J. Math. Soc. Japan
Vol. 65, No. 1 (2013) pp. 1–12
doi: 10.2969/jmsj/06510001

Holomorphic functions on subsets of $\mathbb{C}$

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(Received Apr. 16, 2011)

Abstract. Let Γ be a C^∞ curve in $\mathbb{C}$ containing 0; it becomes Γ_θ after rotation by angle θ about 0. Suppose a C^∞ function f can be extended holomorphically to a neighborhood of each element of the family $\{\Gamma_\theta\}$. We prove that under some conditions on Γ the function f is necessarily holomorphic in a neighborhood of the origin. In case Γ is a straight segment the well known Bochnak-Siciak Theorem gives such a proof for *real analyticity*. We also provide several other results related to testing holomorphy property on a family of certain subsets of a domain in $\mathbb{C}$.

Introduction.

The Bochnak-Siciak Theorem [**Bo**], [**Si**] states the following. Let $f \in C^\infty(D)$, D is a domain, $0 \in D \subset \mathbb{R}^n$. Suppose f is (real) analytic on every line segment through 0. Then f is analytic in the neighborhood of 0 (as a function of n variables). For $n = 2$ this statement can be interpreted as follows. Consider the segment $I = \{(x, y) \mid x \in [-1, 1], y = 0\}$, I_θ its rotation by angle θ about the origin. If f is real analytic on each I_θ then f is real analytic in a neighborhood of the origin as a function of two variables. Here we are interested in examining a similar statement regarding the holomorphic property of f. That is if Γ is a C^∞ curve in $\mathbb{C}$ containing 0, Γ_θ its rotation by angle θ about the origin, and f can be extended holomorphically to a neighborhood of each Γ_θ, then under what condition on Γ can one claim that f is holomorphic in a neighborhood of 0? For Γ real analytic (including $\Gamma = I$) the answer is negative, but for some C^∞ curves the answer is positive.

The questions we are examining here as well as the Bochnak-Siciak Theorem can be considered as solving the Osgood-Hartogs-type problems; here is a quote from [**ST**]: "Osgood-Hartogs-type problems ask for properties of 'objects' whose restrictions to certain 'test-sets' are well known". [**ST**] has a number of examples of such problems. Other meaningful and interesting problems and examples of this type one can find in ([**AM**], [**BM**], [**Bo**], [**LM**], [**Ne**], [**Ne2**], [**Ne3**], [**Re**], [**Sa**], [**Si**], [**Zo**]), and other papers. Most of the research has been devoted to consideration of formal power series and specific classes of functions of several

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2010 *Mathematics Subject Classification.* Primary 30E99, 30C99.
Key Words and Phrases. analytic functions, Hausdorff dimension, Hartogs property.

variables as 'objects' which converge (or, in case of functions, have the property of being smooth) on each curve (or subvariety of lower dimension) of a given family. The property of a series to be convergent (or, for functions, to be smooth) is then proved.

Our work in this paper is also related to another set of specific Osgood-Hartogs-type problems. The famous Hartogs theorem states that a function f in $\mathbb{C}^n$, $n > 1$, is holomorphic if it is holomorphic in each variable separately, that is, f is holomorphic in $\mathbb{C}^n$ if it is holomorphic on every complex line parallel to an axis. So, one can test the holomorphy of a function in $\mathbb{C}^n$ by examining if it is holomorphic on each of the above mentioned complex lines. There is a wide area of interesting results on testing holomorphy on subsets of $\mathbb{C}$, specifically on curves: see [**A1**], [**A2**], [**A3**], [**AG**], [**E**], [**G1**], [**G2**], [**G3**], [**T1**], [**T2**] and references in those articles. Some of these results assume a holomorphic extension into the inside of each closed curve in a given family, others a "Morera-type" property.

In this paper we also consider testing holomorphy on subsets of $\mathbb{C}$. In addition to rotations about a point (when the subset is a curve) as mentioned in the beginning, we will allow some linear transformations to be applied to these subsets. We consider a subset $S \subset \mathbb{C}$ and form a family of "test-sets" by considering all images of S under a (small enough) subset of $\mathcal{L}$, the set of all linear holomorphic automorphisms of $\mathbb{C}$. We then discuss the conditions on S under which a C^∞ function given in a domain will be holomorphic in that domain if it is holomorphic on this specific family of sets. Below is a more precise explanation.

Let $S \subset \mathbb{C}$. We say that $f : S \to \mathbb{C}$ is holomorphic if f is a restriction on S of a function holomorphic in some open neighborhood of S. Let $\mathbb{L}$ be a subset of $\mathcal{L}$.

Definition. The set S has Hartogs property with respect to $\mathbb{L}$ (denoted $S \in \hat{H}(\mathbb{L})$) if the following holds:

Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to \mathbb{C}$ a C^∞ function. Suppose for any $L \in \mathbb{L}$, f restricted to $L(S) \cap \Omega$ is holomorphic. Then f is holomorphic in Ω.

The main question we are addressing here is: which sets S have Hartogs property with respect to a given set of transformations?

We will examine this question depending on $\dim(S)$ – the real Hausdorff dimension of S.

We consider three cases and provide the following answers:

1. $\dim(S) > 1$. We prove that in this case $S \in \hat{H}(\mathbb{T})$, where $\mathbb{T}$ is the group of linear translations (Theorem 1.1).
2. $\dim(S) = 1$. Such a set may or may not have Hartogs property with respect to $\mathbb{T}$. In addition to examples we examine explicitly the case when $S = \Gamma$ is a C^∞ curve, as referred to in the beginning of this introduction. We consider the

set of transformations $\mathbb{T}_1 = \{\sigma \circ \tau : \sigma \in \mathbb{T}, \tau \in \mathbb{U}\}$, where $\mathbb{U}$ is an open subset of the group $\mathbb{C}^*$. Though we do not provide a complete classification of these curves we nevertheless point out the major obstacle for a curve to have Hartogs property: real analyticity. So, in this case we essentially show that if S is a C^∞ curve then $S \in \hat{H}(\mathbb{T}_1)$ if and only if S is not analytic (for exact statements see Proposition 1.5, and Theorem 1.6).

3. $\dim(S) < 1$. As in case 2 such a set may or may not have Hartogs property with respect to $\mathbb{T}$. We specifically examine the situation when S is a sequence with one limit point (so $\dim(S) = 0$), and with a reasonable restriction (a slight change of the definition of a holomorphic function on a sequence), our investigation essentially explains that S has a certain Hartogs property if and only if such a sequence does not eventually end up on an analytic curve (for the precise statement see Theorem 1.9 and the discussions preceding this theorem).

1. Main Results.

1.1. Case 1: dim $(S) > 1$.

Let $S \subset \mathbb{C}$. In this section we prove the following

THEOREM 1.1. *If* $\dim(S) > 1$, *then* $S \in \hat{H}(\mathbb{T})$.

The proof of this theorem follows from several statements below. For all of them S is an arbitrary subset of $\mathbb{C}$. First we consider the following.

Let $p \in S$. A point t in $T := \{z \in \mathbb{C} : |z| = 1\}$ is said to be a limit direction of S at p if there exists a sequence (q_j) in S such that $\lim_j q_j = p$ and $\lim_j \tau(p, q_j) = t$, where $\tau(p, q_j) := (q_j - p)/|q_j - p|$.

LEMMA 1.2. *Let* $\Omega \subset \mathbb{C}$ *be an open set,* $p \in \Omega \cap S$ *and there are at least two limit directions* t_1, t_2 *of* S *at* p. *Suppose a function* $f \in C^1(\Omega)$ *is holomorphic on* $S \cap \Omega$. *If* $t_1 \neq \pm t_2$ *then* $\partial f/\partial \overline{z} = 0$ *at* p.

PROOF. The derivatives of f along linearly independent directions t_1 and t_2 coincide with derivatives of a holomorphic function in the neighborhood of p. The statement now follows from the Cauchy-Riemann equations. □

COROLLARY 1.3. *If a set* $S \subset \mathbb{C}$ *has a point* p *with at least two limit directions* $t_1 \neq \pm t_2$ *of* S *at* p, *then* S *has Hartogs property with respect to* $\mathbb{T}$.

PROOF. Let $\Omega \subset \mathbb{C}$, $f \in C^\infty(\Omega)$. Suppose that for any translation L, f is holomorphic on $L(S) \cap \Omega$. Let $z_0 \in \Omega$. Pick such an L, that $L(p) = z_0$. Since f is holomorphic on $L(S) \cap \Omega$, and (by choice of p) there are at least two limit directions $t_1 \neq \pm t_2$ of $L(S) \cap \Omega$ at z_0, then by Lemma 1.2, $\partial f/\partial \overline{z} = 0$ at z_0. So,

$\partial f/\partial\overline{z} = 0$ everywhere on Ω, and therefore f is holomorphic on Ω. $\square$

For a positive integer N let S_N be the set of points p in S such that S has no more than N distinct limit directions of S at p. Let M_d denote the Hausdorff measure of dimension d. Let $D(p, r)$ denote the closed disc centered at p of radius r.

LEMMA 1.4. *For $d > 1$, $M_d(S_N) = 0$. Hence the Hausdorff dimension of S_N is ≤ 1.*

PROOF. Choose a positive integer K and a positive number ϵ such that

$$B := \frac{2^d N}{K^{d-1}} < 1, \quad D(0,1) \cap \{q : |\tau(0,q) - 1| \leq \epsilon\} \subset \cup_{j=1}^{K} D(j/K, 1/K).$$

For a positive integer n let S_N^n be the set of points p of S such that there exist N directions t_k, $k = 1, \dots, N$, depending on p, satisfying

$$D(p, 1/n) \cap S \subset \cup_{k=1}^{N} \{q \in \mathbb{C} : |\tau(p,q) - t_k| < \epsilon\}.$$

Fix n and consider a disc $D(p', r)$, where $p' \in \mathbb{C}$ and $r \leq 1/(2n)$. If $S_N^n \cap D(p', r)$ is not empty, let p be a point of this intersection. So there exist N directions t_k, $k = 1, \dots, N$, satisfying

$$D(p, 2r) \cap S_N^n \subset D(p, 2r) \cap \cup_{k=1}^{N} \{q \in \mathbb{C} : |\tau(p,q) - t_k| < \epsilon\}.$$

The set on the right side of the above equation can be covered by KN discs of radius $(2r/K)$ with centers

$$p + \frac{2rjt_k}{K}, \quad j = 1, \dots, K, \ k = 1, \dots, N.$$

Hence $D(p', r) \cap S_N^n$ can be covered by KN closed discs of radius $(2r/K)$ provided $r \leq 1/(2n)$.

Now there is a positive integer L such that S_N^n is covered by L discs of radius $1/(2n)$: $S_N^n \subset \cup_{j=1}^{L} D(p_j, 1/(2n))$. Each set $S_N^n \cap D(p_j, 1/(2n))$ is covered by KN discs of radius $1/(nK)$. Hence S_N^n is covered by LKN discs of radius $1/(nK)$. For each of these smaller discs we can proceed with the similar construction. So, continuing this way we see that for any $\nu = 1, 2, \dots,$ the set S_N^n is covered by $L(KN)^\nu$ discs of radius $(1/2n)(2/K)^\nu$. It follows that $M_d(S_N^n) \leq L(KN)^\nu \cdot [(1/2n)(2/K)^\nu]^d = CB^\nu$, where $C = L/(2n)^d$. Hence $M_d(S_N^n) = 0$. Since $S_N \subset$

$\cup_{n=1}^{\infty} S_N^n$, we obtain $M_d(S_N) = 0$. □

Proof of Theorem 1.1. Since $\dim(S) > 1$, then by Lemma 1.4, $S \setminus S_2 \neq \emptyset$. Therefore there is a point $p \in (S \setminus S_2)$, with at least two limit directions $t_1 \neq \pm t_2$ of S at p. Now by Corollary 1.3, S has Hartogs property with respect to $\mathbb{T}$. □

1.2. Case 2: $\dim(S) = 1$.

The most interesting situation in this case is when $S = \Gamma$ is a curve. By using Corollary 1.3 one can easily construct curves that have Hartogs property with respect to $\mathbb{T}$ (any broken curve (not a segment) consisting of two links and forming an angle would be such an example). On the other hand if Γ is a real analytic curve the following statement holds.

Proposition 1.5. *Let $\Gamma \subset \mathbb{C}$ be a real analytic curve. Then Γ does not have Hartogs property with respect to $\mathcal{L}$.*

Proof. Consider a domain $\Omega \subset \mathbb{C}$, say the unit disk, $f = \overline{z} = x - iy$ – a nowhere holomorphic function. We prove that f can be extended holomorphically to a neighborhood of $L(\Gamma) \cap \Omega$ for any $L \in \mathcal{L}$. Without any loss of generality we may assume $L = id$, so we now consider $\Gamma \cap \Omega$. Due to the uniqueness theorem for holomorphic functions we only need to prove the extendability of f locally for any point $z_0 \in \Gamma \cap \Omega$. Again with no loss of generality we may assume that $z_0 = 0$ and that near the origin Γ is described by the equation $y = \varphi(x)$, where $\varphi(x)$ is a real analytic function. Replacing now real coordinates with $z = x + iy$ we get an implicit equation $(1/2i)(z - \overline{z}) = \varphi((1/2)(z + \overline{z}))$, and from here one can locally recover $\overline{z} = \psi(z)$ on Γ, where $\psi(z)$ is holomorphic near the origin. □

We will now concentrate on smooth curves that are not analytic. We start with the following definition.

Let $f(z)$ be a function defined on an open set Ω in the complex plane $\mathbb{C}$ containing the origin. The function f is said to have a Taylor series at 0 if there is a formal power series $g(z, w) = \sum_{jk} a_{jk} z^j w^k \in \mathbb{C}[[z, w]]$ such that for each nonnegative integer n,

$$f(z) - \sum_{j+k \leq n} a_{jk} z^j \overline{z}^k = o(|z|^n).$$

The Taylor series of f at 0 is $g(z, \overline{z}) = \sum_{jk} a_{jk} z^j \overline{z}^k$.

We note that every C^∞ function defined in the neighborhood of 0 has a Taylor series at 0.

Consider a curve of the form $\Gamma := \{t + i\phi(t) : 0 \leq t \leq b\}$, where ϕ is a real-valued continuous function defined on the interval $[0, b]$. The function ϕ is said to have a Taylor series at 0 if there exists an $h(z) := \sum_j b_j z^j \in \mathbb{C}[[z]]$ such that for each nonnegative integer n,

$$\phi(t) - \sum_{j \leq n} b_j t^j = o(|t|^n).$$

Pick an open set $\mathbb{U} \subset \mathbb{C}^*$, and denote $\mathbb{T}_1 = \{\sigma \circ \tau : \sigma \in \mathbb{T}, \tau \in \mathbb{U}\}$.

THEOREM 1.6. *Let $S := \{t + i\phi(t) : 0 \leq t \leq b\}$ be a continuous curve with $\phi(0) = 0$. Suppose ϕ has a Taylor series at 0, and for no $\lambda > 0$ is ϕ analytic on $[0, \lambda)$. Then $S \in \hat{H}(\mathbb{T}_1)$.*

This theorem is a corollary of Theorem 1.8 below.

First some remarks on formal power series. $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$ denotes the set of (formal) power series

$$g(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n \geq 0} a_{k_1 \ldots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

of n variables with complex coefficients. Let $g(0) = g(0, \ldots, 0)$ denote the coefficient $a_{0,\ldots,0}$. A power series equals 0 if all of its coefficients $a_{k_1 \ldots k_n}$ are equal to 0. A power series $g \in \mathbb{C}[[x_1, x_2, \ldots, x_n]]$ is said to be convergent if there is a constant $C = C_g$ such that $|a_{k_1 \ldots k_n}| \leq C^{k_1 + \cdots + k_n}$ for all $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$.

LEMMA 1.7. *Let $g \in \mathbb{C}[[x, y]]$ with $g'_y \neq 0$, let $h \in \mathbb{C}[[x]]$ be a non-zero power series with $h(0) = 0$, let E be a nonempty open set in the complex plane. Suppose that $g(sx, \bar{s}h(x))$ is convergent for each $s \in E$. Then g is convergent and h is convergent.*

PROOF. Pick $s = c \exp(i\alpha) \neq 0$, where $c =| s |$, $s \in E$. We fix α and since E is an open set, there is a non-empty interval (a, b) so for any $c \in [a, b]$, $c \exp(i\alpha) \in E$. Replacing x with $x_1 \exp(-i\alpha)$ we get $g(sx, \bar{s}h(x)) = g(cx_1, ch_1(x_1))$. So, $g(cx_1, ch_1(x_1))$ converges for all $c \in [a, b]$. Using now Theorem 1.2 from [**FM**] we see that $h_1(x)$ converges, implying the convergence of $h(x)$ as well. Now if $h(x)$ is not a monomial of the form $a_1 x$, we apply Theorem 1.1 from [**FM**] to conclude that $g(x, y)$ is convergent as well. For the exceptional case $h(x) = a_1 x$ we need a different selection for the range of $s \in E$. Fix a number $l > 0$ and a non-zero interval $[\beta_1, \beta_2]$, such that $s = l \exp(i\beta) \in E$ for all $\beta \in [\beta_1, \beta_2]$. Then $g(sx, \bar{s}h(x)) = g(s_1 x_1, s_1^{-1} h_1(x_1))$, where $s_1 = \exp(i\beta)$, $x_1 = lx$, and $h_1(x_1) = a_1 x_1$. Applying again Theorem 1.1 from [**FM**] we prove the convergence of $g(x, y)$ in this case as well. $\square$

THEOREM 1.8. *Let $f(z)$ be a continuous function defined on an open connected set Ω in the complex plane $\mathbb{C}$ containing the origin, let $\Gamma := \{t + i\phi(t) : 0 \le t < b\}$ be a continuous curve with $\phi(0) = 0$, and let E be a connected open set in the complex plane. Suppose f and ϕ have a Taylor series at 0, that ϕ is analytic on $[0, \lambda)$ for no $\lambda > 0$, and that for each $s \in E$ with $s \neq 0$ there exists a holomorphic function F_s defined in an open set U_s containing $s^{-1}\Omega \cap \Gamma$ such that $f(sz) = F_s(z)$ for $z \in s^{-1}\Omega \cap \Gamma$. Then f is holomorphic in the open set $\Lambda := \cup_{s\in E}\Gamma_s$, where Γ_s is the connected component of $s\Gamma$ containing the origin.*

PROOF. Let $g(z, \overline{z})$ and $h(t)$ be the Taylor series at 0 of f and ϕ respectively. Let $\gamma(t) = t + i\phi(t)$ and $\omega(t) = t + ih(t)$. Consider an $s \in E$ with $s \neq 0$. Since

$$f(s\gamma(t)) = F_s(\gamma(t)) \tag{1}$$

for $t \in [0, b]$, we see that

$$g(s\omega(t), \overline{s}(2t - \omega(t))) = F_s(\omega(t))$$

as elements in $\mathbb{C}[[t]]$. Let $\psi(t) \in \mathbb{C}[[t]]$ be the inverse of $\omega(t)$ so that $\omega(\psi(t)) = t$. Then

$$g(st, \overline{s}(2\psi(t) - t)) = F_s(t). \tag{2}$$

We claim that $g_w(z, w) \equiv 0$. Suppose that is not the case.

By Lemma 1.7, $g(z, w)$ and $2\psi(t)-t$ are convergent. So $\psi(t)$ is convergent and $\omega(t)$ is convergent. There is a positive number r such that the disk $D(0, r) \subset \Omega$, $g(z, w)$ represents a holomorphic function in $D(0, r) \times D(0, r)$, and $\psi(z)$, $\omega(z)$ represent holomorphic functions in $D(0, r)$. By (1) and (2),

$$f(s\gamma(t)) = g(s\gamma(t), \overline{s}(2\psi(\gamma(t)) - \gamma(t))), \tag{3}$$

provided

$$t \in [0, b], \quad s \in E, \quad |s\gamma(t)| < r, \quad |s(2\psi(\gamma(t)) - \gamma(t))| < r. \tag{4}$$

We choose an open disc $U := D(a, v) \subset E$ with $0 < v < |a|/2$, and a positive number $c < r$, such that (4) and (3) are satisfied for $t \in [0, c]$ and $s \in U$. Fix a $t_0 \in (0, c)$. There is an $s_0 \in U$ such that $g_w(s_0\gamma(t_0), w) \not\equiv 0$. Let $z_0 = s_0\gamma(t_0)$. By (3) we have, for all t sufficiently close to t_0, that

$$f(z_0) = f\left(\frac{z_0}{\gamma(t)} \cdot \gamma(t)\right) = g\left(z_0, \overline{z}_0 \cdot \frac{2\psi(\gamma(t)) - \gamma(t)}{\overline{\gamma(t)}}\right). \tag{5}$$

Since $g_w(z_0, w) \not\equiv 0$, the set $\{w : g(z_0, w) = f(z_0)\}$ is discrete. Hence the function

$$p(t) := \frac{2\psi(\gamma(t)) - \gamma(t)}{\overline{\gamma(t)}}$$

is constant for all t sufficiently close to t_0. It follows that $p(t)$ is constant on $(0, c)$. So there is a complex constant C such that

$$2\psi(\gamma(t)) - \gamma(t) = C\overline{\gamma(t)}, \quad 0 \le t < c. \tag{6}$$

Taking derivatives at 0, we obtain $2\psi'(0)\gamma'(0) - \gamma'(0) = C\overline{\gamma'(0)}$, which forces $C = 1$, since $\psi'(0)\gamma'(0) = 1$ and $2 - \gamma'(0) = \overline{\gamma'(0)}$. From (6) and $\gamma(t) + \overline{\gamma(t)} = 2t$ it follows that

$$\psi(\gamma(t)) = t, \quad 0 \le t < c. \tag{7}$$

The above equation implies that $\gamma(t) = \omega(t)$ for $0 \le t < c$, contradicting the hypothesis that γ is analytic on $[0, \lambda)$ for no $\lambda > 0$. Therefore $g_w(z, w) \equiv 0$.

Now $g(z, w)$ does not depend on w, so $g \in \mathbb{C}[[z]]$, and (2) becomes

$$g(st) = F_s(t),$$

which implies that g is convergent. Hence g represents a holomorphic function in $D(0, r)$ for some $r > 0$. It follows from (1) that

$$f(s\gamma(t))) = g(s\gamma(t)), \tag{8}$$

provided $|s\gamma(t)| < r$. Therefore f is holomorphic in the open set $Q := D(0, r) \cap E\Gamma$.

We now prove that f is holomorphic in Λ. If $0 \in \Lambda$, then $0 \in Q$, and we already know that f is holomphic in a neighborhood of 0. Fix a point $p \in \Lambda$, $p \neq 0$. Then $p \in \Gamma_s$ for some $s \in E$, $s \neq 0$, and $q := p/s$ is a point of Γ, so $q = t_0 + i\phi(t_0)$ for some $t_0 \in (0, b)$. Since Γ_s is the connected component of $s\Gamma$ containing the origin, we see that there is a $\delta \in (0, b - t_0)$ so that $\Gamma' := \{t + i\phi(t) : 0 \le t \le t_0 + \delta\}$ satisfies that $s\Gamma' \subset \Omega$. There is a holomorphic function F_s defined in an open set $U_s \subset s^{-1}\Omega$ containing Γ' such that $f(sz) = F_s(z)$ for $z \in \Gamma'$. Let $V_s = sU_s$ and $G_s(z) = F_s(z/s)$. Then $s\Gamma' \subset V_s \subset \Omega$, G_s is defined on V_s, and $G_s(z) = f(z)$ for $z \in s\Gamma'$. Choose an $\epsilon > 0$ such that the disc $D := D(s, \epsilon)$ is contained in E, D

does not contain the origin, and $D\Gamma' \subset V_s$. We now prove that $f = G_s$ in $D\Gamma'$, hence f is holomorphic in $D\Gamma'$.

Consider a $u \in D$. There is a holomorphic function G_u defined in an open set $V_u \subset \Omega$ containing $u\Gamma'$ such that $f(z) = F_u(z)$ for $z \in u\Gamma'$. Since $V_s \cap V_u$ contains a neighborhood of the origin, $D(0, r) \cap V_s \cap V_u$ is non-empty. By the uniqueness theorem, in the open set $D(0, r) \cap V_s \cap V_u$, the three holomorphic functions g, G_s and G_u are equal. Thus G_s and G_u are equal in the connected component of $V_s \cap V_u$ containing $u\Gamma'$. It follows that $f = G_s$ on $u\Gamma'$ for each $u \in D$. Thus $f = G_s$ in $D\Gamma'$, and f is holomorphic in $D\Gamma'$, which is a neighborhood of p. Therefore f is holomorphic in Λ. □

Proof of Theorem 1.6. Denote $\Gamma = S = \{t + i\phi(t) : 0 \leq t \leq b\}$. Let $\Omega \subset \mathbb{C}$ be a domain, $f \in C^\infty(\Omega)$, $z_0 \in \Omega$. Without any loss of generality we may assume that $0 \in \Omega$, and (since one can use translations to move Γ) $z_0 = 0$. We take $E = \mathbb{U}$ and consider $L_s(\Gamma) = s\Gamma$ for $s \in E$. There is a holomorphic function $G_s(z)$ in the neighborhood of $L_s(\Gamma) \cap \Omega$ that coincides with f on that intersection. Consider $F_s(z) = G_s(sz)$. Then $f(sz) = F_s(z)$ on $s^{-1}\Omega \cap \Gamma$. By Theorem 1.8, $\partial f/\partial \overline{z} = 0$ at z_0. So, $\partial f/\partial \overline{z} = 0$ everywhere on Ω, and therefore f is holomorphic on Ω. □

1.3. Case 3: $\dim(S) < 1$.

In this case an interesting situation to examine is when S is a bounded sequence (z_n) (and therefore $\dim(S) = 0$). By using Corollary 1.3 one can easily construct sequences with one limit point that have Hartogs property with respect to $\mathbb{T}$. On the other hand if one takes a sequence that is located on an analytic curve, and has a limit point on that curve, such a sequence will not have a Hartogs property even with respect to the entire group $\mathcal{L}$. So, a natural hypothesis here is that in order for (z_n) to have Hartogs property with respect to $\mathcal{L}$ there must be no analytic curve Γ that $z_n \in \Gamma$ for large n. However this is not true, and one can construct a counterexample. A similar statement we prove below holds, but it requires a change in the definition of a holomorphic function on a sequence.

We will say that a function f on (z_n) is holomorphic if it can be extended as a holomorphic function to a *connected* open neighborhood of (z_n).

If a set $S = (z_n)$ has Hartogs property with respect to $\mathbb{L}$ and with the above definition of a holomorphic extension, we will denote that by $S \in \hat{H}_0(\mathbb{L})$.

We need another definition for the theorem below.

Consider a sequence (z_n) of complex numbers. Write $z_n = t_n + iu_n$. We assume that $t_n > 0$ and $\lim z_n = 0$. The sequence (z_n) is said to have a Taylor series at 0 if there is an $h(z) = \sum_j b_j z^j \in \mathbb{C}[[z]]$ such that

$$u_n - \sum_{j\le k} b_j t_n^j = o\big(t_n^k\big), \quad n \to \infty,$$

for each nonnegative integer k. Note that h has real coefficients and $b_0 = 0$. We say that (z_n) eventually lies on an analytic curve if there exists a curve $\Gamma = \{(x, y) : y = \varphi(x)\}$, with φ – real analytic function and $\exists N$ such that $z_n \in \Gamma$ for $n \ge N$.

THEOREM 1.9. *Let $S = (z_n)$, $z_1 = 0$, and (z_n) has a Taylor series at 0 of the form $z_n \sim t_n + ih(t_n)$, where t_n are positive real numbers, and $h \in \mathbb{C}[[t]]$ has real coefficients. Suppose that (z_n) does not eventually lie on any analytic curve. Then $S \in \hat{H}_0(\mathbb{T}_2)$, where $\mathbb{T}_2 = \{\sigma \circ \tau : \sigma \in \mathbb{T}, \tau \in C^*\}$.*

This theorem is a corollary of the following

THEOREM 1.10. *Let $f(z)$ be a continuous function defined on the unit disc $D(0, 1)$ in $\mathbb{C}$ that has a Taylor series at 0 and let (z_n) be a sequence with $z_1 = 0$ that has a Taylor series at 0 of the form $z_n \sim t_n + ih(t_n)$, where t_n are positive real numbers, and $h \in \mathbb{C}[[t]]$ has real coefficients. Suppose that (z_n) does not eventually lie on an analytic curve, and that for each $s \in \mathbb{C}$ with $s \neq 0$ there is a holomorphic function $F_s(z)$ defined on a connected neighborhood U_s of the set $Q_s := s^{-1}D(0, 1) \cap \{z_n\}$ such that $f(sz) = F_s(z)$ for $z \in Q_s$. Then f is holomorphic in a neighborhood of the origin.*

PROOF. Let $g(z, \overline{z})$ be the Taylor series of f at 0. Let $\omega(t) = t+ih(t)$. Then

$$g(s\omega(t), \overline{s}(2t - \omega(t))) = F_s(\omega(t)) \tag{9}$$

as elements in $\mathbb{C}[[t]]$. Let $\psi(t) \in \mathbb{C}[[t]]$ be the inverse of $\omega(t)$.

We claim that $g_w(z, w) \equiv 0$. Suppose that is not the case.

Similar to the proof of Theorem 1.8, we see that $h(t)$, $\omega(t)$, $\psi(t)$ are convergent, and

$$g(st, \overline{s}(2\psi(t) - t)) = F_s(t). \tag{10}$$

There is a positive number r such that $D(0, r) \subset \Omega$, $g(z, w)$ represents a holomorphic function in $D(0, r) \times D(0, r)$, and $\psi(z)$, $\omega(z)$ represent holomorphic functions in $D(0, r)$. It follows that

$$f(sz_n) = g(sz_n, \overline{s}(2\psi(z_n) - z_n)), \tag{11}$$

provided

$$|z_n| < r, \quad |sz_n| < r, \quad |s(2\psi(z_n) - z_n| < r. \tag{12}$$

Fix $z_0 \in D(0,r)$ with $z_0 \neq 0$ such that $g_w(z_0, w) \not\equiv 0$. Then the set $\{w : g(z_0, w) = f(z_0)\}$ is discrete. Equation (11) implies that

$$f(z_0) = f\left(\frac{z_0}{z_n} \cdot z_n\right) = g\left(z_0, \overline{z}_0 \cdot \frac{2\psi(z_n) - z_n}{\overline{z}_n}\right) = g(z_0, w_n), \tag{13}$$

where $w_n := \overline{z}_0(2\psi(z_n) - z_n)/\overline{z}_n$. Since the set $\{w : g(z_0, w) = f(z_0)\}$ is discrete, and since $\lim w_n = \overline{z}_0$, we see that there is a positive integer K such that $w_n = \overline{z}_0$ for $n \geq K$. Recall that $z_n \sim t_n + ih(t_n)$. The equation $w_n = \overline{z}_0$ is equivalent to $\psi(z_n) = t_n$, or $z_n = \omega(t_n) = t_n + ih(t_n)$, contradicting the hypothesis that (z_n) does not eventually lie on an analytic curve. Therefore $g_w(z, w) \equiv 0$.

Now $g(z, w)$ does not depend on w, so $g \in \mathbb{C}[[z]]$, and (10) becomes $g(st) = F_s(t)$, which clearly implies that g is convergent. Hence g represents a holomorphic function in $D(0, r)$ for some $r > 0$. Thus $f(sz_n) = g(sz_n)$, provided $|sz_n| < r$. Therefore f is holomorphic in $D(0, r)$. □

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