J. Math. Soc. Japan
Vol. 65, No. 1 (2013) pp. 97–136
doi: 10.2969/jmsj/06510097

Classification of 3-bridge spheres of 3-bridge arborescent links

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(Received May 17, 2011)

Abstract. In this paper, we give an isotopy classification of 3-bridge spheres of 3-bridge arborescent links, which are not Montesinos links. To this end, we prove a certain refinement of a theorem of J. S. Birman and H. M. Hilden [**3**] on the relation between bridge presentations of links and Heegaard splittings of 3-manifolds. In the proof of this result, we also give an answer to a question by K. Morimoto [**23**] on the classification of genus-2 Heegaard splittings of certain graph manifolds.

1. Introduction.

Bridge presentations of links are decompositions of links into two simple parts. Since introduced by Schubert [**31**], bridge numbers and bridge presentations of links have been studied in various references (for example, see [**7**], [**16**], [**17**], [**26**], [**27**], [**28**], [**30**], [**31**] and references therein). The concept of bridge presentations of links is also known as an analogy of a naive way of decomposing 3-manifolds, called Heegaard splittings, via so-called double branched coverings. For example, Birman and Hilden [**3**] gave some relations between bridge presentations of links and Heegaard splittings of 3-manifolds, which contain the one-to-one correspondence between 3-bridge presentations and genus-2 Heegaard splittings (up to homeomorphism). In her previous paper [**17**], the author used the correspondence and a characterization of certain manifolds with genus-2 Heegaard splittings by Kobayashi [**19**] to give a classification of a class of links called arborescent links which have 3-bridge presentations. In this paper, we refine a theorem by Birman and Hilden [**3**] and arguments in [**17**] to give a classification of 3-bridge presentations of the links classified in [**17**]. It is known that bridge presentations of the unknot, 2-bridge links or torus knots are unique by [**26**], [**27**], [**28**], [**30**], [**31**], but some of the links in this paper have multiple 3-bridge presentations. Not only a classification of 3-bridge presentations of the links, we also obtain some results on

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M12.
Key Words and Phrases. 3-bridge spheres, arborescent links.
This research is partially supported by Grant-in-Aid for JSPS Research Fellowships for Young Scientists.

genus-2 Heegaard splittings of certain 3-manifolds which arose in [**23**], by studying bridge presentations and Heegaard splittings together.

An *n-bridge sphere* of a link L in S^3 is a 2-sphere which meets L in $2n$ points and cuts (S^3, L) into n-string trivial tangles (B_1, t_1) and (B_2, t_2). Here, an *n-string trivial tangle* is a pair (B^3, t) of the 3-ball B^3 and n arcs properly embedded in B^3 parallel to the boundary of B^3. We call a link L an *n-bridge link* if L admits an n-bridge sphere and does not admit an $(n-1)$-bridge sphere. Two n-bridge spheres S_1 and S_2 of L are said to be *pairwise isotopic* (*isotopic*, in brief) if there exists a homeomorphism $f : (S^3, L) \to (S^3, L)$ such that $f(S_1) = S_2$ and f is *pairwise isotopic* to the identity, i.e., there is a continuous family of homeomorphisms $f_t : (S^3, L) \to (S^3, L)$ $(0 \leq t \leq 1)$ such that $f_0 = f$ and $f_1 = \mathrm{id}$.

Recall that an *arborescent link* is a link obtained by closing an arborescent tangle with a trivial tangle (see [**11**]), where an *arborescent tangle* is a 2-string tangle possibly with loop components obtained from rational tangles by repeatedly applying the operations in Figure 1. These links form an important family of links which contains 2-bridge links and Montesinos links, and the double branched covering of the 3-sphere S^3 branched over an arborescent link is a graph manifold. Bonahon and Siebenmann [**10**] gave a complete classification of arborescent links (cf. [**14**]).

In [**17**, Theorem 1], we gave the following complete list of 3-bridge arborescent links, where two links are *equivalent* if there exists an orientation-preserving homeomorphism of S^3 which carries one of the two links to the other. (The classification of the links in the list up to equivalence is also given in [**17**, Theorem 2].)

THEOREM 1.1. *Let L be a 3-bridge arborescent link. Then one of the followings holds.*

(1) *L is equivalent to the link $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$ in Figure* 2 (1).
(2) *L is equivalent to the link $L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/\alpha_0), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$ in Figure* 2 (2).

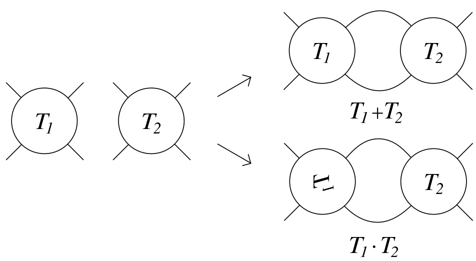
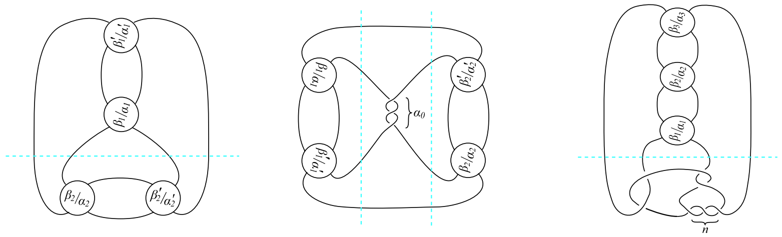


Figure 1.



(1) $L_1((\beta_1/\alpha_1,\ \beta_1'/\alpha_1'),\ (\beta_2/\alpha_2,\ \beta_2'/\alpha_2'))$

(2) $L_2((\beta_1/\alpha_1,\ \beta_1'/\alpha_1'),\ (1/\alpha_0),\ (\beta_2/\alpha_2,\ \beta_2'/\alpha_2'))$

(3) $L_3((\beta_1/\alpha_1,\ \beta_2/\alpha_2,\ \beta_3/\alpha_3),\ (1/2,\ -n/(2n{+}1)))$

Figure 2.

(3) *L is equivalent to the link $L_3((\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3), (1/2, -n/(2n+1)))$ in Figure* 2 (3).
(4) *L is a Montesinos link $L(-b; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$.*

Here, α_i, α_i', β_i, β_i' are integers such that $\alpha_i, \alpha_i' > 1$ and g.c.d.$(\alpha_i, \beta_i) =$ g.c.d.$(\alpha_i', \beta_i') = 1$ $(i = 1, 2, 3)$, *and α_0 and n are integers such that $|\alpha_0| > 1$ and $|2n+1| > 1$. In Figure* 2, *the circle encircling a rational number β/α represents the rational tangle of slope β/α.*

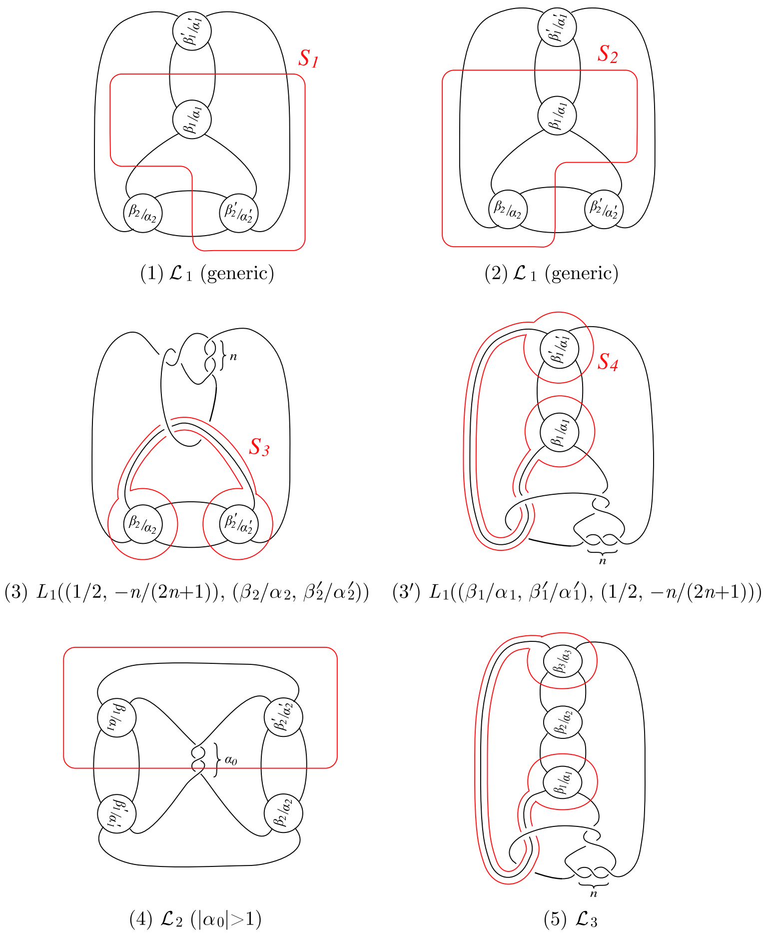
For each $i = 1, 2, 3$, we denote by $\mathcal{L}_i$ the family of links as in (i) in the above theorem. The main purpose of this paper is to give a complete classification of the 3-bridge spheres of the links in $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. We first present a complete list of the 3-bridge spheres.

Theorem 1.2. *Any* 3*-bridge sphere of a link L in $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ is isotopic to one of the* 3*-bridge spheres in Figure* 3*. To be precise, the following hold.*

(i) *If L belongs to $\mathcal{L}_1$, then any* 3*-bridge sphere of L is isotopic to one of the* 3*-bridge spheres S_1, S_2, S_3 and S_4 in* (1), (2), (3) *and* (3') *in Figure* 3.
(ii) *If L belongs to $\mathcal{L}_2$ or $\mathcal{L}_3$, then any* 3*-bridge sphere of L is isotopic to the* 3*-bridge sphere in* (4) *and* (5) *in Figure* 3, *respectively.*

Remark 1.3. In the assertion (i), the word "generic" in (1) and (2) in Figure 3 means that every link $L \in \mathcal{L}_1$ admits the 3-bridge spheres S_1 and S_2, respectively. On the other hand, the 3-bridge spheres S_3 and S_4 in (3) and (3') are possessed only by the links $L_1((1/2, -n/(2n+1)), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$ and $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/2, -n/(2n+1)))$, respectively.

In fact, the list of homeomorphism classes of 3-bridge spheres of the links in $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ is given in [**17**, Proposition 7]. We refine the arguments used in the proof of [**17**, Theorem 1] to obtain the list of isotopy classes of 3-bridge spheres



(1) $\mathcal{L}_1$ (generic) (2) $\mathcal{L}_1$ (generic)

(3) $L_1((1/2,\ -n/(2n{+}1)),\ (\beta_2/\alpha_2,\ \beta_2'/\alpha_2'))$ (3′) $L_1((\beta_1/\alpha_1,\ \beta_1'/\alpha_1'),\ (1/2,\ -n/(2n{+}1)))$

(4) $\mathcal{L}_2$ ($|\alpha_0|>1$) (5) $\mathcal{L}_3$

Figure 3.

in Theorem 1.2. We also use (certain subgroups of) the mapping class groups, obtained in [**17**], of the double branched coverings of S^3 branched along the links.

Next we give a necessary and sufficient condition for any two 3-bridge spheres in Theorem 1.2 to be isotopic. This enables us to classify the 3-bridge spheres of 3-bridge arborescent links which are not Montesinos links. In order to state the result, we recall a notation introduced in [**17**, Notation 1].

Notation 1.4. For rational numbers $s_1, \ldots, s_r$ and $s_1', \ldots, s_r'$,

$$(s_1, \ldots, s_r) \sim (s_1', \ldots, s_r')$$

Table 1. 3-bridge spheres for $L \in \mathcal{L}_1$.

	S_1, S_2	S_3	S_4	μ
(a-1)	1	1	0	2
(a-2)		0	1	2
(a-3)		1	1	3
(a-4)		0	0	1
(b-1)	2	1	0	3
(b-2)		0	1	3
(b-3)		1	1	4
(b-4)		0	0	2

means that $(s_1, \ldots, s_r) = (s'_1, \ldots, s'_r)$ or $(s'_r, \ldots, s'_1)$ in $(\mathbb{Q}/\mathbb{Z})^r$ and $\sum_{i=1}^r s_i = \sum_{i=1}^r s'_i$.

THEOREM 1.5. *Two* 3*-bridge spheres* S_i *and* S_j $(i, j \in \{1, 2, 3, 4\}, i \neq j)$ *for a link* $L_1((\beta_1/\alpha_1, \beta'_1/\alpha'_1), (\beta_2/\alpha_2, \beta'_2/\alpha'_2))$ *are isotopic if and only if* $\{i, j\} = \{1, 2\}$ *and* $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ *for some* $k = 1, 2$*, where* $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$.

From Theorems 1.2 and 1.5, we obtain Table 1, which gives the number μ of isotopy classes of 3-bridge spheres of $L \in \mathcal{L}_1$. In Table 1, $(i$-$j)$ $(i \in \{a, b\}, j \in \{1, 2, 3, 4\})$ means that L satisfies the conditions (i) and (j) as follows.

(a) $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for some $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$,
(b) $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \not\sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for both $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$,
  (1) $(\beta_1/\alpha_1, \beta'_1/\alpha'_1) \sim (1/2, -n/(2n+1))$ for some n and $(\beta_2/\alpha_2, \beta'_2/\alpha'_2) \not\sim (1/2, -m/(2m+1))$ for any m,
  (2) $(\beta_1/\alpha_1, \beta'_1/\alpha'_1) \not\sim (1/2, -n/(2n+1))$ for any n and $(\beta_2/\alpha_2, \beta'_2/\alpha'_2) \sim (1/2, -m/(2m+1))$ for some m,
  (3) $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \sim (1/2, -n/(2n+1))$ for some n for each $k = 1, 2$,
  (4) $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \not\sim (1/2, -n/(2n+1))$ for any n for each $k = 1, 2$.

Arguments used in the proof of Theorem 1.5 enable us to complete the table by Morimoto [**23**] on the numbers of isotopy classes of genus-2 Heegaard splittings of certain manifolds (see Remark 6.7). In particular, we obtain the following corollary, which gives an affirmative answer to a question raised by Morimoto (see [**23**, p. 324]).

COROLLARY 1.6. *The link* $L_1((1/2, -n/2n+1), (1/2, -m/2m+1))$ *with* $|2n+1|, |2m+1| \notin \{1, 3\}$ *admits exactly four* 3*-bridge spheres up to isotopy* (*see Figure* 4).

Unfortunately, our methods do not work for Montesinos links. However, we

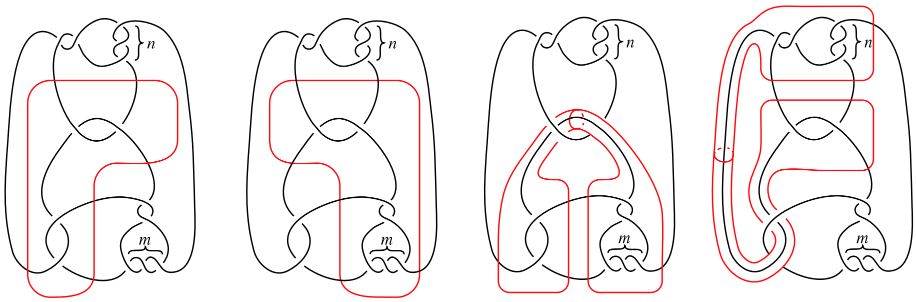


Figure 4.

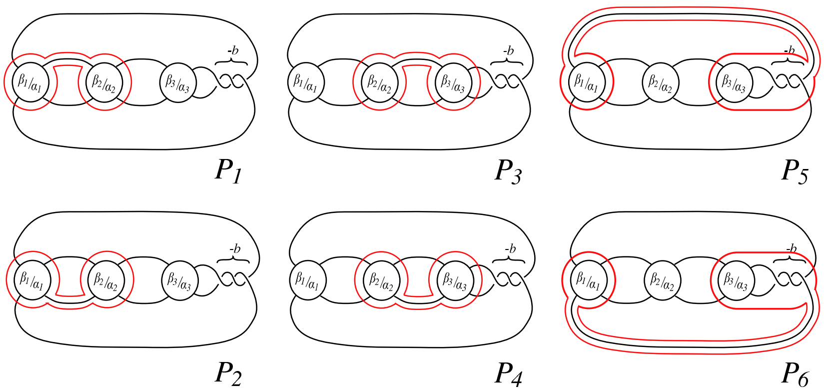


Figure 5.

obtain the following partial result.

THEOREM 1.7. (1) *A 3-bridge nonelliptic Montesinos link admits at most six 3-bridge spheres,* $P_1, \ldots, P_6$*, up to isotopy (see Figure 5).*

(2) *A 3-bridge elliptic Montesinos link admits a unique 3-bridge sphere up to isotopy.*

In the remainder of the introduction, we explain our strategy. For a given 3-bridge link L, we have a map Φ_L from the set of isotopy classes of 3-bridge spheres of L to the set of isotopy classes of genus-2 Heegaard surfaces F of the double branched covering $M_2(L)$, whose hyper-elliptic involutions τ_F are the covering transformation τ_L.

$$\Phi_L : \{\text{3-bridge spheres of } L\}/\sim$$
$$\to \{\text{genus-2 Heegaard surfaces } F \text{ of } M \text{ s.t. } \tau_F = \tau_L\}/\sim.$$

It is obvious that Φ_L is surjective. In Section 2, we prove the following theorem which gives a condition for Φ_L to be injective, by using the results of Boileau and Zimmermann [**8**].

THEOREM 1.8. *Let L be a prime, unsplittable* 3*-bridge link.*

(1) *If L is not a Montesinos link, then Φ_L is bijective.*
(2) *If L is a nonelliptic Montesinos link, then Φ_L is at most* 2-1*.*

REMARK 1.9. It follows from Theorem 1.7 that Φ_L is bijective when L is an elliptic Montesinos link.

By the above theorem, classification of 3-bridge spheres of 3-bridge arborescent links (which are not Montesinos links) is reduced to classification of genus-2 Heegaard surfaces of the double branched coverings. A refinement of the results by Kobayashi [**19**] and Morimoto [**23**] (see [**17**, Theorem 5]) enables us to obtain a complete list of genus-2 Heegaard surfaces. To obtain a classification of the genus-2 Heegaard surfaces, we use their commutator invariants (see Section 6). The commutator invariant turns out to be a complete invariant of genus-2 Heegaard splittings for genus-2 graph manifolds. Namely, when given two 3-bridge spheres of a link cannot be distinguished by the commutator invariants, we can construct an isotopy between them. This completes the classification of 3-bridge arborescent links, which are not Montesinos links, and their 3-bridge spheres up to isotopy.

For Montesinos links, however, the pre-images of the 3-bridge spheres P_i and P_{i+1} $(i = 1, 3, 5)$ are isotopic Heegaard surfaces, and thus we cannot distinguish the 3-bridge spheres by the methods in this paper. We also give, in Remark 7.5, certain sufficient conditions for the 3-bridge spheres in Theorem 1.7 (1) to be mutually isotopic. We conjecture that these conditions actually provide a necessary and sufficient condition.

This paper is organized as follows. In Section 2, we prove Theorem 1.8 on a relation between 3-bridge spheres and genus-2 Heegaard surfaces. In Section 3 we list all 3-bridge spheres of (non-Montesinos) arborescent links up to isotopy by using the results of Kobayashi [**19**] and Morimoto [**23**]. This, together with the results in Section 5, completes the proof of Theorem 1.2. In Section 4, we prove Lemma 3.1 which is used in Section 3. In Section 5, we prove that the " simple exceptional links" admit a unique 3-bridge sphere up to isotopy. In Section 6, we use the commutator invariants of genus-2 Heegaard splittings to distinguish two Heegaard surfaces up to isotopy. In Section 7, we give the list of 3-bridge spheres of 3-bridge Montesinos links and some sufficient conditions for them to be isotopic.

2. 3-bridge spheres and genus-2 Heegaard surfaces.

Let M be a closed orientable 3-manifold of Heegaard genus 2, and let $(V_1, V_2; F)$ be a genus-2 *Heegaard splitting* of M, i.e., V_1 and V_2 are genus-2 handlebodies in M such that $M = V_1 \cup V_2$ and $F = \partial V_1 = \partial V_2 = V_1 \cap V_2$. By [**3**, Proof of Theorem 5], there is an involution τ on M which satisfies the following condition.

(∗) $\tau(V_i) = V_i$ $(i = 1, 2)$ and $\tau|_{V_i}$ is *equivalent* to the standard involution $\mathcal{T}$ on a standard genus-2 handlebody V as illustrated in Figure 6. To be precise, there is a homeomorphism $\psi_i : V_i \to V$ such that $\mathcal{T} = \psi_i(\tau|_{V_i})\psi_i^{-1}$ $(i = 1, 2)$.

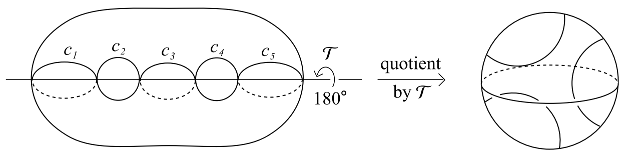


Figure 6.

For each genus-2 Heegaard splitting $(V_1, V_2; F)$, we call an involution of M satisfying the condition (∗) the *hyper-elliptic involution* associated with $(V_1, V_2; F)$ (or associated with F, in brief) and denote it by τ_F. The strong equivalence class of τ_F is uniquely determined by the isotopy class of $(V_1, V_2; F)$ (see [**17**, Proposition 5]). Here, two involutions τ and τ' are said to be *strongly equivalent* if there exists a homeomorphism h on M such that $h\tau h^{-1} = \tau'$ and that h is isotopic to the identity map id_M, and two Heegaard splittings $(V_1, V_2; F)$ and $(W_1, W_2; G)$ of a 3-manifold M are said to be *isotopic* if there exists a self-homeomorphism f of M such that $f(F) = G$ and f is isotopic to the identity map id_M on M.

Let L be a 3-bridge link and let $M_2(L)$ be the double branched covering of S^3 branched over L. Let τ_L be the covering transformation on $M_2(L)$. If S is a 3-bridge sphere of L, its pre-image in $M_2(L)$ is a genus-2 Heegaard surface F such that $\tau_F = \tau_L$. Moreover, the isotopy class of F is uniquely determined by that of S because an isotopy on (S^3, L) lifts to an isotopy on $M_2(L)$. Thus we obtain the following map Φ_L from the set of isotopy classes of 3-bridge spheres of L to the set of isotopy classes of genus-2 Heegaard surfaces of $M_2(L)$, whose hyper-elliptic involutions are τ_L.

$$\Phi_L : \{\text{3-bridge spheres of } L\}/\!\sim \; \to \{\text{genus-2 Heegaard surfaces } F \text{ of } M \text{ s.t. } \tau_F = \tau_L\}/\!\sim.$$

It is obvious that Φ_L is surjective. In the following, we prove Theorem 1.8

which gives a condition for Φ_L to be injective.

To prove Theorem 1.8, we use a result by Boileau and Zimmermann [**8**]. We recall a few concepts introduced in [**8**]. Let $p : \widetilde{M} \to M$ be the universal covering of $M = M_2(L)$. The group $O(L)$ generated by all lifts of τ_L to $\widetilde{M}$ is called the *π-orbifold group* of L. The quotient space $\widetilde{M}/O(L)$ is an orbifold with underlying space S^3 and singular set L. A link in S^3 is said to be *sufficiently complicated* if it is prime, unsplittable and $O(L)$ is infinite.

Remark 2.1. The π-orbifold group $O(L)$ of a link L is isomorphic to the quotient group $\pi_1(S^3 \setminus L)/\langle\langle m^2 \rangle\rangle$, where $\langle\langle m^2 \rangle\rangle$ is the subgroup of $\pi_1(S^3 \setminus L)$ normally generated by the square of the meridian of L (cf. [**8**, p. 187]).

Remark 2.2. Let L be an arborescent link. Then, since $M = M_2(L)$ admits a reduced graph structure, M is irreducible. This implies that L is prime and unsplittable (cf. [**8**, Proposition 1.1]).

We obtain the following lemma from the orbifold theorem [**6**], [**12**] and a result of Dunbar [**13**].

Lemma 2.3. *Let L be a prime, unsplittable link. If $O(L)$ is finite, then L is an elliptic Montesinos link.*

Proof. Let L be a prime, unsplittable link and assume that $O(L)$ is finite. By the equivariant sphere theorem and the branched covering theorem (see [**25**]), the orbifold $\widetilde{M}/O(L)$ is irreducible. By the orbifold theorem [**6**], [**12**], M is geometric and τ_L is an isometry of M. Since $\pi_1(M)$ is finite by the assumption, M is spherical. Hence, the orbifold $\widetilde{M}/O(L)$ with underlying space S^3 and singular set L is a spherical orbifold. By Dunbar's classification of spherical 3-orbifolds (see Table 7 in [**13**]), we see that L is an elliptic Montesinos link. □

Let γ be the homomorphism from the symmetry group $\mathrm{Sym}(S^3, L) = \pi_0 \mathrm{Diff}(S^3, L)$ of L to the outer automorphism group $\mathrm{Out}\, O(L) = \mathrm{Aut}\, O(L)/\mathrm{Inn}\, O(L)$ defined by lifting diffeomorphisms and isotopies from (S^3, L) to $\widetilde{M}$: every lifted diffeomorphism induces an automorphism of $O(L)$ by conjugation.

Proposition 2.4 ([**8**, Theorem 2]). *Let L be a sufficiently complicated link in S^3. Then $\gamma : \mathrm{Sym}(S^3, L) \to \mathrm{Out}\, O(L)$ is an isomorphism.*

We need the following lemma to prove Theorem 1.8.

Lemma 2.5. *Let $M = M_2(L)$, τ_L, $\widetilde{M}$, $O(L)$ be as above, and assume that L is sufficiently complicated. Suppose that φ is a self-homeomorphism of M which* *is homotopic to the identity on M and commutes with τ_L.*

(1) *If M is not a Seifert fibered space such that the center of $\pi_1(M)$ is an infinite cyclic group, then there exists a lift $\widetilde{\varphi}$ of φ to $\widetilde{M}$ which induces by conjugation the identity on $O(L)$.*
(2) *If M is a Seifert fibered space and the center of $\pi_1(M)$ is an infinite cyclic group, then there exists a lift $\widetilde{\varphi}$ of φ to $\widetilde{M}$ which induces by conjugation the identity or α on $O(L)$. Here, α is the automorphism on $O(L)$ given by*

$$\alpha(x) = \begin{cases} x & ([x, h] = 1), \\ xh & ([x, h] \neq 1), \end{cases}$$

where h is an element of $O(L)$ representing a lift of a regular fiber of M.

Proof. This is proved in [**8**, Proposition 4.12] under the additional assumption that M is Haken. This assumption is used only to assure that M is a Seifert fibered space if the center of its fundamental group $\pi_1(M)$ is nontrivial. However, by the affirmative answer to the Seifert fibered space conjecture [**15**, Corollary 8.3], the nontriviality of the center of $\pi_1(M)$ implies that M is a Seifert fibered space even if it is not Haken. □

Proof of Theorem 1.8. Let S and S' be two 3-bridge spheres for a prime, unsplittable 3-bridge link L and set $F := p^{-1}(S)$ and $F' := p^{-1}(S')$, where p is the covering projection. Assume that the Heegaard surfaces F and F' of $M = M_2(L)$ are isotopic, namely, there is a homeomorphism φ on M sending F to F' which is isotopic to the identity map. By the proof of [**3**, Theorem 8] (cf. the proof of [**17**, Proposition 5]), we may assume that φ is τ_L-equivariant. Hence, we have a self-homeomorphism ψ of (S^3, L) which sends S to S' and lifts to φ.

(1) Assume that L is not a Montesinos link. Then L is sufficiently complicated by Lemma 2.3.

If M is not a Seifert fibered space such that $\pi_1(M)$ has an infinite cyclic center, then there exists a lift $\widetilde{\varphi}$ of φ to $\widetilde{M}$ which induces by conjugation the identity map on $O(L)$ by Lemma 2.5 (1). Hence φ induces the identity map on $\mathrm{Out}(O(L))$. By Proposition 2.4, ψ is isotopic to the identity. Hence Φ_L is injective.

If M is a Seifert fibered space such that $\pi_1(M)$ has an infinite cyclic center, then L is a Seifert link, namely, $S^3 \setminus L$ admits a Seifert fibration by circles (see, for example, [**8**, proof of Theorem 1.3]). By [**8**, Corollary 1.4], $O(L)$ has a nontrivial center. Hence, by [**8**, Proposition 4.12], there exists a lift $\widetilde{\varphi}$ of φ to $\widetilde{M}$ which induces by conjugation the identity map on $O(L)$. As in the previous case, we see by using Proposition 2.4 that Φ_L is injective.

(2) Assume that L is a nonelliptic Montesinos link. Then L is sufficiently

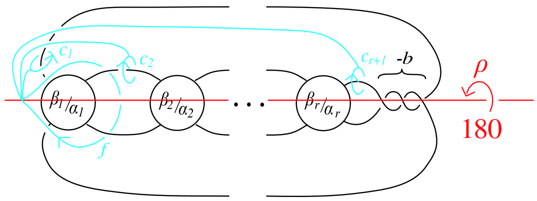


Figure 7.

complicated by Lemma 2.3, and M is a Seifert fibered space. Note that the center of $\pi_1(M)$ is an infinite cyclic group whose generator is a regular fiber. By Lemma 2.5 (2), there exists a lift $\widetilde{\varphi}$ of φ to $\widetilde{M}$ which induces by conjugation the identity map or α, as in Lemma 2.5 (2), on $O(L)$. Hence, by Proposition 2.4, we have at most two 3-bridge spheres up to isotopy for each isotopy class of genus-2 Heegaard surfaces. □

Remark 2.6. We note that the automorphism α of $O(L)$ in the above proof of Theorem 1.8 (2) is induced by a lift of the symmetry ρ of (S^3, L), in Figure 7, to the universal cover $\widetilde{M}$ of $M = M_2(L)$. In fact, if L is a Montesinos link $L(-b; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ (see [**17**, Section 2] for notation), then $O(L) \cong \pi_1(S^3 \setminus L)/\langle\langle m^2\rangle\rangle$ has a group presentation

$$O(L) = \big\langle c_1, c_2, \ldots, c_{r+1}, f \mid c_i^2, c_i f c_i^{-1} f, (c_j c_{j+1})^{\alpha_j} f^{\beta_j}, c_1 c_{r+1} f^b \big\rangle,$$

where c_i and f are represented by the loops c_i and f, respectively, in Figure 7 (cf. [**9**]). Let $\widetilde{\rho}$ be a lift of ρ to $\widetilde{M}$, and let $\iota_{\widetilde{\rho}}$ be the automorphism of $O(L)$ induced by conjugation by $\widetilde{\rho}$. Then we can observe that $\iota_{\widetilde{\rho}}(c_1) = c_1^{-1} f = c_1 f$, $\iota_{\widetilde{\rho}}(c_j) = (c_1 f)(c_j f)(c_1 f)^{-1}$ $(j = 2, \ldots, r+1)$ and $\iota_{\widetilde{\rho}}(f) = (c_1 f) f (c_1 f)^{-1}$. Thus $\iota_{\widetilde{\rho}}$ and α determine the same element of $\operatorname{Out} O(L)$.

3. Proof of Theorem 1.2.

We quickly recall several notations from [**17**]. The symbol $F(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ with $F = D$, A or $M\ddot{o}$ denotes a Seifert fibered space over a disk, an annulus or a Möbius band with singular fibers of indices $\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r$. Each boundary component of $F(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ has a *horizontal loop* which intersects each regular fiber on the boundary component transversely in a single point. See [**17**, Section 2] for precise definition. For a knot or a link K in a manifold, $E(K)$ denotes the exterior of K.

Let L be a 3-bridge arborescent link which belongs to one of the families $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ in Theorem 1.1, and let S be a 3-bridge sphere of L. Then L is prime, unsplittable and is not a Montesinos link (cf. [**17**, Theorem 2]). Thus L satisfies the assumption of Theorem 1.8 (1). Hence the isotopy class of S is uniquely determined by the isotopy class of the genus-2 Heegaard surface F, obtained as the pre-image of S, of the double branched covering $M = M_2(L)$ such that the hyperelliptic involution τ_F associated with F is identified with the covering involution τ_L on M.

Case 1: $L = L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (\beta_2/\alpha_2, \beta_2'/\alpha_2')) \in \mathcal{L}_1$.

Then by the Montesinos trick [**21**] (cf. [**17**, Proposition 7]), we see that the double branched cover M of S^3 branched along L is obtained from the Seifert fibered spaces $M_1 = D(\beta_1/\alpha_1, \beta_1'/\alpha_1')$ and $M_2 = D(\beta_2/\alpha_2, \beta_2'/\alpha_2')$, by gluing them along their boundaries by a homeomorphism so that a horizontal loop and a regular fiber of M_1 are identified with a regular fiber and a horizontal loop of M_2, respectively. Let F_1 and F_2 be the genus-2 Heegaard surfaces of M obtained as the pre-images of the 3-bridge spheres S_1 and S_2 in Figure 3, respectively. When M_1 (resp. M_2) is homeomorphic to $D(1/2, -n/2n+1)$ for some integer n with $|2n+1| > 1$, let F_3 (resp. F_4) be the genus-2 Heegaard surface of M obtained as the pre-image of the 3-bridge spheres S_3 (resp. S_4). Note that F_1 and F_2 are the two genus-2 Heegaard surfaces of M belonging to the family F(1) in [**23**] (cf. [**23**, Proposition 5.2] and [**17**, Section 7, Case 1.1]) and that F_3 and F_4 are genus-2 Heegaard surfaces of M belonging to the families F(2-1) and F(2-2) in [**23**], respectively. By [**23**, Section 5], we see that F is isotopic to F_1, F_2, F_3 or F_4. Hence, by Theorem 1.8, S is isotopic to S_1, S_2, S_3 or S_4.

Case 2: $L = L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/\alpha_0), (\beta_2/\alpha_2, \beta_2'/\alpha_2')) \in \mathcal{L}_2$.

By [**17**, Proposition 4] together with the fact that $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are mutually disjoint (see [**17**, Theorem 2]), one of the following holds.

(i) L is equivalent to the link $L_2((-1/2, 1/2), (1/n), (-1/2, 1/2)) \in \mathcal{L}_2$ in Figure 8. In this case, L is non-simple, i.e., $S^3 \setminus L$ contains an essential torus.
(ii) The double branched covering $M = M_2(L)$ is a graph manifold which admits a nontrivial torus decomposition by separating tori.

The exceptional case where the condition (i) holds is treated in Section 5,

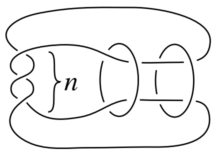


Figure 8. $n \neq 0$.

where we show that L admits a unique 3-bridge sphere up to isotopy (Proposition 5.1). Thus we may assume that M satisfies the condition (ii).

Then the 3-manifold M and its genus-2 Heegaard surface F satisfies the assumption of [**17**, Proposition 7], and we see from the proposition that one of the following conditions (a) and (b) holds.

(a) M belongs to the family M(1-b) and F satisfies the condition (F1) in [**17**, Theorem 5 and Definition 1]. Namely,
  (M1-b) M is obtained by gluing $M_1 = D(\beta_1/\alpha_1, \beta_1'/\alpha_1')$ and $M_2 = E(K) = M\ddot{o}(1/\alpha_0)$, where K is a 1-bridge knot in a lens space, so that a horizontal loop and a regular fiber of M_1 are identified with a regular fiber and a horizontal loop of M_2, respectively, and
  (F1) the intersection of the torus $T := \partial M_1 = \partial M_2$ and each handlebody bounded by F is a single separating essential annulus (see Figure 9 (F1)).

  Moreover,
  - $M_1 \cap F$ is an essential annulus saturated in the Seifert fibration of M_1, and
  - $M_2 \cap F$ is a 2-holed torus which gives a 1-bridge decomposition of the 1-bridge knot K such that $M_2 = E(K)$.

(b) M belongs to the family M(4) and F satisfies the condition (F4) in [**17**, Theorem 5]. Namely,
  (M4) M is obtained by gluing $M_1 = D(\beta_1/\alpha_1, \beta_1'/\alpha_1')$, $M_2 = D(\beta_2/\alpha_2, \beta_2'/\alpha_2')$ and $M_3 = E(S(2\alpha_0, 1)) = A(1/\alpha_0)$, where $S(2\alpha_0, 1)$ is the 2-bridge link of type $(2\alpha_0, 1)$, so that a horizontal loop and a regular fiber of M_i $(i = 1, 2)$ are identified with a regular fiber and a horizontal loop of M_3, respectively, and
  (F4) the intersection of the pair of tori $T := \partial(M_1 \cup M_2) = \partial M_3$ and each handlebody bounded by F consists of two disjoint non-parallel separating essential annuli (see Figure 9 (F4)).

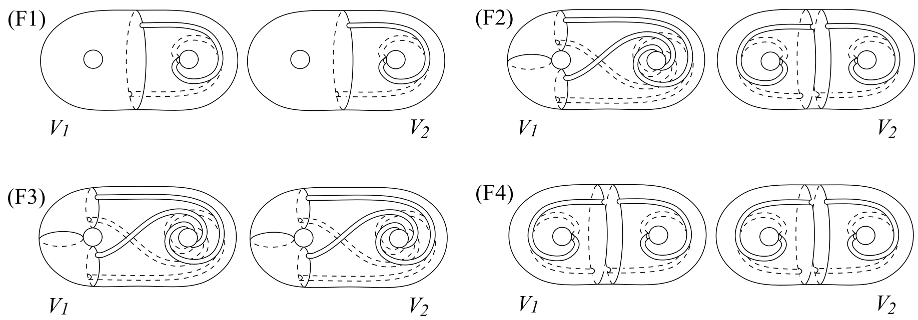


Figure 9.

Moreover,

- $M_i \cap F$ is an essential saturated annulus in M_i $(i = 1, 2)$, and
- $M_3 \cap F$ is a 2-bridge sphere.

Suppose that the condition (a) holds. By [**17**, Lemma 9], we see that $\tau_L|_{M_2}$ is equivalent to the involution g_1 in [**17**, Lemma 4 (2)], where $(M_2, \operatorname{Fix} g_1)/\langle g_1 \rangle$ is the Montesinos pair as illustrated in Figure 10 (1). Recall from [**17**, Remark 6] that the lens space containing K is homeomorphic to $P^2(0; 1/\alpha_0) \cong S^2(\alpha_0; -1/2, 1/2)$ and that K is a regular fiber of $P^2(0; 1/\alpha_0)$ and the meridian of K is a horizontal loop of $M_2 = M\ddot{o}(1/\alpha_0)$. Then g_1 extends to an involution, denoted by the same symbol g_1, of the regular neighborhood $N(K)$ of K in the lens space such that $(N(K), \operatorname{Fix} g_1, K)/\langle g_1 \rangle$ is as illustrated in Figure 10 (2). Here, $K/\langle g_1 \rangle$ is identified with the arc γ in the figure. Since the meridian of K is identified with the horizontal loop of $M_2 = M\ddot{o}(1/\alpha_0)$, the lens space $P^2(0; 1/\alpha_0)$ is the double branched covering of S^3 branched over the link L' in Figure 11 and the image of K by the covering projection is the arc γ in Figure 11. Recall that $F \cap M_2$ is a 2-holed torus which gives a 1-bridge decomposition of K and that $\tau = \tau_F$ preserves M_2 and F (cf. Figure 9 (F1)). Thus $F \cap M_2$ projects to a surface, say $\check{P}$, in $M_2/\langle g_1 \rangle$ such that $\partial \check{P}$ is a simple loop on $\partial(N(K)/\langle g_1 \rangle)$ of "slope" 0. Thus $\partial \check{P}$ bounds a disk in $N(K)/\langle g_1 \rangle$ intersecting γ transversely in a single point. Let P be the union of $\check{P}$ and the disk. Then we have the following lemma, which we prove in Section 4.

Lemma 3.1. *Under the above setting, P is isotopic to the surface P_0 in Figure* 11 *by an isotopy of (S^3, L') preserving γ.*

Thus $\check{P}$ is isotopic to the disk in Figure 12 (1). On the other hand, we can see that $F \cap M_1$ projects to the disk in $(M_1, \operatorname{Fix} \tau|_{M_1})/\langle \tau|_{M_1} \rangle$ as illustrated in Figure 12 (2). Hence, by [**17**, Lemma 7], S is isotopic to the 3-bridge sphere in Figure 13 and hence we obtain the desired result.

Suppose that the condition (b) holds. Let F_0 be the pre-image of the 3-bridge sphere in Figure 3 (4). Then, by the argument in Case 4 in [**17**, Section 7], the

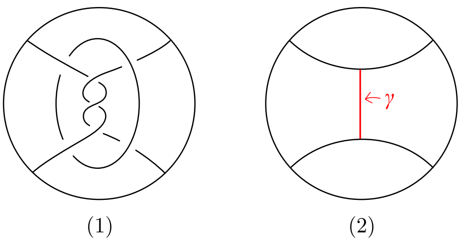


Figure 10.

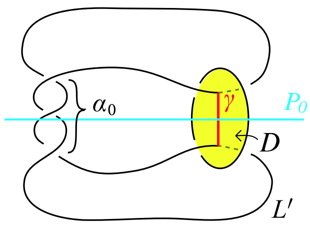


Figure 11.

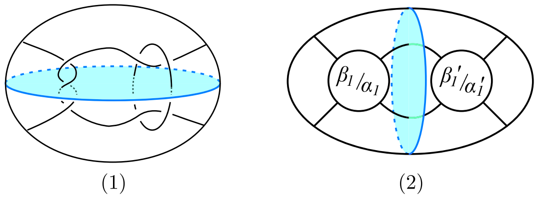


Figure 12.

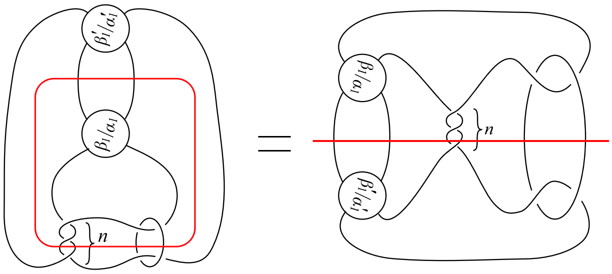


Figure 13.

Heegaard surface F obtained as the pre-image of the given bridge sphere S is isotopic to $(D_\mu^{1/2})^n(F_0)$, the surface obtained from F_0 by applying $n/2$-Dehn twist along $T = \partial M_1 = \partial M_2$ in the direction of the regular fiber μ of M_2 (see [**17**, Section 6] for the precise definition). We may assume that τ_{F_0} is equal to the homeomorphism G_1 in [**17**, Proposition 6 (3)]. Then $\tau_F = D_\mu^n \tau_{F_0}$ by [**17**, Lemma 5]. Since $(D_\mu)^n \neq 1$ whenever $n \neq 0$ by [**17**, Lemma 3 (1)], we see the identity $\tau_F = \tau_L (= \tau_{F_0})$ holds only when $n = 0$. Hence, by Theorem 1.8, S is isotopic to the 3-bridge sphere in Figure 3 (4).

Case 3: $L = L_3((\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3), (1/2, -n/(2n+1))) \in \mathcal{L}_3$.

By [**17**, Proposition 7], M belongs to the family (M2-b) and F satisfies the condition (F2) in [**17**, Theorem 5]. Namely,

(M2-b) M is obtained from $M_1 = D(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ and $M_2 = E(S(2n+1, 1)) \cong D(1/2, -n/(2n+1))$ by gluing their boundary so that a horizontal loop and a regular fiber of M_1 are identified with a regular fiber and a horizontal loop of M_2, respectively, and

(F2) the intersection of the torus $T := \partial M_1 = \partial M_2$ and each handlebody bounded by F consists of two essential annuli as illustrated in Figure 9 (F2). Moreover,

– $M_1 \cap F$ consists of two disjoint essential saturated annuli in M_1 which divide M_1 into three solid tori, and

– the 2-bridge knot corresponding to M_2 is $S(2n+1,1)$, and $M_2 \cap F$ is a 2-bridge sphere.

By [**23**, Theorem 4], a 2-bridge sphere of a 2-bridge knot $S(2n+1,1)$ is unique up to isotopy fixing the knot. So, by [**17**, Lemma 6 (2)], the isotopy type of F is uniquely determined by the isotopy type of $M_1 \cap F$, where the isotopy does not necessarily fix the boundary of M_1.

In order to determine if $\tau_F = \tau_L$, we quickly recall some notations of certain subgroups of the mapping class groups of M and M_1 introduced in [**17**]. Let $\mathcal{M}(M_1)$ be the subgroup of the (orientation-preserving) mapping class group of M_1 which consists of the elements preserving each singular fiber of M_1. (See [**17**, Section 5] for more details.) Let $\mathcal{M}(M)$ be the subgroup of the (orientation-preserving) mapping class group of M which consists of the elements preserving each M_i and each singular fiber of M_i $(i = 1, 2)$. Throughout this paper, we do not distinguish between a self-homeomorphism and its isotopy class: we denote them by the same symbol.

Recall from the condition (F2) that $F \cap M_1$ consists of two disjoint essential saturated annuli in M_1 which divide M_1 into three solid tori. Since $M_1 = D(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ by the condition (M2-b), each of the three solid tori must contain one singular fiber. Moreover, the homeomorphism type of $F \cap M_1$ is determined by the choice of the singular fiber in the solid torus whose boundary contains the two annuli $F \cap M_1$. Thus $F \cap M_1$ is homeomorphic to one of the saturated annuli G_1, G_2 and G_3 obtained as the pre-images of the arcs in the base orbifold illustrated in Figure 14. To be precise, $F \cap M_1$ is isotopic to $f_1(G_i)$ for some $f_1 \in \mathcal{M}(M_1)$ and for some $i = 1, 2, 3$. For each $i = 1, 2, 3$, let F_i be a genus-2 Heegaard surface such that $F_i \cap M_1 = G_i$ and $F_i \cap M_2$ is the 2-bridge sphere of K. Note that we have the unique 2-bridge sphere of K up to isotopy fixing the boundary of M_2 (see [**23**, Theorem 4]). By [**17**, Lemma 6 (2)], any genus-2 Heegaard surface F is isotopic to $f(F_i)$ for some integer n and for some $i = 1, 2, 3$ and for some homeomorphism $f \in \mathcal{M}_0(M)$ of M which is obtained from some $f_1 \in \mathcal{M}(M_1)$ by the rule $f|_{M_1} = f_1 \in \mathcal{M}(M_1)$ and $f|_{M_2} = \mathrm{id}$. Here, $\mathcal{M}_0(M)$ denotes the subgroup of $\mathcal{M}(M)$ consisting of the elements whose restrictions to

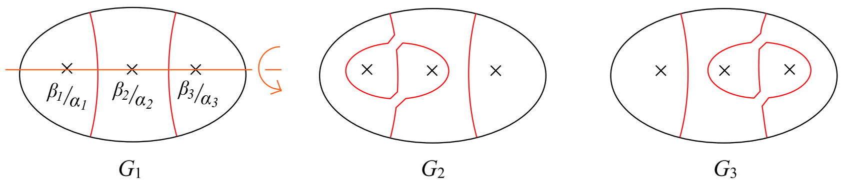


Figure 14.

Table 2.

u \ v	1	x	y
1	1	a^2b	ba^2
x	ba	1	ba^2ba
y	ab	a^2bab	1

M_2 are the identity.

CLAIM 1. *$\tau_{f(F_i)} = \tau_{F_j}$ if and only if $i = j$ and $f = 1$ in $\mathcal{M}(M)$.*

PROOF. Put $\tau := \tau_{F_1}$. Then the hyper-elliptic involution τ_{F_i} associated with F_i is τ, $x\tau x^{-1}(= x^2\tau)$ and $y\tau y^{-1}(= y^2\tau)$ according as $i = 1, 2$ and 3, respectively, and $\tau_{f(F_i)}$ is $f\tau_{F_i}f^{-1}$. Recall from [**17**, Proof of Theorem 2 (3)] that

$$\mathcal{M}(M) \cong \mathcal{M}(M_1) \cong (P_3/\langle(xy)^3\rangle) \rtimes \langle\tau\rangle < (B_3/\langle(xy)^3\rangle) \rtimes \langle\tau\rangle,$$

where P_3 and B_3 are the pure 3-braid group and the 3-braid group, respectively. Recall from [**17**, Claim 1 (2)] that the centralizer $Z(\tau, \mathcal{M}(M))$ of τ is $\{1, \tau\} \cong \mathbb{Z}_2$. Hence, an element $f \in \mathcal{M}_0(M) \subset \mathcal{M}(M)$ commutes with τ only if $f = 1$.

Assume that the hyper-elliptic involution $\tau_{f(F_i)}$ associated with $f(F_i)$ coincides with τ_{F_j} for some $i, j \in \{1, 2, 3\}$. Since the involutions $\tau_{f(F_i)}$ and τ_{F_j} are given by $f(u\tau u^{-1})f^{-1}$ and $v\tau v^{-1}$, respectively, for some $u, v \in \{1, x, y\}$, we have

$$f(u\tau u^{-1})f^{-1} = v\tau v^{-1}.$$

Thus $v^{-1}fu \in \mathcal{M}_0(M)$ commutes with τ in $\mathcal{M}(M)$. Hence, $f = vu^{-1}$. Note that the element vu^{-1} is as in Table 2, and the only element which belongs to P_3 among them is 1 since any other element changes the order of singular points. Hence, f must be 1 and we also have $i = j$. □

Hence, M admits a unique genus-2 Heegaard surface whose hyper-elliptic involution is strongly equivalent to τ_L. By Theorem 1.8, this implies that the 3-bridge sphere in Figure 3 (5) is the unique 3-bridge sphere of L.

4. Proof of Lemma 3.1.

Let L' be the 3-bridge link and P a 3-bridge sphere as in Lemma 3.1. Then P satisfies the following condition (P0).

(P0) the pre-image of P is a 1-bridge torus of K.

This condition is equivalent to the following condition (see [**25**, Theorem 1.2]).

(P0′) P is a 2-bridge sphere of L', i.e., P divides (S^3, L) into two 2-string trivial tangles, (B^3_1, t_1) and (B^3_2, t_2), and moreover, $(B^3_i, t_i, \gamma \cap B^3_i)$ is as illustrated in Figure 15 for $i = 1, 2$.

Let D be the disk bounded by a component of L' containing γ in it as illustrated in Figure 11. Let K_1 be the boundary of D and let K_2 be the other component of L'. Since P meets each K_i in two points and since P meets γ in a single point, one of the following conditions holds.

(i) $D \cap P$ contains an arc δ_1 properly embedded in D which intersects γ transversely in a single point (see Figure 16 (i)), or

(ii) $D \cap P$ contains an arc δ_2 and a loop δ_3, such that δ_2 is disjoint from γ and that δ_3 intersects γ transversely in a single point (see Figure 16 (ii)).

Case (i): The condition (i) holds.

Suppose $D \cap P$ contains a component, c, other than δ_1. Then c is a loop in $D \setminus (\gamma \cup \delta_1)$ and hence it bounds a disk, d_c, in $D \setminus (\gamma \cup \delta_1)$. We may assume c is innermost, i.e., $\mathrm{Int}(d_c) \cap P = \emptyset$. The loop c bounds a disk, d'_c, in P such that $|d'_c \cap L'| \leq 2$. If $|d'_c \cap L'| = 0$, then the 2-sphere $d'_c \cup d_c$ bounds a 3-ball in $S^3 \setminus L'$. Thus P can be isotoped so that c is removed from $D \cap P$. By repeating this deformation, we may assume that $D \cap P$ does not contain a loop bounding a disk in $P \setminus L'$. If $|d'_c \cap L'| = 1$, then the 2-sphere $d'_c \cup d_c$ intersects L' in one point, a contradiction. If $|d'_c \cap L'| = 2$, then c is isotopic in $P \setminus L'$ to the boundary of a regular neighborhood of δ_1 in P, because the disk d'_c is disjoint from the arc $\delta_1 \subset P$. This loop represents the commutator $aba^{-1}b^{-1}$ of the two generators a and b of the (2-bridge) link group of L', where a and b are represented by the meridians of K_1 and K_2 as in Figure 17. Since the loop bounds the disk d_c in $S^3 \setminus L'$, we have $aba^{-1}b^{-1} = 1$. This implies that the link group is a commutative group, which is a contradiction.

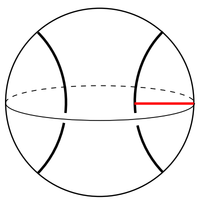


Figure 15.

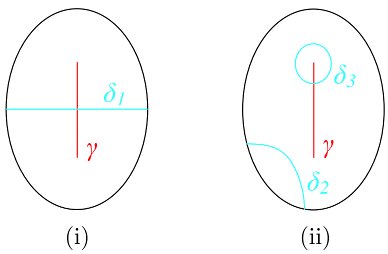


Figure 16.

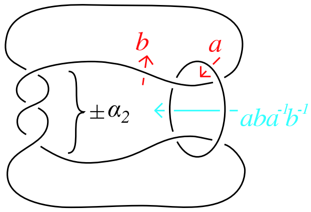


Figure 17.

Hence, we have $P \cap D = \delta_1$.

Cut S^3 along D and close it with two copies of D. Then we have a rational tangle of slope $\pm 1/\alpha$ and the image of P is a disk whose boundary has slope 0. Since such a disk is unique up to isotopy fixing the boundary of the tangle, P can be isotoped to the 2-sphere P_0 in Figure 11 by an isotopy fixing γ.

Case (ii): The condition (ii) holds.

Suppose $D \cap P$ contains a component, c, other than $\delta_2 \cup \delta_3$. By an argument similar to that in the previous case, we may assume that c does not bound a disk in $D \setminus (\gamma \cup \delta_2 \cup \delta_3)$. Then c is a separating loop in $D \setminus \gamma$. Since c is isotopic to K_1 in $S^3 \setminus K_2$, the union $c \cup K_2$ is equivalent to the nontrivial 2-bridge link L'. On the other hand, note that the linking number of c and K_2 is even and that c is isotopic to (a longitude of) K_1 which cannot be parallel to a puncture of $P \cap K_1$ in $P \setminus L'$. These imply that c bounds a disk in $P \setminus L'$ or separates $P \cap K_1$ and $P \cap K_2$. In the former case, c bounds a disk in $S^3 \setminus L'$, which contradicts the fact that $c \cup K_2$ is a nontrivial 2-bridge link. In the latter case, c is isotopic in $P \setminus L'$ to the boundary of a regular neighborhood of δ_2 in P. Since δ_2 bounds a disk in D with an arc on ∂D, we see that c is null-homotopic in $S^3 \setminus L'$, a contradiction. Hence, we have $P \cap D = \delta_2 \cup \delta_3$.

Since P satisfies the condition (P0′), there is a height function $h : S^3 \to [-1, 1]$ such that $P_t := h^{-1}(t)$ satisfies the condition (P0′) when $-1 < t < 1$, and that $P_{\pm 1}$ is an arc meeting K_i $(i = 1, 2)$ in a single point, where $K_2 \cap \gamma = K_2 \cap (P_{+1} \cup P_{-1})$ (see Figure 18). Moreover, we may assume that $P_0 = P$ and that the restriction $g := h|_D$ of h to D has at most one non-degenerate singular point at every level. Thus, for every singular value t_0, $g^{-1}(t_0)$ contains a maximal point, a minimal point or a saddle point. We represent each saddle point in $g^{-1}(t_0)$ by an arc with endpoints on $g^{-1}(t_0 - \varepsilon)$ for sufficiently small $\varepsilon > 0$, as in Figure 19.

Lemma 4.1. *Let t be a regular value of $g(= h|_D)$. Then $g^{-1}(t)$ does not contain a loop separating ∂D and γ in D.*

Proof. Recall that $P_t := h^{-1}(t)$ satisfies the condition (P0′). Hence, $D \cap P_t$

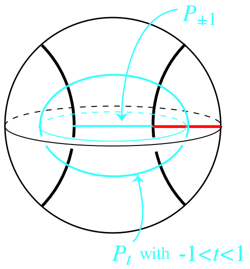


Figure 18.

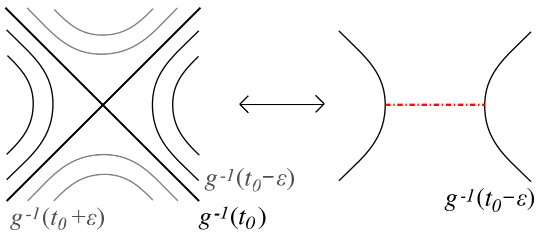


Figure 19.

($= g^{-1}(t)$) satisfies the condition (i) or (ii). In the former case, $D \cap P_t$ does not contain a loop separating ∂D and γ, since $D \cap P_t$ contains a properly embedded arc in D which intersects γ in a single point. In the latter case, we also see that $D \cap P_t$ does not contain a loop separating ∂D and γ by applying the argument at the beginning of Case (ii) to the 3-bridge sphere P_t. □

Let t_0 be a singular value of g and α an arc representing a saddle point in $g^{-1}(t_0)$. Then the arc α is of one of the following three types (see Figure 20):

- α is of *type* 1 if its endpoints are on the same component of $g^{-1}(t_0 - \varepsilon)$, and $g^{-1}(t_0 + \varepsilon)$ contains a loop on D which separates ∂D and γ,
- α is of *type* 2 if its endpoints are on the same component of $g^{-1}(t_0 - \varepsilon)$, and $g^{-1}(t_0 + \varepsilon)$ does not contain a loop on D which separates ∂D and γ, and
- α is of *type* 3 if its endpoints are on different components of $g^{-1}(t_0 - \varepsilon)$.

By Lemma 4.1, we see that an arc of type 1 does not exist. Thus, any arc representing a saddle point of P_{t_0} is of type 2 or of type 3.

Put $X_s := g^{-1}([-1, s])$ for any $s \in [-1, 1]$. Since $P(= P_0)$ cuts D into two disks and an annulus, we may assume that X_0 is the union of the two disks, say X_0^1 and X_0^2. Let X_s^i ($s \in (0, 1]$) be the component of X_s which contains X_0^i

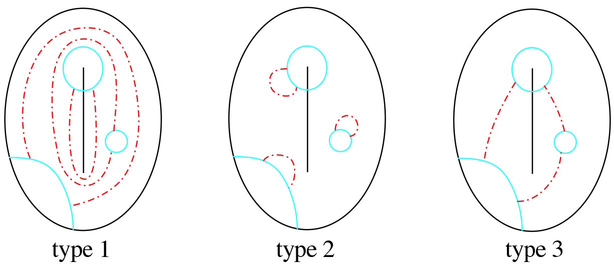


Figure 20. The dashed lines give all possible types of an arc representing a saddle point of g.

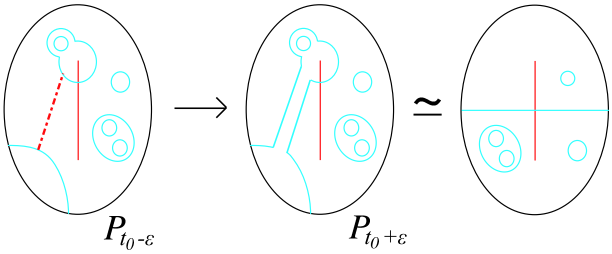


Figure 21.

$(i = 1, 2)$. Since X_1 is connected, there exists a singular value $s_0 \in (0, 1)$ and a sufficiently small $\varepsilon > 0$ such that $X^1_{s_0-\varepsilon} \neq X^2_{s_0-\varepsilon}$ and $X^1_{s_0+\varepsilon} = X^2_{s_0+\varepsilon}$. Then the arc representing the saddle point in $g^{-1}(s_0)$ connects $X^1_{s_0-\varepsilon}$ and $X^2_{s_0-\varepsilon}$. Note that, at any singular point $s'_0(\neq s_0)$, $X^1_{s'_0+\varepsilon} \cup X^2_{s'_0+\varepsilon}$ is homeomorphic to $X^1_{s'_0-\varepsilon} \cup X^2_{s'_0-\varepsilon}$ with some open disks (possibly empty) in it removed. Hence, $X^i_{s_0-\varepsilon}$ is homeomorphic to X^i_0 $(i = 1, 2)$ with some open disks (possibly empty) in it removed. Since the arc representing the saddle point at $t = s_0$ connects the outermost components of $\partial X^1_{s_0-\varepsilon}$ and $\partial X^2_{s_0-\varepsilon}$, which are homeomorphic to ∂X^1_0 and ∂X^2_0, respectively, $P_{s_0+\varepsilon}$ satisfies the condition (i) for the previous case (see Figure 21).

Hence, by the result in Case (i), P can be isotoped to a 2-sphere P_0 in Figure 11 by an isotopy of (S^3, L') preserving γ.

5. 3-bridge spheres of the non-simple exceptional link.

In this section, we show that the exceptional 3-bridge arborescent link L in Figure 8 admits a unique 3-bridge sphere up to isotopy.

Proposition 5.1. *Let L be the link in Figure 8 for some nonzero integer n. Then any 3-bridge sphere of L is isotopic to the 3-bridge sphere S_0 in Figure 22.*

Remark 5.2. Recall from [**17**, Proposition 4] that L is equivalent to $L_2((-1/2, 1/2), (1/n), (-1/2, 1/2))$ or $L_1((-1/2, 1/2-n), (-1/2, 1/2-n))$ according as $|n| > 1$ or $|n| = 1$. Moreover, the 3-bridge sphere S_0 of L is isotopic to the 3-bridge sphere of $L_2((-1/2, 1/2), (1/n), (-1/2, 1/2))$ in Figure 3 (4) when $|n| > 1$,

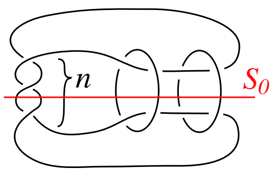


Figure 22.

and isotopic to the 3-bridge sphere $S_1(=S_2)$ of $L_1((-1/2, 1/2-n), (-1/2, 1/2-n))$ in Figure 3 (1) or (2) when $|n| = 1$.

PROOF OF PROPOSITION 5.1. Let L be the link in Figure 8 and S a 3-bridge sphere of L. Let K_1 and K_2 be the two parallel components of L and K_3 the other component. Note that $K_1 \cup K_2$ bounds an annulus, say A, in $S^3 \setminus K_3$. Let D_1 and D_2 be the disjoint disks in S^3 bounded by K_1 and K_2, respectively, such that $D_i \cap A = K_i$ and $D_i \cap K_3$ consists of two points for each $i = 1, 2$ as illustrated in Figure 23. Set $P := A \cup D_1 \cup D_2$. Then P is a 2-sphere which contains $K_1 \cup K_2$ and intersects K_3 in four points. We may assume that S intersects P transversely. Let B_1 and B_2 be the 3-balls in S^3 bounded by P, such that $(B_1, B_1 \cap K_3)$ and $(B_2, B_2 \cap K_3)$ are the tangles as illustrated in Figure 24 (1) and (2), respectively.

Since L consists of 3 components, S intersects each component of L in two points. Hence, one of the following holds.

(A1) $S \cap A$ contains properly embedded non-separating arcs δ_1 and δ_2 in A as in Figure 25 (i), or
(A2) $S \cap A$ contains properly embedded separating arcs δ_3 and δ_4 in A as in Figure 25 (ii).

On the other hand, $S \cap D_i$ $(i = 1, 2)$ satisfies one of the following conditions.

(D1) $S \cap D_i$ contains an arc ε_1^i properly embedded in D_i which separates the two points $D_i \cap K_3$.

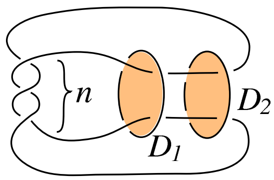


Figure 23.

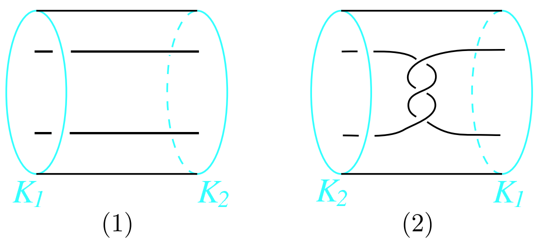


Figure 24.

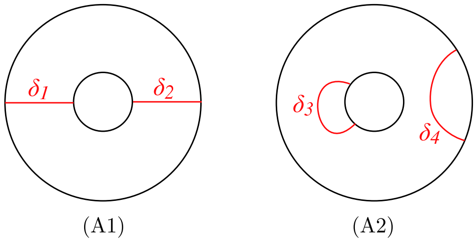


Figure 25.

(D2) $S \cap D_i$ contains an arc ε_2^i properly embedded in D_i which is parallel to the boundary of D_i in $D_i \setminus K_3$.

Case 1: Suppose that the condition (A1) holds.

Case 1.1: Suppose that both $S \cap D_1$ and $S \cap D_2$ satisfy the condition (D1).

Let γ_1 be the loop $\delta_1 \cup \delta_2 \cup \varepsilon_1^1 \cup \varepsilon_1^2$. Then γ_1 bounds two disks, say Δ_1 and Δ_2, in S. We can see that γ_1 is obtained from the loop γ_0 in Figure 26 by applying (half) Dehn twists along the core loop of A. Note that the linking number of γ_1 and K_3 is even, which implies that each Δ_i intersects K_3 in an even number of points. Since S intersects K_3 in two points, one of Δ_1 and Δ_2, say Δ_1, is disjoint from K_3 and the other meets K_3 in two points.

Suppose that $\mathrm{Int}(\Delta_1) \cap P \neq \emptyset$, and pick a (loop) component, c_0, of $\mathrm{Int}(\Delta_1) \cap P$ which is innermost in Δ_1. Since Δ_1 is disjoint from K_3, the disk d_1 bounded by c_0 in Δ_1 is also disjoint from K_3. On the other hand, since c_0 is disjoint from γ_1, c_0 bounds a disk, d_2 in P intersecting K_3 in at most one point. Hence, $d_2 \cap K_3 = \emptyset$. Since L is unsplittable, the 2-sphere $d_1 \cup d_2$ bounds a 3-ball disjoint from L. Thus we may remove the loop component c_0 by an isotopy. By repeating this, we may assume that $\mathrm{Int}(\Delta_1) \cap P$ is empty. Hence, $\Delta_1 \subset B_1$ or $\Delta_1 \subset B_2$.

Recall that $(B_1, B_1 \cap K_3)$ and $(B_2, B_2 \cap K_3)$ are rational tangles of "slopes" $0/1$ and $1/n$, respectively (see Figure 24). Since γ_1 is an essential loop on $P \setminus K_3$ which bounds a disk $\Delta_1 \subset B_1 \setminus K_3$, this implies that γ_1 is isotopic to γ_0, and Δ_1 is isotopic to the disk as in Figure 27 (1).

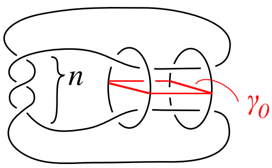


Figure 26.

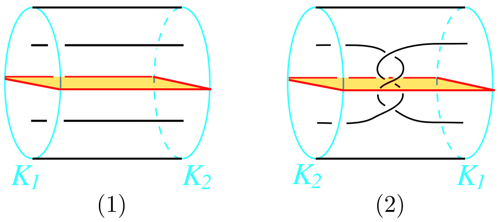


Figure 27.

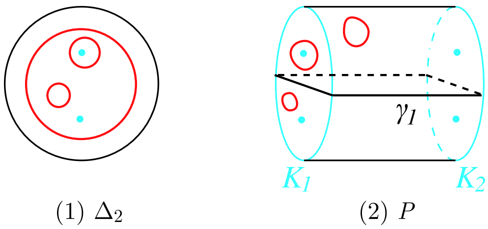


Figure 28.

Note that $\text{Int}(\Delta_2) \cap L = \text{Int}(\Delta_2) \cap K_3$ and it consists of two points. Let c be a component of $\text{Int}(\Delta_2) \cap P$. Then one of the following holds (see Figure 28 (1)).

(i) c bounds a disk in $\text{Int}(\Delta_2) \setminus K_3$,
(ii) c bounds a disk in $\text{Int}(\Delta_2)$ which meets K_3 in a single point,
(iii) c is parallel to $\gamma_1 = \partial\Delta_2$ in $\Delta_2 \setminus K_3$.

On the other hand, c is disjoint from $K_1 \cup K_2 \cap \gamma_1$, and hence bounds a disk in $P \setminus (K_1 \cup K_2 \cup \gamma_1)$ which meets K_3 in at most one point (see Figure 28 (2)).

Let c_1 be a loop satisfying the condition (i) which is innermost in Δ_2. Then c_1 must bound a disk also in $P \setminus (L \cup \gamma_1)$, and hence we can eliminate c_1 from $\text{Int}(\Delta_2) \cap P$ by using the 3-ball bounded by the union of the two disks bounded by c_1. In this way, we can eliminate all loops satisfying the condition (i).

Let c_1 be a loop satisfying the condition (ii) which is innermost in Δ_2. Then c_2 bounds a disk in $\text{Int}(\Delta_2)$ which meets K_3 in a single point, and hence it also bounds a disk in $P \setminus (K_1 \cup K_2 \cup \gamma_1)$ which meets K_3 in one point. The union of the two disks is a 2-sphere in S^3 which meets L in two points. Since L is prime, the 2-sphere bounds a 3-ball in S^3 which meets L in a trivially embedded arc. Hence, we can eliminate c_2 from $\text{Int}(\Delta_2) \cap P$, and we can eliminate all loops satisfying the condition (ii) similarly.

Let c_1 be a loop satisfying the condition (iii). Then c_3 is homotopic to the loop γ_1' in $S^3 \setminus L$ as illustrated in Figure 29. By an argument similar to that in Case (ii) of the proof of Lemma 3.1, we can see that c_3 is not null-homotopic in

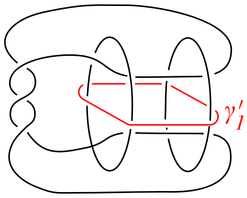


Figure 29.

$S^3 \setminus L$. On the other hand, c_3 bounds a disk in $P \setminus L(\subset S^3 \setminus L)$, a contradiction. Hence, we may assume that $\mathrm{Int}(\Delta_2) \cap P$ is empty, that is, $\Delta_2 \subset B_2$. Since Δ_2 meets K_3 in two points, we see that Δ_2 is isotopic to the disk as in Figure 27 (2).

Therefore, S is isotopic to S_0 in Figure 22.

Case 1.2: Suppose that $S \cap D_1$ and $S \cap D_2$ satisfy (D1) and (D2), respectively.

By an argument similar to that in the previous case, together with the following sublemma, we can see that $S \cap P$ is isotopic to the loop γ_3 as in Figure 30 and that S can be obtained by gluing the two disks in Figure 30 (1) and (2).

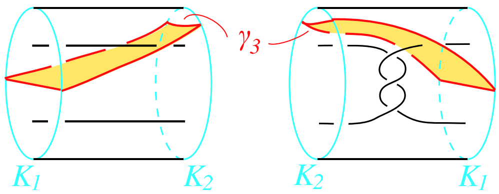


Figure 30.

SUBLEMMA 5.3. *The intersection $S \cap P$ does not contain a loop parallel to K_1 (or K_2) in $P \setminus L$.*

PROOF. Assume on the contrary that $S \cap P$ does not contain a loop c parallel to K_1 (or K_2). Then the union of c and K_3 is equivalent to the sublink $K_1 \cap K_3$ of L. Since $c \cup K_3$ is a nontrivial 2-bridge link with linking number 0 or ± 2, any disk bounded by c meets K_3 in at least two points. Note that c cuts S into two disks. Then the above observation implies that S meets K in at least four points, a contradiction. □

Hence, S can be isotoped to S_0.

Case 1.3: Suppose that both $S \cap D_1$ and $S \cap D_2$ satisfy (D2).

Let $h' : S^3 \to [-1, 1]$ be a height function such that $S_t := h'^{-1}(t)$ is a 2-sphere which meets K_i in two points for each $i = 1, 2, 3$ when $-1 < t < 1$, $S_0 = S$ in particular, and $S_{\pm 1}$ is an arc which meets K_i in one point for each $i = 1, 2, 3$. By applying an argument similar to that for the height function $g = h|_D$ in Case (ii) in the proof of Lemma 3.1 to $h'|_{D_1}$ (or $h'|_{D_2}$), we can see that there exists $t_0 \in (-1, 1)$ such that S_{t_0}, isotopic to S, satisfies the assumption in Case 1.1 or Case 1.2.

Case 2: Suppose that the condition (A2) holds.

Note that every loop component of $A \cap S$ bounds a disk in A or is isotopic to the core loop of A. By Sublemma 6.3, any loop component of $A \cap S$ cannot be isotopic to the core loop of A. Hence, $A \cap S$ consists of only loop components bounding a disk in A. By using an argument similar to that for the "height function" in the previous case, we see that this case can be reduced to Case 1.

This completes the proof of Proposition 5.1. □

6. Classification of 3-bridge spheres.

In this section, we prove Theorem 1.5.

Let L be a 3-bridge arborescent link and suppose that L is not a Montesinos link. By Theorem 1.2 (ii), L admits only one 3-bridge sphere up to isotopy if $L \notin \mathcal{L}_1$. Hence, we focus on the links in $\mathcal{L}_1$. Recall that $\mathcal{L}_1$ is the family of 3-bridge arborescent links in Figure 2 (1). Then, the double branched covering of S^3 branched along a link in $\mathcal{L}_1$ is a 3-manifold obtained from two Seifert fibered spaces $D(\beta_i/\alpha_i, \beta'_i/\alpha'_i)$ $(i = 1, 2)$ over a disk by gluing their boundaries so that a regular fiber and a horizontal loop of M_1 are identified with a horizontal loop and a regular fiber of M_2, respectively.

The proof of Theorem 1.5 is based on the following fact (see [**4**]).

PROPOSITION 6.1. *Let $V_1 \cup_F V_2$ and $W_1 \cup_G W_2$ be genus-g Heegaard splittings of a 3-manifold M such that F and G are isotopic. Assume that the isotopy carries V_1 to W_i for $i = 1$ or 2. Then the generating system $\{x_1, x_2, \ldots, x_g\}$ of $\pi_1(M)$ determined by that of $\pi_1(V_1)$ is Nielsen equivalent to the generating system $\{y_1, y_2, \ldots, y_g\}$ of $\pi_1(M)$ determined by that of $\pi_1(W_i)$. In particular, if $g = 2$ then the commutator $[x_1, x_2]$ is conjugate to $[y_1, y_2]^{\pm 1}$.*

By using this proposition, we distinguish, up to isotopy, the Heegaard surfaces which appear in the proof of Theorem 1.2.

We need the following lemma to solve the conjugacy problems that appear in Lemmas 6.5 and 6.6.

Lemma 6.2. *Let $M = D(\beta_1/\alpha_1, \beta_2/\alpha_2)$ be a Seifert fibered space over a disk with two exceptional fibers ($\alpha_i > 1$), then $\pi_1(M)$ has a group presentation*

$$\pi_1(M) \cong \big\langle c_1, c_2, h \mid [c_j, h], c_j^{\alpha_j} h^{\beta_j} (j = 1, 2)\big\rangle,$$

where $\pi_1(\partial M) = \langle c_1c_2, h\rangle$. For $i = 1, 2$, let η_i be the element of $\pi_1(M)$ represented by the exceptional fiber of M with Seifert index β_i/α_i, namely, $\eta_i = c_i^{\gamma_i} h^{\delta_i}$ for some γ_i and δ_i such that $\alpha_i\delta_i - \beta_i\gamma_i = 1$.

For integers a, b, c and d, let

$$w(a, b, c, d) = \{(c_1c_2)^a h^b\}\eta_1\{(c_1c_2)^c h^d\} \in \pi_1(M).$$

Then the followings are the only solutions of the equation $w(a, b, c, d) = \eta_1^{\pm 1}$ or $\eta_2^{\pm 1}$:

(i) $w(0, b, 0, -b) = \eta_1$,
(ii) $w(\pm 1, b, \pm 1, -b - 2k_1 \pm \beta_2) = \eta_1^{-1}$ *when* $\beta_1 = \pm 1 + k_1\alpha_1$ *and* $\alpha_2 = 2$,
(iii) $w(-1, b, 0, -b - k_1 - k_2) = \eta_2^{\pm 1}$ *when* $\beta_1 = -1 + k_1\alpha_1$ *and* $\beta_2 = \pm 1 + k_2\alpha_2$,
(iv) $w(0, b, 1, -b + k_1 + k_2) = \eta_2^{\pm 1}$ *when* $\beta_1 = 1 + k_1\alpha_1$ *and* $\beta_2 = \mp 1 + k_2\alpha_2$,

where k_i is an integer ($i = 1, 2$).

Let $A * B$ be the free product of two nontrivial groups A and B. A word $w = g_1g_2 \cdots g_n \in A * B$ ($n \geq 0$) is said to be of *normal form* if (i) $g_i \neq 1$, (ii) $g_i \in A$ or $g_i \in B$ ($1 \leq i \leq n$) and (iii) $g_i \in A$ if and only if $g_{i+1} \in B$ ($1 \leq i \leq n-1$). Here, n is called the *length* of $w \in A * B$ and denoted by $|w|$. Then, if a word $w = g_1g_2 \cdots g_n \in A * B$ is of normal form and $n > 1$, then $w \neq 1$ in $A * B$ (see, for example, [**20**, Ch. IV Theorem 1.2]).

To prove Lemma 6.2, we improve the argument in [**24**, Lemma 4.5] and [**16**, Lemma 4.3] which was used to solve certain word problems in the torus knot group.

Proof of Lemma 6.2. We describe only the proof for the case $a \leq -1$. The other cases can be treated by similar arguments.

Consider the quotient group

$$\pi_1(M)/\langle h\rangle \cong \big\langle c_1 \mid c_1^{\alpha_1} = 1\big\rangle * \big\langle c_2 \mid c_2^{\alpha_2} = 1\big\rangle.$$

Suppose $w(a, b, c, d) = \eta_1^{\pm 1}$ or $\eta_2^{\pm 1}$ in $\pi_1(M)$. Then we have $\hat{w}(a, c) = c_1^{\pm\gamma_1}$ or $c_2^{\pm\gamma_2}$ in $\pi_1(M)/\langle h\rangle$, where $\hat{w}(a, c) = (c_1c_2)^a c_1^{\gamma_1}(c_1c_2)^c$ is the element of the quotient group represented by $w(a, b, c, d)$.

Suppose $c \geq 1$. Then we have

$$\hat{w}(a,c) = \left(c_2^{-1}c_1^{-1}\right)^{|a|-1} c_2^{-1}c_1^{\gamma_1}c_2(c_1c_2)^{c-1}.$$

Thus $|\hat{w}(a,c)| > 1$ and hence the equation has no solution.

Suppose $c = 0$. Then we have

$$\hat{w}(a,0) = \left(c_2^{-1}c_1^{-1}\right)^{|a|-1} c_2^{-1}c_1^{\gamma_1-1}.$$

So $|\hat{w}(a,0)| = 1$ if and only if $a = -1$ and $\gamma_1 \equiv 1 \pmod{\alpha_1}$, namely, $\gamma_1 = 1 + k_1'\alpha_1$ for some integer k_1', since the order of c_1 in $\pi_1(M)/\langle h\rangle$ is α_1. Recall that $\alpha_1\delta_1 - \beta_1\gamma_1 = 1$. Thus we have $\beta_1\gamma_1 \equiv -1 \pmod{\alpha_1}$, which implies that $\beta_1 \equiv -1 \pmod{\alpha_1}$, namely, $\beta_1 = -1 + k_1\alpha_1$ for some integer k_1. In this case, $\hat{w}(-1,0) = c_2^{-1}$, and this implies $\pm\gamma_2 \equiv -1 \pmod{\alpha_2}$ and hence $\gamma_2 = \mp 1 + k_2'\alpha_2$ for some integer k_2' and $\beta_2 = \pm 1 + k_2\alpha_2$ for some integer k_2. Moreover, we have $w(-1,b,0,d) = \eta_2^{\pm 1}$, which in turn implies

$$c_2^{\mp\gamma_2-1}c_1^{\gamma_1-1}h^{b+d+\delta_1\mp\delta_2} = 1$$

in $\pi_1(M)$. Since $c_1^{\gamma_1-1} = c_1^{\alpha_1k_1'} = h^{-\beta_1k_1'}$ and $c_2^{\pm\gamma_2-1} = c_2^{\pm\alpha_2k_2'} = h^{\mp\beta_2k_2'}$, we obtain

$$h^{\pm\beta_2k_2'-\beta_1k_1'+b+d+\delta_1\mp\delta_2} = 1,$$

and hence

$$d = -b - \delta_1 \pm \delta_2 + \beta_1k_1' \mp \beta_2k_2'.$$

Since $1 = \alpha_1\delta_1 - \beta_1\gamma_1 = \alpha_1\delta_1 - \beta_1(1+\alpha_1k_1') = \alpha_1(\delta_1 - \beta_1k_1') - (-1+\alpha_1k_1)$, we have $\delta_1 - \beta_1k_1' = k_1$. Similarly, we have $\delta_2 - \beta_2k_2' = \mp k_2$. This implies that

$$d = -b - k_1 - k_2.$$

Thus we obtain the solution (iii).

Suppose $c \le -1$. Then we have

$$\hat{w}(a,0) = \left(c_2^{-1}c_1^{-1}\right)^{|a|-1} c_2^{-1}c_1^{\gamma_1-1}c_2^{-1}c_1^{-1}(c_2^{-1}c_1^{-1})^{|c|-1}.$$

So $|\hat{w}(a,0)| = 1$ if and only if $a = c = -1$, $\gamma_1 \equiv 1 \pmod{\alpha_1}$, namely, $\gamma_1 = 1 + k_1'\alpha_1$ for some integer k_1', and $-2 \equiv 0 \pmod{\alpha_2}$. Since $\gamma_1 \equiv -\beta_1 \pmod{\alpha_1}$, we have $\beta_1 \equiv -1 \pmod{\alpha_1}$, namely, $\beta_1 = -1 + k_1\alpha_1$ for some integer k_1, and we also have

$\alpha_2 = 2$. In this case, $\hat{w}(-1,-1) = c_1^{-1}$, and this implies $w(-1,b,-1,d) = \eta_1^{-1}$ and hence

$$c_2^{-1}c_1^{\gamma_1-1}c_2^{-1}c_1^{\gamma_1-1}h^{b+d+2\delta_1} = 1$$

in $\pi_1(M)$. Since $c_1^{\gamma_1-1} = c_1^{\alpha_1 k_1'} = h^{-\beta_1 k_1'}$ and $c_2^{-2} = c_2^{-\alpha_2} = h^{\beta_2}$, we obtain

$$h^{\beta_2 - 2\beta_1 k_1' + b + d + 2\delta_1} = 1,$$

and hence

$$d = -b - 2\delta_1 + 2\beta_1 k_1' - \beta_2.$$

Since $\delta_1 - \beta_1 k_1' = k_1$, we have

$$d = -b - 2k_1 - \beta_2.$$

Thus we obtain the solution (ii). □

The following lemma can be proved similarly.

LEMMA 6.3. *Let $M = D(\beta_1/\alpha_1, \beta_2, \alpha_2)$, η_i $(i = 1,2)$ and $w(a,b,c,d)$ as in Lemma 6.2. Then the equation $w(a,b,c,d) = 1$ has no solutions (i.e., an exceptional fiber of M is not homotopic to a loop on ∂M).*

LEMMA 6.4. *An unknotting tunnel τ of a nontrivial knot K in S^3 is not homotopic to an arc on $\partial E(K)$.*

PROOF. If τ is homotopic to an arc on $\partial E(K)$, then it follows that the knot group $\pi_1(E(K))$ is generated by the image of $\pi_1(\partial E(K))$, and hence $\pi_1(E(K))$ is abelian. This contradicts the assumption that K is nontrivial. □

Recall from Theorem 1.2 (i) that a link $L \in \mathcal{L}_1$ admits at most four 3-bridge spheres S_1, S_2, S_3 and S_4 in Figure 3 up to isotopy. In the remainder of this section, let F_i $(i = 1,2,3,4)$ be the pre-image of S_i in $M_2(L)$.

The following lemma gives a necessary condition for F_1 and F_2 to be isotopic.

LEMMA 6.5. *Let M be a manifold which belongs to M(1-a) in* [**17**, Theorem 5]*, that is, M is obtained from $M_1 = D(\beta_1/\alpha_1, \beta_1'/\alpha_1')$ and $M_2 = D(\beta_2/\alpha_2, \beta_2'/\alpha_2')$ by gluing their boundaries so that a regular fiber of M_1 is identified with a horizontal loop of M_2, where $\alpha_i, \alpha_i' > 1$ for $i = 1,2$. Let F_1 and F_2 be the two genus-*2

Heegaard surface of M given as above.

Suppose that $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \not\sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for each $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$. Then F_1 and F_2 are not isotopic.

PROOF. Let $U_1 \cup U_2$ be a decomposition of M_1 by a saturated annulus and let $W_1 \cup W_2$ be a one-bridge decomposition of M_2. Put $V_1^1 = U_1 \cup W_1$, $V_2^1 = U_2 \cup W_2$, $V_1^2 = U_1 \cup W_2$ and $V_2^2 = U_2 \cup W_1$. Then we may assume that $F_i = \partial V_1^i = \partial V_2^i$ (see Figure 9 (F1), [**23**] and [**17**, Case 1 in Section 7]).

We describe the generating system of the fundamental group $\pi_1(M)$ of M arising from each handlebody, V_j^i $(i, j \in \{1, 2\})$. Pick a base point x_0 for the fundamental group of M on $T \cap F_1 \cap F_2$, where $T := \partial M_1 = \partial M_2$. Let u_i and v_i be exceptional fibers of M_i whose Seifert indices are β_i/α_i and β'_i/α'_i, respectively. Connect these loops to x_0 by arcs in M_i which does not meet F_i. We denote the generators of $\pi_1(M, x_0)$ obtained from u_i, v_i with the arcs above by u_i, v_i again. Then we have generating systems $\{u_2, u_1\}$ for $\pi_1(V_1^1)$, $\{v_2, v_1\}$ for $\pi_1(V_2^1)$, $\{u_2, v_1\}$ for $\pi_1(V_1^2)$ and $\{v_2, u_1\}$ for $\pi_1(V_2^2)$. This can be seen by using the fact that the 1-bridge decomposition $W_1 \cup W_2$ of M_2 can be chosen so that W_1 is the regular neighborhood in M_2 of the graph obtained by connecting a horizontal loop and the exceptional fiber of M_2 of index β_2/α_2 (see Figure 31 and [**17**, Figure 18]).

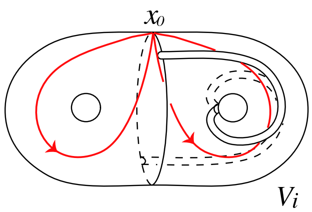


Figure 31.

Suppose that $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \not\sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for both $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$. We prove F_1 and F_2 are not isotopic.

Assume, on the contrary, that F_1 and F_2 are isotopic. Then, by Proposition 6.1, $[u_2, u_1]$ is conjugate to $[u_2, v_1]^{\pm 1}$ or $[v_2, u_1]^{\pm 1}$.

Assume that $[u_2, u_1]$ is conjugate to $[u_2, v_1]$. Let $f_0 : S^1 \to M$ and $f_1 : S^1 \to M$ be maps representing the elements $[u_2, u_1]$ and $[u_2, v_1]$, respectively. By the assumption, f_0 and f_1 are freely homotopic, and hence there is a map

$$\Psi : S^1 (= \mathbb{R}/\mathbb{Z}) \times [0, 1] \longrightarrow M$$

such that $\Psi|_{S^1 \times \{0\}} = f_0$ and $\Psi|_{S^1 \times \{1\}} = f_1$. We may assume that Ψ is transverse to $T (= \partial M_1 = \partial M_2)$ and that $f_i^{-1}(T)$ consists of 4 points $(i = 0, 1)$. Note that the images of the 4 points by f are all equal to the base point x_0 of $\pi_1(M)$. To be precise, we may assume that

$$\Psi\left(\left[0,\frac{1}{4}\right]\times\{0\}\right)=u_2,\qquad \Psi\left(\left[0,\frac{1}{4}\right]\times\{1\}\right)=u_2,$$
$$\Psi\left(\left[\frac{1}{4},\frac{1}{2}\right]\times\{0\}\right)=u_1,\qquad \Psi\left(\left[\frac{1}{4},\frac{1}{2}\right]\times\{1\}\right)=v_1,$$
$$\Psi\left(\left[\frac{1}{2},\frac{3}{4}\right]\times\{0\}\right)=u_2^{-1},\quad \Psi\left(\left[\frac{1}{2},\frac{3}{4}\right]\times\{1\}\right)=u_2^{-1},$$
$$\Psi\left(\left[\frac{3}{4},1\right]\times\{0\}\right)=u_1^{-1},\quad \Psi\left(\left[\frac{3}{4},1\right]\times\{1\}\right)=v_1^{-1}.$$

Since Ψ is transverse to T, $\Psi^{-1}(T)$ is a 1-dimensional submanifold of $S^1\times I$. Since T is incompressible, we may further assume that $\Psi^{-1}(T)$ consists of only arcs. Since u_i and v_i $(i=1,2)$ cannot be homotoped into the boundary (see Lemma 6.3), each component of $\Psi^{-1}(T)$ is an arc joining $S^1\times\{0\}$ and $S^1\times\{1\}$. Then, noting the intersection of $\Psi|_{S^1\times\{i\}}$ and T, we see that the map Ψ is as in Figure 32 (1) or in Figure 32 (2). That is, we may assume

$$\begin{cases}\Psi^{-1}(M_2)=\left(\left[0,\frac{1}{4}\right]\cup\left[\frac{1}{2},\frac{3}{4}\right]\right)\times[0,1],\\[2ex] \Psi^{-1}(M_1)=\left(\left[\frac{1}{4},\frac{1}{2}\right]\cup\left[\frac{3}{4},1\right]\right)\times[0,1],\end{cases}\tag{1}$$

or

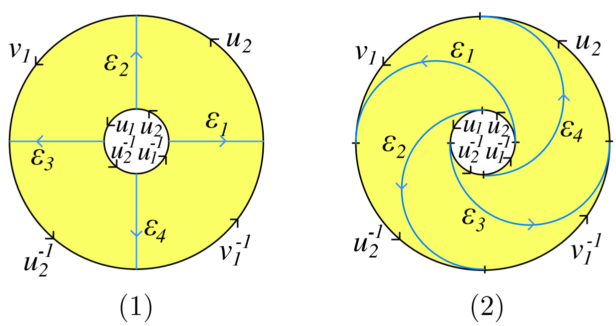


Figure 32.

$$\begin{cases} \Psi^{-1}(M_2) = \left\{ \left(t + \frac{1}{2}s, s\right) \middle| t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right], s \in [0,1] \right\}, \\ \Psi^{-1}(M_1) = \left\{ \left(t + \frac{1}{2}s, s\right) \middle| t \in \left[\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right], s \in [0,1] \right\}. \end{cases} \tag{2}$$

Assume that the identity (1) holds. Let ε_i be the elements of $\pi_1(\partial M_1, x_0)$ represented by $\Psi\,|_{\{(i-1)/4\}\times[0,1]}$ $(i = 1, 2, 3, 4)$. Then

$$\varepsilon_1^{-1} u_2 \varepsilon_2 = u_2 \in \pi_1(M_2), \tag{3}$$

$$\varepsilon_2^{-1} u_1 \varepsilon_3 = v_1 \in \pi_1(M_1), \tag{4}$$

$$\varepsilon_3^{-1} u_2^{-1} \varepsilon_4 = u_2^{-1} \in \pi_1(M_2), \tag{5}$$

$$\varepsilon_4^{-1} u_1^{-1} \varepsilon_1 = v_1^{-1} \in \pi_1(M_1). \tag{6}$$

By Lemma 6.2, equations (3) and (5) have solutions if and only if

$$\varepsilon_1 = \varepsilon_2 = h_2^{n_1} \ (= c_1^{n_1}), \tag{7}$$

$$\varepsilon_3 = \varepsilon_4 = h_2^{n_2} \ (= c_1^{n_2}) \tag{8}$$

for some integers n_1 and n_2, where h_i is a regular fiber of M_i $(i = 1, 2)$ and c_1 is a horizontal loop of M_1, which is identified with h_2. From equation (4), we have

$$c_1^{-n_1} u_1 c_1^{n_2} = v_1.$$

By Lemma 6.2 (iii) and (iv), this equation has a solution if and only if

(i) $\beta_1 = -1 + k_1\alpha_1, \beta_1' = 1 + k_1'\alpha_1'$, $k_1 + k_1' = 0$, $n_1 = 1$ and $n_2 = 0$, or
(ii) $\beta_1 = 1 + k_1\alpha_1, \beta_1' = -1 + k_1'\alpha_1'$, $k_1 + k_1' = 0$, $n_1 = 0$ and $n_2 = 1$.

The first three equalities in (i) (or (ii)) together imply $\beta_1/\alpha_1 + \beta_1'/\alpha_1' = -1/\alpha_1 + 1/\alpha_1'$ (or $\beta_1/\alpha_1 + \beta_1'/\alpha_1' = 1/\alpha_1 + (-1/\alpha_1')$). Hence, by the hypothesis, there do not exist $\{\varepsilon_i\}_{i=1,2,3,4}$ which satisfy equalities $(3), \ldots, (6)$. This is a contradiction.

We can also lead to a contradiction when the equation (2) holds (see Lemma 6.2, (ii), (iii) and (iv)). Hence $[u_2, u_1]$ and $[u_2, v_1]$ are not conjugate.

Similarly, it can be proved that $[u_2, u_1]$ is not conjugate to $[u_2, v_1]^{-1}$ or $[v_2, u_1]^{\pm 1}$. Hence, F_1 and F_2 are not isotopic. □

The following lemma says that any two of F_1, F_2, F_3 and F_4 cannot be isotopic unless they are F_1 and F_2.

LEMMA 6.6. *Let M be a manifold which belongs to* M(1-a) *or* M(2-a) *in* [**17**, Theorem 5]*, that is, M is obtained from $M_1, M_2 \in D[2]$ by gluing their boundaries so that a regular fiber of M_1 is identified with a horizontal loop of M_2. Let $\{G_1, G_2\}$ be a subset of the set $\{F_1, F_2, F_3, F_4\}$ of genus-2 Heegaard surfaces of M, and suppose that $\{G_1, G_2\} \neq \{F_1, F_2\}$. Then G_1 and G_2 are not isotopic.*

PROOF. First, suppose that $G_1 = F_1$ and $G_2 = F_4$. Then $M \in$ M(1-a), and hence M is a union of $M_1 \in D[2]$ and $M_2 = E(S(2n+1, 1)) = D(1/2, -n/(2n+1))$.

Let V_1^i and V_2^i be genus-2 handlebodies in M bounded by G_i. We decompose the handlebodies into several parts as follows (see Figure 9 (F1) and (F2)). Put $U_j := V_j^1 \cap M_1$ and $W_j := V_j^1 \cap M_2$, then $U_1 \cup U_2$ gives a decomposition of M_1 by a saturated annulus and $W_1 \cup W_2$ gives the one-bridge presentation of M_2. Note that either V_1^2 or V_2^2, say V_2^2, is separated into three components by $T := \partial M_1 = \partial M_2$. We put $V_1^2 := U_3 \cup R$ and $V_2^2 := W_3 \cup U_4 \cup W_4$, where $W_3 \cup R \cup W_3$ gives a decomposition of M_1 by two parallel saturated annuli and $U_3 \cup U_4$ gives the two-bridge presentation of M_2.

We describe the generating system of the $\pi_1(M)$ arising from each handlebody, V_j^i $(i, j \in \{1, 2\})$. Pick a base point x_0 for the fundamental group of M on $T \cap G_1$.

Let u_i and v_i be the generators of $\pi_1(M)$ obtained from exceptional fibers of M_i as in the proof of Lemma 6.5. Then the generating systems for $\pi_1(V_1^1, x_0)$ and $\pi_1(V_2^1, x_0)$ are equal to either (i) $\{u_2, u_1\}$ and $\{v_2, v_1\}$, or (ii) $\{u_2, v_1\}$ and $\{v_2, u_1\}$.

Pick a point $x_1 \in W_3 \cap U_4$ and x_2 on $U_4 \cap W_4 \cap G_2$. The generating system of $\pi_1(V_1^2, x_1)$ is $\{\tau_2\tau_1, h_1\}$ and the generating system of $\pi_1(V_2^2, x_1)$ is $\{u_1, \tau_2 v_1' \tau_2^{-1}\}$, where τ_2 is an arc on $U_3 \cap U_4$ joining x_1 to x_2, τ_1 is an arc in R joining x_2 to x_1, h_1 is a regular fiber of M_1 and v_1' is an element of $\pi_1(V_2^2, x_2)$ obtained from v_1 by taking conjugation by an arc τ on T joining x_2 to x_1. We abuse notation to denote the loops $\tau^{-1}\tau_1$ and $\tau_2\tau$ by the symbols τ_1 and τ_2 again. Then the generating systems of $\pi_1(M, x_1)$ arising from V_1^2 and V_2^2 are $\{\tau_2\tau_1, h_1\}$ and $\{u_1, \tau_2 v_1 \tau_2^{-1}\}$, respectively.

Suppose that G_1 and G_2 are isotopic. Then, by Proposition 6.1, $[u_1, \tau_2 v_1 \tau_2^{-1}]$ is conjugate to $[u_2, u_1]^{\pm 1}$ or $[v_2, v_1]^{\pm 1}$. In order to show that this is impossible, recall that $\pi_1(M)$ is the free product of $\pi_1(M_1)$ and $\pi_1(M_2)$ with amalgamated subgroup $\pi_1(T)$. Thus the length of each word of $\pi_1(M)$ with respect to this structure is defined. By using Lemma 6.3, we can see that, for each of $[u_2, u_1]^{\pm 1}$ and $[v_2, v_1]^{\pm 1}$, the minimal length of words conjugate to it is 4. We can also see by using Lemmas 6.3 and 6.4 that the minimal length of words conjugate to $[u_1, \tau_2 v_1 \tau_2^{-1}]$ is 8. Hence, $[u_1, \tau_2 v_1 \tau_2^{-1}]$ is not conjugate to $[u_2, u_1]^{\pm 1}$ or $[v_2, v_1]^{\pm 1}$. Hence, G_1 and G_2 are not isotopic.

Similarly, it can be proved that F_1 or F_2 cannot be isotopic to F_3 or F_4. Moreover, by similar arguments, one can also prove that F_3 and F_4 are not isotopic. □

PROOF OF THEOREM 1.5. Suppose that $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for some $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$. Then the two 3-bridge spheres S_1 and S_2 for L are isotopic by an isotopy illustrated in Figure 33. If $(\beta_k/\alpha_k, \beta'_k/\alpha'_k) \not\sim (\varepsilon_k/\alpha_k, \varepsilon'_k/\alpha'_k)$ for both $k = 1, 2$, where $\varepsilon_k, \varepsilon'_k \in \{\pm 1\}$, then S_i and S_j are not isotopic by Lemmas 6.5, 6.6 and Theorem 1.8. □

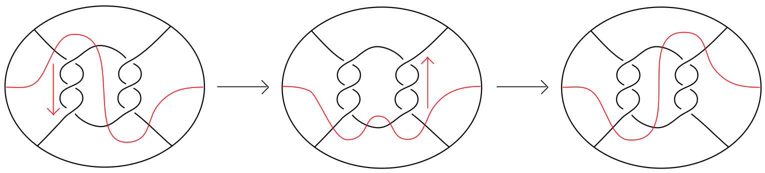


Figure 33.

REMARK 6.7. Lemmas 6.5 and 6.6 enable us to complete Table 5.2 in [**23**]. Except for the following cases, each number μ on the table gives the exact number of Heegaard splittings up to isotopy.

- Let $M_1 = E(S(2n+1, 1)) = D(1/2, -n/(2n+1))$ and $M_2 = D(1/2, -1/3)$, and suppose that a regular fiber of M_2 is identified with a loop $m_1 h_1^a$, where m_1 and h_1 are, respectively, a meridian and a regular fiber of M_1. Then $M = M_1 \cup_f M_2$ admits exactly two genus-2 Heegaard splittings up to isotopy, one of which belongs to F(1) and the other belongs to F(2-2).
- Let $M_1 = D(1/2, -1/3)$ and $M_2 = E(S(2n+1, 1)) = D(1/2, -n/(2n+1))$, and suppose that a regular fiber of M_1 is identified with a loop $m_2 h_2^a$, where m_2 and h_2 are, respectively, a meridian and a regular fiber of M_2. Then $M = M_1 \cup_f M_2$ admits exactly two genus-2 Heegaard splittings up to isotopy, one of which belongs to F(1) and the other belongs to F(2-1).

Moreover, one can obtain the homeomorphism classification of 3-bridge presentations and genus-2 Heegaard splittings by considering the action of the mapping class group of M on the Heegaard surfaces.

7. 3-bridge spheres for Montesinos links.

In this section, we prove Theorem 1.7.

Let $M = S^2(e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ be a Seifert fibered space over S^2 with three exceptional fibers. To describe the results of [**4**], we take two exceptional fibers η_i, η_j $(1 \le i \ne j \le 3)$ and connect them by an arc projected to a simple arc on the base S^2. A regular neighborhood $V(i,j)$ of the graph obtained is a handlebody of genus 2. The closure $W(i,j)$ of the complement is also a handlebody of genus 2 and we obtain a Heegaard surface $F(i,j) = \partial V(i,j) = \partial W(i,j)$ of M.

This is called a *vertical Heegaard surface.*

THEOREM 7.1 ([**4**, Theorem 2.5]). *Let* $M = S^2(e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ *be a Seifert fibered space over* S^2 *with three exceptional fibers.*

(A) *If* $\beta_i \not\equiv \pm 1 \pmod{\alpha_i}$ *for* $i = 1, 2, 3$, *then* M *admits, up to isotopy, exactly three Heegaard surfaces of genus* 2, *namely* $F(1,2)$, $F(2,3)$, $F(3,1)$.
(B) *If* $\beta_i \not\equiv \pm 1 \pmod{\alpha_i}$ *for* $i = 1, 2$ *and* $\beta_3 \equiv \pm 1 \pmod{\alpha_3}$, *then* M *admits, up to isotopy, exactly two Heegaard surfaces of genus* 2, *namely* $F(1,2)$ *and* $F(2,3)(= F(3,1))$.
(C) *If* $\beta_i \equiv \pm 1 \pmod{\alpha_i}$ *for* $i = 2, 3$, *then* M *admits, up to isotopy, a single Heegaard surface of genus* 2, $F(1,2)(= F(2,3) = F(3,1))$, *except when* M *is one of* $S(-1/6a; 1/2, (-a)^{-1}/3, 6^{-1}/a)$ *(a is odd),* $S(-1/6a; (-1)^{-1}/3, (-a)^{-1}/3, 3^{-1}/a)$ *and* $S(-1/4b; 1/2, (-b)^{-1}/4, 4^{-1}/a)$, *where* $a \geq 7$, $b \geq 5$ *and* $\text{g.c.d.}(a,3) = \text{g.c.d.}(b,2) = 1$. *In each exceptional case* M *admits, up to isotopy, a unique additional Heegaard surface of genus* 2 *obtained by presenting* M *as the double branched covering of* S^3 *branched along a* 3*-bridge presentation of a link in Figure* 34. *(See* [**2**], [**5**]*).*

LEMMA 7.2. *The links in Figure* 34 *are not arborescent links.*

PROOF. Suppose that a link, say L, in Figure 34 is an arborescent link. Since L is not hyperbolic, it must be equivalent to a link in Figure 35 by [**10**], [**14**] (cf. [**17**, Proposition 3]). Namely, one of the following holds.

I. L is the boundary of a single unknotted band, i.e., a torus knot or link of type $(2, n)$ for some $n \in \mathbb{Z}$.
II. L has two parallel components, each of which bounds a twice-punctured disk properly embedded in $S^3 \setminus L$.
III. L or its reflection is the pretzel link $P(p, q, r, -1)$, where $p, q, r \geq 2$ and $1/p + 1/q + 1/r \geq 1$.

However, this cannot occur since

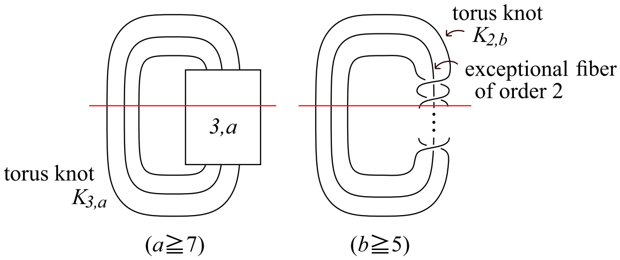


Figure 34.

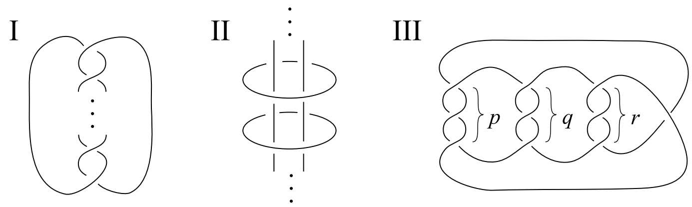


Figure 35.

- a link in I of Figure 35 is either a torus knot $K_{2,n}$ (n is odd) or a torus link $K_{2,n}$ (n is even) which consists of two trivial components,
- a link in II of Figure 35 consists of at least three components,
- $P(2,2,n,-1)$ (n is odd) is a union of a torus knot $K_{2,n}$ and its core of index n,
- $P(2,2,n,-1)$ (n is even) has three components,
- $P(2,3,3,-1)$ is the torus knot $K_{3,4}$,
- $P(2,3,4,-1)$ is a union of the torus knot $K_{2,3}$ and its core of index 2,
- $P(2,3,5,-1)$ is the torus knot $K_{3,5}$,
- $P(2,3,6,-1)$ consists of two components, the torus knot $K_{2,3}$ and the unknot,
- $P(2,4,4,-1)$ has three components,
- $P(3,3,3,-1)$ consists of two trivial components.

Hence we obtain the desired result. □

REMARK 7.3. Let F be an exceptional Heegaard surface of a manifold M in Theorem 7.1 (C) and τ_F the hyper-elliptic involution τ associated with F. Then $(M, \mathrm{Fix}(\tau_F))/\tau_F$ is a links in Figure 34. Hence, for any arborescent link L, the covering involution τ_L of $M_2(L)$ is not equivalent to τ_F.

To prove Theorem 1.7 (2), we need the following proposition.

PROPOSITION 7.4 ([**29**, Theorem 4.1]). *Let L be an elliptic Montesinos link and assume that L is not a 2-bridge link. Then the symmetry group* $\mathrm{Sym}(S^3, L)$ *is as follows according to the type of L. Here, ψ_i ($i = 1,2,3,4$) is a symmetry of (S^3, L) as illustrated in Figure* 36.

(i) *Let $L = L(b; 1/2, 1/2, \beta/\alpha)$ and put $m = (-b+1)\alpha + \beta$.*
(i-1) *If* g.c.d.$(m, 2\alpha) = 1$, *then* $\mathrm{Sym}(S^3, L)$ *is given by*

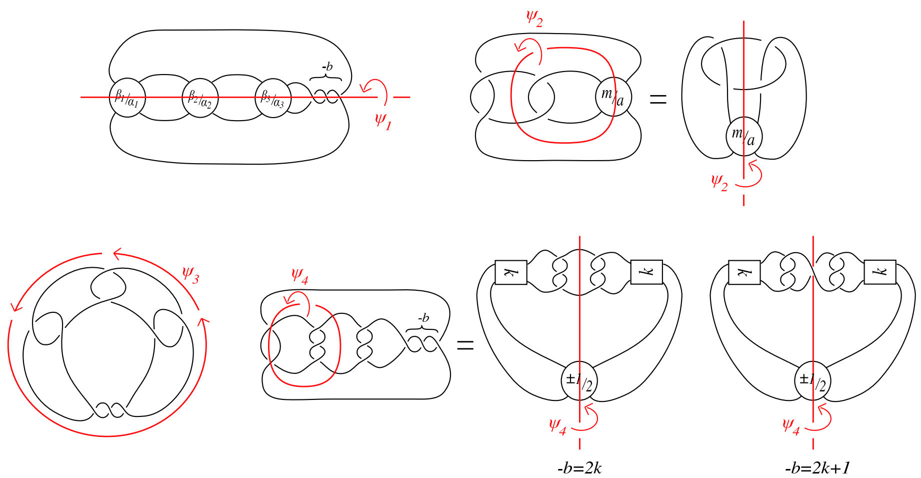


Figure 36.

	$\alpha \geq 3$	$\alpha = 2$
$m \neq 1$	$\langle \psi_1, \psi_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\langle \psi_1, \psi_3 \rangle \cong \mathbb{Z}_2 \oplus D_3$
$m = 1$	$\langle \psi_1 (= \psi_2) \rangle \cong \mathbb{Z}_2$ *if* α *is odd,* $\langle \psi_1, \psi_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *if* α *is even.*	$\langle \psi_1, \psi_3 \rangle \cong \mathbb{Z}_2 \oplus D_3$

(i-2) *If* m *is even and* g.c.d.$(m, \alpha) = 1$, *then*

$$\mathrm{Sym}(S^3, L) = \langle \psi_1, \psi_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(ii) *Let* $L = L(b; 1/2, \beta_2/3, \beta_3/3)$ *and put* $m = -6b + 3 + 2(\beta_2 + \beta_3)$. *Then*

$$\mathrm{Sym}(S^3, L) = \begin{cases} \langle \psi_1, \psi_4 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if g.c.d.}(m, 12) = 1 \text{ and } m \neq 1, \\ \langle \psi_1 \rangle \cong \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

(iii) *If* $L = L(b; 1/2, \beta_2/3, \beta_3/4)$ *or* $L(b; 1/2, \beta_2/3, \beta_3/5)$, *then*

$$\mathrm{Sym}(S^3, L) = \langle \psi_1 \rangle \cong \mathbb{Z}_2.$$

Proof of Theorem 1.7. Let L be a 3-bridge Montesinos link. Let $M_2(L)$ be the double branched covering of S^3 branched over L, and let $p : M_2(L) \to S^3$ be the covering projection.

(1) Suppose that L is nonelliptic, and let P_i $(i = 1, 2, 3, 4, 5, 6)$ be a 3-bridge

sphere of L in Figure 5. Then the pre-images $p^{-1}(P_1)$, $p^{-1}(P_3)$ and $p^{-1}(P_5)$ are isotopic to $F(1,2)$, $F(2,3)$ and $F(1,3)$, respectively. By Theorem 7.1 ([**4**, Theorem 2.5]) and Remark 7.3, these are the only genus-2 Heegaard surfaces of $M_2(L)$ whose hyper-elliptic involutions are strongly equivalent to τ_L. On the other hand, we see $P_{i+1} = \rho(P_i)$ for each $i = 1, 3, 5$, where ρ is the symmetry of (S^3, L) given in Figure 7. Hence, by Theorem 1.8 and Remark 2.6, we see that L admits at most six 3-bridge spheres $P_1,\ldots,P_6$ up to isotopy.

(2) Suppose that L is elliptic. Let P_1 be the 3-bridge sphere of L as illustrated in Figure 5, and let P be any 3-bridge sphere of L. Set $F_1 := p^{-1}(P_1)$ and $F := p^{-1}(P)$. We note that $\tau_{F_1} = \tau_F = \tau_L$. Since $M_2(L)$ admits a unique genus-2 Heegaard surface up to isotopy whose hyper-elliptic involution is τ_L, by Theorem 7.1 ([**4**, Theorem 2.5]) and Remark 7.3, F is isotopic to F_1. Thus, there exists a self-homeomorphism φ of $M_2(L)$ such that $\varphi(F_1) = F$ and φ is isotopic to the identity. By the proof of [**3**, Theorem 8] (cf. the proof of [**17**, Proposition 5]), we may assume that φ is τ_L-equivariant, where τ_L is the covering transformation. So we have a self-homeomorphism ψ of (S^3, L) sending P_1 to P. Hence, it suffices to show that generators of the symmetry group $\mathrm{Sym}(S^3, L)$ preserve P_1 up to isotopy.

We show this only when L satisfies the condition (i-1) of Proposition 7.4, where $m = 1$ and $\alpha \geq 3$. (The other cases can be treated similarly.) In this case, we have $\beta/\alpha = 1/\alpha + (b-1)$ from $(-b+1)\alpha + \beta = 1$, and hence, $L = L(b; 1/2, 1/2, \beta/\alpha)$ is equivalent to $L(0; -1/2, 1/2, 1/\alpha)$. Note that $\psi_2(P_1) = P_1$ and that we can isotope $\psi_1(P_1)$ to P_1 as illustrated in Figure 37. Hence, L admits a unique 3-bridge sphere up to isotopy.

An isotopy between P and P_1 can be constructed similarly for every case. Thus every elliptic Montesinos link admits a unique 3-bridge sphere up to isotopy. □

REMARK 7.5. For nonelliptic Montesinos links, we give some conditions for P_i and P_j $(i, j = 1, \ldots, 6,\ i \neq j)$ to be isotopic by using isotopies as in the proof of Theorem 1.7. The following table gives the conditions for each pair, where (1-k) and (2-k) $(k = 1, 2, 3)$ denote the following conditions.

(1-k) $\beta_k \equiv \pm 1 \pmod{\alpha_k}$ $(k = 1, 2, 3)$,

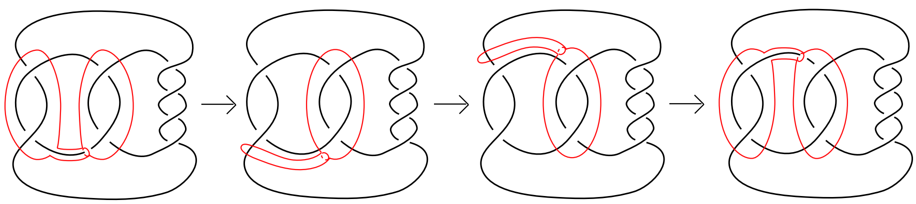


Figure 37.

(2-k) $\alpha_k = 2$ $(k = 1, 2, 3)$.

P_2	P_3	P_4	P_5	P_6	
(2-1) or (2-2)		(1-2)		(1-1)	P_1
	(1-2)		(1-1)		P_2
		(2-2) or (2-3)		(1-3)	P_3
			(1-3)		P_4
				(2-1) or (2-3)	P_5

For example, P_1 and P_2 are isotopic if (2-1) $\alpha_1 = 2$ or (2-2) $\alpha_2 = 2$ holds. Moreover, this implies, for example, P_1 and P_3 are isotopic if (i) (2-1) (or (2-2)) and (1-2) holds, (ii) (1-2) and (2-2) (or (2-3)) holds, or (iii) (1-1) and (1-3) holds. If $\beta_i \equiv \pm 1 \pmod{\alpha_k}$ for all $i = 1, 2, 3$ and $b = \sum(\beta_i/\alpha_i) - \sum(\pm 1/\alpha_i)$, then $P_1, \ldots, P_6$ are mutually isotopic.

Acknowledgments. The author would like to express her appreciation to Professor Makoto Sakuma for his guidance, advices and encouragement. She would also like to thank Kanji Morimoto and Kai Ishihara for their helpful comments.

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