}u^2dx\\
&\le \frac{2}{V_0}(V_0-1)(V_0+1-N)\int_{\boldsymbol{R}^N}(1+R_0+t)^{V_0-2}u^2dx\\
&\le \frac{2}{V_0}(V_0-1)(V_0+1-N)(R+t)^{V_0-2}\|u(t,\cdot)\|^2\\
&\le C_0(V_0,N)(R+t)^{V_0-2}E_\chi(t) \le C_0(V_0,N)(R+t)^{V_0-N+\delta},
\end{aligned}
$$

where $R > R_0+1$, $\chi_t = u$ and the FSP are just used. In the last statement we apply the decay rate of $E_\chi(t)$,

$$
E_\chi(t) \le C(R+t)^{-(N-2-\delta)}, \tag{2.54}
$$

which has been already derived in Proposition 2.2. Then choosing $\delta > 0$ such that $N - V_0 > \delta > 0$ completes the proof of (2.53).

Further, from the equation (2.4) for χ we can derive the energy identity

$$\frac{d}{dt}E_\chi(t) + \int_{\boldsymbol{R}^N} V(x)\chi_t^2 dx = 0, \tag{2.55}$$

and correspondingly the following weighted identity

$$\big[(R+t)^{V_0-2}E_\chi(t)\big]_{t=0}^{t=t} + G(t) = (V_0-2)\int_0^t (R+t)^{V_0-3}E_\chi(t)dt,$$

where

$$G(t) := \int_0^t \int_{\boldsymbol{R}^N} (R+t)^{V_0-2}V(x)\chi_t^2 dxdt.$$

This can be rewritten as

$$(R+t)^{V_0-2}E_\chi(t) + G(t) \le R^{V_0-2}E_\chi(0) + (V_0-2)\int_0^t (R+t)^{V_0-3}E_\chi(t)dt,$$

(recall that $N \ge 3$). Note that $E_\chi(0)$ depends on the function $h(x)$. From Lemma 2.1 one can easily check that $E_\chi(0) < +\infty$. By using the decay rate (2.54) of $E_\chi(t)$ and the choice for δ, namely $N - V_0 - \delta > 0$, we have

$$(R+t)^{V_0-2}E_\chi(t) + G(t) \le C + C(1+t)^{V_0-N+\delta}, \quad (N \ge 3). \tag{2.56}$$

Estimate (2.56) leads to

$$\lim_{t\to+\infty} G(t) < +\infty,$$

so that $I_0 < +\infty$. This completes the proof of Step 1. □

Proof of (Step 2) [in the case $1 \le V_0 < N-1$]. The proof is a slight modification of the proof of Proposition 2.2. In this part, we do not use the function $\chi(t,x)$ (see (2.2)) at all. Instead we directly use the solution $u(t,x)$ to the original problem (1.1)–(1.2), i.e., we set

$$v(t,x) := u(t,x)w(x)^{-1}$$

in (2.6), where again

$$w(x) = \langle x\rangle^{-m}, \quad w_1(x) = k\langle x\rangle^{1-m}.$$

We choose $m = V_0 - 1$ and

$$k = k_+ = \frac{2V_0 + 2\sqrt{V_0^2 - (m+1)^2}}{(m+1)^2} = \frac{2}{V_0}. \tag{2.57}$$

Then we can proceed with the same arguments as in Lemmas 2.7, 2.8, 2.9, 2.10 and in Proposition 2.2 with $\chi(t,x)$ replaced by $u(t,x)$. Similar to Proposition 2.2, we obtain that

$$I_u'(t) \le -CI_u(t)^{(m+2)/(m+1)}, \quad I_u(0) < +\infty,$$

where

$$I_u(t) := \int_t^{+\infty} \int_{\boldsymbol{R}^N} \langle x \rangle^{-1} (u_t^2 + |\nabla u|^2) dx ds.$$

Thus we have

$$E_u(t) \le I_u(t) \le Ct^{-m-1} = Ct^{-V_0}. \qquad \square$$

The last part of this section is the proof of Proposition 2.11. First we prepare the following lemma. The proof is a slight modification of the argument in Radu-Todorova-Yordanov [**21**]. Their result works in the subcritical case $\alpha \in [0,1)$.

Lemma 2.12. *Let $N \ge 1$. Assume that a function $W \in C^2([0,+\infty))$ satisfies*

(i) $W(t) \le \inf_{|x| \le R_0 + t} V(x)$,

(ii) $W''(t) - W'(t)V(x) \ge 0$.

Then the following estimate holds

$$\begin{aligned} &\frac{d}{dt} \int_{\boldsymbol{R}^N} \left(u_t^2 + |\nabla u|^2 + W(t) u_t u + \frac{W(t)V(x) - W'(t)}{2} u^2 \right) dx \\ &\quad + \int_{\boldsymbol{R}^N} (W(t) u_t^2 + W(t) |\nabla u|^2) dx \le 0. \end{aligned}$$

Proof. By multiplying both sides of (1.1) by u_t and integrating over $\boldsymbol{R}^N$ one has

$$\frac{d}{dt} E_u(t) + \int_{\boldsymbol{R}^N} V(x) u_t^2 dx = 0. \tag{2.58}$$

Next, multiplying both sides of (1.1) by $(1/2)W(t)u$ we see

$$\frac{1}{2}W(t)\frac{d}{dt}(u_t,u)-\frac{1}{2}W(t)\|u_t\|^2+\frac{1}{2}W(t)\|\nabla u\|^2+\frac{W(t)}{4}\frac{d}{dt}\int_{\boldsymbol{R}^N}V(x)u^2dx=0,$$

so that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{W(t)(u_t,u)\}-\frac{1}{4}W'(t)\frac{d}{dt}\|u\|^2-\frac{1}{2}W(t)\|u_t\|^2\\&\quad+\frac{1}{2}W(t)\|\nabla u\|^2+\frac{d}{dt}\left\{\frac{W(t)}{4}\int_{\boldsymbol{R}^N}V(x)u^2dx\right\}-\frac{W'(t)}{4}\int_{\boldsymbol{R}^N}V(x)u^2dx=0.\end{aligned}$$

Thus one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{W(t)(u_t,u)\}-\left[\frac{1}{2}\frac{d}{dt}\left\{W'(t)\frac{1}{2}\|u\|^2\right\}-\frac{1}{4}W''(t)\|u\|^2\right]\\&\quad+\left\{\frac{W(t)}{2}\|u_t\|^2-W(t)\|u_t\|^2\right\}+\frac{1}{2}W(t)\|\nabla u\|^2\\&\quad+\frac{d}{dt}\left\{\frac{W(t)}{4}\int_{\boldsymbol{R}^N}V(x)u^2dx\right\}-\frac{W'(t)}{4}\int_{\boldsymbol{R}^N}V(x)u^2dx=0,\end{aligned}$$

so that one has arrived at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{R}^N}\left\{W(t)u_tu-\frac{W'(t)}{2}u^2+\frac{1}{2}W(x)V(x)u^2\right\}dx-\int_{\boldsymbol{R}^N}W(t)u_t^2dx\\&\quad+\frac{1}{2}\int_{\boldsymbol{R}^N}\left\{W(t)u_t^2+W(t)|\nabla u|^2+\frac{W''(t)-W'(t)V(x)}{2}u^2\right\}dx=0.\end{aligned}\tag{2.59}$$

By adding both sides of (2.58) and (2.59) one has

$$\begin{aligned}&\frac{d}{dt}\int_{\boldsymbol{R}^N}\left(u_t^2+|\nabla u|^2+W(t)u_tu+\frac{W(t)V(x)-W'(t)}{2}u^2\right)dx\\&\quad+2\int_{\boldsymbol{R}^N}(V(x)-W(t))u_t^2dx\\&\quad+\int_{\boldsymbol{R}^N}\left\{W(t)u_t^2+W(t)|\nabla u|^2+\frac{1}{2}(W''(t)-W'(t)V(x))u^2\right\}dx=0.\end{aligned}\tag{2.60}$$

The desired estimate can be derived from (2.60) and the assumptions (i) and (ii). □

Set

$$F(t,x) := u_t(t,x)^2 + |\nabla u(t,x)|^2 + W(t)u_t(t,x)u(t,x) + \frac{W(t)V(x) - W'(t)}{2}u(t,x)^2.$$

Then one has the following Lemma.

LEMMA 2.13. *Let $N \geq 1$. Assume that a function $W \in C^2([0,+\infty))$ satisfies*

(iii) $F(t,x) \geq 0$,

(iv) $F(t,x) \leq (1+\varepsilon)u_t(t,x)^2 + |\nabla u(t,x)|^2 + \dfrac{C^*}{\varepsilon}W(t)V(x)u(t,x)^2$,

with large $C^* > 0$ and small $\varepsilon > 0$ specified later. Then for any $\mu > -1$ it is true that

$$\begin{aligned}
&\int_0^t \int_{\boldsymbol{R}^N} (R+s)^{\mu+2} W(s)(u_t^2 + |\nabla u|^2)dxds \\
&\quad \leq C_1 + (\mu+2)(1+\varepsilon)\int_0^t \int_{\boldsymbol{R}^N} (R+s)^{\mu+1}(u_t^2 + |\nabla u|^2)dxds \\
&\qquad + (\mu+2)\frac{C^*}{\varepsilon}\int_0^t \int_{\boldsymbol{R}^N} (R+s)^{\mu+1}W(s)V(x)u^2dxds,
\end{aligned}$$

where $C_1 > 0$ is a constant depending on the initial data (u_0, u_1), $R > 0$, μ, and $W(0)$.

PROOF. It follows from Lemma 2.12 that for $0 \leq t \leq T$,

$$\int_0^T (R+t)^{\mu+2}\frac{d}{dt}\int_{\boldsymbol{R}^N} F(t,x)dxdt + \int_0^T (R+t)^{\mu+2}\int_{\boldsymbol{R}^N} W(t)(u_t^2 + |\nabla u|^2)dxdt \leq 0.$$

By integration by parts with respect to t we see that

$$\begin{aligned}
&(R+T)^{\mu+2}\int_{\boldsymbol{R}^N} F(T,x)dx + \int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+2}W(t)(u_t^2 + |\nabla u|^2)dxdt \\
&\quad \leq R^{\mu+2}\int_{\boldsymbol{R}^N} F(0,x)dx + (\mu+2)\int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+1}F(t,x)dxdt.
\end{aligned}$$

Thus the desired inequality can be derived by the conditions (iii) and (iv) as follows:

$$
\begin{aligned}
&\int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+2} W(t)(u_t^2 + |\nabla u|^2) dxdt \\
&\quad \le C_1 + (\mu+2) \int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+1} \left\{ (1+\varepsilon) u_t^2 + |\nabla u|^2 + \frac{C^*}{\varepsilon} W(t) V(x) u^2 \right\} dxdt \\
&\quad \le C_1 + (\mu+2)(1+\varepsilon) \int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+1} \{ u_t^2 + |\nabla u|^2 \} dxdt \\
&\qquad + (\mu+2) \frac{C^*}{\varepsilon} \int_0^T \int_{\boldsymbol{R}^N} (R+t)^{\mu+1} W(t) V(x) u^2 dxdt,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &:= R^{\mu+2} \int_{\boldsymbol{R}^N} F(0,x) dx \\
&= R^{\mu+2} \int_{\boldsymbol{R}^N} \left(u_1^2 + |\nabla u_0|^2 + W(0) u_1 u_0 + \frac{W(0)V(x) - W'(0)}{2} u_0^2 \right) dx. \quad \square
\end{aligned}
$$

Now we prove Proposition 2.11.

Proof of Proposition 2.11. Choose

$$
W(t) := \frac{V_0}{R+t}. \tag{2.61}
$$

The auxiliary function $W(t)$ satisfies all conditions (i)–(iv) in Lemmas 2.12, 2.13. We postpone the check of these conditions and perform with the proof of the Proposition 2.11. It follows from Lemma 2.13 that

$$
\begin{aligned}
&\{V_0 - (1+\varepsilon)(\mu+2)\} \int_0^t \int_{\boldsymbol{R}^N} (R+s)^{\mu+1} (u_t^2 + |\nabla u|^2) dxds \\
&\quad \le C_1 + (\mu+2) \frac{C^*}{\varepsilon} \int_0^t \int_{\boldsymbol{R}^N} (R+s)^{\mu+1} W(s) V(x) u^2 dxds.
\end{aligned} \tag{2.62}
$$

On the other hand, by multiplying (2.55) by $(R+t)^\mu$ it follows that

$$
\frac{d}{dt}\{(R+t)^\mu E_\chi(t)\} + (R+t)^\mu \int_{\boldsymbol{R}^N} V(x) \chi_t^2 dx = \mu (R+t)^{\mu-1} E_\chi(t).
$$

Integrating over $[0,t]$ we have

$$(R+t)^{\mu}E_{\chi}(t)+\int_0^t\int_{\boldsymbol{R}^N}(R+s)^{\mu}V(x)\chi_t^2dxds$$
$$=R^{\mu}E_{\chi}(0)+\mu\int_0^t(R+s)^{\mu-1}E_{\chi}(s)ds. \tag{2.63}$$

Then $\chi_t=u$, the definition of $W(t)$, and (2.63) lead to

$$\int_0^t\int_{\boldsymbol{R}^N}(R+s)^{\mu+1}W(s)V(x)u^2dxds=V_0\int_0^t\int_{\boldsymbol{R}^N}(R+s)^{\mu}V(x)\chi_t^2dxds$$
$$\le V_0R^{\mu}E_{\chi}(0)+V_0\mu\int_0^t(R+s)^{\mu-1}E_{\chi}(s)ds.$$

Therefore, from (2.62) it follows that

$$2\{V_0-(1+\varepsilon)(\mu+2)\}\int_0^t(R+s)^{\mu+1}E_u(s)ds$$
$$\le C_1+(\mu+2)V_0R^{\mu}E_{\chi}(0)\frac{C^*}{\varepsilon}+(\mu+2)V_0|\mu|\frac{C^*}{\varepsilon}\int_0^t(R+s)^{\mu-1}E_{\chi}(s)ds. \tag{2.64}$$

Now, for fixed $\mu>-1$, we take

$$V_0>\mu+2>1.$$

Furthermore, choose $\varepsilon>0$ so small that

$$\frac{1}{\mu+2}\{V_0-(\mu+2)\}>\varepsilon>0.$$

Then, the estimate in Proposition 2.11 can be derived with

$$C_{00}:=\left(C_1+(\mu+2)V_0R^{\mu}E_{\chi}(0)\frac{C^*}{\varepsilon}\right)(2\{V_0-(1+\varepsilon)(\mu+2)\})^{-1},$$

and

$$C_{10}:=(\mu+2)V_0|\mu|\frac{C^*}{\varepsilon}(2\{V_0-(1+\varepsilon)(\mu+2)\})^{-1}.$$

In the above $C^*>0$ is a large number specified in the sequel in a way that the conditions (iv) of Lemma 2.13 holds for the function $W(t)$.

What left is to check the conditions (i)–(iv) of Lemmas 2.12, 2.13 for the function $W(t)$. The condition (ii) trivially follows from the definition of $W(t)$. Concerning (i), we have

$$V(x) \geq \frac{V_0}{\sqrt{1+|x|^2}} \geq \frac{V_0}{1+|x|} \geq \frac{V_0}{1+R_0+t} \geq \frac{V_0}{R+t} = W(t)$$

for $|x| \leq R_0 + t$, which implies (i).

Since

$$-W(t)u_t u \leq \frac{u_t^2}{2} + \frac{W(t)^2}{2}u^2,$$

we get

$$F(t,x) \geq \frac{1}{2}u_t^2 + |\nabla u|^2 + \frac{W(t)V(x) - W'(t) - W(t)^2}{2}u^2.$$

Therefore, in order to check (iii) it is enough to show that

$$W(t)V(x) - W'(t) - W(t)^2 \geq 0. \tag{2.65}$$

The estimate (2.65) follows from (i) and $W'(t) < 0$:

$$W(t)V(x) - W'(t) - W(t)^2 \geq W(t)V(x) - W(t)^2 = W(t)(V(x) - W(t)) \geq 0.$$

Finally, let us check (iv). Note that (iv) if and only if

$$J(t,x) := \varepsilon u_t^2 + \frac{C^*}{\varepsilon}W(t)V(x)u^2 - W(t)u_t u - \frac{W(t)V(x) - W'(t)}{2}u^2 \geq 0. \tag{2.66}$$

We rewrite $J(t,x)$ as

$$J(t,x) = \left(\sqrt{\varepsilon}u_t - \frac{1}{2\sqrt{\varepsilon}}W(t)u\right)^2 + u^2\left(\frac{W'(t)}{2} - \frac{W(t)V(x)}{2} + \frac{C^*}{\varepsilon}W(t)V(x) - \frac{W(t)^2}{4\varepsilon}\right),$$

and using the FSP we get the estimate

$$
\begin{aligned}
J(t,x) &\geq \left\{ \left(\frac{C^*}{\varepsilon} - \frac{1}{2} \right) W(t)V(x) + \frac{W'(t)}{2} - \frac{W(t)^2}{4\varepsilon} \right\} u^2 \\
&\geq \left\{ \left(\frac{C^*}{\varepsilon} - \frac{1}{2} \right) \frac{V_0}{R+t} \frac{V_0}{1+R_0+t} - \frac{V_0}{2(R+t)^2} - \frac{V_0^2}{4\varepsilon(R+t)^2} \right\} u^2 \\
&= \frac{V_0}{(R+t)^2} \left\{ \left(\frac{C^*}{\varepsilon} - \frac{1}{2} - \frac{1}{4\varepsilon} \right) V_0 - \frac{1}{2} \right\} u^2 \quad (R > R_0 + 1).
\end{aligned} \tag{2.67}
$$

We get $J(t,x) \geq 0$ if we choose $C^* > 0$ large and $\varepsilon > 0$ small such that

$$
\begin{gathered}
C^* > \frac{1}{4}, \quad \frac{(4C^* - 1)}{4\varepsilon} - \frac{1}{2} > 0, \\
1 \geq \frac{1}{2} \left\{ \frac{(4C^* - 1)}{4\varepsilon} - \frac{1}{2} \right\}^{-1}.
\end{gathered} \tag{2.68}
$$

In particular, if (2.68) holds, then

$$
V_0 > \mu + 2 > 1 \geq \frac{1}{2} \left\{ \frac{(4C^* - 1)}{4\varepsilon} - \frac{1}{2} \right\}^{-1},
$$

which implies (see (2.67))

$$
\left(\frac{C^*}{\varepsilon} - \frac{1}{4\varepsilon} - \frac{1}{2} \right) V_0 - \frac{1}{2} > 0.
$$

This completes the proof of Proposition 2.11. □

3. Two-dimensional case.

In this section we prove Theorem 1.3 in order to obtain a sharp result in the two dimensional case. Note that in Lemma 2.9 we used the Hardy-Sobolev inequality. Here we will use the two dimensional Hardy-Sobolev inequality (see Lemma 3.5 below).

In this two dimensional part the weights w, w_1 are much simpler and can be chosen as functions of t only: $w = f(t)$ and $w_1 = g(t)$. Later on these weights $f(t)$ and $g(t)$ will be specified. Now we first multiply the equation (1.1) by $f(t)u_t + g(t)u$, and integrate over $\boldsymbol{R}^N$ and get the following Lemma:

LEMMA 3.1. *Let $N \geq 1$, and let $u \in X_1(0, +\infty)$ be the solution to* (1.1)–(1.2). *Then it is true that*

$$\frac{d}{dt}E(t) + F(t) = 0,$$

where

$$E(t) = \frac{1}{2}\int_{\boldsymbol{R}^N} \left[f(u_t^2 + |\nabla u|^2) + 2guu_t + (Vg - g_t)u^2\right]dx,$$

$$F(t) = \frac{1}{2}\int_{\boldsymbol{R}^N} (2Vf - f_t - 2g)u_t^2\ dx + \frac{1}{2}\int_{\boldsymbol{R}^N} (2g - f_t)|\nabla u|^2\ dx$$
$$+ \frac{1}{2}\int_{\boldsymbol{R}^N} (g_{tt} - Vg_t)u^2\ dx.$$

Proof. The proof is similar to one in [**6**, Lemma 2.1] and we omit it. □

Since the finite speed of propagation applies again to solutions of the corresponding problem (1.1)–(1.2) to estimate the functions $E(t)$ and $F(t)$ it is sufficient to consider the spatial integration over the light cone

$$\Omega(t) = \{x \in \boldsymbol{R}^N : |x| \le R_0 + t\}.$$

Then we have the following lemma.

Lemma 3.2. *Let $N \ge 1$. Assume that the smooth functions $f(t)$ and $g(t)$ satisfy the two conditions below: for $t \ge t_0 \ge 0$,*

(i) $2fV - f_t - 2g \ge 0, \quad x \in \Omega(t)$,
(ii) $2g - f_t \ge 0$.

If $u \in X_1(0, +\infty)$ is the solution to (1.1)–(1.2), *then the following inequality holds*

$$\frac{d}{dt}\{f(t)E_u(t) + g(t)(u_t(t,\cdot), u(t,\cdot))\}$$
$$\le \frac{d}{dt}\left\{\frac{1}{2}\int_{\boldsymbol{R}^N} (g_t - V(x)g)|u(t,x)|^2 dx\right\} + \frac{1}{2}\int_{\boldsymbol{R}^N} V(x)g_t|u(t,x)|^2 dx$$
$$+ \frac{1}{2}\int_{\boldsymbol{R}^N} (-g_{tt})|u(t,x)|^2 dx, \tag{3.1}$$

for $t \ge t_0 \ge 0$.

The proof of this lemma can be easily derived by relying on Lemma 3.1, so we omit it.

Furthermore, we have the estimates.

LEMMA 3.3. *In addition to the assumptions as in Lemma* 3.2 *we further assume that the smooth functions* $f(t)$ *and* $g(t)$ *satisfy the following three conditions*: *for* $t \geq t_0 \geq 0$,

(iii) $-g_{tt} \leq \dfrac{C_1}{1+t}, \quad x \in \boldsymbol{R}^N,$
(iv) $V(x)g_t \leq C_2 V(x), \quad x \in \boldsymbol{R}^N,$
(v) $g_t - V(x)g \leq C_3,$

where $C_i > 0$ $(i = 1, 2, 3)$ *are some constants. If* $u \in X_1(0, +\infty)$ *is the unique solution to* (1.1)–(1.2), *then it is true that*

$$\begin{aligned} & f(t)E_u(t) + g(t)(u_t(t,\cdot), u(t,\cdot)) \\ & \quad \leq f(t_0)E_u(t_0) + g(t_0)(u_t(t_0,\cdot), u(t_0,\cdot)) \\ & \quad\quad + \frac{C_3}{2}\int_{\boldsymbol{R}^N} |u(t,x)|^2 dx - \frac{1}{2}\int_{\boldsymbol{R}^N} (g_t(t_0) - V(x)g(t_0))|u(t_0,x)|^2 dx \\ & \quad\quad + C_2 \int_{t_0}^{t}\int_{\boldsymbol{R}^N} V(x)|u(s,x)|^2 dxds + \frac{C_1}{2V_0}\int_{t_0}^{t}\int_{\boldsymbol{R}^N} \frac{V_0}{1+s}|u(s,x)|^2 dxds, \end{aligned}$$

for each $t \geq t_0 \geq 0$.

PROOF. By using the assumptions (i) and (ii) in Lemma 3.2 we can obtain the inequality (3.1). Then we integrate both sides of (3.1) over $[t_0, t]$, apply the additional conditions (iii)–(v) and get the desired estimate. □

Now we choose the functions $f(t)$ and $g(t)$ in Lemmas 3.2 and 3.3 as follows. In the case $2 < V_0$ we set

$$f(t) = (1+t)^2, \quad g(t) = (1+t), \tag{3.2}$$

and in the case $1 < V_0 \leq 2$, for an arbitrarily fixed $\delta > 0$ we choose

$$f(t) = (1+t)^{V_0 - \delta}, \quad g(t) = \frac{V_0 - \delta}{2}(1+t)^{V_0 - 1 - \delta}. \tag{3.3}$$

LEMMA 3.4. *Let* f *and* g *be defined by* (3.2) *in the case* $2 < V_0$, *and by* (3.3) *in the case* $1 < V_0 \leq 2$. *Then the conditions* (i)–(v) *in Lemmas* 3.2 *and* 3.3 *hold on* $\Omega(t)$ *for* $t \geq t_0 \gg 1$.

Proof. First we check the conditions (i)–(v) in the case $2 < V_0$ (we choose (3.2) in this case). Indeed,

$$\begin{aligned}2fV - f_t - 2g &\geq 2(1+t)^2\frac{V_0}{1+|x|} - 4(1+t)\\ &= 2(1+t)\left\{\frac{V_0(1+t)}{1+|x|} - 2\right\} \geq 2(1+t)\left\{\frac{V_0t+V_0}{(1+R_0)+t} - 2\right\} > 0\end{aligned}$$

for all $x \in \Omega(t)$ with sufficiently large $t \geq t_0 \gg 1$, since

$$\lim_{t\to+\infty}\left(\frac{V_0t+V_0}{(1+R_0)+t} - 2\right) = V_0 - 2 > 0.$$

(ii), (iii), (iv) and (v) follow from elementary calculations.

We omit the check of conditions (i)–(v) in the case $1 < V_0 \leq 2$, since the calculations are straightforward. □

It follows from Lemma 3.3 with large t_0 (defined as in Lemma 3.4) and Lemma 3.4 that

$$\begin{aligned}&f(t)E_u(t) + g(t)(u(t,\cdot), u_t(t,\cdot))\\ &\quad\leq C_4 + \frac{C_3}{2}\int_{\boldsymbol{R}^N}|u(t,x)|^2dx + \frac{C_1}{2V_0}\int_{t_0}^t\int_{\boldsymbol{R}^N}\frac{V_0}{1+s}|u(s,x)|^2dxds\\ &\qquad + C_2\int_{t_0}^t\int_{\boldsymbol{R}^N}V(x)|u(s,x)|^2dxds, \qquad (3.4)\end{aligned}$$

where

$$C_4 = f(t_0)E_u(t_0)+g(t_0)(u(t_0,\cdot), u_t(t_0,\cdot)) - \frac{1}{2}\int_{\boldsymbol{R}^N}(g_t(t_0) - V(x)g(t_0))|u(t_0,x)|^2dx.$$

On the other hand, as in [**4**, Lemma 2.5] we can derive the following Lemma based on the two dimensional Hardy-Sobolev inequality.

Lemma 3.5. *Let $N = 2$, and let $u \in X_1(0,+\infty)$ be the solution to* (1.1)–(1.2). *Then the following estimate holds*

$$\|u(t,\cdot)\|^2 + \int_{t_0}^t\int_{\boldsymbol{R}^2}V(x)|u(s,x)|^2dxds \leq C_{R_0}(\|u_0\|^2 + \|u_1\|^2), \qquad (3.5)$$

for all $t \geq t_0$*, where* $C_{R_0} > 0$ *is a constant depending on* R_0*.*

PROOF. We introduce an auxiliary function

$$\chi(t,x) = \int_0^t u(s,x)ds.$$

Then $\chi(t,x)$ satisfies

$$\chi_{tt} - \Delta\chi + V(x)\chi_t = V(x)u_0 + u_1, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^2, \tag{3.6}$$

$$\chi(0,x) = 0, \quad \chi_t(0,x) = u_0(x), \quad x \in \boldsymbol{R}^2. \tag{3.7}$$

Multiplying (3.6) by χ_t and integrating over $[0,t] \times \boldsymbol{R}^2$ we get

$$\begin{aligned} &\frac{1}{2}(\|\chi_t(t,\cdot)\|^2 + \|\nabla\chi(t,\cdot)\|^2) + \int_0^t \int_{\boldsymbol{R}^2} V(x)\chi_t(s,x)^2 dxds \\ &\quad = \frac{1}{2}\|u_0\|^2 + \int_{\boldsymbol{R}^2} (V(x)u_0(x) + u_1(x))\chi(t,x)dx. \end{aligned} \tag{3.8}$$

Next step is to use the two dimensional Hardy-Sobolev inequality (see [**11**, Lemma 2]):

$$\int_{\boldsymbol{R}^2} \frac{|w(x)|^2}{d(x)^2} dx \leq C \int_{\boldsymbol{R}^2} |\nabla w(x)|^2 dx, \quad w \in H^1(\boldsymbol{R}^2), \tag{3.9}$$

where

$$d(x) := \{1 + \log(1+|x|)\}(1+|x|).$$

The last term of (3.8) can be estimated by using (3.9) and the Schwartz inequality as follows.

$$\begin{aligned} &\int_{\boldsymbol{R}^2} (V(x)u_0(x) + u_1(x))\chi(t,x)dx \\ &\quad \leq \int_{\boldsymbol{R}^2} d(x)(V(x)|u_0(x)| + |u_1(x)|)\frac{|\chi(t,x)|}{d(x)}dx \\ &\quad \leq \left\{ \int_{\boldsymbol{R}^2} d(x)^2(V(x)|u_0(x)| + |u_1(x)|)^2 dx \right\}^{1/2} \left\{ \int_{\boldsymbol{R}^2} \frac{|\chi(t,x)|^2}{d(x)^2} dx \right\}^{1/2} \end{aligned}$$

$$\leq C\left\{\int_{\boldsymbol{R}^2} d(x)^2(V(x)|u_0(x)|+|u_1(x)|)^2dx\right\}^{1/2}\|\nabla\chi(t,\cdot)\|$$
$$\leq \frac{1}{4}\|\nabla\chi(t,\cdot)\|^2 + C_{R_0}\int_{\boldsymbol{R}^2}(|u_0(x)|^2+|u_1(x)|^2)dx, \tag{3.10}$$

where $C_{R_0} > 0$ is a constant depending on $R_0 > 0$ and $V_0 > 0$. Combining (3.8) and (3.10) we derive

$$\frac{1}{2}\|\chi_t(t,\cdot)\|^2 + \frac{1}{4}\|\nabla\chi(t,\cdot)\|^2 + \int_0^t\int_{\boldsymbol{R}^2} V(x)\chi_t(s,x)^2dxds$$
$$\leq \frac{1}{2}\|u_0\|^2 + C_{R_0}(\|u_0\|^2+\|u_1\|^2).$$

The estimate (3.5) follows from above estimate and the fact that $\chi_t = u$. □

As consequence of Lemma 3.5 and FSP, since

$$V(x) \geq \frac{V_0}{1+|x|} \geq \frac{V_0}{1+R_0+t} \geq \frac{1}{(1+R_0)}\frac{V_0}{1+t},$$

we have

$$\int_{t_0}^t\int_{\boldsymbol{R}^2}\frac{V_0}{1+s}|u(s,x)|^2dxds \leq C_{R_0}(\|u_0\|^2+\|u_1\|^2), \tag{3.11}$$

where $C_{R_0} > 0$ is a constant dependent only on R_0.

Now we present the proof of Theorem 1.3.

Proof of Theorem 1.3. It follows from Lemma 3.5, (3.4) and (3.11) that

$$f(t)E_u(t) + g(t)(u(t,\cdot),u_t(t,\cdot)) \leq C_{R_0,\delta},$$

where the constant $C_{R_0,\delta} > 0$ depends on the L^2-norm of the initial data, R_0 and δ (in the case when $1 < V_0 \leq 2$). By using the Schwartz inequality, the definition of the total energy and Lemma 3.5 we get

$$f(t)E_u(t) \leq g(t)\|u(t,\cdot)\|\|u_t(t,\cdot)\| + C \leq Cg(t)\sqrt{E_u(t)} + C, \quad t \geq t_0$$

with a constant $C > 0$ dependent on R_0, δ and the initial data. Further, we set $X(t) = \sqrt{E_u(t)}$ for $t \in [0,+\infty)$ and get

$$f(t)X(t)^2 - Cg(t)X(t) - C \leq 0, \quad t \geq t_0. \tag{3.12}$$

By solving the quadratic inequality (3.12) for $X(t)$ we have

$$\sqrt{E_u(t)} \leq \frac{Cg(t) + \sqrt{C^2 g(t)^2 + 4Cf(t)}}{2f(t)}.$$

This inequality leads to

$$E_u(t) \leq C\left(\frac{g(t)}{f(t)}\right)^2 + C\left(\frac{1}{f(t)}\right), \quad t \geq t_0,$$

which implies the needed decay estimates in both cases for V_0. □

4. One-dimensional case.

In this section we prove Theorem 1.4. The proof is almost the same as the proof of Theorem 1.3 except for Lemma 3.5. We modify the proof of Lemma 3.5 and use the following Morrey inequality instead of the two dimensional Hardy-Sobolev inequality.

Lemma 4.1 (Morrey). *The following estimate holds*

$$|u(x) - u(y)| \leq |x - y|^{1/2} \|u_x\|,$$

for all $u \in H^1(\boldsymbol{R})$.

Relying on Lemma 4.1 we can prove next Lemma (see [**3**]) which is similar to Lemma 3.5.

Lemma 4.2. *Let* $N = 1$, *and suppose* (H1). *Let* $u \in X_1(0, +\infty)$ *be the solution to problem* (1.1)–(1.2). *Then the following bound holds*

$$\|u(t,\cdot)\|^2 + \int_0^t \int_{\boldsymbol{R}} V(x) u(s,x)^2 dx ds \leq C_{R_0}(\|u_0\|^2 + \|u_1\|^2),$$

with some constant $C_{R_0} > 0$ *depending only on* $R_0 > 0$.

Proof. For the solution $u \in X_1([0, +\infty))$ to problem (1.1)–(1.2), we introduce a new function

$$W(t,x) = \int_0^t u(s,x)ds.$$

Then $W(t,x)$ satisfies

$$W_{tt} - W_{xx} + V(x)W_t = V(x)u_0 + u_1, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}, \tag{4.1}$$

$$W(0,x) = 0, \quad W_t(0,x) = u_0(x), \quad x \in \boldsymbol{R}. \tag{4.2}$$

By multiplying both sides of (4.1)–(4.2) by W_t, and integration it over $[0,t] \times \boldsymbol{R}$, one has the identity

$$\begin{aligned}\frac{1}{2}(\|W_t(t,\cdot)\|^2 + \|W_x(t,\cdot)\|^2) &+ \int_0^t \int_{\boldsymbol{R}} V(x)W_t(s,x)^2 dxds \\ &= \frac{1}{2}\|u_0\|^2 + \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))W(t,x)dx. \end{aligned} \tag{4.3}$$

Because of Lemma 4.1, one finds that

$$|W(t,x) - W(t,0)| \le |x|^{1/2}\|W_x(t,\cdot)\| \le \sqrt{(|x|+1)}\|W_x(t,\cdot)\|. \tag{4.4}$$

Now the last term of (4.3) can be estimated as follows by using (H1), (4.4) and the Schwarz inequality:

$$\begin{aligned}&\int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))W(t,x)dx \\ &= \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))(W(t,x) - W(t,0))dx \\ &\quad + \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))W(t,0)dx \\ &= \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))(W(t,x) - W(t,0))dx \\ &\quad + W(t,0)\int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))dx \\ &= \int_{\boldsymbol{R}} (V(x)u_0(x) + u_1(x))(W(t,x) - W(t,0))dx \\ &\le \int_{\boldsymbol{R}} (V(x)|u_0(x)| + |u_1(x)|)|W(t,x) - W(t,0)|dx\end{aligned}$$

$$= \int_{\boldsymbol{R}} \sqrt{1+|x|}(V(x)|u_0(x)| + |u_1(x)|)\frac{|W(t,x)-W(t,0)|}{\sqrt{1+|x|}}dx$$

$$\leq \left(\sup_{x\in\boldsymbol{R}} \frac{|W(t,x)-W(t,0)|}{\sqrt{1+|x|}}\right)\int_{\boldsymbol{R}} \sqrt{1+R_0}(V(x)|u_0(x)| + |u_1(x)|)dx$$

$$\leq \|W_x(t,\cdot)\|\cdot\sqrt{C_{R_0}}\left\{\int_{-R_0}^{R_0}(|u_0(x)|^2+|u_1(x)|^2)dx\right\}^{1/2}$$

$$\leq \frac{1}{4}\|W_x(t,\cdot)\|^2 + C_{R_0}\int_{\boldsymbol{R}}(|u_0(x)|^2+|u_1(x)|^2)dx, \tag{4.5}$$

where $C_{R_0} > 0$ is a constant depending on $R_0 > 0$ and $V_0 > 0$. The statement can be derived by (4.3) and (4.5) because of $W_t = u$. The other parts of the proof are similar to ones in the proof of Theorem 1.3. □

We omit the proof of Proposition 1.5 since it is similar to one in [**6**].

5. Exactness of the decay rate.

In this section we discuss the optimality of decay rates obtained in Theorems 1.1, 1.3 and 1.4. For simplicity of notations, in what follows we use $R > 0$ instead of $R_0 > 0$, where $R_0 > 0$ is the size of support of the initial data (u_0, u_1).

Proof of Theorem 1.6. We use a positive radially symmetric smooth solution $\phi(x)$ to the elliptic problem on $\boldsymbol{R}^N$:

$$\Delta\phi = (1+V(x))\phi \tag{5.1}$$

satisfying (5.10) below. The proof of existence of such solution $\phi(x)$ will be postponed for a while. Next, we set

$$v(t,x) := e^{-t}\phi(x).$$

Then the function $v(t,x)$ solves the following equation with anti-damping term

$$v_{tt} - \Delta v - V(x)v_t = 0, \quad (t,x)\in(0,\infty)\times\boldsymbol{R}^N, \tag{5.2}$$

$$v(0,x) = \phi(x), \quad v_t(0,x) = -\phi(x), \quad x\in\boldsymbol{R}^N. \tag{5.3}$$

Cross multiplying with v_t and u_t we get

$$(u_{tt}, v_t) - (\Delta u, v_t) + (Vu_t, v_t) = 0, \tag{5.4}$$

$$(v_{tt}, u_t) - (\Delta v, u_t) - (Vv_t, u_t) = 0. \tag{5.5}$$

Since

$$\frac{d}{dt}(u_t, v_t) = (u_{tt}, v_t) + (v_{tt}, u_t),$$

$$\frac{d}{dt}\{\nabla u \cdot \nabla v\} = \nabla u \cdot \nabla v_t + \nabla u_t \cdot \nabla v,$$

by adding (5.4) and (5.5), and using integration by parts we obtain

$$\frac{d}{dt}\int_{\boldsymbol{R}^N}(u_t v_t + \nabla u \cdot \nabla v)dx = 0. \tag{5.6}$$

Now, let us choose the initial data $[u_0, u_1] \in C_0^\infty(\boldsymbol{R}^N) \times C_0^\infty(\boldsymbol{R}^N)$ of the original problem (1.1)–(1.2) to satisfy

$$P_0 := -(u_1, \phi) + (\nabla u_0, \nabla\phi) \neq 0. \tag{5.7}$$

The set of initial data $[u_0, u_1]$ satisfying (5.7) is not empty. For example, if $u_1(x) \leq 0$ and $u_0(x) \equiv 0$, since ϕ is positive, we get $P_0 > 0$. Then, by integrating (5.6) over $(0, t)$ we have

$$\int_{\boldsymbol{R}^N}(u_t v_t + \nabla u \cdot \nabla v)dx = P_0 \neq 0. \tag{5.8}$$

Let $\xi(x)$ be the characteristic function of the ball $B(R+t)$:

$$\xi(x) = \begin{cases} 1 & |x| \leq t + R, \\ 0 & |x| > t + R. \end{cases}$$

Then, it follows from the Schwarz inequality that

$$\begin{aligned} &\left|\int_{\boldsymbol{R}^N}(u_t v_t + \nabla u \cdot \nabla v)dx\right| \\ &\quad \leq \int_{B(R+t)}(|u_t||\xi v_t| + |\nabla u||\xi\nabla v|)dx \end{aligned}$$

$$\le 2\int_{B(R+t)}\sqrt{u_t^2+|\nabla u|^2}\sqrt{|\xi v_t|^2+|\xi\nabla v|^2}dx$$

$$\le 2\left\{\int_{\boldsymbol{R}^N}(u_t^2+|\nabla u|^2)dx\right\}^{1/2}\left\{\int_{\boldsymbol{R}^N}(|\xi v_t|^2+|\xi\nabla v|^2)dx\right\}^{1/2},$$

so that from (5.8) we have

$$\frac{P_0^2}{4}\left\{\int_{\boldsymbol{R}^N}(|\xi v_t|^2+|\xi\nabla v|^2)dx\right\}^{-1}\le\int_{\boldsymbol{R}^N}(u_t^2+|\nabla u|^2)dx. \tag{5.9}$$

Now, if we can construct a positive smooth radially symmetric solution to (5.1) satisfying:

$$\begin{aligned}\phi(x)&\sim|x|^{(V_0-N+1)/2}e^{|x|},\\ |\nabla\phi(x)|&\sim|x|^{(V_0-N+1)/2}e^{|x|}|1+\frac{V_0-N+1}{2|x|}|,\quad(|x|\to\infty),\end{aligned} \tag{5.10}$$

for radial $V(x)\sim V_0|x|^{-1}$ as $|x|\to+\infty$ we can show that $v(t,x)=e^{-t}\phi(x)$ satisfies the inequality

$$\|\xi v_t\|_2^2+\|\xi\nabla v\|_2^2\le Ct^{V_0},\quad t\to\infty. \tag{5.11}$$

The estimate (5.11) together with (5.9) will give us the desired estimate $\|u_t\|_2^2+\|\nabla u\|_2^2\ge Ct^{-V_0}$. It remains to check that the problem (5.1) has a solution with the properties (5.10) and (5.11).

Let us first derive (5.11) under the condition (5.1) and (5.10). Indeed we have

$$\begin{aligned}\int_{\boldsymbol{R}^N}(|\xi v_t|^2+|\xi\nabla v|^2)dx&=\int_{|x|\le t+R}(v_t^2+|\nabla v|^2)dx\\ &=e^{-2t}\int_{|x|\le t+R}(\phi(x)^2+|\nabla\phi(x)|^2)dx.\end{aligned} \tag{5.12}$$

The first term in the right hand side of (5.12) can be estimated by using integration by parts and (5.10) as follows:

$$\int_{|x|\le t+R}\phi(x)^2dx\le\int_{|x|\le\rho_0}\phi(x)^2dx+\int_{\rho_0\le|x|\le t+R}e^{2|x|}|x|^{V_0-(N-1)}dx.$$

Further we get

$$\begin{aligned}
&\int_{\rho_0\le|x|\le t+R} e^{2|x|}|x|^{V_0-(N-1)}dx \\
&\quad = \int_{|\omega|=1} d\omega \int_{\rho_0}^{t+R} e^{2r}r^{V_0-(N-1)}r^{N-1}dr = \int_{|\omega|=1} d\omega \int_{\rho_0}^{t+R} e^{2r}r^{V_0}dr \\
&\quad = \int_{|\omega|=1} \left\{ \left[\frac{e^{2r}}{2}r^{V_0}\right]_{\rho_0}^{t+R} - \frac{V_0}{2}\int_{\rho_0}^{t+R} e^{2r}r^{V_0-1}dr \right\} d\omega \le \frac{e^{2R}}{2}C_0e^{2t}(t+R)^{V_0},
\end{aligned}$$

where $\rho_0 > 0$ and $t > 0$ are sufficiently large numbers. Therefore,

$$\int_{|x|\le t+R} \phi(x)^2 dx \le C_1 + C_R e^{2t}(t+R)^{V_0}, \quad (t \gg 1), \tag{5.13}$$

where $C_j > 0$ $(j = 0, 1, R)$ are some constants, which do not depend on t.

For the second term in the right hand side of (5.12), we get by using integration by parts and (5.10) the following estimate:

$$\begin{aligned}
&\int_{|x|\le t+R} |\nabla\phi(x)|^2 dx \\
&\quad \le \int_{|x|\le\rho_0} |\nabla\phi(x)|^2 dx + \int_{\rho_0\le|x|\le t+R} e^{2|x|}|x|^{V_0-(N-1)}\left(1 + \frac{V_0 - N + 1}{2|x|}\right)^2 dx \\
&\quad \le \int_{|x|\le\rho_0} |\nabla\phi(x)|^2 dx + C_3\int_{\rho_0\le|x|\le t+R} e^{2|x|}|x|^{V_0-(N-1)}dx, \quad (\rho_0 \gg 1,\, t \gg 1).
\end{aligned}$$

We can complete similar calculations to (5.13) and get

$$\int_{|x|\le t+R} |\nabla\phi(x)|^2 dx \le C_4 + C_R e^{2t}(t+R)^{V_0}, \tag{5.14}$$

where $C_j > 0$ $(j = 3, 4, R)$ are constants.

The estimate (5.11) can be derived from (5.12), (5.13) and (5.14).

Finally we discuss the existence of a positive radially symmetric solution to (5.1) satisfying (5.10). The idea to construct an asymptotic solution to (5.1) follows from Kato [**8**].

Since (5.1) is equivalent to the following equation in the framework of radial symmetry

$$\phi''(r) + \frac{N-1}{r}\phi'(r) = (1+V(r))\phi(r), \tag{5.15}$$

we want to find a solution to (5.15) with initial data $\phi(0) = 1$ and $\phi'(0) = 1$. The singular coefficient at $r = 0$ does not prevent the above problem from having C^2 solutions. Indeed, ϕ can be obtained as a solution of the integral equation

$$\phi(r) = 1 + \int_0^r K_N(r,s)(1+V(s))\phi(s)\,ds, \quad r > 0,$$

where

$$K_N(r,s) = \begin{cases} s\log\frac{r}{s}, & \text{if } N = 2, \\ \frac{s}{N-2}\left(1 - \frac{s^{N-2}}{r^{N-2}}\right), & \text{if } N \geq 3. \end{cases}$$

It is easy to see that $\phi(r) \geq 0$ and $\phi'(r) \geq 0$ for all r. Hence we only need upper bounds for these functions at large $r \geq 1$.

The substitution of $\phi(r) = e^{r+q(r)}$ into (5.15) yields

$$q''(r) + 2q'(r) + (q'(r))^2 + \frac{N-1}{r}q'(r) = V(r) - \frac{N-1}{r}.$$

We observe that

$$(q'(r))^2 + \frac{N-1}{r}q'(r) \geq -\frac{(N-1)^2}{4r^2},$$

which implies the linear OD inequality

$$q''(r) + 2q'(r) \leq V(r) - \frac{N-1}{r} + \frac{(N-1)^2}{4r^2}. \tag{5.16}$$

We multiply the above ODI by e^{2r} and integrate over $[1,r]$:

$$e^{2r}q'(r) \leq e^2 q'(1) + \int_1^r e^{2s}\left(V(s) - \frac{N-1}{s} + \frac{(N-1)^2}{4s^2}\right)ds, \quad r \geq 1.$$

The right-hand side is asymptotically

$$\frac{V_0 - (N-1)}{2}r^{-1}e^{2r} + O(r^{-2})e^{2r},$$

since $V(r) \sim V_0 r^{-1}$ at large r. Thus, we have

$$q'(r) \le \frac{V_0 - (N-1)}{2} r^{-1} + O(r^{-2}), \quad q(r) \le \frac{V_0 - (N-1)}{2} \log r + O(1).$$

These estimates lead to (5.10), since

$$\phi'(r) = (1 + q'(r)) e^{r+q(r)}, \quad \phi(r) = e^{r+q(r)}. \qquad \square$$

6. Appendix: Weighted $L^2 - L^2$ estimates for $\Delta h = f$.

In the appendix we present the outline of the proof of Lemma 2.1.

There are many estimates for $\Delta h = f$ in $\boldsymbol{R}^N$ when f is a "nice" function; our function f is only $L^2(\boldsymbol{R}^N)$, but vanishes outside a ball with a radius R_0. Unfortunately there is no estimates available that involve powers of $\langle x \rangle$ as weights and hold in all dimensions $N \ge 3$. Therefore, we derive such Hardy and Rellich inequalities by the method of [**1**]. The constants are not sharp but the calculations are simple.

6.1. Preliminary facts.

Proposition 6.1. *The following identities hold for suitable smooth functions Q and h:*

$$\text{(i)} \quad \int_{\boldsymbol{R}^N} h^2 \Delta Q \, dx = -2 \int_{\boldsymbol{R}^N} h \nabla h \cdot \nabla Q \, dx,$$

$$\text{(ii)} \quad \int_{\boldsymbol{R}^N} |\nabla h|^2 Q \, dx = - \int_{\boldsymbol{R}^N} (\Delta h) h Q \, dx + \frac{1}{2} \int_{\boldsymbol{R}^N} h^2 \Delta Q \, dx.$$

We can combine (ii) with $|(\Delta h) h Q| \le \varepsilon h^2 (\Delta Q)/2 + (\Delta h)^2 Q^2 / (2\varepsilon \Delta Q)$.

Proposition 6.2. *Assume that Q and ΔQ are positive a.e. Then*

$$\text{(i)} \quad \int_{\boldsymbol{R}^N} h^2 \Delta Q \, dx \le 4 \int_{\boldsymbol{R}^N} |\nabla h|^2 \frac{|\nabla Q|^2}{\Delta Q} \, dx,$$

$$\text{(ii)} \quad \int_{\boldsymbol{R}^N} |\nabla h|^2 \left(Q - 2(1+\varepsilon) \frac{|\nabla Q|^2}{\Delta Q} \right) dx \le \frac{1}{2\varepsilon} \int_{\boldsymbol{R}^N} (\Delta h)^2 \frac{Q^2}{\Delta Q} \, dx,$$

where $\varepsilon > 0$. Moreover, (i) *and* (ii) *imply a weighted estimate of h in terms of Δh.*

Example. Choose $Q(x) = \langle x \rangle^s$, and let $H(x)$ be the measurable function defined in (2.1) for $f \in L^2(\boldsymbol{R}^N)$ with $f(x) = 0$ for $|x| \gg 1$. Then, since

$$H(x) = O(|x|^{-(N-2)}), \quad |\nabla H(x)| = O(|x|^{-(N-1)}), \quad |x| \to +\infty,$$

one has

$$\int_{|x|=\rho} |\nabla Q(\sigma)||H(\sigma)|^2 dS_\sigma = O(\rho^{-N+2+s}), \quad \rho \to +\infty,$$
$$\int_{|x|=\rho} |Q(\sigma)||H(\sigma)||\nabla H(\sigma)| dS_\sigma = O(\rho^{-N+2+s}), \quad \rho \to +\infty,$$

so that if we assume $0 < s < N - 2$, then Propositions 6.1 and 6.2 hold with $h(x) := H(x)$.

6.2. Applications.

Let $Q(x) = \langle x \rangle^s$. From $\Delta = \partial_r^2 + (N-1)r^{-1}\partial_r$, where $r = |x|$, we have

$$\nabla Q(x) = s\langle x \rangle^{s-2} x, \quad \Delta Q(x) = sN\langle x \rangle^{s-2} + s(s-2)\langle x \rangle^{s-4}|x|^2.$$

We also notice that

$$s(N+s-2)\langle x \rangle^{s-4}|x|^2 \le \Delta Q(x),$$
$$s\langle x \rangle^{s-2} \le \Delta Q(x),$$

if $s > 0$. These estimates and Proposition 6.2 (i) lead to the following Hardy inequality.

Proposition 6.3. *Assume that $N \ge 3$ and $(N-2) > s > 0$. If $h \in H^2_{loc}(\boldsymbol{R}^N)$ satisfies $h(x) = O(r^{-(N-2)})$ and $|\nabla h(x)| = O(r^{-(N-1)})$ as $r \to +\infty$, then*

$$\int_{\boldsymbol{R}^N} h^2(x)\langle x \rangle^{s-2}\, dx \le \frac{4}{N-2} \int_{\boldsymbol{R}^N} |\nabla h(x)|^2 \langle x \rangle^s \, dx.$$

Let us turn to the generalized Rellich inequality. If $s > 0$, we find

$$\frac{|\nabla Q|^2}{\Delta Q} \le \frac{s}{N+s-2}\langle x \rangle^s,$$
$$\frac{Q^2}{\Delta Q} \le \frac{1}{s}\langle x \rangle^{s+2}.$$

The first weight in Proposition 6.2 (ii) satisfies

$$Q - 2(1+\varepsilon)\frac{|\nabla Q|^2}{\Delta Q} \geq \left(1 - \frac{2s(1+\varepsilon)}{N+s-2}\right)\langle x\rangle^s,$$

while the second weight satisfies

$$\frac{1}{2\varepsilon}\frac{Q^2}{\Delta Q} \leq \frac{1}{2s\varepsilon}\langle x\rangle^{s+2}.$$

Choosing $\varepsilon := (1/2)\{(N+s-2)/(2s) - 1\}$, we obtain the following. In this case we note that for $s \in (0, N-2)$

$$\frac{N+s-2}{2s} > 1.$$

Proposition 6.4. *Assume that $N \geq 3$ and $N-2 > s > 0$. Then, under the same assumption as in Proposition* 6.3 *the following estimate holds*

$$\int_{\boldsymbol{R}^N} |\nabla h(x)|^2 \langle x\rangle^s \, dx \leq \frac{4(N-2+s)}{(N-2-s)^2}\int_{\boldsymbol{R}^N} [\Delta h(x)]^2 \langle x\rangle^{s+2} \, dx.$$

Corollary 6.5. *Assume that $N \geq 3$ and $N-2 > s > 0$. Then, under the same assumption as in Proposition* 6.3 *the bound holds*

$$\int_{\boldsymbol{R}^N} h^2(x)\langle x\rangle^{s-2} \, dx \leq \frac{16(N-2+s)}{(N-2)(N-2-s)^2}\int_{\boldsymbol{R}^N} [\Delta h(x)]^2 \langle x\rangle^{s+2} \, dx.$$

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