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On the approximate Jacobian Newton diagrams of an irreducible plane curve

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Abstract. We introduce the notion of an approximate Jacobian Newton diagram which is the Jacobian Newton diagram of the morphism $(f^{(k)}, f)$, where f is a branch and $f^{(k)}$ is a characteristic approximate root of f. We prove that the set of all approximate Jacobian Newton diagrams is a complete topological invariant. This generalizes theorems of Merle and Ephraim about the decomposition of the polar curve of a branch.

1. Introduction.

Every two complex series $f, g \in \boldsymbol{C}\{x, y\}$ such that $f(0,0) = g(0,0) = 0$ define a germ of a holomorphic mapping $(g, f) : (\boldsymbol{C}^2, 0) \longrightarrow (\boldsymbol{C}^2, 0)$. Assume that the curves $f = 0$ and $g = 0$ share no common component. Then the critical locus of this mapping is a germ of an analytic curve and its direct image by (g, f) is also an analytic curve called the *discriminant curve*. Let $D(u, v) = 0$ be an equation of the discriminant curve in the coordinates $(u, v) = (g(x, y), f(x, y))$. We call the Newton diagram of $D(u, v)$ the *Jacobian Newton diagram* of the morphism (g, f) and denote it $\mathcal{N}_J(g, f)$.

Note that if $g = 0$ is a smooth curve transverse to $f = 0$ then $\mathcal{N}_J(g, f)$ is the Jacobian Newton diagram of the curve $f = 0$ introduced in [**Te3**]. With these assumptions Teissier proves in [**Te1**] that $\mathcal{N}_J(g, f)$ depends only on the topological type of the curve $f = 0$.

Merle in [**Me**] studies the case of a smooth curve $g = 0$ transverse to an irreducible singular curve $f = 0$. He gives a description of the Jacobian Newton diagram in terms of other invariants of singularity of a curve $f = 0$. He also shows that the datum of the Jacobian Newton diagram determines the equisingularity class of the curve (or equivalently its embedded topological type). Ephraim in [**Eph**] extends Merle's result to any smooth curve $g = 0$.

Let f be an irreducible Weierstrass polynomial. In this paper we generalize the results of Merle to the family $\{\mathcal{N}_J(f^{(k)}, f)\}_k$, where $f^{(k)}$ is the k-th characteristic

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approximate root of f introduced in [**A-M**]. We prove, in two different ways, that this family is a complete topological invariant of the branch $f=0$. Our computations are based on the decomposition of the critical locus of the mapping $(f^{(k)},f)$, which is analogous to the decomposition of the polar curve obtained by Merle in [**Me**].

2. Plane branches, semigroup and approximate roots.

We mean by the fractional power series the elements of the ring $\boldsymbol{C}\{x\}^*=\bigcup_{n\in\boldsymbol{N}}\boldsymbol{C}\{x^{1/n}\}$. For every two fractional power series δ and δ' we call the number $\mathcal{O}(\delta,\delta')=\mathrm{ord}_x(\delta(x)-\delta'(x))$ the *contact order* between δ and δ'.

Every convergent power series $g(x,y)\in\boldsymbol{C}\{x,y\}$, $g(0,0)=0$ has the Newton-Puiseux factorization

$$g(x,y)=u(x,y)x^N\prod_{i=1}^{d}(y-\gamma_i(x)),$$

where $u(x,y)\in\boldsymbol{C}\{x,y\}$, $u(0,0)\neq 0$, N is a nonnegative integer and $\gamma_i(x)$ are fractional power series of positive order. We will call γ_i the Newton-Puiseux roots of g and denote the set $\{\gamma_1,\dots,\gamma_d\}$ by $\mathrm{Zer}\,g$.

Let $f(x,y)$ be an irreducible power series such that $\mathrm{ord}_y(f(0,y))=n\geq 1$. Then f has a Newton-Puiseux root of the form $\gamma_1(x)=\sum_{i=1}^{\infty}a_ix^{i/n}$. The other Newton-Puiseux roots are $\gamma_j(x)=\sum_{i=1}^{\infty}a_i\omega^{(j-1)i}x^{i/n}$ for $1\leq j\leq n$, where $\omega\in\boldsymbol{C}$ is an n-th primitive root of unity. The contact orders between the elements of $\mathrm{Zer}\,f$ form a set $\{b_1/n,\dots,b_g/n\}$, where $b_1<b_2<\cdots<b_g$ and $\gcd(n,b_1,\dots,b_g)=1$. We put $b_0=n$ and call the sequence $(b_0,b_1,\dots,b_g)$ the *Puiseux characteristic* of f. By convention $b_{g+1}=+\infty$.

Let A and B be finite sets of fractional power series. The *contact* $\mathrm{cont}(A,B)$ is by definition $\max\{\mathcal{O}(\alpha,\beta):\ \alpha\in A,\ \beta\in B\}$. If $\alpha(x)$ is a fractional power series and $f(x,y)$, $g(x,y)$ are irreducible power series co-prime to x then by abuse of notation we will write $\mathrm{cont}(\alpha,f):=\mathrm{cont}(\{\alpha\},\mathrm{Zer}\,f)$ and $\mathrm{cont}(f,g):=\mathrm{cont}(\mathrm{Zer}\,f,\mathrm{Zer}\,g)$.

It is well-known (see for example Lemma 4.3 of [**Ca1**]) that for every Newton-Puiseux root α of f we have $\mathrm{cont}(\alpha,g)=\mathrm{cont}(f,g)$. The contact between irreducible power series has a strong triangle inequality property: if $h_i\in\boldsymbol{C}\{x,y\}$ for $i=1,2,3$ are irreducible power series co-prime to x then $\mathrm{cont}(h_1,h_2)\geq\min(\mathrm{cont}(h_1,h_3),\mathrm{cont}(h_2,h_3))$.

In [**A-M**] the authors introduce the concept of the *approximate root* as a consequence of the following proposition:

Proposition 1. *Let $\boldsymbol{A}$ be an integral domain. If $f(y)\in\boldsymbol{A}[y]$ is monic of* *degree d and p is invertible in $\boldsymbol{A}$ and divides d, then there exists a unique monic polynomial $g(y) \in \boldsymbol{A}[y]$ such that the degree of $f - g^p$ is less than $d - d/p$.*

This allows us to define:

DEFINITION 1. The unique monic polynomial of the preceding proposition is called the p-th approximate root of f.

Let $f \in \boldsymbol{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with Puiseux characteristic $(b_0, \ldots, b_g)$. Put $l_k := \gcd(b_0, \ldots, b_k)$. In particular l_k divides $\deg f = b_0$ for all $k \in \{0, \ldots, g\}$. In the sequel for $k \in \{0, \ldots, g-1\}$ we denote $f^{(k)}$ the l_k-th approximate root of f and we call these polynomials the *characteristic approximate* roots of f. By convention we put $f^{(-1)} = x$.

The following proposition is the main one in [**A-M**] (see also [**G-Pł2**] and [**Po**]):

PROPOSITION 2. *Let $f \in \boldsymbol{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with Puiseux characteristic $(b_0, \ldots, b_g)$. Then the characteristic approximate roots $f^{(k)}$ for $k \in \{0, \ldots, g-1\}$, have the following properties*:

1. *The polynomial $f^{(k)}$ is irreducible with Puiseux characteristic $(b_0/l_k, \ldots, b_k/l_k)$.*
2. *The y-degree of $f^{(k)}$ is equal to b_0/l_k and* $\operatorname{cont}(f, f^{(k)}) = b_{k+1}/b_0$.

EXAMPLE 1. Take the irreducible Weierstrass polynomial $f = (y^3 - 6x^3y - x^4)^2 - 9x^9$ of Puiseux characteristic $(6, 8, 11)$. The characteristic approximate roots of f are $f^{(0)} = y$ and $f^{(1)} = y^3 - 6x^3y - x^4$. The Newton-Puiseux roots of f are of the form $y = \omega^8 x^{4/3} + 2\omega^{10}x^{5/3} + \omega^{11}x^{11/6} + \cdots$, where $\omega^6 = 1$ while the Newton-Puiseux roots of $f^{(1)}$ are $y = \epsilon^4 x^{4/3} + 2\epsilon^5 x^{5/3} - (8/3)x^2 + \cdots$, where $\epsilon^3 = 1$. One can check directly that $\operatorname{cont}(f, f^{(0)}) = 8/6$ and $\operatorname{cont}(f, f^{(1)}) = 11/6$.

3. Jacobian Newton diagrams.

In this section we recall the notion of the Jacobian Newton diagrams and we establish some preliminary results which are necessary for the next.

Write $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} : x \geq 0\}$. Let $f \in \boldsymbol{C}\{x, y\}$, $f(x, y) = \sum a_{i,j} x^i y^j$ be a non-zero convergent power series. Put $\operatorname{supp} f := \{(i, j) : a_{i,j} \neq 0\}$ the *support* of f. By definition the *Newton diagram* of f in the coordinates (x, y) is

$$\Delta_f := \text{Convex Hull}\,(\operatorname{supp} f + \boldsymbol{R}_+^2).$$

An important property of Newton diagrams is that the Newton diagram of a product is the Minkowski sum of Newton diagrams. One has $\Delta_{fg} = \Delta_f + \Delta_g$, where $\Delta_f + \Delta_g = \{a + b : a \in \Delta_f, b \in \Delta_g\}$. In particular if f and g differ by an invertible factor $u \in \boldsymbol{C}\{x, y\}$, $u(0,0) \neq 0$ then $\Delta_f = \Delta_g$. Thus the Newton diagram of a plane analytic curve is well defined because an equation of an analytic curve is given up to invertible factor, where an analytic plane curve is a principal ideal of the ring of convergent power series $\mathbf{C}\{x, y\}$, which we will denote by $f(x, y) = 0$. We will write $\Delta_{f=0}$ for the Newton diagram of the curve $f = 0$.

Following Teissier [**Te2**] we introduce *elementary Newton diagrams.* For $m, n > 0$ we put $\{\frac{n}{m}\} = \Delta_{x^n+y^m}$. We put also $\{\frac{n}{\infty}\} = \Delta_{x^n}$ and $\{\frac{\infty}{m}\} = \Delta_{y^m}$.

Every Newton diagram $\Delta \subsetneq \boldsymbol{R}^2_+$ has a unique representation $\Delta = \sum_{i=1}^{r} \{\frac{L_i}{M_i}\}$, where *inclinations* of successive elementary diagrams form an increasing sequence (by definition the inclination of $\{\frac{L}{M}\}$ is L/M with the conventions that $L/\infty = 0$ and $\infty/M = +\infty$). We shall call this representation the *canonical decomposition* of Δ.

Let $\sigma = (g, f) : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ be an analytic mapping given by $\sigma(x, y) = (g(x, y), f(x, y)) := (u, v)$ and such that $\sigma^{-1}(0,0) = \{(0,0)\}$. Then every local analytic curve $h(x, y) = 0$ has a well-defined *direct image* $\sigma^*(h = 0)$ which is an analytic curve in the target space (see [**Ca2**]). The Newton diagram of the direct image is characterized by two properties:

1. If h is an irreducible power series then $\Delta_{\sigma^*(h=0)} = \left\{\frac{(f,h)_0}{(g,h)_0}\right\}$, where $(r, s)_0$ denotes the intersection multiplicity of the curves $r = 0$ and $s = 0$ at the origin.
2. If $h = h_1 h_2$ then $\Delta_{\sigma^*(h=0)} = \Delta_{\sigma^*(h_1=0)} + \Delta_{\sigma^*(h_2=0)}$.

Let $\mathrm{jac}(g, f) = \partial g/\partial x \cdot \partial f/\partial y - \partial g/\partial y \cdot \partial f/\partial x$ be the Jacobian determinant of the mapping σ. The direct image (see Preliminaries in [**Ca2**]) of $\mathrm{jac}(g, f) = 0$ by σ is called the *discriminant curve.* We will write $\mathcal{N}_J(g, f)$ for the Newton diagram of the discriminant curve and following Teissier (see [**Te3**]) call it the *Jacobian Newton diagram* of the morphism $\sigma = (g, f)$.

4. Approximate Jacobian Newton diagrams of a branch.

In this section we introduce the notion of the approximate Jacobian Newton diagrams of an irreducible plane curve and we compute them. In what follows a branch $f(x, y) = 0$ will be given by an irreducible Weierstrass polynomial.

Let f be an irreducible Weierstrass polynomial and let $f^{(k)}$, for $0 \leq k \leq g-1$, be the characteristic approximate roots of f. The Jacobian Newton diagram $\mathcal{N}_J(f^{(k)}, f)$ is called the k-th *approximate Jacobian Newton diagram of the branch* $f(x, y) = 0$.

The following result about the factorization of the Jacobian $\mathrm{jac}(f^{(k)}, f)$ is the main result of this note:

Theorem 1. *Let $f \in \boldsymbol{C}\{x\}[y]$ be an irreducible Weierstrass polynomial with*

Puiseux characteristic $(b_0, \dots, b_g)$. *Let* $f^{(k)}$, $0 \le k \le g-1$, *be the* k*-th characteristic approximate root of* f. *Then the Jacobian* $\operatorname{jac}(f^{(k)}, f)$ *admits a factorization*

$$\operatorname{jac}(f^{(k)}, f) = \Gamma^{(k+1)} \cdots \Gamma^{(g)},$$

where the factors $\Gamma^{(i)}$ *are not necessary irreducible,* x *is co-prime to the product* $\Gamma^{(k+2)} \cdots \Gamma^{(g)}$ *and such that*

1. *If* α *is a Newton-Puiseux root of* $\Gamma^{(k+1)}$ *then* $\operatorname{cont}(\alpha, f) < b_{k+1}/b_0$.
2. *If* α *is a Newton-Puiseux root of* $\Gamma^{(i)}$, $k+2 \le i \le g$ *then* $\operatorname{cont}(\alpha, f) = b_i/b_0$.
3. *The intersection multiplicity* $(\Gamma^{(i)}, x)_0 = n_1 \cdots n_{i-1}(n_i - 1)$ *for* $k+2 \le i \le g$.

The proof of Theorem 1 will be done in Section 5.

The contacts between Newton-Puiseux roots of $\Gamma^{(k+1)}$ and f are not determined by the Puiseux characteristic of f as the following example shows.

Example 2. Let $f = (y^3 - 6x^3y - x^4)^2 - 9x^9$ be the Weierstrass polynomial from Example 1 and let $g = (y^3 - x^4)^2 + x^9 - x^7y^2$. Both series f and g are irreducible with the same Puiseux characteristic $(6, 8, 11)$. The Jacobian $\operatorname{jac}(f^{(1)}, f) = 243x^8(y^2 - 2x^3)$ has two Newton-Puiseux roots $\alpha_1(x) = \sqrt{2}x^{3/2} + \cdots$, $\alpha_2(x) = -\sqrt{2}x^{3/2} + \cdots$ and $\operatorname{cont}(\alpha_i, f) = 4/3 < b_2/b_0$ for $i = 1, 2$.

On the other hand there are four Newton-Puiseux roots $\beta_1(x) = 0$, $\beta_2(x) = (8/27)x^2 + \cdots$, $\beta_3(x) = \sqrt{(21/27)}x + \cdots$, $\beta_4(x) = -\sqrt{(21/27)}x + \cdots$ of $\operatorname{jac}(g^{(1)}, g) = x^6y(21y^3 - 27x^2y + 8x^4)$ and $\operatorname{cont}(\beta_i, g) = 4/3$ for $i = 1, 2$, but $\operatorname{cont}(\beta_i, g) = 1$ for $i = 3, 4$.

Further we will use the following property of the intersection multiplicity which is a consequence of the Noether's formula (see [**G-Pł2**, Proposition 3.3]):

Property 1. *Let* $g(x, y)$, $h(x, y)$ *be irreducible power series co-prime to* x. *Then for fixed* g, *the function* $h \mapsto (g, h)_0/(x, h)_0$ *depends only on the contact* $\operatorname{cont}(g, h)$ *and is a strictly increasing function of this quantity.*

Corollary 1. *Under assumptions and notations of Theorem* 1 *the Jacobian Newton diagram of the mapping* $(f^{(k)}, f)$ *has the canonical decomposition*

$$\mathcal{N}_J(f^{(k)}, f) = \sum_{i=k+1}^{g} \left\{ \frac{(f, \Gamma^{(i)})_0}{(f^{(k)}, \Gamma^{(i)})_0} \right\}.$$

Proof. We prove that for every irreducible factor h of $\operatorname{jac}(f^{(k)}, f)$ the quotient $(f, h)_0/(f^{(k)}, h)_0$ depends only on the contact $\operatorname{cont}(f, h)$. Indeed there are two cases: if $\operatorname{cont}(f,h) < b_{k+1}/b_0$ then by the strong triangle inequality $\operatorname{cont}(f^{(k)},h) = \operatorname{cont}(f,h)$ hence $(h,f^{(k)})_0/(x,f^{(k)})_0 = (h,f)_0/(x,f)_0$ and we get

$$\frac{(f,h)_0}{(f^{(k)},h)_0} = \frac{(x,f)_0}{(x,f^{(k)})_0}, \tag{1}$$

if $\operatorname{cont}(f,h) > b_{k+1}/b_0$ then also by the strong triangle inequality $\operatorname{cont}(f^{(k)},h) = \operatorname{cont}(f^{(k)},f)$ hence $(f^{(k)},h)_0/(x,h)_0 = (f^{(k)},f)_0/(x,f)_0$ and we get

$$\frac{(f,h)_0}{(f^{(k)},h)_0} = \frac{(x,f)_0}{(f^{(k)},f)_0} \cdot \frac{(f,h)_0}{(x,h)_0}. \tag{2}$$

Fix $i \in \{k+1,\dots,g\}$ and write $\Gamma^{(i)}$ as a product $h_1 \cdots h_r$ of irreducible factors h_j for $1 \le j \le r$. Then the Newton diagram of the direct image of the curve $\Gamma^{(i)} = 0$ is the sum $\sum_{j=1}^{r} \left\{\frac{(f,h_j)_0}{(f^{(k)},h_j)_0}\right\}$. Since all elementary Newton diagrams in the above sum have the same inclination one has

$$\sum_{j=1}^{r} \left\{\frac{(f,h_j)_0}{(f^{(k)},h_j)_0}\right\} = \left\{\frac{\sum_{j=1}^{r}(f,h_j)_0}{\sum_{j=1}^{r}(f^{(k)},h_j)_0}\right\} = \left\{\frac{(f,\Gamma^{(i)})_0}{(f^{(k)},\Gamma^{(i)})_0}\right\}.$$

We proved that the Jacobian Newton diagram $\mathcal{N}_J(f^{(k)},f)$ is the sum of elementary Newton diagrams from the statement of the Corollary. The inclination of the first elementary Newton diagram is given by formula (1) which can be written as $(x,f)_0/(f^{(k)},f)_0 \cdot (f,f^{(k)})_0/(x,f^{(k)})_0$. The inclinations of the remaining elementary Newton diagrams are given by formula (2). By Property 1 these inclinations form a strictly increasing sequence. This finishes the proof. □

Now our aim is to give an arithmetical formula for $\mathcal{N}_J(f^{(k)},f)$.

Put $\overline{b_k} := (f,f^{(k-1)})_0$ for $k \in \{0,1,\dots,g\}$. Following Zariski (see [**Z**]), the set $\{\overline{b_0},\overline{b_1},\dots,\overline{b_g}\}$ is a minimal system of generators of the *semigroup*

$$\Gamma(f) := \{(f,g)_0 : f \text{ is not a factor of } g\}$$

of the branch $f(x,y)=0$. This system of generators is uniquely determined by the Puiseux characteristic of f in the following way: $\overline{b_0} = b_0$, $\overline{b_1} = b_1$ and $\overline{b_q} = n_{q-1}\overline{b_{q-1}} + b_q - b_{q-1}$ for $2 \le q \le g$. Recall that $n_i = l_{i-1}/l_i$, where $l_i = \gcd(b_0,\dots,b_i) = \gcd(\overline{b_0},\dots,\overline{b_i})$.

Remember that the *Milnor number* of a curve $g(x,y)=0$ is by definition the intersection multiplicity $(\partial g/\partial x, \partial g/\partial y)_0$.

Theorem 2. *Let $f = 0$, where f is an irreducible Weierstrass polynomial, be a branch with semigroup $\Gamma(f) = \langle \overline{b_0}, \ldots, \overline{b_g} \rangle$. Then the canonical decomposition of the k-th approximate Jacobian Newton diagram of f is*

$$\mathcal{N}_J(f^{(k)}, f) = \left\{ \frac{l_k(\mu(f^{(k)}) + \overline{m} - 1)}{\mu(f^{(k)}) + \overline{m} - 1} \right\} + \sum_{i=k+2}^{g} \left\{ \frac{(n_i - 1)\overline{b_i}}{\overline{m} n_{k+2} \cdots n_{i-1}(n_i - 1)} \right\},$$

where $\overline{m} = \overline{b_{k+1}}/l_{k+1}$, and $\mu(f^{(k)})$ is the Milnor number of $f^{(k)} = 0$.

Proof. In the course of the proof we shall use the canonical decomposition of $\mathcal{N}_J(f^{(k)}, f)$ from Corollary 1. We shall express all intersection multiplicities $(f, \Gamma^{(i)})_0$ and $(f^{(k)}, \Gamma^{(i)})_0$ for $k + 1 \leq i \leq g$ in terms of the generators of the semigroup $\Gamma(f)$.

First consider $\Gamma^{(i)}$ for $k + 2 \leq i \leq g$. By Theorem 1 the contact of every irreducible factor of $\Gamma^{(i)}$ with f equals b_i/b_0. By Property 1 and Theorem 1:

$$(f, \Gamma^{(i)})_0 = (x, \Gamma^{(i)})_0 \frac{(f, \Gamma^{(i)})_0}{(x, \Gamma^{(i)})_0} = (x, \Gamma^{(i)})_0 \frac{(f, f^{(i-1)})_0}{(x, f^{(i-1)})_0} = (n_i - 1)\overline{b_i}. \tag{3}$$

By Corollary 1 and equality (2)

$$\frac{(f, \Gamma^{(i)})_0}{(f^{(k)}, \Gamma^{(i)})_0} = \frac{(f, f^{(i-1)})_0}{(f^{(k)}, f^{(i-1)})_0} = \frac{(x, f)_0}{(f^{(k)}, f)_0} \cdot \frac{(f, f^{(i-1)})_0}{(x, f^{(i-1)})_0} = \frac{l_{i-1}\overline{b_i}}{\overline{b_{k+1}}}.$$

Hence by (3)

$$(f^{(k)}, \Gamma^{(i)})_0 = \frac{\overline{b_{k+1}}}{l_{i-1}\overline{b_i}} (f, \Gamma^{(i)})_0 = \overline{m} n_{k+2} \cdots n_{i-1}(n_i - 1).$$

In order to compute $(f^{(k)}, \Gamma^{(k+1)})_0$ we use Theorem 3.2 of [**Ca1**]. We get

$$(f^{(k)}, \mathrm{jac}(f^{(k)}, f))_0 = \mu(f^{(k)}) + (f^{(k)}, f)_0 - 1.$$

Since $(f^{(k)}, \mathrm{jac}(f^{(k)}, f))_0 = \sum_{i=k+1}^{g} (f^{(k)}, \Gamma^{(i)})_0$ we have

$$\begin{aligned} (f^{(k)}, \Gamma^{(k+1)})_0 &= \mu(f^{(k)}) + (f^{(k)}, f)_0 - 1 - \sum_{i=k+2}^{g} \overline{m} n_{k+2} \cdots n_{i-1}(n_i - 1) \\ &= \mu(f^{(k)}) + \overline{b_{k+1}} - 1 - \overline{m}(l_{k+1} - 1) = \mu(f^{(k)}) + \overline{m} - 1. \end{aligned}$$

Finally by Corollary 1 and equality (1)

$$\frac{(f,\Gamma^{(k+1)})_0}{(f^{(k)},\Gamma^{(k+1)})_0}=\frac{(x,f)_0}{(x,f^{(k)})_0}=l_k.$$

Hence $(f,\Gamma^{(k+1)})_0=l_k(\mu(f^{(k)})+\overline{m}-1)$. □

Remark 1. In the above proof we compute the inclinations of elementary Newton diagrams of the canonical decomposition of $\mathcal{N}_J(f^{(k)},f)$ which are equal to $(l_{i-1}\overline{b_i})/(\overline{b_{k+1}})$ for $i\in\{k+1,\dots,g\}$. These inclinations are called *Jacobian invariants.*

Example 3. Let $f(x,y)=(y^2-x^3)^2-x^5y$. Then $f=0$ is a branch and $\Gamma(f)=\langle 4,6,13\rangle$. The characteristic approximate roots of f are $f^{(0)}=y$ and $f^{(1)}=y^2-x^3$. The factorization of $\mathrm{jac}(f^{(0)},f)$ described in Theorem 1 is $\mathrm{jac}(f^{(0)},f)=\Gamma^{(1)}\Gamma^{(2)}$, where $\Gamma^{(1)}=x^2$ and $\Gamma^{(2)}=6y^2+5x^2y-6x^3$. We also have $\mathrm{jac}(f^{(1)},f)=x^4(10y^2+3x^3)$. Finally $\mathcal{N}_J(f^{(0)},f)=\{\frac{8}{2}\}+\{\frac{13}{3}\}$ and $\mathcal{N}_J(f^{(1)},f)=\{\frac{28}{14}\}$.

Corollary 2. *The family of the approximate Jacobian Newton diagrams of a branch only depends on its topological type.*

If f is an irreducible Weierstrass polynomial then $f^{(0)}=0$ is a smooth curve. By Smith-Merle-Ephraim (see for example Theorem 2.2 of [**GB-G2**]) the approximate Jacobian Newton diagram $\mathcal{N}_J(f^{(0)},f)$ determines the topological type of the branch $f=0$. Nevertheless we can also obtain the generators of the semigroup of the branch $f=0$ using the whole family of its approximate Jacobian Newton diagrams in an easy way: let $\Gamma(f)=\langle\overline{b_0},\dots,\overline{b_g}\rangle$ be the semigroup of $f=0$. It is clear that $\overline{b_0}$ is the smallest inclination of $\mathcal{N}_J(f^{(0)},f)$. Denote by ι the inclination of the elementary diagram $\mathcal{N}_J(f^{(g-1)},f)$. Put $\mathcal{H}_r$, for $r\in\{0,\dots,g-2\}$, the height of the last elementary diagram of $\mathcal{N}_J(f^{(r)},f)$, that is the height of the elementary diagram of $\mathcal{N}_J(f^{(r)},f)$ which has the biggest inclination. Then $\overline{b}_{r+1}=\iota\mathcal{H}_r/(\iota-1)$ for $r\in\{0,\dots,g-2\}$. Finally $\overline{b}_g=\mathcal{L}/(\iota-1)$, where $\mathcal{L}$ is the length of the last elementary diagram of $\mathcal{N}_J(f^{(g-2)},f)$.

Example 4. Consider the branches $f_i=0$ for $i\in\{1,\dots,4\}$ with semigroups $\Gamma(f_1)=\langle 4,14,31\rangle$, $\Gamma(f_2)=\langle 4,6,35\rangle$, $\Gamma(f_3)=\langle 4,6,37\rangle$ and $\Gamma(f_4)=\langle 6,10,31\rangle$. By Theorem 2 we have $\mathcal{N}_J(f_1^{(1)},f_1)=\mathcal{N}_J(f_2^{(1)},f_2)=\{\frac{72}{36}\}$ and $\mathcal{N}_J(f_3^{(1)},f_3)=\mathcal{N}_J(f_4^{(1)},f_4)=\{\frac{76}{38}\}$.

Given a branch $f=0$, put $\mathcal{F}$ its family of approximate Jacobian Newton diagrams but the first one. The example shows that $\mathcal{F}$ is not a complete topological invariant of a branch. The curves $f_3 = 0$ and $f_4 = 0$ have the same $\mathcal{F}$ but they have different multiplicities at the origin. The curves $f_1 = 0$ and $f_2 = 0$ have the same $\mathcal{F}$ and the same multiplicity at the origin but in spite of it they have different topological type.

5. Proof of Theorem 1.

Let τ be a positive rational number and let $g(x,y) = \sum_{i\in\mathbf{Q},j\in\boldsymbol{N}} a_{ij}x^iy^j \in \boldsymbol{C}\{x\}^*[y]$. Put $w(x) := 1$ and $w(y) := \tau$ the *weights* of the variables x and y. By definition the *weighted order* of g is $\mathrm{ord}_\tau(g) = \min\{i + \tau j : a_{ij} \neq 0\}$ and the *weighted initial part* of g is $\mathrm{in}_\tau(g) = \sum_{i+\tau j=\mathrm{ord}_\tau(g)} a_{ij}x^iy^j$.

LEMMA 1. *Let $g(x,y) = u(x,y) \cdot x^N \prod_{i=1}^{d}(y - \alpha_i(x))$, where $u(0,0) \neq 0$, $N \in \boldsymbol{Q}$, $\alpha_i(x) = c_ix^\tau + \cdots$ for $1 \le i \le k$ and $\mathrm{ord}_x(\alpha_i(x)) < \tau$, for $k+1 \le i \le d$. Then $\mathrm{in}_\tau(g) = cx^M \prod_{i=1}^{k}(y - c_ix^\tau)$ for some $c \in \boldsymbol{C}$ and some $M \in \boldsymbol{Q}$.*

PROOF. Observe that $\mathrm{in}_\tau(y - \alpha_i(x)) = y - c_ix^\tau$ for $1 \le i \le k$ and $\mathrm{in}_\tau(y - \alpha_i(x)) = -\mathrm{in}_\tau\alpha_i(x)$ for $k+1 \le i \le d$. Since the initial part of a product is the product of the initial parts of every factor we get the lemma. $\square$

LEMMA 2. *Let $h_1, h_2 \in \boldsymbol{C}\{x\}^*[y]$ and $\tau \in \boldsymbol{Q}^+$. Assume that the Jacobian $\mathrm{jac}(\mathrm{in}_\tau(h_1), \mathrm{in}_\tau(h_2)) \neq 0$. Then $\mathrm{in}_\tau(\mathrm{jac}(h_1,h_2)) = \mathrm{jac}(\mathrm{in}_\tau(h_1), \mathrm{in}_\tau(h_2))$.*

PROOF. For all monomials $M_1 = x^{i_1}y^{j_1}$ and $M_2 = x^{i_2}y^{j_2}$ we have $\mathrm{jac}(M_1,M_2) = (i_1j_2 - i_2j_1)x^{i_1+i_2-1}y^{j_1+j_2-1}$ hence $\mathrm{ord}_\tau(\mathrm{jac}(M_1,M_2)) = \mathrm{ord}_\tau(M_1) + \mathrm{ord}_\tau(M_2) - 1 - \tau$ provided $i_1j_2 - i_2j_1 \neq 0$. It follows that $\mathrm{jac}(\mathrm{in}_\tau(h_1), \mathrm{in}_\tau(h_2))$ is the sum of monomials of the same weighted order $\mathrm{ord}_\tau(\mathrm{in}_\tau(h_1)) + \mathrm{ord}_\tau(\mathrm{in}_\tau(h_2)) - 1 - \tau$ (that is a quasi-homogeneous polynomial). Moreover $\mathrm{jac}(h_1,h_2) = \mathrm{jac}(\mathrm{in}_\tau(h_1) + (h_1 - \mathrm{in}_\tau(h_1)), \mathrm{in}_\tau(h_2) + (h_2 - \mathrm{in}_\tau(h_2))) = \mathrm{jac}(\mathrm{in}_\tau(h_1),\, \mathrm{in}_\tau(h_2)) +$ *terms of higher weighted order* which proves the lemma. $\square$

Recall that Newton-Puiseux roots of an irreducible Weierstrass polynomial $f \in \boldsymbol{C}\{x\}[y]$, $\deg f = n$ form a cycle: if $\gamma(x) = \sum a_ix^{i/n}$ is a root of f then other roots of f are $\gamma_j(x) = \sum a_i\omega_j^ix^{i/n}$, where ω_j is a n-th root of unity. Moreover $\mathrm{ord}_x(\gamma(x) - \gamma_j(x)) \ge b_{k+1}/b_0$ if and only if ω_j is a l_k-th root of unity (see [**Z**]).

Let $f = \prod_{i=1}^{n}(y - \gamma_i(x))$ be an irreducible Weierstrass polynomial with Puiseux characteristic $(b_0,\ldots,b_g)$ and let $f^{(k)}(x,y) = \prod_{j=1}^{m}(y - \delta_j(x))$, where $n = ml_k$, be the characteristic approximate root of f. Put $J(x,y) := \mathrm{jac}(f^{(k)}, f) = \mathrm{unity} \cdot x^\alpha \prod_l(y - \sigma_l(x))$. In order to prove Theorem 1 we need

LEMMA 3. *Fix $\gamma \in \mathrm{Zer}\, f$ and $\tau \in \boldsymbol{Q}$ such that $\tau \ge b_{k+1}/b_0$. Then*

1. *if* $b_j/b_0 < \tau \le b_{j+1}/b_0$, *where* $j \in \{k+1, \ldots, g\}$ *then* $\sharp\{i : \mathcal{O}(\sigma_i, \gamma) \ge \tau\} = l_j - 1$,
2. *if* $\tau = b_{k+1}/b_0$ *then* $\sharp\{i : \mathcal{O}(\sigma_i, \gamma) \ge \tau\} = n_{k+1}(l_{k+1} - 1)$.

Proof. Let $\tilde{J}(x,y) := J(x, y + \gamma(x))$, $\tilde{f}(x,y) := f(x, y + \gamma(x))$ and $\tilde{f}^{(k)}(x,y) := f^{(k)}(x, y + \gamma(x))$. Clearly $\tilde{J}(x,y) = \text{unity} \cdot x^\alpha \prod_l (y - (\sigma_l(x) - \gamma(x)))$. By Lemma 1 $\sharp\{i : \mathcal{O}(\sigma_i, \gamma) \ge \tau\} = \deg_y(\text{in}_\tau(\tilde{J}(x,y)))$.

Assume first that $\tau > b_{k+1}/b_0$ and $\tau \ne b_j/b_0$ for all $j \in \{k+2, \ldots, g\}$. The weighted initial part of $\tilde{f}(x,y) = \prod_{i=1}^n (y - (\gamma_i(x) - \gamma(x)))$ is equal to $\text{in}_\tau(\tilde{f}(x,y)) = c_1 x^{\alpha_1} y^{d(\tau)}$, where $c_1 \in \boldsymbol{C} \setminus \{0\}$ and $d(\tau) := \sharp\{i : \mathcal{O}(\gamma_i, \gamma) \ge \tau\}$. More precisely if $b_j/b_0 < \tau < b_{j+1}/b_0$ then $d(\tau) = l_j$.

Consider the function $\tilde{f}^{(k)}(x,y) = \prod_{j=1}^m (y - (\delta_j(x) - \gamma(x)))$. Since $\mathcal{O}(\delta_j, \gamma) < \tau$ for every $j \in \{1, \ldots, m\}$, we get by Lemma 1 $\text{in}_\tau \tilde{f}^{(k)}(x,y) = c_2 x^{\alpha_2}$, where $c_2 \in \boldsymbol{C} \setminus \{0\}$.

Using Lemma 2 we get

$$\text{in}_\tau(\tilde{J}(x,y)) = \text{jac}(c_2 x^{\alpha_2}, c_1 x^{\alpha_1} y^{d(\tau)}) = c_1 c_2 \alpha_2 d(\tau) x^{\alpha_1 + \alpha_2 - 1} y^{d(\tau) - 1},$$

so its y-degree is equal to $d(\tau) - 1 = l_j - 1$ for $b_j/b_0 < \tau < b_{j+1}/b_0$.

Let us choose $\tau < b_{j+1}/b_0$ close enough to b_{j+1}/b_0 that no σ_i satisfies $\tau \le \mathcal{O}(\sigma_i, \gamma) < b_{j+1}/b_0$. Then $\sharp\{i : \mathcal{O}(\sigma_i, \gamma) \ge \tau\} = \sharp\{i : \mathcal{O}(\sigma_i, \gamma) \ge b_{j+1}/b_0\}$ and the proof of statement 1 is done.

Assume now that $\tau = b_{k+1}/b_0$. By Lemma 1

$$\begin{aligned}
\text{in}_\tau \tilde{f}(x,y) &= x^{\alpha_3} \prod_{\omega^{l_k} = 1} \left(y - a(\omega^{b_{k+1}} - 1) x^{b_{k+1}/b_0}\right) \\
&= x^{\alpha_3} \prod_{\omega^{l_k} = 1} \left[(y + a x^{b_{k+1}/b_0}) - a\omega^{b_{k+1}} x^{b_{k+1}/b_0}\right] \\
&= x^{\alpha_3} \left[(y + a x^{b_{k+1}/b_0})^{n_{k+1}} - (a x^{b_{k+1}/b_0})^{n_{k+1}}\right]^{l_{k+1}},
\end{aligned}$$

where $\omega \in \boldsymbol{C}$ and a is the coefficient in γ of the term x^{b_{k+1}/b_0}. The last equality follows from the formula $\prod_{\omega^p = 1}(Z - b\omega^q) = (Z^{p/\gcd(p,q)} - b^{p/\gcd(p,q)})^{\gcd(p,q)}$.

Moreover and also using Lemma 1 we have $\text{in}_\tau \tilde{f}^{(k)}(x,y) = x^{\alpha_4}(y + a x^{b_{k+1}/b_0})$ since there is only one Newton-Puiseux root δ_j of $f^{(k)}$ such that $\mathcal{O}(\delta_j, \gamma) \ge b_{k+1}/b_0$ (otherwise if there were two of such roots δ_{j_1}, δ_{j_2} then by the triangular property of the contact order we obtain $\mathcal{O}(\delta_{j_1}, \delta_{j_2}) \ge b_{k+1}/b_0$ which is not possible).

We prove now the equality $\alpha_3 = \alpha_4 l_k$. Note that $\alpha_3 = \sum_{i \in I'} \mathcal{O}(\gamma_i, \gamma)$ and $\alpha_4 = \sum_{j \in J'} \mathcal{O}(\delta_j, \gamma)$, where $I' := \{i : \mathcal{O}(\gamma_i, \gamma) < b_{k+1}/b_0\}$ and $J' := \{j : \mathcal{O}(\delta_j, \gamma) < b_{k+1}/b_0\}$. Using Puiseux characteristic of f and after Section 3 in [**G-Pł3**] we obtain $\alpha_3 = \sum_{i\in I'}\mathcal{O}(\gamma_i,\gamma) = \sum_{l=1}^{k}\sharp\{i : \mathcal{O}(\gamma_i,\gamma) = b_l/b_0\}\cdot b_l/b_0 = (n-l_1)b_1/b_0+\cdots+(l_{k-1}-l_k)b_k/b_0$ and by the same argument $\alpha_4 = \sum_{j\in J'}\mathcal{O}(\delta_j,\gamma) = (n/l_k - l_1/l_k)b_1/b_0+\cdots+(l_{k-1}/l_k - 1)b_k/b_0$.

Finally the initial part of $\tilde{J}$ is

$$\mathrm{in}_\tau(\tilde{J}) = \mathrm{jac}(\mathrm{in}_\tau(\tilde{f}^{(k)}),\mathrm{in}_\tau(\tilde{f})) = \mathrm{jac}(v,(v^{n_{k+1}} - a^{n_{k+1}}u^{\theta})^{l_{k+1}}),$$

where $v = x^{\alpha_4}(y + ax^{b_{k+1}/b_0})$, $u = x$ and $\theta = n_{k+1}(b_{k+1}/b_0 + \alpha_4)$ so $\mathrm{in}_\tau(\tilde{J}) = \partial\mathrm{in}_\tau(\tilde{f})/\partial u\cdot\partial v/\partial y$ and its y-degree is equal to $n_{k+1}(l_{k+1}-1)$. □

Remark 2. The proof of Merle formula in [**G-Pł1**] was based on the equality $\Delta_{\tilde{f}} = \Delta_{\tilde{j}} + \{\frac{\infty}{1}\}$, where $\tilde{j}(x,y) = j(x,y+\gamma(x))$ and $j(x,y) := \mathrm{jac}(x,f)$. Note that the statement of Lemma 3 can be written as $\deg_y \mathrm{in}_\tau(\tilde{J}(x,y)) = \deg_y \mathrm{in}_\tau(\tilde{f}(x,y)) - 1$ for $\tau > b_{k+1}/b_0$. It follows from this equality that $\tilde{\Delta}_{\tilde{f}} = \tilde{\Delta}_{\tilde{J}} + \{\frac{\infty}{1}\}$, where $\tilde{\Delta}_{\tilde{J}}$ and $\tilde{\Delta}_{\tilde{f}}$ are the sums of elementary Newton diagrams in the canonical decompositions of $\Delta_{\tilde{J}}$ and $\Delta_{\tilde{f}}$ respectively with inclinations bigger than b_{k+1}/b_0.

Corollary 3. *Keep the above notations and put* $\tau_i := \mathrm{cont}(\sigma_i, f)$. *Then*

1. *if* $\tau_i \geq b_{k+1}/b_0$ *then* $\tau_i \in \{b_{k+2}/b_0,\ldots,b_g/b_0\}$.
2. *The number* $\sharp\{i : \tau_i = b_j/b_0\} = n_1\cdots n_{j-1}(n_j - 1)$ *for* $j\in\{k+2,\ldots,g\}$.

Proof. First take τ such that $b_j/b_0 < \tau \leq b_{j+1}/b_0$ for $k+1\leq j\leq g$. We shall prove that

$$\sharp\{i : \tau_i \geq \tau\} = n - n_1\cdots n_j. \tag{4}$$

In the set $\mathrm{Zer}\, f$ we define the equivalence relation given by

$$\gamma^*\equiv\gamma' \text{ if and only if } \mathcal{O}(\gamma^*,\gamma') \geq \frac{b_{j+1}}{b_0}.$$

Put $I_\gamma := \{i : \mathcal{O}(\sigma_i,\gamma)\geq\tau\}$ for $\gamma\in\mathrm{Zer}\, f$. By Lemma 3 we get $\sharp I_\gamma = l_j - 1$. Note that $I_{\gamma'} = I_{\gamma^*}$ for $\gamma^*\equiv\gamma'$ and $I_{\gamma'}\cap I_{\gamma^*} = \emptyset$ when $\gamma^*\not\equiv\gamma'$.

Remark that $n_1\cdots n_j$ is the number of cosets in the equivalence relation $\equiv$. Since $\sharp\{i : \tau_i\geq\tau\} = \bigcup_{\gamma\in\mathrm{Zer}\, f} I_\gamma$ we have $\sharp\{i : \tau_i\geq\tau\} = n_1\cdots n_j\cdot\sharp I_\gamma = n_1\cdots n_j(l_j - 1) = n - n_1\cdots n_j$. The equality (4) is proved.

Fix small positive number ϵ such that

$$\sharp\{i : \tau_i = \tau\} = \sharp\{i : \tau_i\geq\tau\} - \sharp\{i : \tau_i\geq\tau+\epsilon\}.$$

If $\tau \neq b_j/b_0$ for all $j \in \{k+2,\dots,g\}$ the above difference is equal to zero. If $\tau = b_j/b_0$ for some $j \in \{k+2,\dots,g\}$, then $\sharp\{i : \tau_i = b_j/b_0\} = (n - n_1\cdots n_{j-1}) - (n - n_1\cdots n_j) = n_1\cdots n_{j-1}(n_j - 1)$.

Finally using the same argument as before (for $\tau = b_{k+1}/b_0$) we have

$$\begin{aligned}\sharp\left\{i : \tau_i = \frac{b_{k+1}}{b_0}\right\} &= \sharp\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0}\right\} - \sharp\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0} + \epsilon\right\} \\ &= \sharp\left\{i : \tau_i \geq \frac{b_{k+1}}{b_0}\right\} - (n - n_1\cdots n_{k+2}) \\ &= n_{k+1}(l_{k+1} - 1)n_1\cdots n_k - (n - n_1\cdots n_{k+1}) = 0. \qquad \square\end{aligned}$$

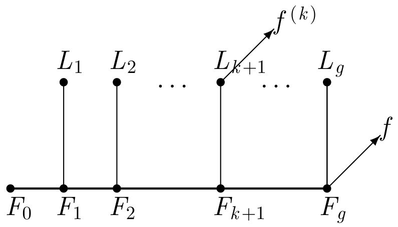
Proof of Theorem 1. Let $k+2 \leq j \leq g$. Put $\Gamma^{(j)} = \prod(y - \sigma_i(x))$, where the product runs over σ_i with $\operatorname{cont}(\sigma_i, f) = b_j/b_0$ and let $\Gamma^{(k+1)} = \operatorname{jac}(f^{(k)}, f)/(\Gamma^{(k+2)}\cdots\Gamma^{(g)})$. It follows from the first statement of Corollary 3 that for every Newton-Puiseux root $\alpha \in \operatorname{Zer}\Gamma^{(k+1)}$ we have $\operatorname{cont}(\alpha, f) < b_{k+1}/b_0$. Finally by the second statement of Corollary 3 we get $(\Gamma^{(i)}, x)_0 = n_1\cdots n_{i-1}(n_i - 1)$ for $k+2 \leq i \leq g$. $\square$

6. Relation with Michel's theorem.

In [**Mi**] the author considered a finite morphism $(f,g) : (X,p) \longrightarrow (\boldsymbol{C}^2, 0)$, where (X,p) is a normal germ of complex surface. Michel determined the Jacobian quotients via a good minimal resolution and pointed out the importance of the multiplicities of the Jacobian quotients. More precisely and following notation of [**Mi**], let R be a good resolution of (f,g) and put $E = R^{-1}(p)$ the exceptional divisor of R. For every irreducible component E_i of E, denote E_i' the set of points of E_i which are smooth points of the total transform $\tilde{E} = R^{-1}((fg)^{-1}(0))$. Denote the order of $f \circ R$ (respectively $g \circ R$) at a generic point of E_i $v(f, E_i)$ (respectively $v(g, E_i)$). The quotient $q_i = v(g,E_i)/v(f,E_i)$ is the *Hironaka number* of E_i.

Let q be a Hironaka number and put $E(q)$ the union of the E_i' such that $q_i = q$ to which we add $E_i \cap E_j$ if $q_i = q_j = q$. Let $\{E^k(q)\}_k$ be the connected components of $E(q)$. By definition a *q-zone* is a connected component of $E(q)$ and a q-zone is a *rupture zone* if there exists in it at least one E_i' with negative Euler characteristic. Then after Theorem 4.8 of [**Mi**] the set of Jacobian invariants of the morphism (f,g) is equal to the set of Hironaka numbers q such that there exists at least one q-zone in E which is a rupture zone.

Consider an irreducible Weierstrass polynomial f with Puiseux characteristic $(b_0, b_1, \dots, b_g)$, where $b_0 < b_1$ (i.e. $x = 0$ is transverse to $f = 0$). Below is the schematic picture of the resolution graph of the curve $f^{(k)}f = 0$.

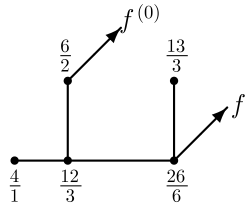


Every Jacobian invariant $q \in \{l_k, l_{k+1}\overline{b_{k+2}}/\overline{b_{k+1}}, \ldots, l_{g-1}\overline{b_g}/\overline{b_{k+1}}\}$ of the morphism $(f^{(k)}, f)$ corresponds to exactly one rupture zone.

The rupture zone for $q = l_k$ is the tree with endpoints F_0, F_{k+1}, $L_1, \ldots, L_k$. It yields the factor $\Gamma^{(k+1)}$ of the Jacobian and by Michel's theorem $(\Gamma^{(k+1)}, h)_0 = \sum_{i=1}^{k+1} v(h, F_i) - \sum_{i=1}^{k} v(h, L_i) - v(h, F_0)$, where $h = f$ or $h = f^{(k)}$.

Every rupture zone for $q = l_{i-1}\overline{b_i}/\overline{b_{k+1}}$, where $k + 2 \leq i \leq g$ is the bamboo with endpoints F_i and L_i. It yields the factor $\Gamma^{(i)}$ of the Jacobian and by Michel's theorem $(\Gamma^{(i)}, h)_0 = v(h, F_i) - v(h, L_i)$ for $k + 2 \leq i \leq g$, where $h = f$ or $h = f^{(k)}$.

As an illustration we draw the resolution graph of $f^{(0)}f = 0$, where f is the Weierstrass polynomial from Example 3. The labels of divisors are Hironaka numbers written in the form $v(f, E_i)/v(f^{(0)}, E_i)$.



There are two rupture zones corresponding to Hironaka numbers 4 and 13/3. It follows from [**Mi**] that $\mathcal{N}_J(f^{(0)}, f) = \{\frac{12}{3}\} - \{\frac{4}{1}\} + \{\frac{26}{6}\} - \{\frac{13}{3}\} = \{\frac{8}{2}\} + \{\frac{13}{3}\}$.

REMARK 3. Remark that Theorem 1 is also true when we change $f^{(k)}$ for any irreducible Weierstrass polynomial with the properties of statement of Proposition 2.

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